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Solving Some Integral and Fractional Differential Equations via Neutrosophic Pentagonal Metric Space

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Abstract: In this paper, we first introduce the notion of neutrosophic pentagonal metric space. We prove several interesting results for some classes contraction mappings and prove some fixed point theorems in neutrosophic pentagonal metric space. Finally, we prove the uniqueness and existence of the integral equation and fractional differential equation to support our main result.

Keywords: neutrosophic metric space; neutrosophic pentagonal metric space; fixed point results; integral equation; fractional differential equation

MSC: 47H10; 54H25

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1. Introduction

A fuzzy set is a category of items with a continuum of membership levels between zero and one. The concept of a fuzzy set was first introduced by Zadeh [1], who also provided a useful starting point for the development of a conceptual framework that, while similar to the framework used for sets in many ways, is more general and may have a wider range of applications, particularly in the fields of pattern classification and information processing. The study of statistical metric spaces and an examination of the continuous characteristics of the distance function were both continued by Schweizer and Sklar in [2]. Fuzzy metric spaces (shortly, Fuzzy MS) were proposed by Kramosil and Michálek in [3], who also expanded on the notion of convergence that is typically used to determine whether a generalization is appropriate. On fuzzy double-controlled MSs, Azmi [4] developed the novel idea of $(\alpha - \Pi)$-fuzzy contractive mappings and illustrated various fixed-point results. In the framework of extended fuzzy b-MSs, various generalized fixed point findings of Banach and Ćirić type are established by Rome et al. [5]. A Hausdorff fuzzy b-MS is described by Batul et al. [6]. A few fixed point results for multivalued mappings in G-complete fuzzy b-MSs that satisfy an appropriate contractiveness criterion are established using the novel idea. Numerous fixed point theorems in fuzzy b-MSs make up Rakić et al. [7]. They provided a necessary condition for a sequence to be Cauchy in the fuzzy b-MS, which was a significant result. By using a control function $\alpha(x, y)$ of the right-hand side of the b-triangle inequality, Mlaiki [8] created a new extension of b-MSs known as controlled metric type spaces. Controlled fuzzy MS is a brand-new development of Sezen's [9] work on fuzzy metrics. Additionally, they demonstrated a new fixed point theorem and a Banach-type fixed point theorem for some fulfilling self-mappings. see [10–16]. Grabiec [17] extended two fixed point theorems of Banach and Edelstein to contractive mappings of complete and compact fuzzy MSs, respectively. In order to established some fixed point theorems, Rehman et al. [18] defined α-admissible and α-Π-fuzzy cone contraction in fuzzy cone MS. Finally, he used theoretical results to show that a nonlinear integral equation has a solution.

Park [19] created an intuitionistic fuzzy MS delta membership and nonmembership functions. The concept of an intuitionistic fuzzy b-MS was first proposed by Konwar [20], who also demonstrated a number of fixed point theorems. Neutosophic MSs, which are initialized to handle membership, nonmembership, and naturalness, were introduced by Kirişci and Simsek in their [21] paper. Simsek and Kirişci [22] demonstrated various fixed point results in the context of neutrosophic MSs. Fixed point findings in neutrosophic MSs were demonstrated by Sowndrarajan et al. [23]. Itoh [24] showed a usage for random differential equations in Banach spaces.

In this article, we introduce the neutrosophic pentagonal MS (also known as NPMS) and demonstrate a few fixed point conclusions. The following are the primary goals of this work:

1. To introduce the notion of neutrosophic pentagonal MS;
2. To prove several fixed-point theorems for contraction mappings;
3. Show the existence of a unique solution of an integral equation;
4. Show the existence of a unique solution of a fractional differential equation.

2. Preliminaries

Here, we'll go over some fundamental terms that will be useful for the key results.

Definition 1 ([19])**.** *Let $\star\colon [0,1]^2 \to [0,1]$ be a binary operation is said to be a continuous triangle norm if:*

(1) $\hat{\omega} \star \check{h} = \check{h} \star \hat{\omega}$, $\forall\, \hat{\omega}, \check{h} \in [0,1]$;
(2) $\star$ *is continuous;*
(3) $\hat{\omega} \star 1 = \hat{\omega}$, $\forall\, \hat{\omega} \in [0,1]$;
(4) $(\hat{\omega} \star \check{h}) \star \tilde{\mu} = \hat{\omega} \star (\check{h} \star \tilde{\mu})$, $\forall\ \hat{\omega}, \check{h}, \tilde{\mu} \in [0,1]$;
(5) *If* $\hat{\omega} \leq \tilde{\mu}$ *and* $\check{h} \leq \varsigma$, $\forall\ \hat{\omega}, \check{h}, \tilde{\mu}, \varsigma \in [0,1]$, *then* $\hat{\omega} \star \check{h} \leq \tilde{\mu} \star \varsigma$.

Definition 2 ([19])**.** *Let $\bullet\colon [0,1]^2 \to [0,1]$ be a binary operation is said to be a continuous triangle co-norm if:*

(1) $\hat{\omega} \bullet \check{h} = \check{h} \bullet \hat{\omega}$, $\forall\ \hat{\omega}, \check{h} \in [0,1]$;
(2) $\bullet$ *is continuous;*
(3) $\hat{\omega} \bullet 0 = 0$, $\forall\ \hat{\omega} \in [0,1]$;
(4) $(\hat{\omega} \bullet \check{h}) \bullet \tilde{\mu} = \hat{\omega} \bullet (\check{h} \bullet \tilde{\mu})$, $\forall\ \hat{\omega}, \check{h}, \tilde{\mu} \in [0,1]$;
(5) *If* $\hat{\omega} \leq \tilde{\mu}$ *and* $\tilde{\mu} \leq \varsigma$, *with* $\hat{\omega}, \check{h}, \tilde{\mu}, \varsigma \in [0,1]$, *then* $\hat{\omega} \bullet \check{h} \leq \tilde{\mu} \bullet \varsigma$.

Definition 3 ([24])**.** *Let $\chi, \mathbf{F}\colon K \times K \to [1,+\infty)$ be given two non-comparable functions, if $\partial\colon K \times K \to [0,+\infty)$ satisfies axioms:*

(a) $\partial(U, \mathcal{W}) = 0$ *iff* $U = \mathcal{W}$;
(b) $\partial(U, \mathcal{W}) = \partial(\mathcal{W}, U)$;
(c) $\partial(U, \mathcal{W}) \leq \chi(U, \pounds)\partial(U, \pounds) + \mathbf{F}(\pounds, \mathcal{W})\partial(\pounds, \mathcal{W})$;

$\forall\ U, \mathcal{W}, \pounds \in K$, *then,* (K, ∂) *is known to be a double controlled MS (shortly, DCMS).*

Definition 4 ([25])**.** *Let $K \neq \emptyset$ and $\chi, \mathbf{F}\colon K \times K \to [1,+\infty)$ be known non-comparable functions. And $\star$ is a continuous t-norm also Λ be a fuzzy set on $K \times K \times (0,+\infty)$ is said to be fuzzy double controlled metric on K, $\forall\ U, \mathcal{W}, \pounds \in K$ if:*

(i) $\Lambda(U, \mathcal{W}, 0) = 0$;
(ii) $\Lambda(U, \mathcal{W}, \Gamma) = 1\ \forall\ \Gamma > 0, \Leftrightarrow U = \mathcal{W}$;
(iii) $\Lambda(U, \mathcal{W}, \Gamma) = \Lambda(\mathcal{W}, U, \Gamma)$;

(iv) $\Lambda(\mathcal{U},\pounds,\Gamma+\Gamma) \geq \Lambda\left(\mathcal{U},\mathcal{W},\frac{\Gamma}{\chi(\mathcal{U},\mathcal{W})}\right) \star \Lambda\left(\mathcal{W},\pounds,\frac{\Gamma}{\mathbf{F}(\mathcal{W},\pounds)}\right)$;

(v) $\Lambda(\mathcal{U},\mathcal{W},\cdot)\colon (0,+\infty) \to [0,1]$ *is continuous on left.*

Then, $(K,\Lambda,\mathcal{N},\star)$ *is known to be a fuzzy DCMS.*

Definition 5 ([20]). *Let* $K \neq \emptyset$. *Let* $\star$, $\bullet$ *are the continuous t-norm, continuous t-co-norm respectively,* $\flat \geq 1$ *and* $\Lambda,\mathcal{N}$ *be fuzzy sets on* $K \times K \times (0,+\infty)$. *If* $(K,\Lambda,\mathcal{N},\star,\bullet)$ *fullfils all* $\mathcal{U},\mathcal{W} \in K$ *and* $\breve{g},\Gamma > 0$:

(I) $\Lambda(\mathcal{U},\mathcal{W},\Gamma)+\Pi(\mathcal{U},\mathcal{W},\Gamma) \leq 1$;
(II) $0 < \Lambda(\mathcal{U},\mathcal{W},\Gamma)$;
(III) $\Lambda(\mathcal{U},\mathcal{W},\Gamma) = 1 \Leftrightarrow \mathcal{U} = \mathcal{W}$;
(IV) $\Lambda(\mathcal{U},\mathcal{W},\Gamma) = \Lambda(\mathcal{W},\mathcal{U},\Gamma)$;
(V) $\Lambda(\mathcal{U},\pounds,\flat(\Gamma+\breve{g})) \geq \Lambda(\mathcal{U},\mathcal{W},\Gamma) \star \Lambda(\mathcal{W},\pounds,\breve{g})$;
(VI) $\Lambda(\mathcal{U},\mathcal{W},\cdot)$ *is a non-decreasing function of* $\mathbb{R}^+$ *and* $\lim_{\Gamma\to+\infty}\Lambda(\mathcal{U},\mathcal{W},\Gamma) = 1$;
(VII) $0 < \Pi(\mathcal{U},\mathcal{W},\Gamma)$;
(VIII) $\Pi(\mathcal{U},\mathcal{W},\Gamma) = 0 \Leftrightarrow \mathcal{U} = \mathcal{W}$;
(IX) $\Pi(\mathcal{U},\mathcal{W},\Gamma) = \Pi(\mathcal{W},\mathcal{U},\Gamma)$;
(X) $\Pi(\mathcal{U},\pounds,\flat(\Gamma+\breve{g})) \leq \Pi(\mathcal{U},\mathcal{W},\Gamma) \bullet \Pi(\mathcal{W},\pounds,\breve{g})$;
(XI) $\Pi(\mathcal{U},\mathcal{W},\cdot)$ *is a non-increasing function of* $\mathbb{R}^+$ *and* $\lim_{\Gamma\to+\infty}\Pi(\mathcal{U},\mathcal{W},\Gamma) = 0$,

Then, $(K,\Lambda,\Pi,\star,\bullet)$ *is an intuitionistic fuzzy* $\flat$*-MS.*

Definition 6 ([21]). *Let* $K \neq \emptyset$, $\star$, $\bullet$ *are the continuous t-norm, continuous t-co-norm respectively, and* $\Lambda,\Pi,\mathcal{S}$ *are neutrosophic sets (shortly, N-sets) on* $K \times K \times (0,+\infty)$ *is known to be a neutrosophic metric on K, if for all* $\mathcal{U},\mathcal{W},\pounds \in K$, *the following axioms are fulfilled:*

(1) $\Lambda(\mathcal{U},\mathcal{W},\Gamma)+\Pi(\mathcal{U},\mathcal{W},\Gamma)+\mathcal{S}(\mathcal{U},\mathcal{W},\Gamma) \leq 3$;
(2) $0 < \Lambda(\mathcal{U},\mathcal{W},\Gamma)$;
(3) $\Lambda(\mathcal{U},\mathcal{W},\Gamma) = 1 \ \forall \ \Gamma > 0, \Leftrightarrow \mathcal{U} = \mathcal{W}$;
(4) $\Lambda(\mathcal{U},\mathcal{W},\Gamma) = \Lambda(\mathcal{W},\mathcal{U},\Gamma)$;
(5) $\Lambda(\mathcal{U},\pounds,\Gamma+\breve{g}) \geq \Lambda(\mathcal{U},\mathcal{W},\Gamma) \star \Lambda(\mathcal{W},\pounds,\breve{g})$;
(6) $\Lambda(\mathcal{U},\mathcal{W},\cdot)\colon (0,+\infty) \to [0,1]$ *is continuous and* $\lim_{\Gamma\to+\infty}\Lambda(\mathcal{U},\mathcal{W},\Gamma) = 1$;
(7) $1 < \Pi(\mathcal{U},\mathcal{W},\Gamma)$;
(8) $\Pi(\mathcal{U},\mathcal{W},\Gamma) = 0 \ \forall \ \Gamma > 0, \Leftrightarrow \mathcal{U} = \mathcal{W}$;
(9) $\Pi(\mathcal{U},\mathcal{W},\Gamma) = \Pi(\mathcal{W},\mathcal{U},\Gamma)$;
(10) $\Pi(\mathcal{U},\pounds,\Gamma+\breve{g}) \leq \Pi(\mathcal{U},\mathcal{W},\Gamma) \bullet \Pi(\mathcal{W},\pounds,\breve{g})$;
(11) $\Pi(\mathcal{U},\mathcal{W},\cdot)\colon (0,+\infty) \to [0,1]$ *is continuous and* $\lim_{\Gamma\to+\infty}\Pi(\mathcal{U},\mathcal{W},\Gamma) = 0$;
(12) $1 < \mathcal{S}(\mathcal{U},\mathcal{W},\Gamma)$;
(13) $\mathcal{S}(\mathcal{U},\mathcal{W},\Gamma) = 0 \ \forall \ \Gamma > 0, \Leftrightarrow \mathcal{U} = \mathcal{W}$;
(14) $\mathcal{S}(\mathcal{U},\mathcal{W},\Gamma) = \mathcal{S}(\mathcal{W},\mathcal{U},\Gamma)$;
(15) $\mathcal{S}(\mathcal{U},\pounds,\Gamma+\breve{g}) \leq \mathcal{S}(\mathcal{U},\mathcal{W},\Gamma) \bullet \mathcal{S}(\mathcal{W},\pounds,\breve{g})$;
(16) $\mathcal{S}(\mathcal{U},\mathcal{W},\cdot)\colon (0,+\infty) \to [0,1]$ *is continuous and* $\lim_{\Gamma\to+\infty}\mathcal{S}(\mathcal{U},\mathcal{W},\Gamma) = 0$;
(17) *If* $\Gamma \leq 0$, *then* $\Lambda(\mathcal{U},\mathcal{W},\Gamma) = 0, \Pi(\mathcal{U},\mathcal{W},\Gamma) = 0$;

Then, $(K,\Lambda,\Pi,\mathcal{S},\star,\bullet)$ *is known to be a neutrosophic MS.*

In this article, we define NPMS and demonstrate fixed point theorems.

3. Main Results

This section presents NPMS and illustrates some fixed-point theorems.

