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Characterization of Non-Linear Bi-Skew Jordan n-Derivations on Prime $*$-Algebras

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Abstract: Let $\mathfrak{A}$ be a prime $*$-algebra. A product defined as $U \bullet V = UV^* + VU^*$ for any $U, V \in \mathfrak{A}$, is called a bi-skew Jordan product. A map $\xi : \mathfrak{A} \to \mathfrak{A}$, defined as $\xi\left(p_n\left(U_1, U_2, \ldots, U_n\right)\right) = \sum_{k=1}^{n} p_n\left(U_1, U_2, ..., U_{k-1}, \xi(U_k), U_{k+1}, \ldots, U_n\right)$ for all $U_1, U_2, ..., U_n \in \mathfrak{A}$, is called a non-linear bi-skew Jordan n-derivation. In this article, it is shown that ξ is an additive $*$-derivation.

Keywords: $*$-derivation; bi-skew Jordan n-derivation; prime $*$-algebras

MSC: 47C10; 16W25

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Citation: Ali, A.; Alali, A.S.; Tasleem, M. Characterization of Non-Linear Bi-Skew Jordan n-Derivations on Prime $*$-Algebras. *Axioms* **2023**, *12*, 753. https://doi.org/10.3390/axioms12080753

Academic Editor: Florin Felix Nichita

Received: 8 June 2023
Revised: 25 July 2023
Accepted: 27 July 2023
Published: 30 July 2023

1. Introduction

Let $\mathfrak{A}$ be an associative $*$-algebra. Recall that a map $\xi : \mathfrak{A} \to \mathfrak{A}$, is called an additive derivation if $\xi(U + V) = \xi(U) + \xi(V)$ and $\xi(UV) = \xi(U)V + U\xi(V)$ for all $U, V \in \mathfrak{A}$. Let $U * V = UV + VU^*$ and $[U, V]_* = UV - VU^*$ denote the skew Jordan product and skew Lie product of elements $U, V \in \mathfrak{A}$, respectively. These products are also called $*$-Jordan product and $*$-Lie product, respectively. The difficulty of the representability of quadratic functionals by sesqui-linear functionals on left-modules over $*$-algebras is greatly impacted by the existence of such Jordan bracket-based products in regard to the so-called Jordan $*$-derivations (see [1–3]). We say a map $\xi : \mathfrak{A} \to \mathfrak{A}$, without considering the linearity assumption, is called a multiplicative skew (or $*$)-Jordan derivation if

$$\xi(U * V) = \xi(U) * V + U * \xi(V)$$

for all $U, V \in \mathfrak{A}$. Furthermore, without the linearity assumption, a map $\xi : \mathfrak{A} \to \mathfrak{A}$ is called a multiplicative skew or $*$-Jordan triple derivation if it satisfies

$$\xi(U * V * W) = \xi(U) * V * W + U * \xi(V) * W + U * V * \xi(W)$$

for all $U, V, W \in \mathfrak{A}$. A map $\xi : \mathfrak{A} \to \mathfrak{A}$ is said to be an additive $*$-derivation if it is an additive derivation and satisfies $\xi(U^*) = \xi(U)^*$ for all $U \in \mathfrak{A}$. Many authors investigated the structure of skew Jordan derivations and skew Jordan triple derivations on different algebras, see, e.g., [2–6]. For instance, Taghavi et al. [5] showed that a non-linear $*$-Jordan derivation on a factor von Neumann algebra is an additive $*$-derivation. Zhao and Li [6] proved that every non-linear $*$-Jordan triple derivation on a von Neumann algebra with no central summands of type I_1 is an additive $*$-derivation. A lot of work was also carried out by considering Lie product ($[U, V] = UV - VU$) and $*$-Lie product ($[U, V]_* = UV - VU^*$) (see [7–16]). In [15], Yu and Zhang proved that every non-linear Lie derivation on triangular algebras has the standard form, i.e., it is a sum of an additive derivation and a central valued map. Furthermore, the authors of [7,8], respectively, established that a non-linear Lie triple derivation on triangular algebras and a non-linear Lie type derivation on von

Neumann algebras have the standard form. The structure of non-linear $*$-Lie derivation on factor von Neumann algebra was also explored by Yu and Zhang [16], and they proved that such a map is an additive $*$-derivation. On similar grounds, the characterization of non-linear skew Lie triple derivations on factor von Neumann algebras [11], non-linear $*$-Lie derivations on standard operator algebras [9], non-linear $*$-Lie-type derivations on von Neumann algebras [12] and non-linear $*$-Lie type derivations on standard operator algebras [13] is performed, and they are proven to be additive $*$-derivations.

Let us recall the definition of a prime $*$-algebra. A prime $*$-algebra is an algebra $\mathfrak{A}$ with involution $*$, in which $U\mathfrak{A}V$ equates to (0), gives either $U = 0$ or $V = 0$. The class of prime $*$-algebras is very important and has numerous applications in various disciplines. In the context of operator theory and quantum mechanics, prime $*$-algebras are used to study the behavior of operators on Hilbert spaces and provide insights into the nature of physical observables and symmetries in quantum systems. Prime $*$-algebras are a larger class containing factor von Neumann algebras and standard operator algebras. Therefore, it would be of great importance to characterize a map on prime $*$-algebras. In recent years, some mathematicians focus to explore the structure of $*$-Jordan type derivations on prime $*$-algebras, see [17,18]. Inspired by skew Jordan product, very recently, Kong and Li [19] introduced a new product, namely, bi-skew Jordan product, as $U \bullet V = UV^* + VU^*$ for all $U, V \in \mathfrak{A}$. They proved that every non-linear/multiplicative bi-skew Jordan derivation, i.e., a map ξ from $\mathfrak{A}$ to itself, (where $\mathfrak{A}$ is a prime $*$-algebra) satisfying $\xi(U \bullet V) = \xi(U) \bullet V + U \bullet \xi(V)$ for all $U, V \in \mathfrak{A}$, is an additive $*$-derivation on $\mathfrak{A}$ provided $dim(\mathfrak{A}) \geq 2$. Later, Khan and Alhazmi [20] extended the results of Kong and Li [19] to multiplicative bi-skew Jordan triple derivation and proved that every multiplicative bi-skew Jordan triple derivation, i.e., a map $\xi : \mathfrak{A} \to \mathfrak{A}$ satisfying $\xi(U \bullet V \bullet W) = \xi(U) \bullet V \bullet W + U \bullet \xi(V) \bullet W + U \bullet V \bullet \xi(W)$ for all $U, V, W \in \mathfrak{A}$, is an additive $*$-derivation. We can naturally develop them further when bi-skew Jordan derivations and bi-skew Jordan triple derivations are taken into account. Let's assume that $n \geq 2$ is a fixed positive integer and see the list of polynomials with involution.

