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On the Realization of Exact Upper Bounds of the Best Approximations on the Classes $H^{1,1}$ by Favard Sums

Dmytro Bushev and Inna Kal'chuk *

Faculty of Information Technologies and Mathematics, Lesya Ukrainka Volyn National University, 43025 Lutsk, Ukraine; bushev-d@ukr.net
* Correspondence: k.inna.80@gmail.com

Abstract: In this paper, we find the sets of all extremal functions for approximations of the Hölder classes of H^1 2π-periodic functions of one variable by the Favard sums, which coincide with the set of all extremal functions realizing the exact upper bounds of the best approximations of this class by trigonometric polynomials. In addition, we obtain the sets of all of extremal functions for approximations of the class H^1 by linear methods of summation of Fourier series. Furthermore, we receive the set of all extremal functions for the class H^1 in the Korneichuk–Stechkin lemma and its analogue, the Stepanets lemma, for the Hölder class $H^{1,1}$ functions of two variables being 2π-periodic in each variable.

Keywords: Favard sums; best approximation; exact upper bounds; extremal functions; uniform metric

MSC: 41A52; 42A10

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1. Introduction

The exact values of approximation characteristics are especially valued in the theory of function approximation. Finding the exact values of approximation characteristics even for functions and classes of functions of one variable is a rare phenomenon. The exact values of approximation characteristics in the theory of approximation of functions and classes of functions of many variables being 2π-periodic in each variable, except the result of the work [1], are unknown.

In the theory of function approximation, as in other branches of mathematics, it is difficult to formulate the problem and attract the attention of specialists to it. The problem of finding the exact values of approximation characteristics for functions and classes of functions of many variables remains relevant. The exact values of approximation characteristics even for the simplest classes of functions of many variables have not been found. Forty years ago, the famous Ukrainian mathematician Oleksandr Stepanets called its solution the problem of the twenty-first century.

Let H^1, $H^{1,1}$ be the classes of functions $f(x)$ and $f(x,y)$ that are 2π-periodic in the variable x and the variables x, y, for which the following conditions hold, respectively:

$$|f(x) - f(x')| \leq |x - x'|, \quad |f(x,y) - f(x',y')| \leq |x - x'| + |y - y'|. \tag{1}$$

Let

$$E_n(f) = \inf_{T_{n-1}} \|f(x) - T_{n-1}(x)\|_C$$

be the best approximation of the function $f(x)$ by the trigonometric polynomials $T_{n-1}(x)$ of the degree $(n-1)$, where C is the space of 2π-periodic continuous functions with the uniform norm $\|f\|_C = \max_t |f(t)|$.

Let

$$E_{n,m}(f) := \inf_{T_{n-1,m-1}} \|f(x,y) - T_{n-1,m-1}(x,y)\|_C$$

be the best approximation of the function $f(x,y)$ by the trigonometric polynomials $T_{n-1,m-1}(x,y)$ of the degree $(n-1)$ in the variable x and the degree $(m-1)$ in the variable y in the uniform metric.

Let

$$F_n(u) = \frac{1}{2} + \sum_{k=1}^{n-1} \frac{k\pi}{2n} \cot \frac{k\pi}{2n} \cos ku$$

be the Favard kernel, and

$$F_n(f,x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) F_n(t-x) dt,$$

$$F_{n,m}(f,x,y) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(t,z) F_n(t-x) F_m(z-y) dt dz$$

be Favard sums of the degree $(n-1)$ and double rectangular Favard sums of the degree $(n-1)$ in the variable x and the degree $(m-1)$ in the variable y, respectively.

Favard proved in 1936 that

$$\mathcal{E}_n = \sup_{f \in H^1} \|f(x) - F_n(f,x)\|_C = \frac{\pi}{2n} = E_n(H^1) := \sup_{f \in H^1} E_n(f),$$

i.e., the Favard method implements the exact upper bound of the best approximations on the class H^1. In the work [1], the exact value of approximations of classes $H^{1,1}$ by Favard sums was found, namely, for $n,m \geq 2$

$$\mathcal{E}_{n,m} := \sup_{f \in H^{1,1}} \|f(x,y) - F_{n,m}(f,x,y)\|_C$$

$$= \frac{\pi}{2n} + \frac{\pi}{2m} + \frac{8}{\pi^2} \int_0^{\frac{\pi}{n}} \Phi_n(x) \Phi_m(x) dx, \tag{2}$$

where $\Phi_k(x) = \sum_{i=1}^{k-1} \overline{\Phi}_i^k(x)$ is the sum of permutations in descending order of the functions $\Phi_i^k = \left| \int_{i\pi/k}^{x} F_k(t) dt \right|$ (for definition of the permutation, see, e.g., [2] (p. 130)).

2. Main Result

Theorem 1. *For any natural numbers n and m, $n, m \geq 2$, it is asserted that*

$$\mathcal{E}_{n,m} > E_{n,m}\left(H^{1,1}\right) := \sup_{f \in H^{1,1}} E_{n,m}(f).$$

Theorem 1 was formulated without proof in [3]. We should note that the exact value of $E_{n,m}\left(H^{1,1}\right)$, as well as the best linear approximation method reflecting the class $H^{1,1}$ into the space of all trigonometric polynomials $T_{n-1,m-1}(x,y)$ of the degree at most $(n-1)$ in the variable x and $(m-1)$ in the variable y are unknown. However, it was found that $E_{n,m}\left(H^{1,1}\right) \geq \frac{\pi}{2n} + \frac{\pi}{2m}$. According to the result of J. Mairhuber [4], the polynomial of the best approximation $T_{n-1,m-1}(x,y)$ for the function $f(x,y)$ is not unique, which makes it difficult to find this polynomial.

Let us denote by $W^1_{[a,b]}$ and $W^{1,1}_p$ the classes of functions $f(x)$ and $f(x,y)$ defined on the segment $[a,b]$ and the rectangle $P = [a,b] \times [a_1,b_1]$ satisfying conditions (1). The summable function $\psi(x) \in \bigvee_{a,b}^{c}$ if almost everywhere on (a,c) $(a < c < b)$ $\psi(x) > 0$ $(\psi(x) < 0)$, almost everywhere on (c,b) $\psi(x) < 0$ $(\psi(x) > 0)$ and $\int_a^b \psi(t) dt = 0$.