Definition 7. *Let* $K \neq \emptyset$ *and function* $\chi\colon K \times K \to [1,+\infty)$ *be non-comparable,* $\star$, $\bullet$ *are the continuous t-norm, continuous t-co-norm respectively, and* $\Lambda,\Pi,\mathcal{M}$ *be N-sets on* $K \times K \times (0,+\infty)$ *is known to be a neutrosophic pentagonal metric on K, if for any* $\mathcal{U},\pounds \in K$ *and all distinct* $\check{x},\mathcal{W},\pounds \in K$, *the following axioms are fulfilled:*

(i) $\Lambda(\mathcal{U},\mathcal{W},\Gamma)+\Pi(\mathcal{U},\mathcal{W},\Gamma)+\mathcal{M}(\mathcal{U},\mathcal{W},\Gamma) \leq 3$;

(ii) $0 < \Lambda(\mathcal{U},\mathcal{W},\Gamma)$;
(iii) $\Lambda(\mathcal{U},\mathcal{W},\Gamma) = 1 \;\forall\; \Gamma > 0, \Leftrightarrow \mathcal{U} = \mathcal{W}$;
(iv) $\Lambda(\mathcal{U},\mathcal{W},\Gamma) = \Lambda(\mathcal{W},\mathcal{U},\Gamma)$;
(v) $\Lambda(\mathcal{U},\pounds,\Gamma + \breve{g} + \check{k} + \breve{e}) \geq \Lambda(\mathcal{U},\mathcal{W},\Gamma) \star \Lambda(\mathcal{W},\breve{x},\breve{g}) \star \Lambda(\breve{x},\breve{y},\check{k}) \star \Lambda(\breve{y},\pounds,\breve{e})$;
(vi) $\Lambda(\mathcal{U},\mathcal{W},\cdot)\colon (0,+\infty) \to [0,1]$ *is continuous and* $\lim_{\Gamma\to+\infty} \Lambda(\mathcal{U},\mathcal{W},\Gamma) = 1$;
(vii) $1 < \Pi(\mathcal{U},\mathcal{W},\Gamma)$;
(viii) $\Pi(\mathcal{U},\mathcal{W},\Gamma) = 0 \;\forall\; \Gamma > 0, \Leftrightarrow \mathcal{U} = \mathcal{W}$;
(ix) $\Pi(\mathcal{U},\mathcal{W},\Gamma) = \Pi(\mathcal{W},\mathcal{U},\Gamma)$;
(x) $\Pi(\mathcal{U},\pounds,\Gamma + \breve{g} + \check{k} + \breve{e}) \leq \Pi(\mathcal{U},\mathcal{W},\Gamma) \bullet \Pi(\mathcal{W},\breve{x},\breve{g}) \bullet \Pi(\breve{x},\breve{y},\check{k}) \bullet \Pi(\breve{y},\pounds,\breve{e})$;
(xi) $\Pi(\mathcal{U},\mathcal{W},\cdot)\colon (0,+\infty) \to [0,1]$ *is continuous and* $\lim_{\Gamma\to+\infty} \Pi(\mathcal{U},\mathcal{W},\Gamma) = 0$;
(xii) $1 < \mathcal{M}(\mathcal{U},\mathcal{W},\Gamma)$;
(xiii) $\mathcal{M}(\mathcal{U},\mathcal{W},\Gamma) = 0 \;\forall\; \Gamma > 0, \Leftrightarrow \mathcal{U} = \mathcal{W}$;
(xiv) $\mathcal{M}(\mathcal{U},\mathcal{W},\Gamma) = \mathcal{M}(\mathcal{W},\mathcal{U},\Gamma)$;
(xv) $\mathcal{M}(\mathcal{U},\pounds,\Gamma + \breve{g} + \check{k} + \breve{e}) \leq \mathcal{M}(\mathcal{U},\mathcal{W},\Gamma) \bullet \mathcal{M}(\mathcal{W},\breve{x},\breve{g}) \bullet \mathcal{M}(\breve{x},\breve{y},\check{k}) \bullet \mathcal{M}(\breve{y},\pounds,\breve{e})$;
(xvi) $\mathcal{M}(\mathcal{U},\mathcal{W},\cdot)\colon (0,+\infty) \to [0,1]$ *is continuous and* $\lim_{\Gamma\to+\infty} \mathcal{M}(\mathcal{U},\mathcal{W},\Gamma) = 0$;
(xvii) *If* $\Gamma \leq 0$, *then* $\Lambda(\mathcal{U},\mathcal{W},\Gamma) = 0, \Pi(\mathcal{U},\mathcal{W},\Gamma) = 1$ *and* $\mathcal{S}(\mathcal{U},\mathcal{W},\Gamma) = 1$.

Then, $(K,\Lambda,\Pi,\mathcal{M},\star,\bullet)$ *is said to be a NPMS.*

Example 1. *Let* $K = \{1,2,3,4\}$. *Define* $\Lambda,\Pi,\mathcal{M}\colon K \times K \times (0,+\infty) \to [0,1]$ *as*

$$\Lambda(\mathcal{U},\mathcal{W},\Gamma) = \begin{cases} 1, & \text{if} \quad \mathcal{U} = \mathcal{W} \\ \frac{\Gamma}{\Gamma+\max\{\mathcal{U},\mathcal{W}\}}, & \text{otherwise}, \end{cases}$$

$$\Pi(\mathcal{U},\mathcal{W},\Gamma) = \begin{cases} 0, & \text{if} \quad \mathcal{U} = \mathcal{W} \\ \frac{\max\{\mathcal{U},\mathcal{W}\}}{\Gamma+\max\{\mathcal{U},\mathcal{W}\}}, & \text{otherwise}, \end{cases}$$

and

$$\mathcal{M}(\mathcal{U},\mathcal{W},\Gamma) = \begin{cases} 0, & \text{if} \quad \mathcal{U} = \mathcal{W} \\ \frac{\max\{\mathcal{U},\mathcal{W}\}}{\Gamma}, & \text{otherwise}, \end{cases}$$

Then , $(K,\Lambda,\Pi,\mathcal{M},\star,\bullet)$ *is a NPMS with continuous t-norm* $\hat{\omega} \star \check{h} = \hat{\omega}\check{h}$ *and continuous t-conorm,* $\hat{\omega} \bullet \bar{f} = \max\{\hat{\omega},\bar{f}\}$.

Proof. Now, we prove the conditions (v), (x) and (xv) others are obvious.

Let $\mathcal{U} = 1, \mathcal{W} = 2, \breve{x} = 3, \pounds = 4$ and $\rho = 5$. Then

$$\Lambda(1,5,\Gamma + \breve{g} + \check{k} + \breve{q}) = \frac{\Gamma + \breve{g} + \check{k} + \breve{q}}{\Gamma + \breve{g} + \check{k} + \breve{q} + \max\{1,5\}} = \frac{\Gamma + \breve{g} + \check{k} + \breve{q}}{\Gamma + \breve{g} + \check{k} + \breve{q} + 5}.$$

On the other hand,

$$\Lambda(1,2,\Gamma) = \frac{\Gamma}{\Gamma + \max\{1,2\}} = \frac{\Gamma}{\Gamma + 2},$$

$$\Lambda(2,3,\breve{g}) = \frac{\breve{g}}{\breve{g} + \max\{2,3\}} = \frac{\breve{g}}{\breve{g} + 3},$$

$$\Lambda(3,4,\check{k}) = \frac{\check{k}}{\check{k} + \max\{3,4\}} = \frac{\check{k}}{\check{k} + 4}$$

and

$$\Lambda(4,5,\check{q}) = \frac{\check{q}}{\check{q} + \max\{4,5\}} = \frac{\check{q}}{\check{q}+5}.$$

i.e.,

$$\frac{\Gamma + \check{g} + \check{k} + \check{q}}{\Gamma + \check{g} + \check{k} + \check{q} + 5} \geq \frac{\Gamma}{\Gamma+2} \frac{\check{g}}{\check{g}+3} \frac{\check{k}}{\check{k}+4} \frac{\check{q}}{\check{q}+5}.$$

Then it satisfies all $\Gamma, \check{g}, \check{k}, \check{q} > 0$. Hence,

$$\Lambda(U,\rho,\Gamma + \check{g} + \check{k} + \check{q}) \geq \Lambda(U,\mathcal{W},\Gamma) \star \Lambda(\mathcal{W},\check{x},\check{g}) \star \Lambda(\check{x},\pounds,\check{k}) \star \Lambda(\pounds,\rho,\check{q}).$$

Now,

$$\Pi(1,5,\Gamma + \check{g} + \check{k} + \check{q}) = \frac{\max\{1,5\}}{\Gamma + \check{g} + \check{k} + \check{q} + \max\{1,5\}} = \frac{5}{\Gamma + \check{g} + \check{k} + \check{q} + 5}.$$

On the other hand,

$$\Pi(1,2,\Gamma) = \frac{\max\{1,2\}}{\Gamma + \max\{1,2\}} = \frac{2}{\Gamma+2},$$

$$\Pi(2,3,\check{g}) = \frac{\max\{2,3\}}{\check{g} + \max\{2,3\}} = \frac{3}{\check{g}+3},$$

$$\Pi(3,4,\check{k}) = \frac{\max\{3,4\}}{\check{k} + \max\{3,4\}} = \frac{4}{\check{k}+4}$$

and

$$\Pi(4,5,\check{q}) = \frac{\max\{4,5\}}{\check{q} + \max\{4,5\}} = \frac{5}{\check{q}+5}$$

i.e.,

$$\frac{5}{\Gamma + \check{g} + \check{k} + \check{q} + 5} \leq \max\left\{\frac{2}{\Gamma+2}, \frac{3}{\check{g}+3}, \frac{4}{\check{k}+4}, \frac{5}{\check{q}+5}\right\}.$$

Then it satisfies all $\Gamma, \check{g}, \check{k}, \check{q} > 0$. Hence,

$$\Pi(U,\rho,\Gamma + \check{g} + \check{k} + \check{q}) \leq \Pi(U,\mathcal{W},\Gamma) \bullet \Pi(\check{x},\pounds,\hat{s}) \bullet \Pi(\check{x},\pounds,\hat{w}) \bullet \Pi(\pounds,\rho,\hat{y}).$$

Now,

$$\mathcal{M}(1,5,\Gamma + \check{g} + \check{k} + \check{q}) = \frac{\max\{1,5\}}{\Gamma + \check{g} + \check{k} + \check{q}} = \frac{5}{\Gamma + \check{g} + \check{k} + \check{q}}.$$

On the other hand,

$$\mathcal{M}(1,2,\Gamma) = \frac{\max\{1,2\}}{\Gamma} = \frac{2}{\Gamma},$$

$$\mathcal{M}(2,3,\check{g}) = \frac{\max\{2,3\}}{\check{g}} = \frac{3}{\check{g}},$$

$$\mathcal{M}(3,4,\check{k}) = \frac{\max\{3,4\}}{\check{k}} = \frac{4}{\check{k}}$$

and

$$\mathcal{M}(4,5,\check{q}) = \frac{\max\{4,5\}}{\check{q}} = \frac{5}{\check{q}}$$

i.e.,

$$\frac{5}{\Gamma + \breve{g} + \check{k} + \check{q}} \le \max\left\{\frac{2}{\Gamma}, \frac{3}{\breve{g}}, \frac{4}{\check{k}}, \frac{5}{\check{q}}\right\}.$$

Then it satisfies all $\Gamma, \breve{g}, \check{k}, \check{q} > 0$. Hence,

$$\mathcal{M}(U,\rho,\Gamma + \breve{g} + \check{k} + \check{q}) \le \mathcal{M}(U,\mathcal{W},\Gamma) \bullet \mathcal{M}(\mathcal{W},\check{x},\breve{g}) \bullet \mathcal{M}(\check{x},£,\check{k}) \bullet \mathcal{M}(£,\rho,\check{k}).$$

Hence, $(K,\Lambda,\Pi,\mathcal{M},\star,\bullet)$ is a NPMS. □

Remark 1. *The above example satisfies for continuous t-norm* $\hat{\omega} \star \bar{f} = \min\{\hat{\omega}, \bar{f}\}$ *and continuous t-co-norm* $\hat{\omega} \bullet \bar{f} = \max\{\hat{\omega}, \bar{f}\}$.

Definition 8. *Let* $(K,\Lambda,\Pi,\mathcal{M},\star,\bullet)$ *is a NPMS, an open ball is then defined* $\mathcal{M}(U,\mathtt{r},\Gamma)$ *with center* U*, radius* $\mathtt{r}, 0 < \mathtt{r} < 1$ *and* $\Gamma > 0$ *as follows:*

$$\mathcal{M}(U,\mathtt{r},\Gamma) = \{\mathcal{W} \in K\colon \Lambda(U,\mathcal{W},\Gamma) > 1 - \mathtt{r}, \Pi(U,\mathcal{W},\Gamma) < \mathtt{r}, \mathcal{M}(U,\mathcal{W},\Gamma) < \mathtt{r}\}.$$

Definition 9. *Let* $(K,\Lambda,\Pi,\mathcal{M},\star,\bullet)$ *is a NPMS and* $\{U_{\beta_1}\}$ *be a sequence in* K*. Then* $\{U_{\beta_1}\}$ *is called:*

(a) *a convergent if exists* $U \in K$ *such that*

$$\lim_{\beta_1 \to +\infty} \Lambda(U_{\beta_1}, U, \Gamma) = 1, \lim_{\beta_1 \to +\infty} \Pi(U_{\beta_1}, U, \Gamma) = 0, \lim_{\beta_1 \to +\infty} \mathcal{M}(U_{\beta_1}, U, \Gamma) = 0,\ \forall\ \Gamma > 0,$$

(b) *a Cauchy sequence, if for each* $\bar{f} > 0, \Gamma > 0$*, exists* $\beta_{1_0} \in \mathbb{N}$ *such that*

$$\Lambda(U_{\beta_1}, U_{\beta_1+\mathfrak{q}}, \Gamma) \ge 1 - \bar{f}, \Pi(U_{\beta_1}, U_{\beta_1+\mathfrak{q}}, \Gamma) \le \bar{f}, \Pi(U_{\beta_1}, U_{\beta_1+\mathfrak{q}}, \Gamma) \le \bar{f}, \text{for all } \beta_1, \mathfrak{m} \ge \beta_{1_0},$$

If every Cauchy sequence convergent in K*, then* $(K,\Lambda,\Pi,\mathcal{M},\star,\bullet)$ *is said to be complete NPMS.*

Lemma 1. *Let* $\{U_{\beta_1}\}$ *be a Cauchy sequence in NPMS* $(K,\Lambda,\Pi,\mathcal{M},\star,\bullet)$ *such that* $U_{\beta_1} \ne U_{\mathfrak{m}}$ *whenever* $\mathfrak{m}, \beta_1 \in \mathbb{N}$ *with* $\beta_1 \ne \mathfrak{m}$*. Then the sequence* $\{U_{\beta_1}\}$ *can converge to, at most, one limit point.*

Proof. Contrarily, suppose that $U_{\beta_1} \to U$, $U_{\beta_1} \to \mathcal{W}$, and $U \ne \mathcal{W}$.
Then, $\lim_{\beta_1 \to +\infty} \Lambda(U_{\beta_1}, U, \Gamma) = 1$, $\lim_{\beta_1 \to +\infty} \Pi(U_{\beta_1}, U, \Gamma) = 0$,
$\lim_{\beta_1 \to +\infty} \mathcal{M}(U_{\beta_1}, U, \Gamma) = 0$, and $\lim_{\beta_1 \to +\infty} \Lambda(U_{\beta_1}, \mathcal{W}, \Gamma) = 1$,
$\lim_{\beta_1 \to +\infty} \Pi(U_{\beta_1}, \mathcal{W}, \Gamma) = 0$, $\lim_{\beta_1 \to +\infty} \mathcal{M}(U_{\beta_1}, \mathcal{W}, \Gamma) = 0$, for all $\Gamma > 0$. Suppose

$$\Lambda(\mathcal{U},\mathcal{W},\Gamma) \geq \Lambda\left(\mathcal{U},\mathcal{U}_{\beta_1},\frac{\Gamma}{4}\right) \star \Lambda\left(\mathcal{U}_{\beta_1},\mathcal{U}_{\beta_1+1},\frac{\Gamma}{4}\right) \star \Lambda\left(\mathcal{U}_{\beta_1+1},\mathcal{U}_{\beta_1+2},\frac{\Gamma}{4}\right) \star \Lambda\left(\mathcal{U}_{\beta_1+2},\mathcal{W},\frac{\Gamma}{4}\right)$$
$$\to 1 \star 1 \star 1 \star 1, \quad \text{as} \quad \beta_1 \to +\infty,$$
$$\Pi(\mathcal{U},\mathcal{W},\Gamma) \leq \Pi\left(\mathcal{U},\mathcal{U}_{\beta_1},\frac{\Gamma}{4}\right) \bullet \Pi\left(\mathcal{U}_{\beta_1},\mathcal{U}_{\beta_1+1},\frac{\Gamma}{4}\right) \bullet \Pi\left(\mathcal{U}_{\beta_1+1},\mathcal{U}_{\beta_1+2},\frac{\Gamma}{4}\right) \bullet \Pi\left(\mathcal{U}_{\beta_1+2},\mathcal{W},\frac{\Gamma}{4}\right)$$
$$\to 0 \bullet 0 \bullet 0 \bullet 0, \quad \text{as} \quad \beta_1 \to +\infty,$$
$$\mathcal{M}(\mathcal{U},\mathcal{W},\Gamma) \leq \mathcal{M}\left(\mathcal{U},\mathcal{U}_{\beta_1},\frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(\mathcal{U}_{\beta_1},\mathcal{U}_{\beta_1+1},\frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(\mathcal{U}_{\beta_1+1},\mathcal{U}_{\beta_1+2},\frac{\Gamma}{4}\right)$$
$$\bullet \mathcal{M}\left(\mathcal{U}_{\beta_1+2},\mathcal{W},\frac{\Gamma}{4}\right)$$
$$\to 0 \bullet 0 \bullet 0 \bullet 0, \quad \text{as} \quad \beta_1 \to +\infty.$$

That is $\Lambda(\mathcal{U},\mathcal{W},\Gamma) \geq 1 \star 1 \star 1 \star 1 = 1, \Pi(\mathcal{U},\mathcal{W},\Gamma) \leq 0 \bullet 0 \bullet 0 \bullet 0 = 0$ and $\mathcal{M}(\mathcal{U},\mathcal{W},\Gamma) \leq 0 \bullet 0 \bullet 0 \bullet 0 = 0$. Hence $\mathcal{U} = \mathcal{W}$, i.e., the sequence converges to at most one limit point. □

Lemma 2. *Let $(K,\Lambda,\Pi,\mathcal{M},\star,\bullet)$ is a NPMS. If for some $0 < \natural < 1$ and for any $\mathcal{U},\mathcal{W} \in K$, $\Gamma > 0$,*

$$\Lambda(\mathcal{U},\mathcal{W},\Gamma) \geq \Lambda\left(\mathcal{U},\mathcal{W},\frac{\Gamma}{\natural}\right), \Pi(\mathcal{U},\mathcal{W},\Gamma) \leq \Pi\left(\mathcal{U},\mathcal{W},\frac{\Gamma}{\natural}\right), \mathcal{M}(\mathcal{U},\mathcal{W},\Gamma) \leq \mathcal{M}\left(\mathcal{U},\mathcal{W},\frac{\Gamma}{\natural}\right) \tag{1}$$

then $\mathcal{U} = \mathcal{W}$.