$$\begin{aligned}
p_1(U_1) &= U_1, \\
p_2(U_1, U_2) &= p_1(U_1) \bullet U_2 = U_1 \bullet U_2 = U_1 U_2^* + U_2 U_1^*, \\
p_3(U_1, U_2, U_3) &= p_2(U_1, U_2) \bullet U_3 = U_1 \bullet U_2 \bullet U_3, \\
p_4(U_1, U_2, U_3, U_4) &= p_3(U_1, U_2, U_3) \bullet U_4 = U_1 \bullet U_2 \bullet U_3 \bullet U_4 \\
&\ldots, \\
p_n(U_1, U_2, U_3 \ldots, U_n) &= p_{n-1}(U_1, U_2, \ldots, U_{n-1}) \bullet U_n \\
&= U_1 \bullet U_2 \bullet U_3 \bullet \ldots U_{n-1} \bullet U_n
\end{aligned}$$

Accordingly, a multiplicative bi-skew Jordan n-derivation is a mapping $\xi : \mathfrak{A} \to \mathfrak{A}$, satisfying the condition

$$\xi\Big(p_n\Big(U_1, U_2, \ldots, U_n\Big)\Big) = \sum_{k=1}^{n} p_n\Big(U_1, \ldots, U_{k-1}, \xi(U_k), U_{k+1}, \ldots, U_n\Big),$$

for all $U_1, U_2, \ldots, U_n \in \mathfrak{A}$. This is the best way to define multiplicative bi-skew Jordan n-derivations, using this notion. Every multiplicative bi-skew Jordan derivation is a multiplicative bi-skew Jordan 2-derivation according to the definition, and every multiplicative bi-skew Jordan triple derivation is a multiplicative bi-skew Jordan 3-derivation. One can easily check that every multiplicative bi-skew Jordan derivation on any $*$-algebra is a multiplicative bi-skew Jordan triple derivation but the converse is not true, in general. Multiplicative bi-skew Jordan-type derivations refer to the multiplicative bi-skew Jordan 2-, multiplicative bi-skew Jordan 3- and multiplicative bi-skew Jordan n-derivations. Inspired by the above mentioned work in this article, we focus our study on multiplicative bi-skew Jordan type derivations on prime $*$-algebras.

2. Preliminaries

We need to give some preliminaries in order to state and prove our main theorem. Throughout the work, $\mathfrak{A}$ represents a prime $*$-algebra and $\mathbb{C}$ denotes the field of complex numbers. Let H be a complex Hilbert space. We denote by $\mathcal{B}(H)$ the algebra of all bounded linear operators on H. An operator $P \in \mathcal{B}(H)$ is called a projection provided $P^* = P$ and $P^2 = P$. Any operator $U \in \mathcal{B}(H)$ can be expressed as $U = \Re U + i\Im U$, where i is the imaginary unit, $\Re U = \frac{U+U^*}{2}$ and $\Im U = \frac{U-U^*}{2i}$. Note that both $\Re U$ and $\Im U$ are self-adjoint. Let $P = P_1 \in \mathfrak{A}$ be a projection. Write $P_2 = I - P_1$ and $\mathfrak{A}_{ij} = P_i\mathfrak{A}P_j$. Then, $\mathfrak{A} = \mathfrak{A}_{11} + \mathfrak{A}_{12} + \mathfrak{A}_{21} + \mathfrak{A}_{22}$. Let $\mathcal{M} = \{M \in \mathfrak{A} | \ M^* = M\}$ and $\mathcal{N} = \{N \in \mathfrak{A} | \ N^* = -N\}$, $\mathcal{M}_{12} = \{P_1MP_2 + P_2MP_1 | \ M \in \mathcal{M}\}$ and $\mathcal{M}_{ii} = P_i\mathcal{M}P_i$ $(i = 1, 2)$. Thus, for every $M \in \mathcal{M}$, $M = M_{11} + M_{12} + M_{22}$ for every $M_{12} \in \mathcal{M}_{12}$ and $M_{ii} \in \mathcal{M}_{ii}$ $(i = 1, 2)$.

In proving our main theorem, we frequently use the following lemma and remark.

Lemma 1. *For any* $U \in \mathfrak{A}$, $p_n\left(U, \frac{1}{2}I, \frac{1}{2}I, \ldots, \frac{1}{2}I\right) = \frac{1}{2}(U + U^*)$.

Proof. By doing the recursive calculation, we obtain

$$\begin{aligned} p_n\left(U, \frac{1}{2}I, \frac{1}{2}I, \ldots, \frac{1}{2}I\right) &= p_{n-1}\left(\frac{1}{2}(U+U^*), \frac{1}{2}I, \ldots, \frac{1}{2}I\right) \\ &= p_{n-2}\left(\frac{1}{2}(U+U^*), \frac{1}{2}I, \ldots, \frac{1}{2}I\right) \\ &= \ldots = p_2\left(\frac{1}{2}(U+U^*), \frac{1}{2}I\right) \\ &= \frac{1}{2}(U+U^*). \end{aligned}$$

□

Remark 1. *If* $U \in \mathcal{M}$, *i.e.,* $U^* = U$, *then*

$$p_n\left(U, \frac{1}{2}I, \frac{1}{2}I, \ldots, \frac{1}{2}I\right) = U.$$

3. Main Result

Theorem 1. *Let* $\mathfrak{A}$ *be a prime* $*$*-algebra with* $dim(\mathfrak{A}) \geq 2$, *containing the identity element* I *and a non-trivial projection* P. *A map* $\xi : \mathfrak{A} \to \mathfrak{A}$ *is a multiplicative bi-skew Jordan-type derivation if and only if it is an additive* $*$*-derivation.*

Only the necessity needs to be established. The proof of the theorem is demonstrated in a series of claims, which are as follows.

Claim 1. $\xi(0) = 0$.

Proof. It follows that

$$\begin{aligned} \xi(0) &= \xi\Big(p_n\Big(0, 0, \ldots, 0\Big)\Big) = p_n\Big(\xi(0), 0, \ldots, 0, \ldots, 0\Big) + p_n\Big(0, \xi(0), \ldots, 0, \ldots, 0\Big) \\ &+ \ldots + p_n\Big(0, 0, \ldots, \xi(0), \ldots, 0\Big) + \ldots + p_n\Big(0, 0, \ldots, 0, \ldots, \xi(0)\Big) \\ &= 0. \end{aligned}$$

□

Claim 2. $\xi(M)^* = \xi(M)$ *for every* $M \in \mathcal{M}$.