Let $\psi(x) \in \bigvee_{a,b}^{c}$, $\varphi(y) \in \bigvee_{a_1,b_1}^{c_1}$ and $t = \rho(x)$, $z = \delta(y)$ be the functions defined by the equalities

$$\int_a^x \psi(t)dt = \int_a^{\rho(x)} \psi(t)dt, \quad x \in [a,c],\ \rho(x) \in [c,b],$$

$$\int_{a_1}^y \varphi(z)dz = \int_{a_1}^{\delta(y)} \varphi(z)dz, \quad y \in [a_1,c_1],\ \delta(y) \in [c_1,b_1],$$

and $\rho^{-1}(x)$ and $\delta^{-1}(x)$ be the inverse functions to $\rho(x)$ and $\delta(x)$.

M.P. Korneichuk [2] (pp. 190–198) for the class $W^1_{[a,b]}$ and O.I. Stepanets [5] (p. 52) for the class $W_p^{1,1}$ proved the following statements.

Lemma K [2]. *The following equalities hold*

$$\sup_{f\in W^1_{[a,b]}} \left| \int_a^b f(x)\psi(x)dx \right| = \int_a^c |\psi(t)|(\rho(t)-t)dt = \int_c^b |\psi(t)|(t-\rho^{-1}(t))dt$$

$$= \left| \int_a^b f^*(x)\psi(x)dx \right|. \tag{3}$$

In this case, the upper bound in (3) is implemented by functions from the class $W^1_{[a,b]}$ of the form $f^*(x) = K \pm x$, where K is arbitrary constant.

Lemma S [5]. *The following equalities hold*

$$\sup_{f\in W_p^{1,1}} \left| \int_a^b \int_{a_1}^{b_1} f(x,y)\psi(x)\varphi(y)dxdy \right|$$

$$= 2\int_a^c \int_{a_1}^{c_1} |\psi(t)\varphi(z)| \min\{\rho(t)-t, \delta(z)-z\} dt\, dz = \left| \int_a^b \int_{a_1}^{b_1} f^*(x,y)\psi(x)\varphi(y)dx\, dy \right|, \tag{4}$$

and the exact upper bound in (4) is realized by the function $f^*(x,y)$ specified in this lemma (see [5] (pp. 52–54)).

Let us denote by $\gamma^*_{nm}(x,y)$, $f^*(x)$, $f^*(x,y)$ the arbitrary extremal functions from the classes $H^{1,1}$, $W^1_{[a,b]}$, $W_P^{1,1}$ implementing exact upper bounds in (2)–(4), respectively, i.e., such that

$$\mathcal{E}_{n,m} = \|\gamma^*_{nm}(x,y) - F_{n,m}(\gamma^*_{nm}, x, y)\|_C,$$

$$\sup_{f\in W^1_{[a,b]}} \left| \int_a^b f(x)\psi(x)dx \right| = \left| \int_a^b f^*(x)\psi(x)dx \right|,$$

$$\sup_{f\in W_p^{1,1}} \left| \int_a^b \int_{a_1}^{b_1} f(x,y)\psi(x)\varphi(y)dxdy \right| = \int_a^b \int_{a_1}^{b_1} f^*(x,y)\psi(x)\varphi(y)dxdy.$$

Let us prove that all extremal functions $\gamma^*_{nm}(x,y)$ realizing the exact upper bound in (2) have the same oscillations equal to $\pi/n + \pi/m$. To do this, we have to establish that if two arbitrary extremal functions realizing the exact upper bound in (4) coincide on one of the larger sides of P, then they coincide on the entire rectangle and have the same oscillations. The proof of the last statement is based on the description of the set of all extremal functions that realize the exact upper bound in (3).

Lemma 1. *The set of all extremal functions $f^*(x)$ realizing the exact upper bound in (3) is the set of functions of the form $f^*(x) = K \pm x$, where K is an arbitrary constant.*

Proof. If for the arbitrary extremal function almost everywhere on $[a,b]$ $f^{*\prime}(x) = \pm 1$, then due to the absolute continuity of all functions of the class $W^1_{[a,b]}$ (see [5] (pp. 15–16)), $f^*(x) = \pm x + K$.

Let us prove that almost everywhere on $[a,b]$ $f^{*\prime}(x) = \pm 1$. To do this, we have to establish that any extremal function $f^*(x)$ satisfies the equalities

$$f^*(x) - f^*(\rho(x)) = \rho(x) - x, \tag{5}$$

or

$$f^*(x) - f^*(\rho(x)) = -(\rho(x) - x) \tag{6}$$

for $x \in [a,c]$ and almost everywhere on $[a,c]$

$$f^{*\prime}(x) = f^{*\prime}(\rho(x)). \tag{7}$$

Since $f^*(x)$ is absolutely continuous on $[a,b]$, and therefore, differentiable almost everywhere on $[a,b]$ (see [6] (p. 229)), $\rho(x)$ is absolutely continuous on $[a,c]$ (see [5] (p. 19)) and $c \le \rho(x) \le b$, then $f^*(\rho(x))$ is differentiable almost everywhere on $[a,c]$. From (5) and (6) we then get that almost everywhere on $[a,c]$

$$f^{*\prime}(x) - f^{*\prime}(\rho(x))\rho'(x) = \rho'(x) - 1, \tag{8}$$

or

$$f^{*\prime}(x) - f^{*\prime}(\rho(x))\rho'(x) = -\rho'(x) + 1. \tag{9}$$

Using (7)–(9), we have almost everywhere on $[a,c]$ $f^{*\prime}(x) = -1$ or $f^{*\prime}(x) = 1$. Let us prove that $f^*(x)$ satisfies equalities (5) and (6). If $f^*(x)$ is an extremal function, then, performing transformations such as in the proof of Theorem 3.1 (see [5] (p. 20)), we obtain

$$\left| \int_a^b f^*(x)\psi(x)dx \right| = \left| \int_a^c (f^*(t) - f^*(\rho(t)))\psi(t)dt \right|$$

$$= \int_a^c (\rho(t) - t)|\psi(t)|dt. \tag{10}$$

Without loss of generality, we may assume that $\psi(x) > 0$ almost everywhere on $[a,c]$. It then follows from (10) that

$$\int_a^c \psi(t)((\rho(t) - t) + f^*(t) - f^*(\rho(t)))dt = 0$$

or

$$\int_a^c \psi(t)((\rho(t) - t) - (f^*(t) - f^*(\rho(t))))dt = 0. \tag{11}$$

Since $c \le \rho(t) \le b$ and $f^* \in W^1_{[a,b]}$ for $t \in [a,c]$, then $\rho(t) - t \ge |f^*(t) - f^*(\rho(t))|$, whence $\rho(t) - t \pm (f^*(t) - f^*(\rho(t))) \ge 0$ for $t \in [a,c]$. From (11), due to the non-negativity and summability of functions $\psi(t)((\rho(t) - t) \pm (f^*(t) - f^*(\rho(t)))$ (see [6] (Theorem 6, p. 131)), it follows that equalities (5) and (6) are valid almost everywhere on $[a,c]$. Since these functions are continuous, equalities (5) and (6) are valid for $x \in [a,c]$.