Proof. By (1) follows that

$$\Lambda(\mathcal{U},\mathcal{W},\Gamma) \geq \Lambda\left(\mathcal{U},\mathcal{W},\frac{\Gamma}{\natural^{\beta_1}}\right), \Pi(\mathcal{U},\mathcal{W},\Gamma) \leq \Pi\left(\mathcal{U},\mathcal{W},\frac{\Gamma}{\natural^{\beta_1}}\right),$$
$$\mathcal{M}(\mathcal{U},\mathcal{W},\Gamma) \leq \mathcal{M}\left(\mathcal{U},\mathcal{W},\frac{\Gamma}{\natural^{\beta_1}}\right), \beta_1 \in \mathbb{N}, \Gamma > 0.$$

Now

$$\Lambda(\mathcal{U},\mathcal{W},\Gamma) \geq \lim_{\beta_1 \to +\infty} \Lambda\left(\mathcal{U},\mathcal{W},\frac{\Gamma}{\natural^{\beta_1}}\right) = 1,$$
$$\Pi(\mathcal{U},\mathcal{W},\Gamma) \leq \lim_{\beta_1 \to +\infty} \Pi\left(\mathcal{U},\mathcal{W},\frac{\Gamma}{\natural^{\beta_1}}\right) = 0,$$
$$\mathcal{M}(\mathcal{U},\mathcal{W},\Gamma) \leq \lim_{\beta_1 \to +\infty} \mathcal{M}\left(\mathcal{U},\mathcal{W},\frac{\Gamma}{\natural^{\beta_1}}\right) = 0, \Gamma > 0.$$

Also, by definition of (iii), (viii), (xiii), i.e., $\mathcal{U} = \mathcal{W}$. □

Theorem 1. *Suppose $(K,\Lambda,\Pi,\mathcal{M},\star,\bullet)$ is a complete NPMS, $\natural \in (0,1)$ and assume that*

$$\lim_{\Gamma \to +\infty} \Lambda(\mathcal{U},\mathcal{W},\Gamma) = 1, \lim_{\Gamma \to +\infty} \Pi(\mathcal{U},\mathcal{W},\Gamma) = 0 \quad \text{and} \quad \lim_{\Gamma \to +\infty} \mathcal{M}(\mathcal{U},\mathcal{W},\Gamma) = 0, \tag{2}$$

for all $\mathcal{U},\mathcal{W} \in K$ and $\Gamma > 0$. Let $\wp\colon K \to K$ be a mapping satisfying

$$\Lambda(\wp\mathcal{U},\wp\mathcal{W},\natural\Gamma) \geq \Lambda(\mathcal{U},\mathcal{W},\Gamma),$$
$$\Pi(\wp\mathcal{U},\wp\mathcal{W},\natural\Gamma) \leq \Pi(\mathcal{U},\mathcal{W},\Gamma) \quad \text{and} \quad \mathcal{M}(\wp\mathcal{U},\wp\mathcal{W},\natural\Gamma) \leq \mathcal{M}(\mathcal{U},\mathcal{W},\Gamma), \tag{3}$$

for all $\mathcal{U},\mathcal{W} \in K$ and $\Gamma > 0$. Then $\wp$ has a unique fixed point (shortly, ufp).

Proof. Consider a point U_0 of K and define a sequence U_{β_1} by $U_{\beta_1} = \wp^{\beta_1} U_0 = \wp U_{\beta_1 - 1}$, $\beta_1 \in \mathbb{N}$.
By utilising (3) for all $\Gamma > 0$, we obtain

$$\begin{aligned}
\Lambda(U_{\beta_1}, U_{\beta_1+1}, \natural\Gamma) &= \Lambda(\wp U_{\beta_1-1}, \wp U_{\beta_1}, \natural\Gamma) \geq \Lambda(U_{\beta_1-1}, U_{\beta_1}, \Gamma) \geq \Lambda\left(U_{\beta_1-2}, U_{\beta_1-1}, \frac{\Gamma}{\natural}\right) \\
&\geq \Lambda\left(U_{\beta_1-3}, U_{\beta_1-2}, \frac{\Gamma}{\natural^2}\right) \geq \cdots \geq \Lambda\left(U_0, U_1, \frac{\Gamma}{\natural^{\beta_1-1}}\right), \\
\Pi(U_{\beta_1}, U_{\beta_1+1}, \natural\Gamma) &= \Pi(\wp U_{\beta_1-1}, \wp U_{\beta_1}, \natural\Gamma) \leq \Pi(U_{\beta_1-1}, U_{\beta_1}, \Gamma) \leq \Pi\left(U_{\beta_1-2}, U_{\beta_1-1}, \frac{\Gamma}{\natural}\right) \\
&\leq \Pi\left(U_{\beta_1-3}, U_{\beta_1-2}, \frac{\Gamma}{\natural^2}\right) \leq \cdots \leq \Pi\left(U_0, U_1, \frac{\Gamma}{\natural^{\beta_1-1}}\right).
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}(U_{\beta_1}, U_{\beta_1+1}, \natural\Gamma) &= \mathcal{M}(\wp U_{\beta_1-1}, \wp U_{\beta_1}, \Gamma) \leq \mathcal{M}(U_{\beta_1-1}, U_{\beta_1}, \Gamma) \leq \mathcal{M}\left(U_{\beta_1-2}, U_{\beta_1-1}, \frac{\Gamma}{\natural}\right) \\
&\leq \mathcal{M}\left(U_{\beta_1-3}, U_{\beta_1-2}, \frac{\Gamma}{\natural^2}\right) \leq \cdots \leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{\natural^{\beta_1-1}}\right).
\end{aligned}$$

We obtain

$$\begin{gathered}
\Lambda(U_{\beta_1}, U_{\beta_1+1}, \natural\Gamma) \geq \Lambda\left(U_0, U_1, \frac{\Gamma}{\natural^{\beta_1-1}}\right), \\
\Pi(U_{\beta_1}, U_{\beta_1+1}, \natural\Gamma) \leq \Pi\left(U_0, U_1, \frac{\Gamma}{\natural^{\beta_1-1}}\right) \quad \text{and} \quad \mathcal{M}(U_{\beta_1}, U_{\beta_1+1}, \natural\Gamma) \leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{\natural^{\beta_1-1}}\right).
\end{gathered} \tag{4}$$

Consequently,

$$\begin{aligned}
\Lambda(U_{\beta_1}, U_{\beta_1+2}, \natural\Gamma) &= \Lambda(\wp U_{\beta_1-1}, \wp U_{\beta_1+1}, \natural\Gamma) \geq \Lambda(U_{\beta_1-1}, U_{\beta_1+1}, \Gamma) \geq \Lambda\left(U_{\beta_1-2}, U_{\beta_1}, \frac{\Gamma}{\natural}\right) \\
&\geq \Lambda\left(U_{\beta_1-3}, U_{\beta_1-1}, \frac{\Gamma}{\natural^2}\right) \geq \cdots \geq \Lambda\left(U_0, U_2, \frac{\Gamma}{\natural^{\beta_1-1}}\right), \\
\Pi(U_{\beta_1}, U_{\beta_1+2}, \natural\Gamma) &= \Pi(\wp U_{\beta_1-1}, \wp U_{\beta_1+1}, \natural\Gamma) \leq \Pi(U_{\beta_1-1}, U_{\beta_1+1}, \Gamma) \leq \Pi\left(U_{\beta_1-2}, U_{\beta_1}, \frac{\Gamma}{\natural}\right) \\
&\leq \Pi\left(U_{\beta_1-3}, U_{\beta_1-1}, \frac{\Gamma}{\natural^2}\right) \leq \cdots \leq \Pi\left(U_0, U_2, \frac{\Gamma}{\natural^{\beta_1-1}}\right).
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}(U_{\beta_1}, U_{\beta_1+2}, \natural\Gamma) &= \mathcal{M}(\wp U_{\beta_1-1}, \wp U_{\beta_1+1}, \Gamma) \leq \mathcal{M}(U_{\beta_1-1}, U_{\beta_1+1}, \Gamma) \leq \mathcal{M}\left(U_{\beta_1-2}, U_{\beta_1}, \frac{\Gamma}{\natural}\right) \\
&\leq \mathcal{M}\left(U_{\beta_1-3}, U_{\beta_1-1}, \frac{\Gamma}{\natural^2}\right) \leq \cdots \leq \mathcal{M}\left(U_0, U_2, \frac{\Gamma}{\natural^{\beta_1-1}}\right).
\end{aligned}$$

We obtain

$$\begin{gathered}
\Lambda(U_{\beta_1}, U_{\beta_1+2}, \natural\Gamma) \geq \Lambda\left(U_0, U_2, \frac{\Gamma}{\natural^{\beta_1-1}}\right), \\
\Pi(U_{\beta_1}, U_{\beta_1+2}, \natural\Gamma) \leq \Pi\left(U_0, U_2, \frac{\Gamma}{\natural^{\beta_1-1}}\right) \quad \text{and} \quad \mathcal{M}(U_{\beta_1}, U_{\beta_1+2}, \natural\Gamma) \leq \mathcal{M}\left(U_0, U_2, \frac{\Gamma}{\natural^{\beta_1-1}}\right).
\end{gathered} \tag{5}$$

It follows that

$$\Lambda(U_{\beta_1}, U_{\beta_1+3}, \natural\Gamma) = \Lambda(\wp U_{\beta_1-1}, \wp U_{\beta_1+2}, \natural\Gamma) \geq \Lambda(U_{\beta_1-1}, U_{\beta_1+2}, \Gamma) \geq \Lambda\left(U_{\beta_1-2}, U_{\beta_1+1}, \frac{\Gamma}{\natural}\right)$$
$$\geq \Lambda\left(U_{\beta_1-3}, U_{\beta_1}, \frac{\Gamma}{\natural^2}\right) \geq \cdots \geq \Lambda\left(U_0, U_3, \frac{\Gamma}{\natural^{\beta_1-1}}\right),$$
$$\Pi(U_{\beta_1}, U_{\beta_1+3}, \natural\Gamma) = \Pi(\wp U_{\beta_1-1}, \wp U_{\beta_1+2}, \natural\Gamma) \leq \Pi(U_{\beta_1-1}, U_{\beta_1+2}, \Gamma) \leq \Pi\left(U_{\beta_1-2}, U_{\beta_1+1}, \frac{\Gamma}{\natural}\right)$$
$$\leq \Pi\left(U_{\beta_1-3}, U_{\beta_1}, \frac{\Gamma}{\natural^2}\right) \leq \cdots \leq \Pi\left(U_0, U_3, \frac{\Gamma}{\natural^{\beta_1-1}}\right).$$

and

$$\mathcal{M}(U_{\beta_1}, U_{\beta_1+3}, \natural\Gamma) = \mathcal{M}(\wp U_{\beta_1-1}, \wp U_{\beta_1+2}, \Gamma) \leq \mathcal{M}(U_{\beta_1-1}, U_{\beta_1+2}, \Gamma) \leq \mathcal{M}\left(U_{\beta_1-2}, U_{\beta_1+1}, \frac{\Gamma}{\natural}\right)$$
$$\leq \mathcal{M}\left(U_{\beta_1-3}, U_{\beta_1}, \frac{\Gamma}{\natural^2}\right) \leq \cdots \leq \mathcal{M}\left(U_0, U_3, \frac{\Gamma}{\natural^{\beta_1-1}}\right).$$

We obtain

$$\Lambda(U_{\beta_1}, U_{\beta_1+3}, \natural\Gamma) \geq \Lambda\left(U_0, U_3, \frac{\Gamma}{\natural^{\beta_1-1}}\right),$$
$$\Pi(U_{\beta_1}, U_{\beta_1+3}, \natural\Gamma) \leq \Pi\left(U_0, U_3, \frac{\Gamma}{\natural^{\beta_1-1}}\right) \quad \text{and} \quad \mathcal{M}(U_{\beta_1}, U_{\beta_1+3}, \natural\Gamma) \leq \mathcal{M}\left(U_0, U_3, \frac{\Gamma}{\natural^{\beta_1-1}}\right).$$

Similarly, for $\mathrm{j} = 1, 2, 3, \ldots$, we obtain

$$\Lambda(U_{\beta_1}, U_{\beta_1+3\mathrm{j}+1}, \natural\Gamma) \geq \Lambda\left(U_0, U_{3\mathrm{j}+1}, \frac{\Gamma}{\natural^{\beta_1-1}}\right), \Pi(U_{\beta_1}, U_{\beta_1+3\mathrm{j}+1}, \natural\Gamma) \leq \Pi\left(U_0, U_{3\mathrm{j}+1}, \frac{\Gamma}{\natural^{\beta_1-1}}\right)$$
$$\text{and} \quad \mathcal{M}(U_{\beta_1}, U_{\beta_1+3\mathrm{j}+1}, \natural\Gamma) \leq \mathcal{M}\left(U_0, U_{3\mathrm{j}+1}, \frac{\Gamma}{\natural^{\beta_1-1}}\right), \tag{6}$$

$$\Lambda(U_{\beta_1}, U_{\beta_1+3\mathrm{j}+2}, \natural\Gamma) \geq \Lambda\left(U_0, U_{3\mathrm{j}+2}, \frac{\Gamma}{\natural^{\beta_1-1}}\right), \Pi(U_{\beta_1}, U_{\beta_1+3\mathrm{j}+2}, \natural\Gamma) \leq \Pi\left(U_0, U_{3\mathrm{j}+2}, \frac{\Gamma}{\natural^{\beta_1-1}}\right)$$
$$\text{and} \quad \mathcal{M}(U_{\beta_1}, U_{\beta_1+3\mathrm{j}+2}, \natural\Gamma) \leq \mathcal{M}\left(U_0, U_{3\mathrm{j}+2}, \frac{\Gamma}{\natural^{\beta_1-1}}\right),$$

$$\Lambda(U_{\beta_1}, U_{\beta_1+3\mathrm{j}+3}, \natural\Gamma) \geq \Lambda\left(U_0, U_{3\mathrm{j}+3}, \frac{\Gamma}{\natural^{\beta_1-1}}\right), \Pi(U_{\beta_1}, U_{\beta_1+3\mathrm{j}+3}, \natural\Gamma) \leq \Pi\left(U_0, U_{3\mathrm{j}+3}, \frac{\Gamma}{\natural^{\beta_1-1}}\right)$$
$$\text{and} \quad \mathcal{M}(U_{\beta_1}, U_{\beta_1+3\mathrm{j}+3}, \natural\Gamma) \leq \mathcal{M}\left(U_0, U_{3\mathrm{j}+3}, \frac{\Gamma}{\natural^{\beta_1-1}}\right). \tag{7}$$