Proof. For any $M \in \mathcal{M}$, observe that $M = p_n\Big(M, \frac{1}{2}I, \dots, \frac{1}{2}I\Big)$. Thus,

$$\begin{aligned}
\xi(M) &= \xi\Big(p_n\Big(M, \frac{1}{2}I, \dots, \frac{1}{2}I\Big)\Big) \\
&= p_n\Big(\xi(M), \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_n\Big(M, \xi(\frac{1}{2}I), \dots, \frac{1}{2}I\Big) \\
&+ \ldots + p_n\Big(M, \frac{1}{2}I, \dots, \xi(\frac{1}{2}I)\Big) \\
&= p_{n-1}\Big(\frac{1}{2}(\xi(M) + \xi(M)^*), \frac{1}{2}I, \dots, \frac{1}{2}I\Big) \\
&+ p_{n-1}\Big(M\xi(\frac{1}{2}I)^* + \xi(\frac{1}{2}I)M, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) \\
&+ \ldots + p_{n-1}\Big(M, \frac{1}{2}I, \dots, \xi(\frac{1}{2}I)\Big) \\
&= \frac{1}{2}\Big(\xi(M) + \xi(M)^*\Big) + (n-1)\Big(M\xi(\frac{1}{2}I)^* + \xi(\frac{1}{2}I)M\Big).
\end{aligned}$$

This implies that

$$\xi(M) = \xi(M)^* + 2(n-1)\Big(M\xi(\frac{1}{2}I)^* + \xi(\frac{1}{2}I)M\Big). \tag{1}$$

It follows that

$$\xi(M)^* = \xi(M) + 2(n-1)\Big(\xi(\frac{1}{2}I)M + M\xi(\frac{1}{2}I)^*\Big). \tag{2}$$

Combining (1) and (2), we obtain $\xi(M)^* = \xi(M)$. This completes the proof. □

Claim 3. *For any $U_{11} \in \mathcal{M}_{11}, V_{12} \in \mathcal{M}_{12}$ and $W_{22} \in \mathcal{M}_{22}$, we have*
(i) $\xi(U_{11} + V_{12}) = \xi(U_{11}) + \xi(V_{12})$;
(ii) $\xi(V_{12} + W_{22}) = \xi(V_{12}) + \xi(W_{22})$.

Proof. (i) Let $T = \xi(U_{11} + V_{12}) - \xi(U_{11}) - \xi(V_{12})$. It is obvious from Claim 2 that $T \in \mathcal{M}$, i.e., $T^* = T$. Our aim is to show that $T = T_{11} + T_{12} + T_{22} = 0$. In view of $p_n\Big(P_2, U_{11}, P_1, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) = 0$ and Claim 1, we have

$$\begin{aligned}
&\xi\Big(p_n\Big(P_2, U_{11} + V_{12}, P_1, \frac{1}{2}I, \dots, \frac{1}{2}I\Big)\Big) \\
= &\xi\Big(p_n\Big(P_2, U_{11}, P_1, \frac{1}{2}I, \dots, \frac{1}{2}I\Big)\Big) + \xi\Big(p_n\Big(P_2, V_{12}, P_1, \frac{1}{2}I, \dots, \frac{1}{2}I\Big)\Big) \\
= &p_n\Big(\xi(P_2), U_{11}, P_1, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_n\Big(P_2, \xi(U_{11}), P_1, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) \\
+ &p_n\Big(P_2, U_{11}, \xi(P_1), \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_n\Big(P_2, U_{11}, P_1, \xi(\frac{1}{2}I), \dots, \frac{1}{2}I\Big) \\
+ &\ldots + p_n\Big(P_2, U_{11}, P_1, \frac{1}{2}I, \dots, \xi(\frac{1}{2}I)\Big) + p_n\Big(\xi(P_2), V_{12}, P_1, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) \\
+ &p_n\Big(P_2, \xi(V_{12}), P_1, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_n\Big(P_2, V_{12}, \xi(P_1), \frac{1}{2}I, \dots, \frac{1}{2}I\Big) \\
+ &p_n\Big(P_2, V_{12}, P_1, \xi(\frac{1}{2}I), \dots, \frac{1}{2}I\Big) + \ldots + p_n\Big(P_2, V_{12}, P_1, \frac{1}{2}I, \dots, \xi(\frac{1}{2}I)\Big) \\
= &p_n\Big(\xi(P_2), U_{11} + V_{12}, P_1, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_n\Big(P_2, \xi(U_{11}) + \xi(V_{12}), P_1, \frac{1}{2}I, \dots, \frac{1}{2}I\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ \quad p_n\Big(P_2, U_{11}+V_{12}, \xi(P_1), \frac{1}{2}I, ...\frac{1}{2}I\Big) + p_n\Big(P_2, U_{11}+V_{12}, P_1, \xi(\frac{1}{2}I), \dots, \frac{1}{2}I\Big) \\
&+ \quad ... + p_n\Big(P_2, U_{11}+V_{12}, P_1, \frac{1}{2}I, \dots, \xi(\frac{1}{2}I)\Big).
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
&\quad \xi\Big(p_n\Big(P_2, U_{11}+V_{12}, P_1, \frac{1}{2}I, \dots, \frac{1}{2}I\Big)\Big) \\
&= p_n\Big(\xi(P_2), U_{11}+V_{12}, P_1, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_n\Big(P_2, \xi(U_{11}+V_{12}), P_1, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) \\
&+ p_n\Big(P_2, U_{11}+V_{12}, \xi(P_1), \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_n\Big(P_2, U_{11}+V_{12}, P_1, \xi(\frac{1}{2}I), \dots, \frac{1}{2}I\Big) \\
&+ ... + p_n\Big(P_2, U_{11}+V_{12}, P_1, \frac{1}{2}I, \dots, \xi(\frac{1}{2}I)\Big).
\end{aligned}
$$