Let us prove that $f^*(x)$ satisfies the relation (7). Since $f^* \in W^1_{[a,b]}$, then for $x, x + \Delta x, \rho(x), (\rho(x) + \Delta x) \in [a,b]$, using (5) and (6), we have

$$|f^*(x + \Delta) - f^*(\rho(x) + \Delta x)| \le \rho(x) - x = |f^*(x) - f^*(\rho(x))|. \tag{12}$$

As a result of the continuity of $f^*(x)$, for $\Delta x \to 0$ the sign of $(f^*(x) - f^*(\rho(x)))$ coincides with the sign of $(f^*(x + \Delta x) - f^*(\rho(x) + \Delta x))$. Therefore, from (12) it follows

$$f^*(x + \Delta x) - f^*(x) \le f^*(\rho(x) + \Delta x) - f^*(\rho(x)), \tag{13}$$

or

$$f^*(x+\Delta x) - f^*(x) \geq f^*(\rho(x)+\Delta x) - f^*(\rho(x)). \tag{14}$$

Using (13) and (14) we have

$$f^{*\prime}(x+0) \leq f^{*\prime}(\rho(x)+0) \quad \text{and} \quad f^{*\prime}(x-0) \geq f^{*\prime}(\rho(x)-0),$$

or

$$f^{*\prime}(x+0) \geq f^{*\prime}(\rho(x)+0) \quad \text{and} \quad f^{*\prime}(x-0) \leq f^{*\prime}(\rho(x)-0).$$

Therefore, due to the differentiability of the function $f^*(x)$, we obtain that $f^{*\prime}(x) = f^{*\prime}(\rho(x))$ almost everywhere on $[a,c]$. In a similar way, we prove that $f^{*\prime}(x) = \pm 1$ almost everywhere on $[c,b]$. Lemma 1 has been proved. □

Corollary 1. *Let $\varphi(y)$ be the function that is summable and sign-preserving almost everywhere on $[a_1,b_1]$. Then*

$$\sup_{f\in W_p^{1,1}} \left| \int_a^b \int_{a_1}^{b_1} \psi(x)\varphi(y) f(x,y)\,dx\,dy \right| = \left| \int_{a_1}^{b_1} \varphi(y) \int_a^c \psi(t)(\rho(t)-t)\,dt\,dy \right|, \tag{15}$$

where $\psi(x)$, $\rho(x)$ are the same functions as in Lemma K. Moreover, the set of all extremal functions $f^*(x,y) \in W_p^{1,1}$ realizing the exact upper bound in (15) has the set of functions of the form

$$f^*(x,y) = \pm x + g(y),$$

where $g(y)$ is the arbitrary function from the class $W^1_{[a_1,b_1]}$.

Proof. The relation (15) was proved in [5] (Lemma 5.1, p. 54). Just as it was done in the proof of Lemma 5.1, using Lemma 1 and the fact that $\int_a^b \psi(x)g(y)dx = 0$ for the arbitrary function $g(y)$, we get that

$$f^*(x,y) = \pm x + g(y),$$

where $g(y) \in W^1_{[a_1,b_1]}$. The corollary has been proved. □

Let

$$\mathfrak{E}^* = \left\{ f_n^*(x) \in H^1 : \sup_{f\in H^1} \|f(x) - F_n(f,x)\|_C = \frac{\pi}{2n} = \|f_n^*(x) - F_n(f_n^*,x)\|_C \right\}$$

be the set of all extremal functions for the Favard method on the class H^1. The following statement is then true.

Theorem 2. *The set $\mathfrak{E}^*$ is the set of functions of the form*

$$f_n^*(x) = \pm\varphi_n(x-x_0) + C,$$

where $\varphi_n(t)$ is the $2\pi/n$-periodic even function, $\varphi_n(t) = t$ for $t \in [0,\pi/n]$, x_0 and C are arbitrary constants.

Proof. We can prove that

$$\sup_{f\in H^1} \|f(x) - F_n(f,x)\|_C = \frac{2}{\pi} \sup_{f\in H} \left| \int_0^\pi f(t)F_n(t)\,dt \right|,$$

where H is the subset of even functions $f(x)$ from the class H^1 such that

$$\|f(x) - F_n(f,x)\|_C = |f(0) - F_n(f,0)| = |F_n(f,0)|.$$

Moreover, the arbitrary extremal function $f_n^*(x)$ can be obtained from the arbitrary extremal function

$$\varphi_n(t) \in H : \frac{2}{\pi} \sup_{f \in H} \left| \int_0^\pi f(t) F_n(t) dt \right| = \frac{2}{\pi} \left| \int_0^\pi \varphi_n(t) F_n(t) dt \right|$$

by shifting its graph parallel to the OX- and OY-axes, i.e.,

$$f_n^*(x) = \varphi_n(x - x_0) + C.$$

Let us prove that the extremal function $\varphi_n(t) \in H$ is unique up to a sign. It is clear that

$$\sup_{f \in H} \frac{2}{\pi} \left| \int_0^\pi f(t) F_n(t) dt \right|$$

$$\leq \frac{2}{\pi} \left(\sup_{f \in H} \left| \int_0^{\pi/n} f(t) F_n(t) dt \right| + \sum_{k=1}^{n-1} \sup_{f \in H} \left| \int_{k\pi/n}^{(k+1)\pi/n} f(t) F_n(t) dt \right| \right). \tag{16}$$

Since $F_n(t) > 0$ on $[0, \frac{\pi}{n}]$ and $f(t) \in H$, then

$$\sup_{f \in H} \left| \int_0^{\pi/n} f(t) F_n(t) dt \right| = \int_0^{\pi/n} t F_n(t) dt. \tag{17}$$

Since (see [7]) $\int_{k\pi/n}^{(k+1)\pi/n} F_n(t) dt = 0$ then applying Lemma K for each segment $[k\pi/n, (k+1)\pi/n]$ we get

$$\sup_{f \in H} \left| \int_{k\pi/n}^{(k+1)\pi/n} f(t) F_n(t) dt \right| = \int_{k\pi/n}^{(k+1)\pi/n} ((-1)^k t + C_k) F_n(t) dt. \tag{18}$$

From (16)–(18), due to the continuity of the extremal function $\varphi_n(t)$, it follows that $\varphi_n(t)$ is $2\pi/n$-periodic even function, $\varphi_n(t) = t$ for $t \in [0, \pi/n]$ and

$$\sup_{f \in H} \frac{2}{\pi} \left| \int_0^\pi f(t) F_n(t) dt \right| = \frac{2}{\pi} \int_0^\pi \varphi_n(t) F_n(t) dt. \tag{19}$$