By using (4), we obtain for each $\mathrm{j} = 1, 2, 3, \ldots,$

$$\Lambda(U_0, U_{3\mathrm{j}+1}, \Gamma) \geq \Lambda\left(U_0, U_1, \frac{\Gamma}{4}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^2}\right)$$
$$\star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^3}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^4}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^5}\right)$$
$$\star \cdots \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^{3\mathrm{j}-1}}\right),$$

$$\Pi(U_0, U_{3j+1}, \Gamma) \leq \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^2}\right)$$
$$\bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^3}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^4}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^5}\right)$$
$$\bullet \cdots \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{3j-1}}\right)$$

and

$$\mathcal{M}(U_0, U_{3j+1}, \Gamma) \leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^2}\right)$$
$$\bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^3}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^4}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^5}\right)$$
$$\bullet \cdots \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{3j-1}}\right).$$

Now, from (6), we get

$$\begin{aligned}\Lambda(U_{\beta_1}, U_{\beta_1+3j+1}, \natural\Gamma) &\geq \Lambda\left(U_0, U_{3j+1}, \frac{\Gamma}{\natural^{\beta_1-1}}\right) \\ &\geq \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1}}\right) \\ &\star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+1}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+2}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+3}}\right) \\ &\star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+4}}\right) \star \cdots \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^{3j+\beta_1-2}}\right), \end{aligned} \tag{8}$$

$$\begin{aligned}\Pi(U_{\beta_1}, U_{\beta_1+3j+1}, \natural\Gamma) &\leq \Pi\left(U_0, U_{3j+1}, \frac{\Gamma}{\natural^{\beta_1-1}}\right) \\ &\leq \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1}}\right) \\ &\bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+1}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+2}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+3}}\right) \\ &\bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+4}}\right) \bullet \cdots \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{3j+\beta_1-2}}\right) \end{aligned} \tag{9}$$

and

$$\begin{aligned}\mathcal{M}(U_{\beta_1}, U_{\beta_1+3j+1}, \natural\Gamma) &\leq \mathcal{M}\left(U_0, U_{3j+1}, \frac{\Gamma}{\natural^{\beta_1-1}}\right) \\ &\leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1}}\right) \\ &\bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+1}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+2}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+3}}\right) \\ &\bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+4}}\right) \bullet \cdots \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{3j+\beta_1-2}}\right). \end{aligned} \tag{10}$$

By using (4) and (5), we obtain for each $\mathrm{j} = 1, 2, 3, \ldots,$

$$\Lambda(U_0, U_{3\mathrm{j}+2}, \Gamma) \geq \Lambda\left(U_0, U_1, \frac{\Gamma}{4}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^2}\right)$$
$$\star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^3}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^4}\right) \star \Lambda\left(U_0, U_2, \frac{\Gamma}{4\natural^5}\right)$$
$$\star \cdots \star \Lambda\left(U_0, U_2, \frac{\Gamma}{4\natural^{3\mathrm{j}-1}}\right),$$

$$\Pi(U_0, U_{3\mathrm{j}+2}, \Gamma) \leq \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^2}\right)$$
$$\bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^3}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^4}\right) \bullet \Pi\left(U_0, U_2, \frac{\Gamma}{4\natural^5}\right)$$
$$\bullet \cdots \bullet \Pi\left(U_0, U_2, \frac{\Gamma}{4\natural^{3\mathrm{j}-1}}\right)$$

and

$$\mathcal{M}(U_0, U_{3\mathrm{j}+2}, \Gamma) \leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^2}\right)$$
$$\bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^3}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^4}\right) \bullet \mathcal{M}\left(U_0, U_2, \frac{\Gamma}{4\natural^5}\right)$$
$$\bullet \cdots \bullet \mathcal{M}\left(U_0, U_2, \frac{\Gamma}{4\natural^{3\mathrm{j}-1}}\right).$$

Now, from (6), we get

$$\begin{aligned}\Lambda(U_{\beta_1}, U_{\beta_1+3\mathrm{j}+2}, \natural\Gamma) &\geq \Lambda\left(U_0, U_{3\mathrm{j}+2}, \frac{\Gamma}{\natural^{\beta_1-1}}\right) \\ &\geq \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1}}\right) \\ &\star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+1}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+2}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+3}}\right) \\ &\star \Lambda\left(U_0, U_2, \frac{\Gamma}{4\natural^{\beta_1+4}}\right) \star \cdots \star \Lambda\left(U_0, U_2, \frac{\Gamma}{4\natural^{3\mathrm{j}+\beta_1-2}}\right), \end{aligned} \tag{11}$$

$$\begin{aligned}\Pi(U_{\beta_1}, U_{\beta_1+3\mathrm{j}+2}, \natural\Gamma) &\leq \Pi\left(U_0, U_{3\mathrm{j}+2}, \frac{\Gamma}{\natural^{\beta_1-1}}\right) \\ &\leq \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1}}\right) \\ &\bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+1}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+2}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+3}}\right) \\ &\bullet \Pi\left(U_0, U_2, \frac{\Gamma}{4\natural^{\beta_1+4}}\right) \bullet \cdots \bullet \Pi\left(U_0, U_2, \frac{\Gamma}{4\natural^{3\mathrm{j}+\beta_1-2}}\right) \end{aligned} \tag{12}$$

and

$$
\begin{aligned}
\mathcal{M}(U_{\beta_1}, U_{\beta_1+3\mathrm{j}+2}, \natural\Gamma) &\leq \mathcal{M}\left(U_0, U_{3\mathrm{j}+2}, \frac{\Gamma}{\natural^{\beta_1-1}}\right) \\
&\leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1}}\right) \\
&\bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+1}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+2}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+3}}\right) \\
&\bullet \mathcal{M}\left(U_0, U_2, \frac{\Gamma}{4\natural^{\beta_1+4}}\right) \bullet \cdots \bullet \mathcal{M}\left(U_0, U_2, \frac{\Gamma}{4\natural^{3\mathrm{j}+\beta_1-2}}\right).
\end{aligned} \tag{13}
$$

By using (4) and (5), we obtain for each $\mathrm{j} = 1, 2, 3, \ldots,$

$$
\begin{aligned}
\Lambda(U_0, U_{3\mathrm{j}+3}, \Gamma) \geq \Lambda\left(U_0, U_1, \frac{\Gamma}{4}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^2}\right) \\
\star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^3}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^4}\right) \star \Lambda\left(U_0, U_3, \frac{\Gamma}{4\natural^5}\right) \\
\star \cdots \star \Lambda\left(U_0, U_3, \frac{\Gamma}{4\natural^{3\mathrm{j}-1}}\right),
\end{aligned}
$$

$$
\begin{aligned}
\Pi(U_0, U_{3\mathrm{j}+3}, \Gamma) \leq \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^2}\right) \\
\bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^3}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^4}\right) \bullet \Pi\left(U_0, U_3, \frac{\Gamma}{4\natural^5}\right) \\
\bullet \cdots \bullet \Pi\left(U_0, U_3, \frac{\Gamma}{4\natural^{3\mathrm{j}-1}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{M}(U_0, U_{3\mathrm{j}+3}, \Gamma) \leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^2}\right) \\
\bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^3}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^4}\right) \bullet \mathcal{M}\left(U_0, U_3, \frac{\Gamma}{4\natural^5}\right) \\
\bullet \cdots \bullet \mathcal{M}\left(U_0, U_3, \frac{\Gamma}{4\natural^{3\mathrm{j}-1}}\right).
\end{aligned}
$$

Now, from (6), we get

$$
\begin{aligned}
\Lambda(U_{\beta_1}, U_{\beta_1+3\mathrm{j}+3}, \natural\Gamma) &\geq \Lambda\left(U_0, U_{3\mathrm{j}+3}, \frac{\Gamma}{\natural^{\beta_1-1}}\right) \\
&\geq \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1}}\right) \\
&\star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+1}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+2}}\right) \star \Lambda\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+3}}\right) \\
&\star \Lambda\left(U_0, U_3, \frac{\Gamma}{4\natural^{\beta_1+4}}\right) \star \cdots \star \Lambda\left(U_0, U_3, \frac{\Gamma}{4\natural^{3\mathrm{j}+\beta_1-2}}\right),
\end{aligned} \tag{14}
$$

$$\begin{aligned}\Pi(U_{\beta_1}, U_{\beta_1+3j+3}, \natural\Gamma) &\leq \Pi\left(U_0, U_{3j+3}, \frac{\Gamma}{\natural^{\beta_1-1}}\right)\\ &\leq \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1}}\right)\\ &\bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+1}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+1}}\right) \bullet \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+3}}\right)\\ &\bullet \Pi\left(U_0, U_3, \frac{\Gamma}{4\natural^{\beta_1+4}}\right) \bullet \cdots \bullet \Pi\left(U_0, U_3, \frac{\Gamma}{4\natural^{3j+\beta_1-2}}\right) \end{aligned} \tag{15}$$

and

$$\begin{aligned}\mathcal{M}(U_{\beta_1}, U_{\beta_1+3j+3}, \natural\Gamma) &\leq \mathcal{M}\left(U_0, U_{3j+3}, \frac{\Gamma}{\natural^{\beta_1-1}}\right)\\ &\leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1}}\right)\\ &\bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+1}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+2}}\right) \bullet \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1+3}}\right)\\ &\bullet \mathcal{M}\left(U_0, U_2, \frac{\Gamma}{4\natural^{\beta_1+4}}\right) \bullet \cdots \bullet \mathcal{M}\left(U_0, U_2, \frac{\Gamma}{4\natural^{3j+\beta_1-2}}\right). \end{aligned} \tag{16}$$

Using (8)–(16), for each case $\beta_1 \to +\infty$, we deduce that

$$\lim_{\beta_1\to+\infty} \Lambda(U_{\beta_1}, U_{\beta_1+i}, \Gamma) = 1 \star 1 \star \cdots \star 1 = 1,$$
$$\lim_{\beta_1\to+\infty} \Pi(U_{\beta_1}, U_{\beta_1+i}, \Gamma) = 0 \bullet 0 \bullet \cdots \bullet 0 = 0$$

and

$$\lim_{\beta_1\to+\infty} \mathcal{M}(U_{\beta_1}, U_{\beta_1+i}, \Gamma) = 0 \bullet 0 \bullet \cdots \bullet 0 = 0.$$

Which implies that $\{U_{\beta_1}\}$ is a Cauchy sequence. Since $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ is a complete NPMS, we have

$$\lim_{\beta_1\to+\infty} U_{\beta_1} = U.$$

Now, U is a fixed point of $\wp$, using conditions (v), (x), (xv) and Equation (2), we obtained

$$\begin{aligned}\Lambda(U, \wp U, \Gamma) &\geq \Lambda\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \star \Lambda\left(U_{\beta_1}, U_{\beta_1+1}, \frac{\Gamma}{4}\right) \star \Lambda\left(U_{\beta_1+1}, U_{\beta_1+2}, \frac{\Gamma}{4}\right) \star \Lambda\left(U_{\beta_1+2}, \wp U, \frac{\Gamma}{4}\right)\\ &= \Lambda\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \star \Lambda\left(\wp U_{\beta_1-1}, \wp U_{\beta_1}, \frac{\Gamma}{4}\right) \star \Lambda\left(\wp U_{\beta_1}, \wp U_{\beta_1+1}, \frac{\Gamma}{4}\right)\\ &\star \Lambda\left(\wp U_{\beta_1+1}, \wp U, \frac{\Gamma}{4}\right)\\ &\geq \Lambda\left(U, U_{\beta_1}, \frac{\Gamma}{4}\right) \star \Lambda\left(U_{\beta_1-1}, U_{\beta_1}, \frac{\Gamma}{4\natural^{\beta_1-2}}\right) \star \Lambda\left(U_{\beta_1}, U_{\beta_1+1}, \frac{\Gamma}{4\natural^{\beta_1-1}}\right)\\ &\star \Lambda\left(U_{\beta_1+1}, U, \frac{\Gamma}{4\natural}\right)\\ &\to 1 \star 1 \star 1 \star 1 = 1 \quad \text{as} \quad \beta_1 \to +\infty, \end{aligned}$$

$$\begin{aligned}\Pi(U,\wp U,\Gamma) &\le \Pi\left(U,U_{\beta_1},\frac{\Gamma}{4}\right)\bullet\Pi\left(U_{\beta_1},U_{\beta_1+1},\frac{\Gamma}{4}\right)\bullet\Pi\left(U_{\beta_1+1},U_{\beta_1+2},\frac{\Gamma}{4}\right)\bullet\Pi\left(U_{\beta_1+2},\wp U,\frac{\Gamma}{4}\right)\\ &= \Pi\left(U,U_{\beta_1},\frac{\Gamma}{4}\right)\bullet\Pi\left(\wp U_{\beta_1-1},\wp U_{\beta_1},\frac{\Gamma}{4}\right)\bullet\Pi\left(\wp U_{\beta_1},\wp U_{\beta_1+1},\frac{\Gamma}{4}\right)\\ &\bullet\Pi\left(\wp U_{\beta_1+1},\wp U,\frac{\Gamma}{4}\right)\\ &\le \Pi\left(U,U_{\beta_1},\frac{\Gamma}{4}\right)\bullet\Pi\left(U_{\beta_1-1},U_{\beta_1},\frac{\Gamma}{4\natural^{\beta_1-2}}\right)\bullet\Pi\left(U_{\beta_1},U_{\beta_1+1},\frac{\Gamma}{4\natural^{\beta_1-1}}\right)\\ &\bullet\Pi\left(U_{\beta_1+1},U,\frac{\Gamma}{4\natural}\right)\\ &\to 0\bullet 0\bullet 0\bullet 0=0 \quad \text{as} \quad \beta_1\to+\infty,\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}(U,\wp U,\Gamma) &\le \mathcal{M}\left(U,U_{\beta_1},\frac{\Gamma}{4}\right)\bullet\mathcal{M}\left(U_{\beta_1},U_{\beta_1+1},\frac{\Gamma}{4}\right)\\ &\bullet\mathcal{M}\left(U_{\beta_1+1},U_{\beta_1+2},\frac{\Gamma}{4}\right)\bullet\mathcal{M}\left(U_{\beta_1+2},\wp U,\frac{\Gamma}{4}\right)\\ &= \mathcal{M}\left(U,U_{\beta_1},\frac{\Gamma}{4}\right)\bullet\mathcal{M}\left(\wp U_{\beta_1-1},\wp U_{\beta_1},\frac{\Gamma}{4}\right)\bullet\mathcal{M}\left(\wp U_{\beta_1},\wp U_{\beta_1+1},\frac{\Gamma}{4}\right)\\ &\bullet\mathcal{M}\left(\wp U_{\beta_1+1},\wp U,\frac{\Gamma}{4}\right)\\ &\le \mathcal{M}\left(U,U_{\beta_1},\frac{\Gamma}{4}\right)\bullet\mathcal{M}\left(U_{\beta_1-1},U_{\beta_1},\frac{\Gamma}{4\natural^{\beta_1-2}}\right)\bullet\mathcal{M}\left(U_{\beta_1},U_{\beta_1+1},\frac{\Gamma}{4\natural^{\beta_1-1}}\right)\\ &\bullet\mathcal{M}\left(U_{\beta_1+1},U,\frac{\Gamma}{4\natural}\right)\\ &\to 0\bullet 0\bullet 0\bullet 0=0 \quad \text{as} \quad \beta_1\to+\infty,\end{aligned}$$