From the last two expressions, we conclude that $p_n\Big(P_2, T, P_1, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) = 0$. Using the primeness of $\mathfrak{A}$, we obtain $T_{12} = 0$. Furthermore, since $p_n\Big(P_2 - P_1, V_{12}, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) = 0$, we can write

$$
\begin{aligned}
&\quad p_n\Big(\xi(P_2-P_1), U_{11}+V_{12}, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_n\Big(P_2-P_1, \xi(U_{11}+V_{12}), \frac{1}{2}I, \dots, \frac{1}{2}I\Big) \\
&+ p_n\Big(P_2-P_1, U_{11}+V_{12}, \xi(\frac{1}{2}I), \dots, \frac{1}{2}I\Big) + ... + p_n\Big(P_2-P_1, U_{11}+V_{12}, \frac{1}{2}I, \dots, \xi(\frac{1}{2}I)\Big) \\
&= \xi\Big(p_n\Big(P_2-P_1, U_{11}+V_{12}, \frac{1}{2}I, \dots, \frac{1}{2}I\Big)\Big) \\
&= \xi\Big(p_n\Big(P_2-P_1, U_{11}, \frac{1}{2}I, \dots, \frac{1}{2}I\Big)\Big) + \xi\Big(p_n\Big(P_2-P_1, V_{12}, \frac{1}{2}I, \dots, \frac{1}{2}I\Big)\Big) \\
&= p_n\Big(\xi(P_2-P_1), U_{11}, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_n\Big(P_2-P_1, \xi(U_{11}), \frac{1}{2}I, \dots, \frac{1}{2}I\Big) \\
&+ p_n\Big(P_2-P_1, U_{11}, \xi(\frac{1}{2}I), \dots, \frac{1}{2}I\Big) + ... + p_n\Big(P_2-P_1, U_{11}, \frac{1}{2}I, \dots, \xi(\frac{1}{2}I)\Big) \\
&+ p_n\Big(\xi(P_2-P_1), V_{12}, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_n\Big(P_2-P_1, \xi(V_{12}), \frac{1}{2}I, \dots, \frac{1}{2}I\Big) \\
&+ p_n\Big(P_2-P_1, V_{12}, \xi(\frac{1}{2}I), \dots, \frac{1}{2}I\Big) + ... + p_n\Big(P_2-P_1, V_{12}, \frac{1}{2}I, \dots, \xi(\frac{1}{2}I)\Big) \\
&= p_n\Big(\xi(P_2-P_1), U_{11}+V_{12}, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_n\Big(P_2-P_1, \xi(U_{11})+\xi(V_{12}), \frac{1}{2}I, \dots, \frac{1}{2}I\Big) \\
&+ p_n\Big(P_2-P_1, U_{11}+V_{12}, \xi(\frac{1}{2}I), \dots, \frac{1}{2}I\Big) + ... + p_n\Big(P_2-P_1, U_{11}+V_{12}, \frac{1}{2}I, \dots, \xi(\frac{1}{2}I)\Big).
\end{aligned}
$$

From this, we obtain $p_n(P_2 - P_1, T, \frac{1}{2}I, \dots, \frac{1}{2}I) = 0$. Using Claim 2, we obtain $P_2T + TP_2 - P_1T - TP_1 = 0$. Multiplying this equation by P_1 and P_2, respectively, on both sides, we obtain $T_{11} = T_{22} = 0$. Therefore, $T = 0$. In a similar manner, we can establish (ii) . Thereby the proof is completed. □

Claim 4. *For any $U_{11} \in \mathcal{M}_{11}, V_{12} \in \mathcal{M}_{12}$ and $W_{22} \in \mathcal{M}_{22}$, we have*

$$\xi(U_{11} + V_{12} + W_{22}) = \xi(U_{11}) + \xi(V_{12}) + \xi(W_{22}).$$

Proof. We show that $T = \xi(U_{11} + V_{12} + W_{22}) - \xi(U_{11}) - \xi(V_{12}) - \xi(W_{22}) = 0$. In view of Claim 3 and $p_n(P_1, W_{22}, P_1, \frac{1}{2}I, \dots, \frac{1}{2}I) = 0$, we have

$$\xi\Big(p_n\Big(P_1, U_{11} + V_{12} + W_{22}, P_1, \frac{1}{2}I, \dots, \frac{1}{2}I\Big)\Big)$$

$$
\begin{aligned}
&= \xi\Big(p_n\Big(P_1, U_{11}+V_{12}, P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big)\Big) + \xi\Big(p_n\Big(P_1, W_{22}, P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big)\Big) \\
&= p_n\Big(\xi(P_1), U_{11}+V_{12}, P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) + p_n\Big(P_1, \xi(U_{11}+V_{12}), P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) \\
&+ p_n\Big(P_1, U_{11}+V_{12}, \xi(P_1), \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) + p_n\Big(P_1, U_{11}+V_{12}, P_1, \xi(\frac{1}{2}I), \ldots, \frac{1}{2}I\Big) \\
&+ \ldots + p_n\Big(P_1, U_{11}+V_{12}, P_1, \frac{1}{2}I, \ldots, \xi(\frac{1}{2}I)\Big) + p_n\Big(\xi(P_1), W_{22}, P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) \\
&+ p_n\Big(P_1, \xi(W_{22}), P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) + p_n\Big(P_1, W_{22}, \xi(P_1), \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) \\
&+ p_n\Big(P_1, W_{22}, P_1, \xi(\frac{1}{2}I), \ldots, \frac{1}{2}I\Big) + \ldots + p_n\Big(P_1, W_{22}, P_1, \frac{1}{2}I, \ldots, \xi(\frac{1}{2}I)\Big) \\
&= p_n\Big(\xi(P_1), U_{11}+V_{12}, P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) + p_n\Big(P_1, \xi(U_{11})+\xi(V_{12}), P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) \\
&+ p_n\Big(P_1, U_{11}+V_{12}, \xi(P_1), \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) + p_n\Big(P_1, U_{11}+V_{12}, P_1, \xi(\frac{1}{2}I), \ldots, \frac{1}{2}I\Big) \\
&+ \ldots + p_n\Big(P_1, U_{11}+V_{12}, P_1, \frac{1}{2}I, \ldots, \xi(\frac{1}{2}I)\Big) + p_n\Big(\xi(P_1), W_{22}, P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) \\
&+ p_n\Big(P_1, \xi(W_{22}), P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) + p_n\Big(P_1, W_{22}, \xi(P_1), \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) \\
&+ p_n\Big(P_1, W_{22}, P_1, \xi(\frac{1}{2}I), \ldots, \frac{1}{2}I\Big) + \ldots + p_n\Big(P_1, W_{22}, P_1, \frac{1}{2}I, \ldots, \xi(\frac{1}{2}I)\Big) \\
&= p_n\Big(\xi(P_1), U_{11}+V_{12}+W_{22}, P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) + p_n\Big(P_1, \xi(U_{11})+\xi(V_{12})+\xi(W_{22}), \\
&\quad P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) + p_n\Big(P_1, U_{11}+V_{12}+W_{22}, \xi(P_1), \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) \\
&+ p_n\Big(P_1, U_{11}+V_{12}+W_{22}, P_1, \xi(\frac{1}{2}I), \ldots, \frac{1}{2}I\Big) + \ldots \\
&+ p_n\Big(P_1, U_{11}+V_{12}+W_{22}, P_1, \frac{1}{2}I, \ldots, \xi(\frac{1}{2}I)\Big).
\end{aligned}
$$