We assume that there is another extremal function $\overline{\varphi}_n(t) \in H$. Then

$$0 = \frac{2}{\pi} \int_0^\pi \varphi_n(t) F_n(t) dt - \frac{2}{\pi} \int_0^\pi \overline{\varphi}_n(t) F_n(t) dt$$

$$= \frac{2}{\pi} \left(\int_0^{\pi/n} \varphi_n(t) F_n(t) dt - \int_0^{\pi/n} \overline{\varphi}_n(t) F_n(t) dt \right) \tag{20}$$

$$+ \sum_{k=1}^{n-1} \left(\int_{k\pi/n}^{(k+1)\pi/n} \varphi_n(t) F_n(t) dt - \int_{k\pi/n}^{(k+1)\pi/n} \overline{\varphi}_n(t) F_n(t) dt \right).$$

From (16)–(19) it follows

$$\int_0^{\pi/n} \varphi_n(t) F_n(t) dt - \int_0^{\pi/n} \overline{\varphi}_n(t) F_n(t) dt \geq 0, \tag{21}$$

$$\int_{k\pi/n}^{(k+1)\pi/n} \varphi_n(t) F_n(t) dt - \int_{k\pi/n}^{(k+1)\pi/n} \overline{\varphi}_n(t) F_n(t) dt \geq 0, \; k = \overline{1, n-1}. \tag{22}$$

In the inequality (21), the equal sign is possible only if $\overline{\varphi}_n(t) = \varphi_n(t) = t$ for $t \in [0, \pi/n]$.

Since $\varphi_n(t)$ is the extremal function of Lemma K on each segment $[k\pi/n, (k+1)\pi/n]$, then by Lemma 1 the equal sign in (22) is possible only if $\overline{\varphi}_n(t) = \varphi_n(t) + C_k$ for $t \in [k\pi/n, (k+1)\pi/n]$. In order to justify the equal sign present in (20), it must take place in (21) and (22). Therefore, due to the continuity of functions $\overline{\varphi}_n(t)$ and $\varphi_n(t)$, the equality $\overline{\varphi}_n(t) = \varphi_n(t)$ holds on $[0, \pi/n]$ and $[k\pi/n, (k+1)\pi/n]$. As a result of the parity and 2π-periodicity of these functions, the equality $\overline{\varphi}_n(t) = \varphi_n(t)$ holds on the entire real axis.

Therefore, $\varphi_n(t)$ is the unique extremal function from the class H up to a sign. The theorem has been proved. □

In a similar way, we can describe the set of all extremal functions for the arbitrary linear approximation method

$$U_n(\Lambda, f, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) U_n(\lambda, t - x)dt,$$

where $U_n(\lambda, t) = \frac{1}{2} + \sum_{k=1}^{n-1} \lambda_k^{(n)} \cos kt$ is the kernel of the method (approximation properties of linear methods studied, for example, in [8–11]). Since any trigonometric polynomial of the order $(n-1)$ has at most $2n-2$ roots on $[-\pi, \pi)$ (see, e.g., [12] (p. 214)), then the function $\Phi(x) = \int_x^{\pi} U_n(\lambda, t)dt$ can have at most n roots on $[0, \pi]$. Let $\Phi(x) = \int_x^{\pi} U_n(\lambda, t)dt$ have exactly m roots x_k $(k = \overline{1, m})$ on $[0, \pi]$, $0 \le m \le n$, and the function $f^*_{u_n}(x) \in H^1$ is such that

$$\sup_{f \in H^1} \|f(x) - U_n(\Lambda, f, x)\|_C = \left\|f^*_{u_n}(x) - U_n(\Lambda, f^*_{u_n}, x)\right\|_C,$$

i.e., it is the arbitrary extremal function for the $U_n(\Lambda, f, x)$ on the class H^1. Then, analogously to the proof of Theorem 2, we can prove the following statement.

Theorem 3. *The set of all extremal functions $f^*_{u_n}(x)$ for the method $U_n(\Lambda, f, x)$ on the class H^1 is the set of functions of the form*

$$f^*_{u_n}(x) = \pm\varphi_{u_n}(x - x_0) + K,$$

where x_0 and K are arbitrary constants and $\varphi_{u_n}(t)$ is the even 2π-periodic continuous function such that $\varphi'_{u_n}(t) = 1$ for $t \in [0, x_1]$ and $\varphi'_{u_n}(t) = (-1)^k$ for $t \in (x_k, x_{k+1})$, i.e.,

$$\varphi_{u_n}(t) = \begin{cases} t, \ t \in [0, x_1], \\ (-1)^k t + 2\sum_{i=1}^{k}(-1)^{i+1}x_i, \ t \in (x_k, x_{k+1}), \end{cases}$$

$$k = \overline{1, m}, \ \ 0 \le m \le n.$$

Let $\widehat{\mathfrak{E}} = \left\{\widehat{f}_n(x) \in H^1 : E_n(H^1)_C = \frac{\pi}{2n} = E_n(\widehat{f}_n)_C\right\}$ be the set of all extremal functions realizing the exact upper bound of the best approximations on the class H^1.

Theorem 4. *The set $\widehat{\mathfrak{E}} = \mathfrak{E}^*$ and for each function from these sets the best approximation polynomials are constants.*

Proof. According to Theorem 2 and the Chebyshev criterion (see, e.g., [2] (p. 46)), for any function $f^*_n(x) \in \mathfrak{E}^*$ it follows that

$$E_n(f^*_n) = E_n(\pm\varphi_n(x - x_0) + C) = E_n(\varphi_n) = \|\varphi_n\|_C = \frac{\pi}{2n} = E_n(H^1).$$

These relations imply that for any function $f_n^*(x) \in \mathfrak{E}^*$ the polynomials of the best approximation are constants and $\mathfrak{E}^* \subseteq \widehat{\mathfrak{E}}$. For any function $\widehat{f}_n(x) \in \widehat{\mathfrak{E}}$, it follows that

$$E_n(\widehat{f}_n) = \frac{\pi}{2n} = \left\| \widehat{f}_n(x) - T^*_{n-1}(\widehat{f}_n, x) \right\|_C$$
$$\leq \left\| \widehat{f}_n(x) - F_n(\widehat{f}_n, x) \right\|_C \leq \sup_{f \in H^1} \|f(x) - F_n(f,x)\|_C = \frac{\pi}{2n},$$

where $T^*_{n-1}(\widehat{f}_n, x)$ is the best approximation polynomial of the degree $(n-1)$ of the function $\widehat{f}_n(x)$. This means that $\left\| \widehat{f}_n(x) - F_n(\widehat{f}_n, x) \right\|_C = \frac{\pi}{2n}$, i.e., $\widehat{f}_n(x) \in \mathfrak{E}^*$. So $\mathfrak{E}^* \supseteq \widehat{\mathfrak{E}}$. Taking into account that $\mathfrak{E}^* \subseteq \widehat{\mathfrak{E}}$, the theorem has been proved. □