Hence, $\wp U = U$. Let $\wp\tilde{\mu} = \tilde{\mu}$ for some $\tilde{\mu} \in K$, then

$$\begin{aligned}1 \ge \Lambda(\tilde{\mu},U,\Gamma) = \Lambda(\wp\tilde{\mu},\wp U,\Gamma) &\ge \Lambda\left(\tilde{\mu},U,\frac{\Gamma}{\natural}\right) = \Lambda\left(\wp\tilde{\mu},\wp U,\frac{\Gamma}{\natural}\right)\\ &\ge \Lambda\left(\tilde{\mu},U,\frac{\Gamma}{\natural^2}\right) \ge \cdots \ge \Lambda\left(\tilde{\mu},U,\frac{\Gamma}{\natural^{\beta_1}}\right) \to 1 \quad \text{as} \quad \beta_1\to+\infty,\\ 0 \le \Pi(\tilde{\mu},U,\Gamma) = \Pi(\wp\tilde{\mu},\wp U,\Gamma) &\le \Pi\left(\tilde{\mu},U,\frac{\Gamma}{\natural}\right) = \Pi\left(\wp\tilde{\mu},\wp U,\frac{\Gamma}{\natural}\right)\\ &\le \Pi\left(\tilde{\mu},U,\frac{\Gamma}{\natural^2}\right) \le \cdots \le \Pi\left(\tilde{\mu},U,\frac{\Gamma}{\natural^{\beta_1}}\right) \to 0 \quad \text{as} \quad \beta_1\to+\infty,\end{aligned}$$

and

$$\begin{aligned}0 \le \mathcal{M}(\tilde{\mu},U,\Gamma) = \mathcal{M}(\wp\tilde{\mu},\wp U,\Gamma) &\le \mathcal{M}\left(\tilde{\mu},U,\frac{\Gamma}{\natural}\right) = \mathcal{M}\left(\wp\tilde{\mu},\wp U,\frac{\Gamma}{\natural}\right)\\ &\le \mathcal{M}\left(\tilde{\mu},U,\frac{\Gamma}{\natural^2}\right) \le \cdots \le \mathcal{M}\left(\tilde{\mu},U,\frac{\Gamma}{\natural^{\beta_1}}\right) \to 0 \quad \text{as} \quad \beta_1\to+\infty,\end{aligned}$$

using by (iii), (viii) and (xiii), $U = \tilde{\mu}$. □

Definition 10. *Let* $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ *be a NPMS. A function* $\wp\colon K \to K$ *is an NPC (neutrosophic pentagonal contraction) if* $\exists\ 0 < \natural < 1$, *such that*

$$\frac{1}{\Lambda(\mathcal{P}U, \mathcal{P}\mathcal{W}, \Gamma)} - 1 \leq \natural\left[\frac{1}{\Lambda(U, \mathcal{W}, \Gamma)} - 1\right] \tag{17}$$

$$\Pi(\mathcal{P}U, \mathcal{P}\mathcal{W}, \Gamma) \leq \natural\Pi(U, \mathcal{W}, \Gamma), \tag{18}$$

and

$$\mathcal{M}(\mathcal{P}U, \mathcal{P}\mathcal{W}, \Gamma) \leq \natural\mathcal{M}(U, \mathcal{W}, \Gamma), \tag{19}$$

for all $U, \mathcal{W} \in K$ *and* $\Gamma > 0$.

Theorem 2. *Let* $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ *be a complete NPMS with* $\chi\colon K \times K \to [1, +\infty)$ *and assume that*

$$\lim_{\Gamma\to+\infty} \mathcal{M}(U, \mathcal{W}, \Gamma) = 0, \lim_{\Gamma\to+\infty} \Pi(U, \mathcal{W}, \Gamma) = 0 \quad \text{and} \quad \lim_{\Gamma\to+\infty} \Lambda(U, \mathcal{W}, \Gamma) = 1, \tag{20}$$

for all $U, \mathcal{W} \in K$ *and* $\Gamma > 0$. *Let* $\wp\colon K \to K$ *be a ND-controlled contraction. Furthermore, assume that for an arbitrary* $U_0 \in K$, *and* $\beta_1, \mathfrak{q} \in \mathbb{N}$, *where* $U_{\beta_1} = \wp^{\beta_1} U_0 = \wp U_{\beta_1 - 1}$. *Then,* $\wp$ *has a ufp.*

Proof. Suppose U_0 be a point of K and define a sequence $\{U_{\beta_1}\}$ by $U_{\beta_1} = \wp^{\beta_1} U_0 = \wp U_{\beta_1 - 1}, \beta_1 \in \mathbb{N}$. Using by (17)–(19) for all $\Gamma > 0, \beta_1 > \mathfrak{q}$, we deduce

$$\begin{aligned}\frac{1}{\Lambda(U_{\beta_1}, U_{\beta_1+1}, \Gamma)} - 1 &= \frac{1}{\Lambda(\wp U_{\beta_1-1}, \wp U_{\beta_1}, \Gamma)} - 1 \\ &\leq \natural\left[\frac{1}{\Lambda(U_{\beta_1-1}, U_{\beta_1}, \Gamma)}\right] = \frac{\natural}{\Lambda(U_{\beta_1-1}, U_{\beta_1}, \Gamma)} - \natural.\end{aligned}$$

This implies

$$\begin{aligned}\frac{1}{\Lambda(U_{\beta_1}, U_{\beta_1+1}, \Gamma)} &\leq \frac{\natural}{\Lambda(U_{\beta_1-1}, U_{\beta_1}, \Gamma)} + (1 - \natural) \\ &\leq \frac{\natural^2}{\Lambda(U_{\beta_1-2}, U_{\beta_1-1}, \Gamma)} + \natural(1 - \natural) + (1 - \natural).\end{aligned}$$

In this manner, we conclude that

$$\begin{aligned}\frac{1}{\Lambda(U_{\beta_1}, U_{\beta_1+1}, \Gamma)} &\leq \frac{\natural^{\beta_1}}{\Lambda(U_0, U_1, \Gamma)} + \natural^{\beta_1-1}(1 - \natural) + \natural^{\beta_1-2}(1 - \natural) \\ &+ \cdots + \natural(1 - \natural) + (1 - \natural) \\ &\leq \frac{\natural^{\beta_1}}{\Lambda(U_0, U_1, \Gamma)} + (\natural^{\beta_1-1} + \natural^{\beta_1-2} + \cdots + 1)(1 - \natural) \\ &\leq \frac{\natural^{\beta_1}}{\Lambda(U_0, U_1, \Gamma)} + (1 - \natural^{\beta_1}).\end{aligned}$$

We obtain

$$\frac{1}{\frac{\natural^{\beta_1}}{\Lambda(U_0, U_1, \Gamma)} + (1 - \natural^{\beta_1})} \leq \Lambda(U_{\beta_1}, U_{\beta_1+1}, \Gamma)$$

$$\Pi(U_{\beta_1}, U_{\beta_1+1}, \Gamma) = \Pi(\wp U_{\beta_1-1}, \wp U_{\beta_1}, \Gamma) \leq \natural \Pi(U_{\beta_1-1}, U_{\beta_1}, \Gamma) = \Pi(\wp U_{\beta_1-2}, \wp U_{\beta_1-1}, \Gamma)$$
$$\leq \natural^2 \Pi(U_{\beta_1-2}, U_{\beta_1-1}, \Gamma) \leq \cdots \leq \natural^{\beta_1} \Pi(U_0, U_1, \Gamma)$$

and

$$\mathcal{M}(U_{\beta_1}, U_{\beta_1+1}, \Gamma) = \mathcal{M}(\wp U_{\beta_1-1}, \wp U_{\beta_1}, \Gamma) \leq \natural \mathcal{M}(U_{\beta_1-1}, U_{\beta_1}, \Gamma) = \mathcal{M}(\wp U_{\beta_1-2}, \wp U_{\beta_1-1}, \Gamma)$$
$$\leq \natural^2 \mathcal{M}(U_{\beta_1-2}, U_{\beta_1-1}, \Gamma) \leq \cdots \leq \natural^{\beta_1} \mathcal{M}(U_0, U_1, \Gamma). \quad (21)$$

It again follows that

$$\begin{aligned} \frac{1}{\Lambda(U_{\beta_1}, U_{\beta_1+2}, \Gamma)} - 1 &= \frac{1}{\Lambda(\wp U_{\beta_1-1}, \wp U_{\beta_1+1}, \Gamma)} - 1 \\ &\leq \natural \left[\frac{1}{\Lambda(U_{\beta_1-1}, U_{\beta_1+1}, \Gamma)} \right] \\ &= \frac{\natural}{\Lambda(U_{\beta_1-1}, U_{\beta_1+1}, \Gamma)} - \natural \\ \Rightarrow \frac{1}{\Lambda(U_{\beta_1}, U_{\beta_1+2}, \Gamma)} &\leq \frac{\natural}{\Lambda(U_{\beta_1-1}, U_{\beta_1+1}, \Gamma)} + (1-\natural) \\ &\leq \frac{\natural^2}{\Lambda(U_{\beta_1-2}, U_{\beta_1}, \Gamma)} + \natural(1-\natural) + (1-\natural). \end{aligned}$$

In this manner, we conclude that

$$\begin{aligned} \frac{1}{\Lambda(U_{\beta_1}, U_{\beta_1+2}, \Gamma)} &\leq \frac{\natural^{\beta_1}}{\Lambda(U_0, U_2, \Gamma)} + \natural^{\beta_1-1}(1-\natural) + \natural^{\beta_1-2}(1-\natural) \\ &+ \cdots + \natural(1-\natural) + (1-\natural) \\ &\leq \frac{\natural^{\beta_1}}{\Lambda(U_0, U_2, \Gamma)} + (\natural^{\beta_1-1} + \natural^{\beta_1-2} + \cdots + 1)(1-\natural) \\ &\leq \frac{\natural^{\beta_1}}{\Lambda(U_0, U_2, \Gamma)} + (1-\natural^{\beta_1}). \end{aligned}$$

We obtain

$$\frac{1}{\frac{\natural^{\beta_1}}{\Lambda(U_0, U_2, \Gamma)} + (1-\natural^{\beta_1})} \leq \Lambda(U_{\beta_1}, U_{\beta_1+2}, \Gamma)$$

$$\Pi(U_{\beta_1}, U_{\beta_1+2}, \Gamma) = \Pi(\wp U_{\beta_1-1}, \wp U_{\beta_1+1}, \Gamma) \leq \natural \Pi(U_{\beta_1-1}, U_{\beta_1+1}, \Gamma) = \Pi(\wp U_{\beta_1-2}, \wp U_{\beta_1}, \Gamma)$$
$$\leq \natural^2 \Pi(U_{\beta_1-2}, U_{\beta_1}, \Gamma) \leq \cdots \leq \natural^{\beta_1} \Pi(U_0, U_2, \Gamma)$$

and

$$\mathcal{M}(U_{\beta_1}, U_{\beta_1+2}, \Gamma) = \mathcal{M}(\wp U_{\beta_1-1}, \wp U_{\beta_1+1}, \Gamma) \leq \natural \mathcal{M}(U_{\beta_1-1}, U_{\beta_1+1}, \Gamma) = \mathcal{M}(\wp U_{\beta_1-2}, \wp U_{\beta_1}, \Gamma)$$
$$\leq \natural^2 \mathcal{M}(U_{\beta_1-2}, U_{\beta_1}, \Gamma) \leq \cdots \leq \natural^{\beta_1} \mathcal{M}(U_0, U_2, \Gamma).$$

Consequently,

$$\begin{aligned}\frac{1}{\Lambda(U_{\beta_1}, U_{\beta_1+3}, \Gamma)} - 1 &= \frac{1}{\Lambda(\wp U_{\beta_1-1}, \wp U_{\beta_1+2}, \Gamma)} - 1 \\ &\leq \natural\left[\frac{1}{\Lambda(U_{\beta_1-1}, U_{\beta_1+2}, \Gamma)}\right] \\ &= \frac{\natural}{\Lambda(U_{\beta_1-1}, U_{\beta_1+2}, \Gamma)} - \natural.\end{aligned}$$

This implies

$$\begin{aligned}\frac{1}{\Lambda(U_{\beta_1}, U_{\beta_1+3}, \Gamma)} &\leq \frac{\natural}{\Lambda(U_{\beta_1-1}, U_{\beta_1+2}, \Gamma)} + (1-\natural) \\ &\leq \frac{\natural^2}{\Lambda(U_{\beta_1-2}, U_{\beta_1+1}, \Gamma)} + \natural(1-\natural) + (1-\natural).\end{aligned}$$

In this manner, we conclude that

$$\begin{aligned}\frac{1}{\Lambda(U_{\beta_1}, U_{\beta_1+3}, \Gamma)} &\leq \frac{\natural^{\beta_1}}{\Lambda(U_0, U_3, \Gamma)} + \natural^{\beta_1-1}(1-\natural) + \natural^{\beta_1-2}(1-\natural) \\ &\quad + \cdots + \natural(1-\natural) + (1-\natural) \\ &\leq \frac{\natural^{\beta_1}}{\Lambda(U_0, U_3, \Gamma)} + (\natural^{\beta_1-1} + \natural^{\beta_1-2} + \cdots + 1)(1-\natural) \\ &\leq \frac{\natural^{\beta_1}}{\Lambda(U_0, U_3, \Gamma)} + (1-\natural^{\beta_1}).\end{aligned}$$

We obtain

$$\frac{1}{\frac{\natural^{\beta_1}}{\Lambda(U_0,U_3,\Gamma)} + (1-\natural^{\beta_1})} \leq \Lambda(U_{\beta_1}, U_{\beta_1+3}, \Gamma),$$

$$\begin{aligned}\Pi(U_{\beta_1}, U_{\beta_1+3}, \Gamma) &= \Pi(\wp U_{\beta_1-1}, \wp U_{\beta_1+2}, \Gamma) \leq \natural\Pi(U_{\beta_1-1}, U_{\beta_1+2}, \Gamma) = \Pi(\wp U_{\beta_1-2}, \wp U_{\beta_1+1}, \Gamma) \\ &\leq \natural^2\Pi(U_{\beta_1-2}, U_{\beta_1+1}, \Gamma) \leq \cdots \leq \natural^{\beta_1}\Pi(U_0, U_3, \Gamma)\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}(U_{\beta_1}, U_{\beta_1+3}, \Gamma) &= \mathcal{M}(\wp U_{\beta_1-1}, \wp U_{\beta_1+2}, \Gamma) \leq \natural\mathcal{M}(U_{\beta_1-1}, U_{\beta_1+2}, \Gamma) = \mathcal{M}(\wp U_{\beta_1-2}, \wp U_{\beta_1+1}, \Gamma) \\ &\leq \natural^2\mathcal{M}(U_{\beta_1-2}, U_{\beta_1+1}, \Gamma) \leq \cdots \leq \natural^{\beta_1}\mathcal{M}(U_0, U_3, \Gamma).\end{aligned}$$

Similarly, for $j = 1, 2, 3, \ldots$, we obtain

$$\frac{1}{\frac{\natural^{\beta_1}}{\Lambda(U_0,U_{3j+1},\Gamma)} + (1-\natural^{\beta_1})} \leq \Lambda(U_{\beta_1}, U_{\beta_1+3j+1}, \Gamma)$$

$$\Pi(U_{\beta_1}, U_{\beta_1+3j+1}, \natural\Gamma) \leq \natural^{\beta_1}\Pi(U_0, U_{3j+1}, \Gamma) \quad \text{and} \quad \mathcal{M}(U_{\beta_1}, U_{\beta_1+3j+1}, \natural\Gamma) \leq \natural^{\beta_1}\mathcal{M}(U_0, U_{3j+1}, \Gamma),$$

$$\frac{1}{\frac{\natural^{\beta_1}}{\Lambda(U_0,U_{3j+2},\Gamma)} + (1-\natural^{\beta_1})} \leq \Lambda(U_{\beta_1}, U_{\beta_1+3j+2}, \Gamma)$$

$$\Pi(U_{\beta_1}, U_{\beta_1+3j+2}, \natural\Gamma) \leq \natural^{\beta_1}\Pi(U_0, U_{3j+2}, \Gamma) \quad \text{and} \quad \mathcal{M}(U_{\beta_1}, U_{\beta_1+3j+2}, \natural\Gamma) \leq \natural^{\beta_1}\mathcal{M}(U_0, U_{3j+2}, \Gamma),$$

$$\frac{1}{\frac{\natural^{\beta_1}}{\Lambda(U_0,U_{3j+3},\Gamma)}+(1-\natural^{\beta_1})} \leq \Lambda(U_{\beta_1},U_{\beta_1+3j+3},\Gamma)$$

$$\Pi(U_{\beta_1},U_{\beta_1+3j+3},\natural\Gamma) \leq \natural^{\beta_1}\Pi(U_0,U_{3j+3},\Gamma) \quad \text{and} \quad \mathcal{M}(U_{\beta_1},U_{\beta_1+3j+3},\natural\Gamma) \leq \natural^{\beta_1}\mathcal{M}(U_0,U_{3j+3},\Gamma).$$