Furthermore, we can write

$$
\begin{aligned}
&\xi\Big(p_n\Big(P_1, U_{11}+V_{12}+W_{22}, P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big)\Big) \\
&= p_n\Big(\xi(P_1), U_{11}+V_{12}+W_{22}, P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) + p_n\Big(P_1, \xi(U_{11}+V_{12}+W_{22}), P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) \\
&+ p_n\Big(P_1, U_{11}+V_{12}+W_{22}, \xi(P_1), \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) + p_n\Big(P_1, U_{11}+V_{12}+W_{22}, P_1, \xi(\frac{1}{2}I), \ldots, \frac{1}{2}I\Big) \\
&+ \ldots + p_n\Big(P_1, U_{11}+V_{12}+W_{22}, P_1, \frac{1}{2}I, \ldots, \xi(\frac{1}{2}I)\Big).
\end{aligned}
$$

Equating the above two relations, we have $p_n(P_1, T, P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I) = 0$. The primeness of $\mathfrak{A}$ and $T^* = T$ imply that $T_{11} = T_{12} = 0$. It remains to show that $T_{22} = 0$. Observe that $p_n(P_2, U_{11}, P_2, \frac{1}{2}I, \ldots, \frac{1}{2}I) = 0$. Reasoning as above, we obtain $T_{22} = 0$ and, hence, $T = 0$. □

Claim 5. *For any $U_{12}, V_{12} \in \mathcal{M}_{12}$, we have*

$$\xi(U_{12}+V_{12}) = \xi(U_{12}) + \xi(V_{12}).$$

Proof. For any $A_{12}, B_{12} \in \mathfrak{A}_{12}$, assume that $U_{12} = A_{12} + A_{12}^* \in \mathcal{M}_{12}$ and $V_{12} = B_{12} + B_{12}^* \in \mathcal{M}_{12}$. Thus,

$$\begin{aligned} & p_n(P_1 + A_{12} + A_{12}^*, P_2 + B_{12} + B_{12}^*, \frac{1}{2}I, \ldots, \frac{1}{2}I) \\ = \; & (A_{12} + A_{12}^*) + (B_{12} + B_{12}^*) + (A_{12}B_{12}^* + A_{12}^*B_{12} + B_{12}A_{12}^* + B_{12}^*A_{12}) \\ = \; & U_{12} + V_{12} + U_{12}V_{12}^* + V_{12}U_{12}^*. \end{aligned}$$

Note that $U_{12}V_{12}^* + V_{12}U_{12}^* = A_{12}B_{12}^* + B_{12}A_{12}^* + A_{12}^*B_{12} + B_{12}^*A_{12} = W_{11} + X_{22}$, where $W_{11} = A_{12}B_{12}^* + B_{12}A_{12}^* \in \mathcal{M}_{11}$ and $X_{22} = A_{12}^*B_{12} + B_{12}^*A_{12} \in \mathcal{M}_{22}$. Since $A_{12} + A_{12}^*, B_{12} + B_{12}^* \in \mathcal{M}_{12}$, it follows from Claims 3 and 4 that

$$\begin{aligned} & \xi(U_{12} + V_{12}) + \xi(W_{11}) + \xi(X_{22}) \\ = \; & \xi(U_{12} + V_{12} + W_{11} + X_{22}) \\ = \; & \xi(U_{12} + V_{12} + U_{12}V_{12}^* + V_{12}U_{12}^*) \\ = \; & \xi\Big(p_n\Big(P_1 + A_{12} + A_{12}^*, P_2 + B_{12} + B_{12}^*, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big)\Big) \\ = \; & p_n\Big(\xi(P_1) + \xi(A_{12} + A_{12}^*), P_2 + B_{12} + B_{12}^*, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) \\ + \; & p_n\Big(P_1 + A_{12} + A_{12}^*, \xi(P_2) + \xi(B_{12} + B_{12}^*), \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) \\ + \; & p_n\Big(P_1 + A_{12} + A_{12}^*, P_2 + B_{12} + B_{12}^*, \xi(\frac{1}{2}I), \ldots, \frac{1}{2}I\Big) + \ldots \\ + \; & p_n\Big(P_1 + A_{12} + A_{12}^*, P_2 + B_{12} + B_{12}^*, \frac{1}{2}I, \ldots, \xi(\frac{1}{2}I)\Big) \\ = \; & \xi\Big(p_n\Big(P_1, P_2, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big)\Big) + \xi\Big(p_n\Big(P_1, B_{12} + B_{12}^*, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big)\Big) \\ + \; & \xi\Big(p_n\Big(A_{12} + A_{12}^*, P_2, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big)\Big) + \xi\Big(p_n\Big(A_{12} + A_{12}^*, B_{12} + B_{12}^*, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big)\Big) \\ = \; & \xi(U_{12}) + \xi(V_{12}) + \xi(U_{12}V_{12}^* + V_{12}U_{12}^*) \\ = \; & \xi(U_{12}) + \xi(V_{12}) + \xi(W_{11}) + \xi(X_{22}). \end{aligned}$$

Therefore, we have

$$\xi(U_{12} + V_{12}) = \xi(U_{12}) + \xi(V_{12}).$$

□

Claim 6. *For every $U_{ii}, V_{ii} \in \mathcal{M}_{ii}$ $(i = 1, 2)$, we have*

$$\xi(U_{ii} + V_{ii}) = \xi(U_{ii}) + \xi(V_{ii}).$$

Proof. We will prove for $i = 1$, the other case can be proven analogously. To prove this, we show that $T = \xi(U_{11} + V_{11}) - \xi(U_{11}) - \xi(V_{11}) = 0$. We have