Corollary 2. *If $n-1 > 0$ and $T^*_{n-1}(f,x)$ is the polynomial of the best approximation of the function $f(x) \in H^1$ then $E_n(f)_C < \pi/2n$.*

Proof. For each function $f(x) \in H^1$ the inequality $E_n(f)_C \leq \pi/2n$ is true. If $E_n(f) = \pi/2n$, then using Theorem 4 we get $\deg T^*_{n-1}(f,x) = 0$ that contradicts the condition of the Corollary 2. The corollary has been proved. □

Corollary 3. *If the approximation method is different from the Favard method, i.e., $U_n(\Lambda, f, x) \neq F_n(f,x)$, then*

$$\sup_{f \in H^1} \|f(x) - U_n(\Lambda, f, x)\|_C > \sup_{f \in H^1} \|f(x) - F_n(f,x)\|_C = \frac{\pi}{2n}. \tag{23}$$

Moreover, the set of all extremal functions $f^*_{u_n}(x)$ for the method $U_n(\Lambda, f, x)$ on the class H^1 does not intersect with the set of extremal functions $f_n^*(x)$ for the Favard method on this class.

Proof. If $f(x) \in H^1$, then

$$f(x) - U_n(\Lambda, f, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(D_1(t) - \sum_{k=1}^{n-1} \frac{\lambda_k^{(n)}}{k} \sin kt \right) f'(x-t)dt,$$

where $D_1(u) = \sum_{k=1}^{\infty} \frac{\sin ku}{k}$ is the 2π-periodic Bernoulli function (see, e.g., [2] (pp. 109–111)). Since the function $f(x)$ belongs to the class H^1 and the Bernoulli kernel $D_1(u)$ has a unique polynomial of the best approximation in the metric L (see, for example, [2] (p. 59–69)), we prove that the Favard method presents the unique best approximation method on the class H^1. Therefore, the relations (23) hold.

Let the extremal function $f^*_{u_n}(x)$ for the method $U_n(\Lambda, f, x)$ belong to the set $\mathfrak{E}^*$. So, according to Theorem 2 we have

$$f^*_{u_n}(x) = \pm\varphi_n(x - x_0) + C$$

and as a result of the $2\pi/n$-periodicity of the function $\varphi_n(t)$ (see, e.g., [2] (p. 61)) we get

$$U_n(\Lambda, f^*_{u_n}, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} (\varphi_n(t) + C)dt = \frac{\pi}{2n} + C.$$

Then

$$\|f^*_{u_n}(x) - U_n(\Lambda, f^*_{u_n}, x)\|_C = \frac{\pi}{2n}$$

that contradicts the fact proved above. The corollary has been proved. □

Lemma 2. *Let $f^*(x,y) \in W_p^{1,1}$ be an arbitrary extremal function of Lemma S, K_{f^*} be the oscillation of the function $f^*(x,y)$ on P, $b-a \le b_1 - a_1$ and $y_0 \in [a_1, c_1]$ such that $\delta(y_0) - y_0 = b - a$. Then*

$$b - a \le K_{f^*} = \max\{\max_{a_1 \le y \le y_0} f^*(a,y), \max_{\delta(y_0) \le y \le b_1} f^*(b,y)\}$$
$$- \min\{\min_{\delta(y_0) \le y \le b_1} f^*(a,y), \min_{a_1 \le y \le y_0} f^*(b,y)\} \le b_1 - a_1.$$

Moreover, if two arbitrary extremal functions coincide on one of the larger sides of the rectangle P, then they coincide over the entire rectangle.

Proof. Without loss of generality, we may assume that $\psi(x) > 0$ almost everywhere on $[a,c]$ and $\psi(x) < 0$, almost everywhere on $[c,b]$, $\varphi(y) > 0$ almost everywhere on $[a_1, c_1]$ and $\varphi(y) < 0$ almost everywhere on $[c_1, b_1]$. Let us break P into sets E_i $(i = \overline{1,8})$:

$$E_1 = \{(x,y) \in [a,c] \times [a_1,c_1] : \rho(x) - x \le \delta(y) - y\},$$
$$E_2 = \{(x,y) \in [c,b] \times [a_1,c_1] : x - \rho^{-1}(x) \le \delta(y) - y\},$$
$$E_3 = \{(x,y) \in [c,b] \times [a_1,c_1] : \delta(y) - y \le x - \rho^{-1}(x)\},$$
$$E_4 = \{(x,y) \in [c,b] \times [c_1,b_1] : y - \delta^{-1}(y) \le x - \rho^{-1}(x)\},$$
$$E_5 = \{(x,y) \in [c,b] \times [c_1,b_1] : x - \rho^{-1}(x) \le y - \delta^{-1}(y)\},$$
$$E_6 = \{(x,y) \in [a,c] \times [c_1,b_1] : \rho(x) - x \le y - \delta^{-1}(y)\},$$
$$E_7 = \{(x,y) \in [a,c] \times [c_1,b_1] : y - \delta^{-1}(y) \le \rho(x) - x\},$$
$$E_8 = \{(x,y) \in [a,c] \times [a_1,c_1] : \delta(y) - y \le \rho(x) - x\}.$$

Let us prove that the arbitrary extremal function $f^*(x,y)$ satisfies the relations:

$$f^*(x,y) = -x + K_1(y), \qquad (x,y) \in E_1 \cup E_2, \tag{24}$$

$$f^*(x,y) = x + K_2(y), \qquad (x,y) \in E_6 \cup E_5, \tag{25}$$

$$f^*(x,y) = -y + v_1(x), \qquad (x,y) \in E_8 \cup E_7, \tag{26}$$

$$f^*(x,y) = y + v_2(x), \qquad (x,y) \in E_3 \cup E_4. \tag{27}$$

Here, $K_1(y) \in W^1_{[a_1,c_1]}$ if $(x,y) \in E_1 \cup E_2$ for each fixed x, $K_2(y) \in W^1_{[c_1,b_1]}$ if $(x,y) \in E_6 \cup E_5$ for each fixed x, $v_1(x) \in W^1_{[a,c]}$ if $(x,y) \in E_8 \cup E_7$ for each fixed y and $v_2(x) \in W^1_{[c,b]}$ if $(x,y) \in E_3 \cup E_4$ for each fixed y. Applying the same transformations as in the proof of Lemma S and Lemma 1, we establish that the arbitrary extremal function $f^*(x,y)$ on $[a,c] \times [a_1,c_1]$ satisfies the equality

$$f^*(x,y) - f^*(\rho(x),y) - f^*(x,\delta(y)) + f^*(\rho(x),\delta(y))$$
$$= 2\min\{\rho(x) - x, \delta(y) - y\}.$$