By using (21), we obtain for each $j = 1,2,3,....$,

$$\begin{aligned}
\Lambda(U_0,U_{3j+1},\Gamma) \geq{} & \frac{1}{\frac{1}{\Lambda(U_0,U_1,\frac{\Gamma}{4})}} \star \frac{1}{\frac{\natural}{\Lambda(U_0,U_1,\frac{\Gamma}{4})}+(1-\natural)} \star \frac{1}{\frac{\natural^2}{\Lambda(U_0,U_1,\frac{\Gamma}{4})}+(1-\natural^2)} \\
& \star \frac{1}{\frac{\natural^3}{\Lambda(U_0,U_1,\frac{\Gamma}{4})}+(1-\natural^3)} \star \frac{1}{\frac{\natural^4}{\Lambda(U_0,U_1,\frac{\Gamma}{4})}+(1-\natural^4)} \\
& \star \frac{1}{\frac{\natural^5}{\Lambda(U_0,U_1,\frac{\Gamma}{4})}+(1-\natural^5)} \star \frac{1}{\frac{\natural^6}{\Lambda(U_0,U_1,\frac{\Gamma}{4})}+(1-\natural^6)} \\
& \star \cdots \star \frac{1}{\frac{\natural^{3j}}{\Lambda(U_0,U_1,\frac{\Gamma}{4})}+(1-\natural^{3j})}
\end{aligned}$$

$$\begin{aligned}
\Pi(U_0,U_{3j+1},\Gamma) \leq{} & \Pi\left(U_0,U_1,\frac{\Gamma}{4}\right) \bullet \natural\Pi\left(U_0,U_1,\frac{\Gamma}{4}\right) \bullet \natural^2\Pi\left(U_0,U_1,\frac{\Gamma}{4}\right) \\
& \bullet \natural^3\Pi\left(U_0,U_1,\frac{\Gamma}{4}\right) \bullet \natural^4\Pi\left(U_0,U_1,\frac{\Gamma}{4}\right) \bullet \natural^5\Pi\left(U_0,U_1,\frac{\Gamma}{4}\right) \\
& \bullet \natural^6\Pi\left(U_0,U_1,\frac{\Gamma}{4}\right) \bullet \cdots \bullet \natural^{3j}\Pi\left(U_0,U_1,\frac{\Gamma}{4}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}(U_0,U_{3j+1},\Gamma) \leq{} & \mathcal{M}\left(U_0,U_1,\frac{\Gamma}{4}\right) \bullet \natural\mathcal{M}\left(U_0,U_1,\frac{\Gamma}{4}\right) \bullet \natural^2\mathcal{M}\left(U_0,U_1,\frac{\Gamma}{4}\right) \\
& \bullet \natural^3\mathcal{M}\left(U_0,U_1,\frac{\Gamma}{4}\right) \bullet \natural^4\mathcal{M}\left(U_0,U_1,\frac{\Gamma}{4}\right) \bullet \natural^5\mathcal{M}\left(U_0,U_1,\frac{\Gamma}{4}\right) \\
& \bullet \natural^6\mathcal{M}\left(U_0,U_1,\frac{\Gamma}{4}\right) \bullet \cdots \bullet \natural^{3j}\mathcal{M}\left(U_0,U_1,\frac{\Gamma}{4}\right).
\end{aligned}$$

Now, from (21), we get

$$\begin{aligned}
\Lambda(U_{\beta_1},U_{\beta_1+3j+1},\natural\Gamma) \geq{} & \Lambda\left(U_0,U_{3j+1},\frac{\Gamma}{\natural^{\beta_1-1}}\right) \\
\geq{} & \frac{1}{\frac{1}{\Lambda(U_0,U_1,\frac{\Gamma}{4\natural^{\beta_1-1}})}} \star \frac{1}{\frac{\natural}{\Lambda(U_0,U_1,\frac{\Gamma}{4\natural^{\beta_1-1}})}+(1-\natural)} \\
& \star \frac{1}{\frac{\natural^2}{\Lambda(U_0,U_1,\frac{\Gamma}{4\natural^{\beta_1-1}})}+(1-\natural^2)} \\
& \star \frac{1}{\frac{\natural^3}{\Lambda(U_0,U_1,\frac{\Gamma}{4\natural^{\beta_1-1}})}+(1-\natural^3)} \star \frac{1}{\frac{\natural^4}{\Lambda(U_0,U_1,\frac{\Gamma}{4\natural^{\beta_1-1}})}+(1-\natural^4)} \\
& \star \frac{1}{\frac{\natural^5}{\Lambda(U_0,U_1,\frac{\Gamma}{4\natural^{\beta_1-1}})}+(1-\natural^5)} \star \frac{1}{\frac{\natural^6}{\Lambda(U_0,U_1,\frac{\Gamma}{4\natural^{\beta_1-1}})}+(1-\natural^6)} \\
& \star \cdots \star \frac{1}{\frac{\natural^{3j}}{\Lambda(U_0,U_1,\frac{\Gamma}{4\natural^{\beta_1-1}})}+(1-\natural^{3j})},
\end{aligned} \tag{22}$$

$$\begin{aligned}\Pi(U_{\beta_1}, U_{\beta_1+3j+1}, \natural\Gamma) &\leq \Pi\left(U_0, U_{3j+1}, \frac{\Gamma}{\natural^{\beta_1-1}}\right)\\ &\leq \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural^2\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right)\\ &\bullet \natural^3\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural^4\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right)\\ &\bullet \natural^5\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural^6\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right)\\ &\bullet \cdots \bullet \natural^{3j}\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right)\end{aligned} \tag{23}$$

and

$$\begin{aligned}\mathcal{M}(U_{\beta_1}, U_{\beta_1+3j+1}, \natural\Gamma) &\leq \mathcal{M}\left(U_0, U_{3j+1}, \frac{\Gamma}{\natural^{\beta_1-1}}\right)\\ &\leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right)\\ &\bullet \natural^2\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural^3\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right)\\ &\bullet \natural^4\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural^5\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right)\\ &\bullet \natural^6\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \cdots \bullet \natural^{3j}\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right).\end{aligned}$$

By using (21), we obtain for each $j = 1, 2, 3, \ldots,$

$$\begin{aligned}\Lambda(U_0, U_{3j+2}, \Gamma) \geq &\frac{1}{\frac{1}{\Lambda(U_0,U_1,\frac{\Gamma}{4})}} \star \frac{1}{\frac{\natural}{\Lambda(U_0,U_1,\frac{\Gamma}{4})} + (1-\natural)} \star \frac{1}{\frac{\natural^2}{\Lambda(U_0,U_1,\frac{\Gamma}{4})} + (1-\natural^2)}\\ &\star \frac{1}{\frac{\natural^3}{\Lambda(U_0,U_1,\frac{\Gamma}{4})} + (1-\natural^3)} \star \frac{1}{\frac{\natural^4}{\Lambda(U_0,U_1,\frac{\Gamma}{4})} + (1-\natural^4)}\\ &\star \frac{1}{\frac{\natural^5}{\Lambda(U_0,U_1,\frac{\Gamma}{4})} + (1-\natural^5)} \star \frac{1}{\frac{\natural^6}{\Lambda(U_0,U_2,\frac{\Gamma}{4})} + (1-\natural^6)}\\ &\star \cdots \star \frac{1}{\frac{\natural^{3j}}{\Lambda(U_0,U_1,\frac{\Gamma}{4})} + (1-\natural^{3j})}\end{aligned}$$

$$\begin{aligned}\Pi(U_0, U_{3j+2}, \Gamma) \leq &\Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural\Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^2\Pi\left(U_0, U_1, \frac{\Gamma}{4}\right)\\ &\bullet \natural^3\Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^4\Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^5\Pi\left(U_0, U_1, \frac{\Gamma}{4}\right)\\ &\bullet \natural^6\Pi\left(U_0, U_2, \frac{\Gamma}{4}\right) \bullet \cdots \bullet \natural^{3j}\Pi\left(U_0, U_2, \frac{\Gamma}{4}\right)\end{aligned}$$

and

$$\mathcal{M}(U_0, U_{3j+2}, \Gamma) \leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^2\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^3\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^4\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^5\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^6\mathcal{M}\left(U_0, U_2, \frac{\Gamma}{4}\right) \bullet \cdots \bullet \natural^{3j}\mathcal{M}\left(U_0, U_2, \frac{\Gamma}{4}\right).$$

Now, from (21), we get

$$\begin{aligned}\Lambda(U_{\beta_1}, U_{\beta_1+3j+2}, \natural\Gamma) &\geq \Lambda\left(U_0, U_{3j+2}, \frac{\Gamma}{\natural^{\beta_1-1}}\right)\\ &\geq \frac{1}{\frac{1}{\Lambda(U_0,U_1,\frac{\Gamma}{4\natural^{\beta_1-1}})}} \star \frac{1}{\frac{\natural}{\Lambda(U_0,U_1,\frac{\Gamma}{4\natural^{\beta_1-1}})} + (1-\natural)}\\ &\star \frac{1}{\frac{\natural^2}{\Lambda(U_0,U_1,\frac{\Gamma}{4\natural^{\beta_1-1}})} + (1-\natural^2)} \star \frac{1}{\frac{\natural^3}{\Lambda(U_0,U_1,\frac{\Gamma}{4\natural^{\beta_1-1}})} + (1-\natural^3)}\\ &\star \frac{1}{\frac{\natural^4}{\Lambda(U_0,U_1,\frac{\Gamma}{4\natural^{\beta_1-1}})} + (1-\natural^4)} \star \frac{1}{\frac{\natural^5}{\Lambda(U_0,U_1,\frac{\Gamma}{4\natural^{\beta_1-1}})} + (1-\natural^5)}\\ &\star \frac{1}{\frac{\natural^6}{\Lambda(U_0,U_2,\frac{\Gamma}{4\natural^{\beta_1-1}})} + (1-\natural^6)} \star \cdots \star \frac{1}{\frac{\natural^{3j}}{\Lambda(U_0,U_1,\frac{\Gamma}{4\natural^{\beta_1-1}})} + (1-\natural^{3j})}\end{aligned} \tag{24}$$

$$\begin{aligned}\Pi(U_{\beta_1}, U_{\beta_1+3j+2}, \natural\Gamma) &\leq \Pi\left(U_0, U_{3j+2}, \frac{\Gamma}{\natural^{\beta_1-1}}\right)\\ &\leq \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right)\\ &\bullet \natural^2\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural^3\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right)\\ &\bullet \natural^4\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural^5\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right)\\ &\bullet \natural^6\Pi\left(U_0, U_2, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \cdots \bullet \natural^{3j}\Pi\left(U_0, U_2, \frac{\Gamma}{4\natural^{\beta_1-1}}\right)\end{aligned} \tag{25}$$

and

$$\begin{aligned}\mathcal{M}(U_{\beta_1}, U_{\beta_1+3j+2}, \natural\Gamma) &\leq \mathcal{M}\left(U_0, U_{3j+2}, \frac{\Gamma}{\natural^{\beta_1-1}}\right)\\ &\leq \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right)\\ &\bullet \natural^2\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural^3\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right)\\ &\bullet \natural^4\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural^5\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right)\\ &\bullet \natural^6\mathcal{M}\left(U_0, U_2, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \cdots \bullet \natural^{3j}\mathcal{M}\left(U_0, U_2, \frac{\Gamma}{4\natural^{\beta_1-1}}\right).\end{aligned} \tag{26}$$

By using (21), we obtain for each $\mathrm{j} = 1, 2, 3, \ldots,$

$$\begin{aligned}
\Lambda(U_0, U_{3\mathrm{j}+3}, \Gamma) \geq{}& \frac{1}{\frac{1}{\Lambda(U_0,U_1,\frac{\Gamma}{4})}} \star \frac{1}{\frac{\natural}{\Lambda(U_0,U_1,\frac{\Gamma}{4})} + (1-\natural)} \star \frac{1}{\frac{\natural^2}{\Lambda(U_0,U_1,\frac{\Gamma}{4})} + (1-\natural^2)} \\
&\star \frac{1}{\frac{\natural^3}{\Lambda(U_0,U_1,\frac{\Gamma}{4})} + (1-\natural^3)} \star \frac{1}{\frac{\natural^4}{\Lambda(U_0,U_1,\frac{\Gamma}{4})} + (1-\natural^4)} \\
&\star \frac{1}{\frac{\natural^5}{\Lambda(U_0,U_1,\frac{\Gamma}{4})} + (1-\natural^5)} \star \frac{1}{\frac{\natural^6}{\Lambda(U_0,U_3,\frac{\Gamma}{4})} + (1-\natural^6)} \\
&\star \cdots \star \frac{1}{\frac{\natural^{3\mathrm{j}}}{\Lambda(U_0,U_3,\frac{\Gamma}{4})} + (1-\natural^{3\mathrm{j}})}
\end{aligned}$$

$$\begin{aligned}
\Pi(U_0, U_{3\mathrm{j}+3}, \Gamma) \leq{}& \Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural\Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^2\Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \\
&\bullet \natural^3\Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^4\Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^5\Pi\left(U_0, U_1, \frac{\Gamma}{4}\right) \\
&\bullet \natural^6\Pi\left(U_0, U_3, \frac{\Gamma}{4}\right) \bullet \cdots \bullet \natural^{3\mathrm{j}}\Pi\left(U_0, U_3, \frac{\Gamma}{4}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}(U_0, U_{3\mathrm{j}+3}, \Gamma) \leq{}& \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^2\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \\
&\bullet \natural^3\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^4\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \bullet \natural^5\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4}\right) \\
&\bullet \natural^6\mathcal{M}\left(U_0, U_3, \frac{\Gamma}{4}\right) \bullet \cdots \bullet \natural^{3\mathrm{j}}\mathcal{M}\left(U_0, U_3, \frac{\Gamma}{4}\right).
\end{aligned}$$

Now, from (21), we get

$$\begin{aligned}
\Lambda(U_{\beta_1}, U_{\beta_1+3\mathrm{j}+3}, \natural\Gamma) \geq{}& \Lambda\left(U_0, U_{3\mathrm{j}+3}, \frac{\Gamma}{\natural^{\beta_1-1}}\right) \\
\geq{}& \frac{1}{\frac{1}{\Lambda(U_0,U_1,\frac{\Gamma}{4\natural^{\beta_1-1}})}} \star \frac{1}{\frac{\natural}{\Lambda(U_0,U_1,\frac{\Gamma}{4\natural^{\beta_1-1}})} + (1-\natural)} \\
&\star \frac{1}{\frac{\natural^2}{\Lambda(U_0,U_1,\frac{\Gamma}{4\natural^{\beta_1-1}})} + (1-\natural^2)} \star \frac{1}{\frac{\natural^3}{\Lambda(U_0,U_1,\frac{\Gamma}{4\natural^{\beta_1-1}})} + (1-\natural^3)} \\
&\star \frac{1}{\frac{\natural^4}{\Lambda(U_0,U_1,\frac{\Gamma}{4\natural^{\beta_1-1}})} + (1-\natural^4)} \star \frac{1}{\frac{\natural^5}{\Lambda(U_0,U_1,\frac{\Gamma}{4\natural^{\beta_1-1}})} + (1-\natural^5)} \\
&\star \frac{1}{\frac{\natural^6}{\Lambda(U_0,U_3,\frac{\Gamma}{4\natural^{\beta_1-1}})} + (1-\natural^6)} \\
&\star \cdots \star \frac{1}{\frac{\natural^{3\mathrm{j}}}{\Lambda(U_0,U_3,\frac{\Gamma}{4\natural^{\beta_1-1}})} + (1-\natural^{3\mathrm{j}})},
\end{aligned} \tag{27}$$

$$\begin{aligned}\Pi(U_{\beta_1}, U_{\beta_1+3j+3}, \natural\Gamma) &\le \Pi\left(U_0, U_{3j+3}, \frac{\Gamma}{\natural^{\beta_1-1}}\right) \\ &\le \Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \\ &\bullet \natural^2\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural^3\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \\ &\bullet \natural^4\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural^5\Pi\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \\ &\bullet \natural^6\Pi\left(U_0, U_3, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \cdots \bullet \natural^{3j}\Pi\left(U_0, U_3, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \end{aligned} \quad (28)$$

and

$$\begin{aligned}\mathcal{M}(U_{\beta_1}, U_{\beta_1+3j+3}, \natural\Gamma) &\le \mathcal{M}\left(U_0, U_{3j+3}, \frac{\Gamma}{\natural^{\beta_1-1}}\right) \\ &\le \mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \\ &\bullet \natural^2\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural^3\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \\ &\bullet \natural^4\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \natural^5\mathcal{M}\left(U_0, U_1, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \\ &\bullet \natural^6\mathcal{M}\left(U_0, U_3, \frac{\Gamma}{4\natural^{\beta_1-1}}\right) \bullet \cdots \bullet \natural^{3j}\mathcal{M}\left(U_0, U_3, \frac{\Gamma}{4\natural^{\beta_1-1}}\right). \end{aligned} \quad (29)$$