$$\begin{aligned} & \xi\Big(p_n\Big(P_2, U_{11} + V_{11}, P_2, \frac{1}{2}I, \dots, \frac{1}{2}I\Big)\Big) \\ = & \xi\Big(p_n\Big(P_2, U_{11}, P_2, \frac{1}{2}I, \dots, \frac{1}{2}I\Big)\Big) + \xi\Big(p_n\Big(P_2, V_{11}, P_2, \frac{1}{2}I, \dots, \frac{1}{2}I\Big)\Big) \\ = & p_n\Big(\xi(P_2), U_{11}, P_2, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_n\Big(P_2, \xi(U_{11}), P_2, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) \\ + & p_n\Big(P_2, U_{11}, \xi(P_2), \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_n\Big(P_2, U_{11}, P_2, \xi(\frac{1}{2}I), \dots, \frac{1}{2}I\Big) \\ + & ... + p_n\Big(P_2, U_{11}, P_2, \frac{1}{2}I, \dots, \xi(\frac{1}{2}I)\Big) + p_n\Big(\xi(P_2), V_{11}, P_2, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) \\ + & p_n\Big(P_2, \xi(V_{11}), P_2, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_n\Big(P_2, V_{11}, \xi(P_2), \frac{1}{2}I, \dots, \frac{1}{2}I\Big) \\ + & p_n\Big(P_2, V_{11}, P_2, \xi(\frac{1}{2}I), \dots, \frac{1}{2}I\Big) + ... + p_n\Big(P_2, V_{11}, P_2, \frac{1}{2}I, \dots, \xi(\frac{1}{2}I)\Big) \\ = & p_n\Big(\xi(P_2), U_{11} + V_{11}, P_2, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_n\Big(P_2, \xi(U_{11}) + \xi(V_{11}), P_2, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) \\ + & p_n\Big(P_2, U_{11} + V_{11}, \xi(P_2), \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_n\Big(P_2, U_{11} + V_{11}, P_2, \xi(\frac{1}{2}I), \dots, \frac{1}{2}I\Big) \\ + & ... + p_n\Big(P_2, U_{11} + V_{11}, P_2, \frac{1}{2}I, \dots, \xi(\frac{1}{2}I)\Big). \end{aligned}$$

Apparently, we can have

$$\begin{aligned} & \xi\Big(p_n\Big(P_2, U_{11} + V_{11}, P_2, \frac{1}{2}I, \dots, \frac{1}{2}I\Big)\Big) \\ = & p_n\Big(\xi(P_2), U_{11} + V_{11}, P_2, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_n\Big(P_2, \xi(U_{11} + V_{11}), P_2, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) \\ + & p_n\Big(P_2, U_{11} + V_{11}, \xi(P_2), \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_n\Big(P_2, U_{11} + V_{11}, P_2, \xi(\frac{1}{2}I), \dots, \frac{1}{2}I\Big) \\ + & ... + p_n\Big(P_2, U_{11} + V_{11}, P_2, \frac{1}{2}I, \dots, \xi(\frac{1}{2}I)\Big). \end{aligned}$$

From the last two expressions, we have $p_n(P_2, T, P_2, \frac{1}{2}I, \dots, \frac{1}{2}I) = 0$, and thus, the primeness of $\mathfrak{A}$ gives that $T_{12} = T_{22} = 0$. Now, to show that $T_{11} = 0$, assume that $W = A_{12} + A_{12}^* \in \mathcal{M}_{12}$ for $A_{12} \in \mathfrak{A}_{12}$. Then $p_n\Big(W, U_{11}, \frac{1}{2}I, \dots, \frac{1}{2}I\Big), p_n\Big(W, V_{11}, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) \in \mathcal{M}_{12}$. Therefore, from Claim 5, we can write

$$\begin{aligned} & p_n\Big(\xi(W), U_{11} + V_{11}, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_n\Big(W, \xi(U_{11} + V_{11}), \frac{1}{2}I, \dots, \frac{1}{2}I\Big) \\ + & p_n\Big(W, U_{11} + V_{11}, \xi(\frac{1}{2}I), \dots, \frac{1}{2}I\Big) + ... + p_n\Big(W, U_{11} + V_{11}, \frac{1}{2}I, \dots, \xi(\frac{1}{2}I)\Big) \\ = & \xi\Big(p_n\Big(W, U_{11} + V_{11}, \frac{1}{2}I, \dots, \frac{1}{2}I\Big)\Big) \\ = & \xi\Big(p_n\Big(W, U_{11}, \frac{1}{2}I, \dots, \frac{1}{2}I\Big)\Big) + \xi\Big(p_n\Big(W, V_{11}, \frac{1}{2}I, \dots, \frac{1}{2}I\Big)\Big) \\ = & p_n\Big(\xi(W), U_{11} + V_{11}, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) + p_n\Big(W, \xi(U_{11}) + \xi(V_{11}), \frac{1}{2}I, \dots, \frac{1}{2}I\Big) \\ + & p_n\Big(W, U_{11} + V_{11}, \xi(\frac{1}{2}I), \dots, \frac{1}{2}I\Big) + ... + p_n\Big(W, U_{11} + V_{11}, \frac{1}{2}I, \dots, \xi(\frac{1}{2}I)\Big). \end{aligned}$$

Thus, we obtain $p_n\Big(W, T, \frac{1}{2}I, \dots, \frac{1}{2}I\Big) = 0$. This gives $T_{11} = 0$. Hence, the proof is completed. □

Remark 2. *It follows from Claims 3–6 that ξ is additive on $\mathcal{M}$.*

Claim 7. $\xi(I) = 0$.

Proof. In view of Claim 2 and Remark 2, we have

$$\begin{aligned} \xi(P_1) &= \xi\Big(p_n\Big(P_1, \frac{1}{2}I, \ldots, \frac{1}{2}I\Big)\Big) \\ &= p_n\Big(\xi(P_1), \frac{1}{2}I, \ldots, \frac{1}{2}I\Big) + p_n\Big(P_1, \xi(\frac{1}{2}I), \ldots, \frac{1}{2}I\Big) \\ &+ \ldots + p_n\Big(P_1, \frac{1}{2}I, \ldots, \xi(\frac{1}{2}I)\Big) \\ &= \xi(P_1) + (n-1)\Big(P_1\xi(\frac{1}{2}I) + \xi(\frac{1}{2}I)P_1\Big) \end{aligned}$$

This implies

$$P_1\xi(\frac{1}{2}I) + \xi(\frac{1}{2}I)P_1 = 0.$$

Multiplying the above equation by P_2 from left, right and by P_1 on both sides, respectively, we obtain $P_2\xi(\frac{1}{2}I)P_1 = 0, P_1\xi(\frac{1}{2}I)P_2 = 0$ and $P_1\xi(\frac{1}{2}I)P_1 = 0$. By replacing P_1 with P_2 in the above calculation, we can obtain $P_2\xi(\frac{1}{2}I)P_2 = 0$. Therefore, we obtain $\xi(\frac{1}{2}I) = 0$, and thus, using Remark 2, we obtain $\xi(I) = 0$. □

Claim 8. $\xi(N)^* = -\xi(N)$, *for every* $N \in \mathcal{N}$.