This equality is equivalent to equalities:

$$f^*(x,y) - f^*(\rho(x),y) = \rho(x) - x, \qquad (x,y) \in E_1, \tag{28}$$

$$f^*(x,\delta(y)) - f^*(\rho(x),\delta(y)) = -(\rho(x) - x), \qquad (x,y) \in E_1, \tag{29}$$

$$f^*(x,y) - f^*(x,\delta(y)) = \delta(y) - y, \qquad (x,y) \in E_8, \tag{30}$$

$$f^*(\rho(x),y) - f^*(\rho(x),\delta(y)) = -(\delta(y) - y), \qquad (x,y) \in E_8. \tag{31}$$

Substituting $x = \rho^{-1}(t)$ and $t = x$ in (28), we get $f^*(x,y) - f^*(\rho^{-1}(x),y) = \rho^{-1}(x) - x$, if $(x,y) \in E_2$ because E_1 maps to E_2 after the replacement. Therefore, on $E_1 \cup E_2$ the extremal function $f^*(x,y)$ for each fixed $y(a_1 \le y \le c_1)$ satisfies the equalities $f^*(x,y) - f^*(\rho(x),y) = \rho(x) - x$ if $(x,y) \in E_1$, $f^*(x,y) - f^*(\rho^{-1}(x),y) = \rho^{-1}(x) - x$ if $(x,y) \in E_2$.

Thinking in the same way as in the proof of Lemma 1 and Corollary 1, we conclude that the arbitrary extremal function $f^*(x,y)$ on $E_1 \cup E_2$ satisfies relation (24). Similarly, using (29)–(31), we prove that equalities (25)–(27) hold, respectively. Taking into account the definiteness of the extremal function on each of the sets E_i and its continuity, we write it on the sides of the rectangle:

$$f^*(a,y) = \begin{cases} -y + v_1(a),\ y_0 \le y \le \delta(y_0), \\ u_1(y),\ a_1 \le y \le y_0, \\ u_2(y),\ \delta(y_0) \le y \le b_1, \end{cases}$$

$$f^*(b,y) = \begin{cases} y + v_2(b),\ y_0 \le y \le \delta(y_0), \\ u_1(y) - (b-a),\ a_1 \le y \le y_0, \\ u_2(y) + (b-a),\ \delta(y_0) \le y \le b_1, \end{cases} \tag{32}$$

where

$$u_1(y) = -a + K_1(y), u_2(y) = a + K_2(y),$$
$$u_1(y_0) = -y_0 + v_1(a), u_2(\delta(y_0)) = -\delta(y_0) + v_1(a),$$
$$f^*(x,a_1) = -x + K_1(a_1), f^*(x,b_1) = x + K_2(b_1).$$

Let us prove that

$$K_{f^*} = \max_{a_1 \le y \le b_1} \{f^*(a,y), f^*(b,y)\} - \ min_{a_1 \le y \le b_1} \{f^*(a,y), f^*(b,y)\}.$$

We have to prove that

$$\forall(\alpha,\beta) \in P \min_{a_1 \le y \le b_1} \{f^*(a,y), f^*(b,y)\} \le f^*(\alpha,\beta)$$

$$\le \max_{a_1 \le y \le b_1} \{f^*(a,y), f^*(b,y)\}. \tag{33}$$

Let $y_0 \le \beta \le \delta(y_0)$. Let us prove that

$$f^*(b,y_0) = f^*(a,\delta(y_0)) \le f^*(x,\beta) \le f^*(a,y_0) = f^*(b,\delta(y_0))$$

for $x \in [a,b]$.

Since $f^*(a,y_0) = u_1(y_0) = -y_0 + v_1(a)$ and $f^*(b,\delta(y_0)) = u_2(\delta(y_0)) + (b-a) = -\delta(y_0) + v_1(a) + (b-a)$ then, taking into account that $b - a = \delta(y_0) - y_0$, we get

$$f^*(a,y_0) = f^*(b,\delta(y_0)). \tag{34}$$

Similarly, we can prove that

$$f^*(b,y_0) = f^*(a,\delta(y_0)). \tag{35}$$

If $x - a \le \beta - y_0$, then $f^*(x,\beta) \le f^*(a,y_0)$. Indeed,

$$f^*(a,y_0) - f^*(x,\beta) = f^*(a,y_0) - f^*(a,\beta) + f^*(a,\beta)) - f^*(x,\beta).$$

Taking into account relation (32) for the function $f^*(a, y)$, we get

$$f^*(a, y_0) - f^*(a, \beta) = \beta - y_0.$$

Since the function $f^*(x, \beta)$ belongs to the class $W^1_{[a,b]}$, we then get

$$f^*(a, \beta) - f^*(x, \beta) \geq -(x - a),$$

hence

$$f^*(x, \beta) \leq f^*(a, y_0). \tag{36}$$

If $x - a \geq \beta - y_0$ then, taking into account definition (32) of the extremal function $f^*(b, y)$ and the fact that $f^*(x, \beta)$ belongs to the class $W^1_{[a,b]}$, we get

$$\begin{aligned} &f^*(b, \delta(y_0)) - f^*(x, \beta) \\ &= f^*(b, \delta(y_0)) - f^*(b, \beta) + f^*(b, \beta) - f^*(x, \beta) \\ &= \delta(y_0) - \beta + f^*(b, \beta) - f^*(x, \beta) \\ &= \delta(y_0) - y_0 - (\beta - y_0) + f^*(b, \beta) - f^*(x, \beta) \end{aligned}$$

$$\geq b - a - (\beta - y_0) - (b - x) = (x - a) - (\beta - y_0) \geq 0. \tag{37}$$