Using (22)–(29), for each case $\beta_1 \to +\infty$, we deduce

$$\lim_{\beta_1 \to +\infty} \Lambda(U_{\beta_1}, U_{\beta_1+q}, \Gamma) = 1 \star 1 \star \cdots \star 1 = 1,$$
$$\lim_{\beta_1 \to +\infty} \Pi(U_{\beta_1}, U_{\beta_1+q}, \Gamma) = 0 \bullet 0 \bullet \cdots \bullet 0 = 0,$$

and

$$\lim_{\beta_1 \to +\infty} \mathcal{M}(U_{\beta_1}, U_{\beta_1+q}, \Gamma) = 0 \bullet 0 \bullet \cdots \bullet 0 = 0.$$

It follows that $\{U_{\beta_1}\}$ is a Cauchy sequence. Since $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ be a complete NPMS, there exists

$$\lim_{\beta_1 \to +\infty} U_{\beta_1} = U.$$

Since (v), (x) and (xv), we get

$$\begin{aligned}\frac{1}{\Lambda(\wp U_{\beta_1}, \wp U, \Gamma)} - 1 &\le \natural\left[\frac{1}{\Lambda(U_{\beta_1}, U, \Gamma)} - 1\right] = \frac{\natural}{\Lambda(U_{\beta_1}, U, \Gamma)} - \natural \\ &\Rightarrow \frac{1}{\frac{\natural}{\Lambda(U_{\beta_1}, U, \Gamma)} + (1-\natural)} \le \Lambda(\wp U_{\beta_1}, \wp U, \Gamma).\end{aligned}$$

By the above inequality, we have

$$
\begin{aligned}
\Lambda(U,\wp U,\Gamma) &\geq \Lambda\left(U,U_{\beta_1},\frac{\Gamma}{4}\right)\star\Lambda\left(U_{\beta_1},U_{\beta_1+1},\frac{\Gamma}{4}\right)\star\Lambda\left(U_{\beta_1+1},\wp U_{\beta_1+2},\frac{\Gamma}{4}\right)\\
&\star\Lambda\left(U_{\beta_1+2},\wp U,\frac{\Gamma}{4}\right)\\
&\geq \Lambda\left(U,U_{\beta_1},\frac{\Gamma}{4}\right)\star\Lambda\left(\wp U_{\beta_1-1},\wp U_{\beta_1},\frac{\Gamma}{4}\right)\star\Lambda\left(\wp U_{\beta_1},\wp U_{\beta_1+1},\frac{\Gamma}{4}\right)\\
&\star\Lambda\left(\wp U_{\beta_1+1},\wp U,\frac{\Gamma}{4}\right)\\
&\geq \Lambda\left(U,U_{\beta_1},\frac{\Gamma}{4}\right)\star\frac{1}{\frac{\natural^{\beta_1-1}}{\Lambda(U_{\beta_1-1},U_{\beta_1},\frac{\Gamma}{4})}+(1-\natural^{\beta_1-1})}\star\frac{1}{\frac{\natural^{\beta_1}}{\Lambda(U_{\beta_1},U_{\beta_1+1},\frac{\Gamma}{4})}+(1-\natural^{\beta_1})}\\
&\star\frac{1}{\frac{\natural}{\Lambda(U_{\beta_1+1},U,\frac{\Gamma}{4})}+(1-\natural)}\\
&\to 1\star 1\star 1\star 1=1 \text{ as } \beta_1\to+\infty,
\end{aligned}
$$

$$
\begin{aligned}
\Pi(U,\wp U,\Gamma) &\leq \Pi\left(U,U_{\beta_1},\frac{\Gamma}{4}\right)\bullet\Pi\left(U_{\beta_1},U_{\beta_1+1},\frac{\Gamma}{4}\right)\bullet\Pi\left(U_{\beta_1+1},\wp U_{\beta_1+2},\frac{\Gamma}{4}\right)\\
&\bullet\Pi\left(U_{\beta_1+2},\wp U,\frac{\Gamma}{4}\right)\\
&\leq \Pi\left(U,U_{\beta_1},\frac{\Gamma}{4}\right)\bullet\Pi\left(\wp U_{\beta_1-1},\wp U_{\beta_1},\frac{\Gamma}{4}\right)\bullet\Pi\left(\wp U_{\beta_1},\wp U_{\beta_1+1},\frac{\Gamma}{4}\right)\\
&\bullet\Pi\left(\wp U_{\beta_1+1},\wp U,\frac{\Gamma}{4}\right)\\
&\leq \Pi\left(U,U_{\beta_1},\frac{\Gamma}{4}\right)\bullet\natural^{\beta_1-1}\Pi\left(U_{\beta_1-1},U_{\beta_1},\frac{\Gamma}{4}\right)\bullet\natural^{\beta_1}\Pi\left(U_{\beta_1},U_{\beta_1+1},\frac{\Gamma}{4}\right)\\
&\bullet\natural\Pi\left(U_{\beta_1+1},U,\frac{\Gamma}{4}\right)\\
&\to 0\bullet 0\bullet 0\bullet 0=0 \text{ as } \beta_1\to+\infty
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{M}(U,\wp U,\Gamma) &\leq \mathcal{M}\left(U,U_{\beta_1},\frac{\Gamma}{4}\right)\bullet\mathcal{M}\left(U_{\beta_1},U_{\beta_1+1},\frac{\Gamma}{4}\right)\bullet\mathcal{M}\left(U_{\beta_1+1},U_{\beta_1+2},\frac{\Gamma}{4}\right)\\
&\bullet\mathcal{M}\left(U_{\beta_1+2},\wp U,\frac{\Gamma}{4}\right)\\
&\leq \mathcal{M}\left(U,U_{\beta_1},\frac{\Gamma}{4}\right)\bullet\mathcal{M}\left(\wp U_{\beta_1-1},\wp U_{\beta_1},\frac{\Gamma}{4}\right)\bullet\mathcal{M}\left(\wp U_{\beta_1},\wp U_{\beta_1+1},\frac{\Gamma}{4}\right)\\
&\bullet\mathcal{M}\left(\wp U_{\beta_1},\wp U,\frac{\Gamma}{4}\right)\\
&\leq \mathcal{M}\left(U,U_{\beta_1},\frac{\Gamma}{4}\right)\bullet\natural^{\beta_1-1}\mathcal{M}\left(U_{\beta_1-1},U_{\beta_1},\frac{\Gamma}{4}\right)\bullet\natural^{\beta_1}\mathcal{M}\left(U_{\beta_1},U_{\beta_1+1},\frac{\Gamma}{4}\right)\\
&\bullet\natural\mathcal{M}\left(U_{\beta_1},U,\frac{\Gamma}{4}\right)\\
&\to 0\bullet 0\bullet 0\bullet 0=0 \text{ as } \beta_1\to+\infty.
\end{aligned}
$$

Hence, $\wp U = U$. Let $\wp\tilde{\mu} = \tilde{\mu}$ for some $\tilde{\mu} \in K$, then

$$\begin{aligned}\frac{1}{\Lambda(U,\tilde{\mu},\Gamma)} - 1 &= \frac{1}{\Lambda(\wp U,\wp\tilde{\mu},\Gamma)} - 1\\ &\leq \natural\left[\frac{1}{\Lambda(U,\tilde{\mu},\Gamma)} - 1\right] < \frac{1}{\Lambda(U,\tilde{\mu},\Gamma)} - 1,\end{aligned}$$

which is a contradiction.

$$\Pi(U,\tilde{\mu},\Gamma) = \Pi(\wp U,\wp\tilde{\mu},\Gamma) \leq \natural\Pi(U,\tilde{\mu},\Gamma) < \Pi(U,\tilde{\mu},\Gamma),$$

which is a contradiction and

$$\mathcal{M}(U,\tilde{\mu},\Gamma) = \mathcal{M}(\wp U,\wp\tilde{\mu},\Gamma) \leq \natural\mathcal{M}(U,\tilde{\mu},\Gamma) < \mathcal{M}(U,\tilde{\mu},\Gamma),$$

which is a contradiction. Therefore, we obtain $\Lambda(U,\tilde{\mu},\Gamma) = 1, \Pi(U,\tilde{\mu},\Gamma) = 0$ and $\mathcal{M}(U,\tilde{\mu},\Gamma) = 0$, hence, $U = \tilde{\mu}$. □

Example 2. *Let* $K = [0,1]$. *Define* $\Lambda, \Pi, \mathcal{M}\colon K \times K \times (0,+\infty) \to [0,1]$ *as*

$$\begin{aligned}\Lambda(U,\mathcal{W},\Gamma) &= \frac{\Gamma}{\Gamma + |U - \mathcal{W}|},\\ \Pi(U,\mathcal{W},\Gamma) &= \frac{|U - \mathcal{W}|}{\Gamma + |U - \mathcal{W}|},\\ \mathcal{M}(U,\mathcal{W},\Gamma) &= \frac{|U - \mathcal{W}|}{\Gamma}.\end{aligned}$$

Then, $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ *is a complete NPMS with continuous t-norm and t-co-norm, i.e.,* $\hat{\omega} \star \check{h} = \hat{\omega}\check{h}$ *and* $\hat{\omega} \bullet \check{h} = \max\{\hat{\omega}, \check{h}\}$.
Define $\wp\colon K \to K$ *by* $\wp(U) = \frac{1-3^{-U}}{11}$ *and set* $\natural \in [\frac{1}{2}, 1)$, *then*

$$\begin{aligned}\Lambda(\wp U,\wp\mathcal{W},\natural\Gamma) &= \Lambda\left(\frac{1-3^{-U}}{11}, \frac{1-3^{-\mathcal{W}}}{11}, \natural\Gamma\right)\\ &= \frac{\natural\Gamma}{\natural\Gamma + \left|\frac{1-3^{-U}}{11} - \frac{1-3^{-\mathcal{W}}}{11}\right|} = \frac{\natural\Gamma}{\natural\Gamma + \frac{|3^{-U}-3^{-\mathcal{W}}|}{11}}\\ &\geq \frac{\natural\Gamma}{\natural\Gamma + \frac{|U-\mathcal{W}|}{11}} = \frac{11\natural\Gamma}{11\natural\Gamma + |U - \mathcal{W}|} \geq \frac{\Gamma}{\Gamma + |U - \mathcal{W}|} = \Lambda(U,\mathcal{W},\Gamma),\end{aligned}$$

$$\begin{aligned}\Pi(\wp U,\wp\mathcal{W},\natural\Gamma) &= \Pi\left(\frac{1-3^{-U}}{11}, \frac{1-3^{-\mathcal{W}}}{11}, \natural\Gamma\right)\\ &= \frac{\left|\frac{1-3^{-U}}{11} - \frac{1-3^{-\mathcal{W}}}{11}\right|}{\natural\Gamma + \left|\frac{1-3^{-U}}{11} - \frac{1-3^{-\mathcal{W}}}{11}\right|} = \frac{\frac{|3^{-U}-3^{-\mathcal{W}}|}{11}}{\natural\Gamma + \frac{|3^{-U}-3^{-\mathcal{W}}|}{11}}\\ &= \frac{|3^{-U} - 3^{-\mathcal{W}}|}{11\natural\Gamma + |3^{-U} - 3^{-\mathcal{W}}|} \leq \frac{|U - \mathcal{W}|}{11\natural\Gamma + |U - \mathcal{W}|} \leq \frac{|U - \mathcal{W}|}{\Gamma + |U - \mathcal{W}|} = \Pi(U,\mathcal{W},\Gamma)\end{aligned}$$

and

$$
\begin{aligned}
\mathcal{M}(\wp U, \wp \mathcal{W}, \natural\Gamma) &= \mathcal{M}\left(\frac{1-3^{-U}}{11}, \frac{1-3^{-\mathcal{W}}}{11}, \natural\Gamma\right) \\
&= \frac{\left|\frac{1-3^{-U}}{11} - \frac{1-3^{-\mathcal{W}}}{11}\right|}{\natural\Gamma} = \frac{\frac{|3^{-U}-3^{-\mathcal{W}}|}{11}}{\natural\Gamma} \\
&= \frac{|3^{-U} - 3^{-\mathcal{W}}|}{11\natural\Gamma} \leq \frac{|U-\mathcal{W}|}{11\natural\Gamma} \leq \frac{|U-\mathcal{W}|}{\Gamma} = \mathcal{M}(U, \mathcal{W}, \Gamma).
\end{aligned}
$$

As a result, all of Theorem 1 criteria are satisfied, and 0 *is the only fixed point for* $\wp$*.*

4. Applications

4.1. Application to Fredholm Integral Equation

Let $K = \mathcal{C}([\jmath, \delta], \mathbb{R})$ be the set of real value continuous functions on $[\jmath, \delta]$. Consider the integral equation:

$$
U(\aleph) = \wedge(\aleph) + \ell \int_{\jmath}^{\delta} \mho(\aleph, \kappa) U(\aleph) \varsigma\kappa \quad \text{for} \quad \aleph, \kappa \in [\jmath, \delta], \tag{30}
$$

where $\wedge(\kappa)$ is a fuzzy function of $\kappa \in [\jmath, \delta]$, $\ell > 0$ and $\mho \colon \mathcal{C}([\jmath, \delta] \times \mathbb{R}) \to \mathbb{R}^+$. Define Λ, Π and $\mathcal{M}$ by means of

$$
\Lambda(U(\aleph), \mathcal{W}(\aleph), \Gamma) = \sup_{\aleph \in [\jmath, \delta]} \frac{\Gamma}{\Gamma + |U(\aleph) - \mathcal{W}(\aleph)|} \quad \forall \quad U, \mathcal{W} \in K \text{ and } \Gamma > 0,
$$

$$
\Pi(U(\aleph), \mathcal{W}(\aleph), \Gamma) = 1 - \sup_{\aleph \in [\jmath, \delta]} \frac{\Gamma}{\Gamma + |U(\aleph) - \mathcal{W}(\aleph)|} \quad \forall \quad U, \mathcal{W} \in K \text{ and } \Gamma > 0,
$$

and

$$
\mathcal{M}(U(\aleph), \mathcal{W}(\aleph), \Gamma) = \sup_{\aleph \in [\jmath, \delta]} \frac{|U(\aleph) - \mathcal{W}(\aleph)|}{\Gamma} \quad \forall \quad U, \mathcal{W} \in K \text{ and } \Gamma > 0,
$$

by continuous t-norm and continuous t-co-norm define by $\hat{\omega} \star \check{h} = \hat{\omega} \cdot \check{h}$ and $\hat{\omega} \bullet \check{h} = \max\{\hat{\omega}, \check{h}\}$. Then $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ is a complete NPMS. Suppose that $|\mho(\aleph, \kappa) U(\aleph) - \mho(\aleph, \kappa) \mathcal{W}(\aleph)| \leq |U(\aleph) - \mathcal{W}(\aleph)|$ for $U, \mathcal{W} \in K, 0 < \natural < 1$ and $\forall \aleph, \kappa \in [\jmath, \delta]$. Let $\mho(\aleph, \kappa)(\ell \int_{\jmath}^{\delta} d\kappa) \leq \natural < 1$. Then Equation (30) has a unique solution.