Proof. Observe that $p_n\Big(N, I, \ldots, I\Big) = 0$ for any $N \in \mathcal{N}$; therefore, from Claim 7, we have

$$\begin{aligned} 0 &= \xi\Big(p_n\Big(N, I, \ldots, I\Big)\Big) = p_n(\xi(N), I, \ldots, I) \\ &= 2^{n-2}\Big(\xi(N) + \xi(N)^*\Big). \end{aligned}$$

Thus, $\xi(N)^* = -\xi(N)$ for all $N \in \mathcal{N}$. □

Claim 9. $\xi(iI) \in \mathcal{Z}(\mathfrak{A})$.

Proof. Let $M \in \mathcal{M}$. Then, from Claim 8, we have

$$0 = \xi\Big(p_n\Big(M, iI, I, \ldots, I\Big)\Big) = 2^{n-2}\Big(\xi(iI)M - M\xi(iI)\Big).$$

This gives $\xi(iI)M = M\xi(iI)$. Since for any $U \in \mathfrak{A}$, $U = M_1 + iM_2$ for $M_1, M_2 \in \mathcal{M}$. Therefore, $U\xi(iI) = \xi(iI)U$ for all $U \in \mathfrak{A}$, and, hence, $\xi(iI) \in \mathcal{Z}(\mathfrak{A})$. □

Claim 10. *For any* $N \in \mathcal{N}$, $\xi(iN) = i\xi(N) + \xi(iI)N$.

Proof. It follows from Claims 2, 7 and 8 that

$$\begin{aligned} \xi\Big(p_n\Big(N, iI, I, \ldots, I\Big)\Big) &= p_n\Big(\xi(N), iI, I, \ldots, I\Big) \\ &+ p_n\Big(N, \xi(iI), \ldots, I\Big) \\ &= -2^{n-1}\Big(\xi(iI)N + i\xi(N)\Big). \end{aligned} \tag{3}$$

Furthermore, from Remark 2, we have

$$\xi\Big(p_n\Big(N, iI, I, \ldots, I\Big)\Big) = -2^{n-1}\Big(\xi(iN)\Big) \tag{4}$$

Equations (3) and (4) lead to

$$\xi(iN) = i\xi(N) + \xi(iI)N.$$

□

Claim 11. *ξ is additive on $\mathcal{N}$.*

Proof. Let $N_1, N_2 \in \mathcal{N}$. Then, from Claim 10 and Remark 2, we have

$$\begin{aligned} & i\xi(N_1+N_2) + \xi(iI)(N_1+N_2) \\ &= \xi(i(N_1+N_2) = \xi(iN_1) + \xi(iN_2) \\ &= i(\xi(N_1)+\xi(N_2)) + \xi(iI)(N_1+N_2). \end{aligned}$$

This gives

$$\xi(N_1+N_2) = \xi(N_1) + \xi(N_2).$$

□

Claim 12. *ξ is additive on $\mathfrak{A}$.*

Proof. Let $N, N' \in \mathcal{N}$. In view of Remark 2, Claims 7, 9 and 10, we have

$$\begin{aligned} & 2^{n-1}\Big(i\xi(N') + \xi(iI)N'\Big) = 2^{n-1}\xi(iN') = \xi(2^{n-1}iN') \\ &= \xi\Big(p_n\Big(I, I, \ldots, (N+iN')\Big)\Big) \\ &= 2^{n-2}\xi(N+iN')^* + 2^{n-2}\xi(N+iN') \end{aligned} \tag{5}$$

and

$$\begin{aligned} & -2^{n-1}\Big(i\xi(N) + \xi(iI)N\Big) = -2^{n-1}\xi(iN) = \xi(-2^{n-1}iN) \\ = \;& \xi\Big((N+iN'), iI, I, \ldots, I\Big) \\ = \;& -2^{n-2}i\xi(N+iN') + 2^{n-2}i\xi(N+iN')^* - 2^{n-1}\xi(iI)N. \end{aligned} \tag{6}$$

So, we have from Equations (5) and (6) that

$$\xi(N+iN') = \xi(N) + i\xi(N') + \xi(iI)N' \tag{7}$$

for all $N, N' \in \mathcal{N}$. Now let $U, V \in \mathfrak{A}$ such that $U = U_1 + iU_2$ and $V = V_1 + iV_2$ for all $U_1, U_2, V_1, V_2 \in \mathcal{N}$. Using Equation (7) and Claim 11, we have

$$\begin{aligned} \xi(U+V) &= \xi\Big((U_1+V_1) + i(U_2+V_2)\Big) \\ &= \xi(U_1+V_1) + i\xi(U_2+V_2) + \xi(iI)(U_2+V_2) \\ &= \Big(\xi(U_1) + i\xi(U_2) + \xi(iI)U_2\Big) \\ &+ \Big(\xi(V_1) + i\xi(V_2) + \xi(iI)V_2\Big) \\ &= \xi(U_1+iU_2) + \xi(V_1+iV_2) \\ &= \xi(U) + \xi(V). \end{aligned}$$

□

Claim 13. *$\xi(U^*) = \xi(U)^*$ for all $U \in \mathfrak{A}$.*

Proof. We know that any element $U \in \mathfrak{A}$ can be expressed as $U = U_1 + iU_2$ for $U_1, U_2 \in \mathcal{N}$, so it follows from Equation (7) and Claim 8 that

$$\begin{aligned} \xi(U)^* &= \xi(U_1 + iU_2)^* \\ &= \Big(\xi(U_1) + i\xi(U_2) + \xi(iI)U_2\Big)^* \\ &= \xi(-U_1) + i\xi(U_2) + \xi(iI)U_2. \end{aligned} \tag{8}$$

On the other hand,

$$\xi(U^*) = \xi(-U_1 + iU_2) = \xi(-U_1) + i\xi(U_2) + \xi(iI)U_2. \tag{9}$$

From Equations (8) and (9), we obtain $\xi(U^*) = \xi(U)^*$. □

Claim 14. *ξ is a derivation on $\mathcal{N}$.*

Proof. Since for any $N_1, N_2 \in \mathcal{N}$, $N_1N_2 - N_2N_1 \in \mathcal{N}$, it follows from Claim 10 that

$$\begin{aligned} -2^{n-2}\xi\Big(N_1N_2 + N_2N_1\Big) &= \xi\Big(p_n\Big(N_1, N_2, I, \ldots, I\Big)\Big) \\ &= p_n\Big(\xi(N_1), N_2, I, \ldots, I\Big) \\ &+ p_n\Big(N_1, \xi(N_2), I, \ldots, I\Big) \\ &= -2^{n-2}\xi(N_1)N_2 - 2^{n-2}N_2\xi(N_1) \\ &- 2^{n-2}N_1\xi(N_2) - 2^{n-2}\xi(N_2)N_1. \end{aligned} \tag{10}$$