From relations (34), (36) and (37), it follows that

$$f^*(x, \beta) \leq f^*(a, y_0) = f^*(b, \delta(y_0)). \tag{38}$$

If $x - a \leq \delta(y_0) - \beta$ then similarly we prove that

$$f^*(x, \beta) \geq f^*(a, \delta(y_0)) = f^*(b, y_0). \tag{39}$$

If $x - a \geq \delta(y_0) - \beta$ then we prove that

$$f^*(x, \beta) \geq f^*(b, y_0) = f^*(a, \delta(y_0)). \tag{40}$$

Let $a_1 \leq \beta \leq y_0$. Then, according to the definitions of the function $\delta(y)$ and the sets E_1, E_2, we get $\delta(\beta) - \beta \geq \delta(y_0) - y_0 = b - a$, $(x, \beta) \in E_1 \cup E_2$ and $f^*(x, \beta) = -x + K_1(\beta)$. According to (32) $K_1(\beta) = u_1(\beta) + a$. This is why

$$f^*(x, \beta) = (-x + a) + u_1(\beta) \leq u_1(\beta) = f^*(a, \beta) \leq \max_{a_1 \leq y \leq y_0} f^*(a, y)$$

$$\leq \max_{a_1 \leq y \leq b_1} f^*(a, y) \leq \max_{a_1 \leq y \leq b_1} \{f^*(a, y), f^*(b, y)\}. \tag{41}$$

Similarly, we prove that

$$f^*(x, \beta) \geq \min_{a_1 \leq y \leq y_0} \{f^*(b, y)\} \geq \min_{a_1 \leq y \leq b_1} \{f^*(a, y), f^*(b, y)\}. \tag{42}$$

Let $\delta(y_0) \leq \beta \leq b_1$. So, $(x, \beta) \in E_6 \cup E_5$ and $f^*(x, \beta) = x + K_2(\beta)$. Therefore, we prove that

$$\min_{a_1 \leq y \leq b_1} \{f^*(a, y), f^*(b, y)\} \leq \min_{\delta(y_0) \leq y \leq b_1} \{f^*(a, y)\} \leq f^*(x, \beta)$$

$$\leq \max_{\delta(y_0) \leq y \leq b_1} \{f^*(b, y)\} \leq \max_{a_1 \leq y \leq b_1} \{f^*(a, y), f^*(b, y)\}. \tag{43}$$

Relations (38)–(43) imply equality (33). Taking into account the definition (32) of functions $f^*(a,y)$ and $f^*(b,y)$, from (33), we obtain

$$K_{f^*} = \max\left\{ \max_{a_1 \le y \le y_0} f^*(a,y), \max_{\delta(y_0) \le y \le b_1} f^*(b,y) \right\}$$

$$- \min\left\{ \min_{\delta(y_0) \le y \le b_1} f^*(a,y), \min_{a_1 \le y \le y_0} f^*(b,y) \right\}.$$

The points where the extreme values of the function $f^*(x,y)$ (extreme points) are reached, lie on one of the larger sides of the rectangle or on both sides. If the extremal points lie on one of the larger sides of the rectangle, then, given the definition of the extremal function on the larger sides and the fact that functions $f^*(a,y)$ and $f^*(b,y)$ belong to the class $W^1_{[a_1,b_1]}$, we conclude that

$$b - a \le K_{f^*} \le b_1 - a_1. \tag{44}$$

If the extreme points lie on both larger sides, then (32) implies that

$$K_{f^*} = \max_{a_1 \le y \le y_0} u_1(y) - \min_{a_1 \le y \le y_0} (u_1(y) - (b-a)),$$

or

$$K_{f^*} = \max_{\delta(y_0)) \le y \le b_1} (u_2(y) + (b-a)) - \min_{\delta(y_0) \le y \le b_1} u_2(y).$$

So,

$$b - a \le K_{f^*} \le b - a + y_0 - a_1 < b_1 - a_1,$$

or

$$b - a \le K_{f^*} \le b - a + b_1 - \delta(y_0) < b_1 - a_1. \tag{45}$$

From (44) and (45), it follows that $b - a \le K_{f^*} \le b_1 - a_1$.

Let $f_1^*(x,y)$ and $f_2^*(x,y)$ be arbitrary extremal functions coinciding on one of the larger sides of the rectangle P, i.e., $f_1^*(a,y) \equiv f_2^*(a,y)$, or $f_1^*(b,y) \equiv f_2^*(b,y)$. Then

$$f_1^*(a,y) = \begin{cases} -y + v_1^1(a), \; y_0 \le y \le \delta(y_0), \\ u_1^1(y), \; a_1 \le y \le y_0, \\ u_2^1(y), \; \delta(y_0) \le y \le b_1, \end{cases}$$

$$f_2^*(a,y) = \begin{cases} -y + v_2^1(a), \; y_0 \le y \le \delta(y_0), \\ u_1^2(y), \; a_1 \le y \le y_0, \\ u_2^2(y), \; \delta(y_0) \le y \le b_1, \end{cases}$$

where $u_1^1(y) = -a + K_1^1(y)$, $u_2^1(y) = a + K_2^1(y)$ and $u_1^2(y) = -a + K_1^2(y)$, $u_2^2(y) = a + K_2^2(y)$, $u_1^1(y) = u_1^2(y)$, $u_2^1(y) = u_2^2(y)$.

Taking into account the definition of the extremal function $f^*(x,y)$ on $E_1 \cup E_2$ and on $E_6 \cup E_5$ and the fact that $f_1^*(a,y) = f_2^*(a,y)$, we get $f_1^*(x,y) = f_2^*(x,y)$ on $E_1 \cup E_2$ and $E_6 \cup E_5$. On the set $E_8 \cup E_7$ $f_1^*(x,y) = -y + v_1^1(x)$, and $f_2^*(x,y) = -y + v_1^2(x)$. Let $y = l_1(x)$ be the line separating the sets E_1 and E_8, i.e., $\rho(x) - x = \delta(l_1(x)) - l_1(x)$ for $x \in [a,c]$. Since $f^*(x,y)$ is continuous on $y = l_1(x)$, then, taking into account the definition of the extremal function on E_1 and E_8, we get: $-x + K_1^1(l_1(x)) = -l_1(x) + v_1^1(x)$ and $-x + K_1^2(l_1(x)) = -l_1(x) + v_1^2(x)$. Since $K_1^1(l_1(x)) = K_1^2(l_1(x))$, then $v_1^1(x) = v_1^2(x)$ and $f_1^*(x,y) = f_2^*(x,y)$ by $E_8 \cup E_7$. We prove, similarly, that $f_1^*(x,y) = f_2^*(x,y)$ on $E_3 \cup E_4$. So, $f_1^*(x,y) = f_2^*(x,y)$ on the entire rectangle P. The lemma has been proved. □

Lemma 3. *The set of all extremal functions for the Favard method on the class $H^{1,1}$ is the set of functions given by relations*

$$\gamma^*_{nm}(x,y) = \pm f^*_{nm}(x-x_0, y-y_0) + K,$$

where $f^*_{nm}(x,y)$ is the extremal function constructed in [1], x_0, y_0, K are arbitrary constants.