Proof. Define $\wp \colon K \to K$ by

$$
\wp U(\aleph) = \wedge(\aleph) + \ell \int_{\jmath}^{\delta} \mho(\aleph, \kappa) U(\aleph) \varsigma\kappa, \text{ for all } \aleph, \kappa \in [\jmath, \delta].
$$

Now, for all $U, \mathcal{W} \in K$, we deduce

$$
\begin{aligned}
\Lambda(\wp U(\aleph), \wp \mathcal{W}(\aleph), \natural\Gamma) &= \sup_{\aleph \in [\jmath, \delta]} \frac{\natural\Gamma}{\natural\Gamma + |\wp U(\aleph) - \wp \mathcal{W}(\aleph)|} \\
= \sup_{\aleph \in [\jmath, \delta]} &\frac{\natural\Gamma}{\natural\Gamma + | \wedge(\aleph) + \ell \int_{\jmath}^{\delta} \mho(\aleph, \kappa) U(\aleph) \varsigma\kappa - \wedge(\aleph) - \ell \int_{\jmath}^{\delta} \mho(\aleph, \kappa) U(\aleph) \varsigma\kappa|} \\
= \sup_{\aleph \in [\jmath, \delta]} &\frac{\natural\Gamma}{\natural\Gamma + |\ell \int_{\jmath}^{\delta} \mho(\aleph, \kappa) U(\aleph) \varsigma\kappa - \ell \int_{\jmath}^{\delta} \mho(\aleph, \kappa) U(\aleph) \varsigma\kappa|}
\end{aligned}
$$

$$
\begin{aligned}
&= \sup_{\aleph\in[\jmath,\delta]} \frac{\natural\Gamma}{\natural\Gamma + |\mho(\aleph,\kappa)U(\aleph) - \mho(\aleph,\kappa)\mathcal{W}(\aleph)|(\ell\int_\jmath^\delta \varsigma\kappa)} \\
&\geq \sup_{\aleph\in[\jmath,\delta]} \frac{\Gamma}{\Gamma + |U(\aleph) - \mathcal{W}(\aleph)|} \\
&\geq \Lambda(U(\aleph), \mathcal{W}(\aleph), \Gamma),
\end{aligned}
$$

$$
\begin{aligned}
\Pi(\wp U(\aleph), \wp\mathcal{W}(\aleph), \natural\Gamma) &= 1 - \sup_{\aleph\in[\jmath,\delta]} \frac{\natural\Gamma}{\natural\Gamma + |\wp U(\aleph) - \wp\mathcal{W}(\aleph)|} \\
&= 1 - \sup_{\aleph\in[\jmath,\delta]} \frac{\natural\Gamma}{\natural\Gamma + |\wedge(\aleph) + \ell\int_\jmath^\delta \mho(\aleph,\kappa)U(\aleph)\varsigma\kappa - \wedge(\aleph) - \ell\int_\jmath^\delta \mho(\aleph,\kappa)U(\aleph)\varsigma\kappa|} \\
&= 1 - \sup_{\aleph\in[\jmath,\delta]} \frac{\natural\Gamma}{\natural\Gamma + |\ell\int_\jmath^\delta \mho(\aleph,\kappa)U(\aleph)\varsigma\kappa - \ell\int_\jmath^\delta \mho(\aleph,\kappa)U(\aleph)\varsigma\kappa|} \\
&= 1 - \sup_{\aleph\in[\jmath,\delta]} \frac{\natural\Gamma}{\natural\Gamma + |\mho(\aleph,\kappa)U(\aleph) - \mho(\aleph,\kappa)\mathcal{W}(\aleph)|(\ell\int_\jmath^\delta \varsigma\kappa)} \\
&\leq 1 - \sup_{\aleph\in[\jmath,\delta]} \frac{\Gamma}{\Gamma + |U(\aleph) - \mathcal{W}(\aleph)|} \\
&\leq \Pi(U(\aleph), \mathcal{W}(\aleph), \Gamma),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{M}(\wp U(\aleph), \wp\mathcal{W}(\aleph), \natural\Gamma) &= \sup_{\aleph\in[\jmath,\delta]} \frac{|\wp U(\aleph) - \wp\mathcal{W}(\aleph)|}{\natural\Gamma} \\
&= \sup_{\aleph\in[\jmath,\delta]} \frac{|\wedge(\aleph) + \ell\int_\jmath^\delta \mho(\aleph,\kappa)U(\aleph)\varsigma\kappa - \wedge(\aleph) - \ell\int_\jmath^\delta \mho(\aleph,\kappa)U(\aleph)\varsigma\kappa|}{\natural\Gamma} \\
&= \sup_{\aleph\in[\jmath,\delta]} \frac{|\ell\int_\jmath^\delta \mho(\aleph,\kappa)U(\aleph)\varsigma\kappa - \ell\int_\jmath^\delta \mho(\aleph,\kappa)U(\aleph)\varsigma\kappa|}{\natural\Gamma} \\
&= \sup_{\aleph\in[\jmath,\delta]} \frac{|\mho(\aleph,\kappa)U(\aleph) - \mho(\aleph,\kappa)\mathcal{W}(\aleph)|(\ell\int_\jmath^\delta \varsigma\kappa)}{\natural\Gamma} \\
&\leq \sup_{\aleph\in[\jmath,\delta]} \frac{|U(\aleph) - \mathcal{W}(\aleph)|}{\Gamma} \\
&\leq \mathcal{M}(U(\aleph), \mathcal{W}(\aleph), \Gamma),
\end{aligned}
$$

As a result, $\wp$ has a ufp and all of the requirements of Theorem 1 are fulfilled. It is obvious that the Equation (30) has only one solution. □

Example 3. *Consider the non-linear integral equation.*

$$U(\aleph) = |\cos\aleph| + \frac{1}{11}\int_0^1 \kappa U(\kappa)\varsigma\kappa, \textit{ for all } \kappa \in [0,1].$$

Then it has a solution in K.

Proof. Let $\wp\colon K \to K$ be defined by

$$\wp U(\aleph) = |\cos\aleph| + \frac{1}{11}\int_0^1 \kappa U(\kappa)\varsigma\kappa,$$

and set $\mho(\aleph,\kappa)U(\aleph) = \frac{1}{11}\kappa U(\kappa)$ and $\mho(\aleph,\kappa)\mathcal{W}(\aleph) = \frac{1}{11}\kappa\mathcal{W}(\kappa)$, where $U,\mathcal{W} \in K$, and $\forall\, \aleph,\kappa \in [0,1]$. Then we obtain

$$\begin{aligned}|\mho(\aleph,\kappa)U(\aleph) - \mho(\aleph,\kappa)\mathcal{W}(\aleph)| &= |\frac{1}{11}\kappa U(\kappa) - \frac{1}{11}\kappa\mathcal{W}(\kappa)| \\ &= \frac{\kappa}{11}|U(\kappa) - \mathcal{W}(\kappa)| \leq |U(\kappa) - \mathcal{W}(\kappa)|.\end{aligned}$$

Furthermore, see that $\frac{1}{11}\int_0^1 \kappa\varsigma\kappa = \frac{1}{11}(\frac{(1)^2}{2} - \frac{(0)^2}{2}) = \frac{1}{11} = \natural \leq 1$, where $\ell = \frac{1}{11}$. Then, it follows that all criteria of the above application are easily verified and the above problem has a solution in K. □

4.2. Application to Fractional Differential Equations

In order to start, we need to review some basic definitions from the theory of fractional calculus.
For a function $U \in \mathcal{C}[0,1]$, the Reiman-Liouville fractional derivative of order $\ell > 0$ is given by

$$\frac{1}{\mathbf{F}(\beta_1 - \ell)}\frac{d^{\beta_1}}{d\aleph^{\beta_1}}\int_0^{\aleph}\frac{U(\jmath)d\jmath}{(\aleph - \jmath)^{\ell-\beta_1+1}} = \mathcal{D}^{\ell}U(\aleph),$$

For as long as the right hand side is pointwise defined on $[0,1]$, $[\ell]$ is the integer portion of the number ℓ, $\mathbf{F}$ is the Euler gamma function.
Consider the following fractional differential equation

$$\begin{aligned}&{}^{\jmath}\mathcal{D}^{\chi}U(\aleph) + \mathfrak{f}(\aleph, U(\aleph)) = 0, \quad 1 \leq \aleph \leq 0, \quad 2 \leq \chi > 1; \\ &U(0) = U(1) = 0,\end{aligned} \tag{31}$$

where $\mathfrak{f}$ is a continuous function from $[0,1] \times \mathbb{R}$ to $\mathbb{R}$ and ${}^{\jmath}\mathcal{D}^{\chi}$ represents the Caputo fractional derivative of order χ and it is defined by

$${}^{\jmath}\mathcal{D}^{\chi} = \frac{1}{\mathbf{F}(\beta_1 - \chi)}\int_0^{\aleph}\frac{U^{\beta_1}(\jmath)d\jmath}{(\aleph - \jmath)^{\chi-\beta_1+1}}.$$

The given fractional differential Equation (31) is equivalent

$$U(\aleph) = \int_0^1 \Upsilon(\aleph,\jmath)\mathfrak{f}(\aleph, U(\jmath))d\jmath,$$

for all $U \in \mathcal{Y}$ and $\aleph \in [0,1]$, where

$$\Upsilon(\aleph,\jmath) = \begin{cases} \frac{[\aleph(1-\jmath)]^{\chi-1} - (\aleph-\jmath)^{\chi-1}}{\mathbf{F}(\chi)}, & 0 \leq \jmath \leq \aleph \leq 1, \\ \frac{[\aleph(1-\jmath)]^{\chi-1}}{\mathbf{F}(\chi)}, & 0 \leq \aleph \leq \jmath \leq 1. \end{cases}$$

Let $\mathcal{C}([0,1],\mathbb{R}) = K$ be the space of all continuous functions defined on $[0,1]$. Define Λ, Π and $\mathcal{M}$ by means of

$$\Lambda(U(\aleph),\mathcal{W}(\aleph),\Gamma) = \sup_{\aleph\in[0,1]}\frac{\Gamma}{\Gamma + |U(\aleph) - \mathcal{W}(\aleph)|}, \text{ for all } U,\mathcal{W} \in K \text{ and } \Gamma > 0,$$

$$\Pi(U(\aleph),\mathcal{W}(\aleph),\Gamma) = 1 - \sup_{\aleph\in[0,1]}\frac{\Gamma}{\Gamma + |U(\aleph) - \mathcal{W}(\aleph)|}, \text{ for all } U,\mathcal{W} \in K \text{ and } \Gamma > 0$$

and

$$\mathcal{M}(U(\aleph), \mathcal{W}(\aleph), \Gamma) = \sup_{\aleph \in [0,1]} \frac{|U(\aleph) - \mathcal{W}(\aleph)|}{\Gamma}, \text{ for all } U, \mathcal{W} \in K \text{ and } \Gamma > 0,$$

with continuous t-norm and continuous t-co-norm define by $\hat{\omega} \star \check{h} = \hat{\omega} \cdot \check{h}$ and $\hat{\omega} \bullet \check{h} = \max\{\hat{\omega}, \check{h}\}$. Then $(K, \Lambda, \Pi, \mathcal{M}, \star, \bullet)$ is a complete NPMS.

Theorem 3. *Consider the nonlinear fractional differential Equation (31). Let us assume that the following claims are true:*

(i) *there exist* $\aleph \in [0,1]$ *and* $U, \mathcal{W} \in \mathcal{C}([0,1], \mathbb{R})$ *such that*

$$|\mathfrak{f}(\aleph, U) - \mathfrak{f}(\aleph, \mathcal{W})| \leq |U(\aleph) - \mathcal{W}(\aleph)|;$$

(ii) $\sup_{\aleph \in [0,1]} \int_0^1 \Upsilon(\aleph, \jmath) \varsigma \aleph \leq \natural < 1.$

Then the fractional differential Equation (31) has a unique solution in K.

Proof. Let us consider the function $\wp \colon K \to K$ defined by

$$KU(\aleph) = \int_0^1 \Upsilon(\aleph, \jmath) \mathfrak{f}(\aleph, U(\jmath)) d\jmath.$$

It is obvious that U^* is a solution to the problem (31) if U^* is a fixed point of $\wp$.
Now, for all $U, \mathcal{W} \in K$, we deduce

$$\begin{aligned}
\Lambda(\wp U(\aleph), \wp \mathcal{W}(\aleph), \natural\Gamma) &= \sup_{\aleph \in [0,1]} \frac{\natural\Gamma}{\natural\Gamma + |\wp U(\aleph) - \wp \mathcal{W}(\aleph)|} \\
&= \sup_{\aleph \in [0,1]} \frac{\natural\Gamma}{\natural\Gamma + |\int_0^1 \Upsilon(\aleph, \jmath) \mathfrak{f}(\aleph, U(\jmath)) d\jmath - \int_0^1 \Upsilon(\aleph, \jmath) \mathfrak{f}(\aleph, \mathcal{W}(\jmath)) d\jmath|} \\
&= \sup_{\aleph \in [0,1]} \frac{\natural\Gamma}{\natural\Gamma + \int_0^1 \Upsilon(\aleph, \jmath) |\mathfrak{f}(\aleph, U(\jmath)) - \mathfrak{f}(\aleph, \mathcal{W}(\jmath))| d\jmath} \\
&\geq \sup_{\aleph \in [0,1]} \frac{\Gamma}{\Gamma + |U(\aleph) - \mathcal{W}(\aleph)|} \\
&\geq \Lambda(U(\aleph), \mathcal{W}(\aleph), \Gamma),
\end{aligned}$$

$$\begin{aligned}
\Pi(\wp U(\aleph), \wp \mathcal{W}(\aleph), \natural\Gamma) &= 1 - \sup_{\aleph \in [0,1]} \frac{\natural\Gamma}{\natural\Gamma + |\wp U(\aleph) - \wp \mathcal{W}(\aleph)|} \\
&= 1 - \sup_{\aleph \in [0,1]} \frac{\natural\Gamma}{\natural\Gamma + |\int_0^1 \Upsilon(\aleph, \jmath) \mathfrak{f}(\aleph, U(\jmath)) d\jmath - \int_0^1 \Upsilon(\aleph, \jmath) \mathfrak{f}(\aleph, \mathcal{W}(\jmath)) d\jmath|} \\
&= 1 - \sup_{\aleph \in [0,1]} \frac{\natural\Gamma}{\natural\Gamma + \int_0^1 \Upsilon(\aleph, \jmath) |\mathfrak{f}(\aleph, U(\jmath)) - \mathfrak{f}(\aleph, \mathcal{W}(\jmath))| d\jmath} \\
&\leq 1 - \sup_{\aleph \in [0,1]} \frac{\Gamma}{\Gamma + |U(\aleph) - \mathcal{W}(\aleph)|} \\
&\leq \Pi(U(\aleph), \mathcal{W}(\aleph), \Gamma)
\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}(\wp U(\aleph), \wp \mathcal{W}(\aleph), \natural\Gamma) &= \sup_{\aleph\in[0,1]} \frac{|\wp U(\aleph) - \wp \mathcal{W}(\aleph)|}{\natural\Gamma} \\ &= \sup_{\aleph\in[0,1]} \frac{|\int_0^1 \Upsilon(\aleph,\jmath)\mathfrak{f}(\aleph,U(\jmath))d\jmath - \int_0^1 \Upsilon(\aleph,\jmath)\mathfrak{f}(\aleph,\mathcal{W}(\jmath))d\jmath|}{\natural\Gamma} \\ &= \sup_{\aleph\in[0,1]} \frac{\int_0^1 \Upsilon(\aleph,\jmath)|\mathfrak{f}(\aleph,U(\jmath)) - \mathfrak{f}(\aleph,\mathcal{W}(\jmath))|d\jmath}{\natural\Gamma} \\ &\leq \sup_{\aleph\in[0,1]} \frac{|U(\aleph) - \mathcal{W}(\aleph)|}{\Gamma} \\ &\leq \mathcal{M}(U(\aleph), \mathcal{W}(\aleph), \Gamma).\end{aligned}$$

As a result, $\wp$ has a ufp and all of Theorem 1 requirements are satisfied. It implies that there is only one solution to the Equation (30). □

5. Conclusions

We proposed the idea of neutrosophic pentagonal MS in this study and proved new varieties of fixed-point theorems. By applying a new methodology to an application based on the literature, we have shown that it outperforms our results.

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