Moreover,

$$\begin{aligned} &2^{n-2}i\xi\Big(N_1N_2 - N_2N_1\Big) + 2^{n-2}\xi(iI)\Big(N_1N_2 - N_2N_1\Big) \\ = \;& \xi\Big(2^{n-2}i(N_1N_2 - N_2N_1)\Big) \\ = \;& \xi\Big(p_n\Big(N_1, iN_2, I, \ldots, I\Big)\Big) \\ = \;& p_n\Big(\xi(N_1), iN_2, I, \ldots, I\Big) + p_n\Big(N_1, \xi(iN_2), I, \ldots, I\Big) \\ = \;& 2^{n-2}i\xi(N_1)N_2 - 2^{n-2}iN_2\xi(N_1) + 2^{n-2}iN_1\xi(N_2) \\ - \;& 2^{n-2}i\xi(N_2)N_1 + 2^{n-2}\xi(iI)(N_1N_2 - N_2N_1) \end{aligned} \tag{11}$$

for all $N_1, N_2 \in \mathcal{N}$. Equations (10) and (11) conclude that

$$\xi(N_1N_2) = \xi(N_1)N_2 + N_1\xi(N_2)$$

for all $N_1, N_2 \in \mathcal{N}$. Hence, the proof. □

Claim 15. $\xi(iI) = 0$.

Proof. We know from Claim 7 that $\xi(I) = 0$. Thus, by Remark 2 and Claim 14, we have

$$0 = \xi(-I) = \xi\Big((iI)(iI)\Big) = \xi(iI)(iI) + (iI)\xi(iI) = 2i\xi(iI).$$

Thus, $\xi(iI) = 0$. □

Claim 16. $\xi(iU) = i\xi(U)$ *for all* $U \in \mathfrak{A}$.

Proof. From Claims 10 and 15, we obtain $\xi(iN) = i\xi(N)$ for all $N \in \mathcal{N}$. Since for any $U \in \mathfrak{A}$, we can write $U = N_1 + iN_2$ for $N_1, N_2 \in \mathcal{N}$. It follows from Claim 12 that

$$\xi(iU) = \xi(i(N_1 + iN_2)) = i\Big(\xi(N_1) + i\xi(N_2)\Big) = i\xi(U).$$

Hence, the result. □

Proof of Theorem 1. By Claims 12 and 13, ξ is additive with $\xi(U^*) = \xi(U)^*$. The final step in the proof is to demonstrate that ξ is a derivation on $\mathfrak{A}$.

For any $U, V \in \mathfrak{A}$, assume that $U = U_1 + iU_2$ and $V = V_1 + iV_2$ for all $U_1, U_2, V_1, V_2 \in \mathcal{N}$. Thus, it follows from Claims 14–16 that

$$\begin{aligned} \xi(UV) &= \xi((U_1 + iU_2)(V_1 + iV_2)) \\ &= \xi(U_1V_1 + iU_1V_2 + iU_2V_1 - U_2V_2) \\ &= \xi(U_1)V_1 + U_1\xi(V_1) + i\xi(U_1)V_2 \\ &+ iU_1\xi(V_2) + i\xi(U_2)V_1 + iU_2\xi(V_1) \\ &- \xi(U_2)V_2 - U_2\xi(V_2) \end{aligned} \tag{12}$$

On the other hand,

$$\begin{aligned} \xi(U)V + U\xi(V) &= \xi(U_1 + iU_2)(V_1 + iV_2) \\ &+ (U_1 + iU_2)\xi(V_1 + iV_2) \\ &= (\xi(U_1) + i\xi(U_2))(V_1 + iV_2) \\ &+ (U_1 + iU_2)(\xi(V_1) + i\xi(V_2)) \\ &= \xi(U_1)V_1 + U_1\xi(V_1) + i\xi(U_1)V_2 \\ &+ iU_1\xi(V_2) + i\xi(U_2)V_1 + iU_2\xi(V_1) \\ &- \xi(U_2)V_2 - U_2\xi(V_2) \end{aligned} \tag{13}$$

Comparing Equations (12) and (13), we conclude that ξ is a derivation on $\mathfrak{A}$. This completes the theorem's proof. □

4. Discussion

Previously, the authors studied the structures of multiplicative/non-linear bi-skew Jordan (i.e., $n = 2$) and Jordan triple (i.e., $n = 3$) derivations on prime $*$-algebras. In this article, we have given a characterization of multiplicative/non-linear bi-skew Jordan n-derivations (i.e., for any $n \geq 2$) on prime $*$-algebras. Therefore, our result is more general. In particular, one can easily obtain the result for $n = 2$ (respectively, for $n = 3$) easily in the case of multiplicative bi-skew Jordan (respectively Jordan triple) derivations on prime $*$-algebras.

5. Conclusions

In this article we explored the structure of non-linear bi-skew Jordan n-derivation (ξ) acting on a prime $*$-algebra $\mathfrak{A}$. Indeed, we proved that such a map is additive derivation preserving the $*$-structure of algebra $\mathfrak{A}$, i.e., $\xi(U^*) = \xi(U)^*$ for all $U \in \mathfrak{A}$. One can further investigate the structure of non-linear bi-skew Jordan n-derivations on different algebras such as triangular algebras, generalized matrix algebras, incidence algebras, etc.

Author Contributions: Supervision, A.A.; conceptualization, A.A., A.S.A. and M.T.; methodology, A.A., A.S.A. and M.T.; writing—original draft, M.T.; writing—review and editing, A.A. and A.S.A.; Validation, A.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Princess Nourah Bint Abdulrahman, University grant number PNURSP2023R231.

Data Availability Statement: Not applicable.

Acknowledgments: The second author extends his appreciation to Princess Nourah Bint Abdulrahman university for funding this research under Researchers Supporting Project number (PNURSP2023R231), Princess Nourah Bint Abdulrahman University, Riyadh, Saudi Arabia. The authors are highly indebted to the reviewers for their valuable comments and suggestions.

Conflicts of Interest: According to the authors, there is no conflict of interest.

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