Proof. From [1] it follows that

$$f^*_{nm}(x,y) = \begin{cases} x+y, (x+y) \in [0, \frac{\pi}{n}] \times [0, \frac{\pi}{m}], \\ x+\varphi(y), (x,y) \in [0, \frac{\pi}{n}] \times [0, \pi], \\ y+\psi(x), (x,y) \in [0, \pi] \times [0, \frac{\pi}{m}], \\ (-1)^{(k+1)(i+1)} F_{k,i}(x,y) + C_{k,i} + r(y),\ (x,y) \in \\ \in [\frac{k\pi}{n}, \frac{(k+1)\pi}{n}] \times [\frac{i\pi}{m}, \frac{(i+1)\pi}{m}], k = \overline{1, n-1}, i = \overline{1, m-1}. \end{cases}$$

Here $\varphi(y)$ is the $2\pi/m$-periodic even function, $\varphi(y) = y$ for $y \in [0, \pi/m]$, $\psi(x)$ is the even, $2\pi/n$-periodic function, $\psi(x) = x$ for $x \in [0, \pi/n]$, and $F_{k,i}(x,y) \in W^{1,1}_{P_{k,i}}$ such that

$$\sup_{f \in W^{1,1}_{P_{k,i}}} \left| \int_{k\frac{\pi}{n}}^{(k+1)\frac{\pi}{n}} \int_{i\frac{\pi}{m}}^{(i+1)\frac{\pi}{m}} f(x,y) F_n(x) F_m(y) dxdy \right|$$

$$= \int_{k\frac{\pi}{n}}^{(k+1)\frac{\pi}{n}} \int_{i\frac{\pi}{m}}^{(i+1)\frac{\pi}{m}} (-1)^{(k+1)(i+1)} F_{k,i}(x,y) F_n(x) F_m(y) dxdy,$$

i.e., $F_{k,i}(x,y)$ are the extremal functions of Lemma S for the class $W^{1,1}_{P_{k,i}}$ on the rectangles $P_{k,i} = [k\frac{\pi}{n}, (k+1)\frac{\pi}{n}] \times [i\frac{\pi}{m}, (i+1)\frac{\pi}{m}]$, $C_{k,i}$ are constants, which are chosen so that $f^*_{nm}(x,y)$ is continuous on $[\frac{\pi}{n}, \pi] \times [\frac{\pi}{m}, \pi]$, $r(y) = f^*(\frac{\pi}{n}, y) - (F_{1,i}(\frac{\pi}{n}, y) + C_{1,i})$ is the function that guarantees the continuity of $f^*_{nm}(x,y)$ on the line $x = \pi/n$ if $n \geq m$. We can prove that

$$\sup_{f \in H^{1,1}} \|f(x,y) - F_{nm}(f,x,y)\|_C$$

$$= \frac{4}{\pi^2} \sup_{f \in H_0} \left| \int_0^\pi \int_0^\pi f(t,z) F_n(t) F_m(z) dtdz \right|,$$

where H_0 is the subset of functions from the class $H^{1,1}$ that are even in each of the variables, such that

$$\|f(x,y) - F_{nm}(f,x,y)\|_C = |f(0,0) - F_{nm}(f,0,0)| = |F_{nm}(f,0,0)|.$$

Moreover, if $\varphi^*_{nm}(x,y) \in H_0$ is such that

$$\frac{4}{\pi^2} \sup_{f \in H_0} \left| \int_0^\pi \int_0^\pi f(x,y) F_n(x) F_m(y) dxdy \right|$$

$$= \frac{4}{\pi^2} \int_0^\pi \int_0^\pi \varphi^*_{nm}(x,y) F_n(x) F_m(y) dxdy,$$

i.e., the arbitrary extremal function from the class H_0, then

$$\gamma^*_{nm}(x,y) = \pm \varphi^*_{nm}(x-x_0, y-y_0) + K.$$

Let us prove that the extremal function $\varphi^*_{nm}(x,y) \in H_0$ is unique and coincides with $f^*_{nm}(x,y) \in H_0$. We suppose that there exists another extremal function $\overline{f}^*_{nm}(x,y) \in H_0$, different from $f^*_{nm}(x,y)$. Then

$$
\begin{aligned}
0 = {} & \frac{4}{\pi^2}\left(\int_0^{\pi}\int_0^{\pi} f^*_{nm}(t,z)F_n(t)F_m(z)dtdz \right. \\
& \left. - \int_0^{\pi}\int_0^{\pi} \overline{f}^*_{nm}(t,z)F_n(t)F_m(z)dtdz\right) \\
= {} & \frac{4}{\pi^2}\left(\left(\int_0^{\frac{\pi}{n}}\int_0^{\frac{\pi}{m}} f^*_{nm}(t,z)F_n(t)F_m(z)dtdz \right.\right. \\
& \left. - \int_0^{\frac{\pi}{n}}\int_0^{\frac{\pi}{m}} \overline{f}^*_{nm}(t,z)F_n(t)F_m(z)dtdz\right) \\
& + \left(\sum_{i=1}^{m-1}\left(\int_0^{\pi/n}\int_{i\pi/n}^{(i+1)\pi/m} f^*_{nm}(t,z)F_n(t)F_m(z)dtdz \right.\right. \\
& \left. - \int_0^{\pi/n}\int_{i\pi/n}^{(i+1)\pi/m} \overline{f}^*_{nm}(t,z)F_n(t)F_m(z)dtdz\right) \\
& + \sum_{k=1}^{n-1}\left(\int_{k\pi/n}^{(k+1)\pi/n}\int_0^{\pi/m} f^*_{nm}(t,z)F_n(t)F_m(z)dtdz \right. \\
& \left. - \int_{k\pi/n}^{(k+1)\pi/n}\int_0^{\pi/m} \overline{f}^*_{nm}(t,z)F_n(t)F_m(z)dtdz\right) \\
& + \sum_{k=1}^{n-1}\sum_{i=1}^{m-1}\left(\int_{k\pi/n}^{(k+1)\pi/n}\int_{i\pi/m}^{(i+1)\pi/m} f^*_{nm}(t,z)F_n(t)F_m(z)dtdz \right. \\
& \left. - \int_{k\pi/n}^{(k+1)\pi/n}\int_{i\pi/m}^{(i+1)\pi/m} \overline{f}^*_{nm}(t,z)F_n(t)F_m(z)dtdz\right).
\end{aligned} \tag{46}
$$

Taking into account that $f^*_{nm}(x,y)$ belongs to the class H_0 and its construction, similarly as it was done in Theorem 2, we get:

$$
\begin{aligned}
& \int_0^{\pi/n}\int_0^{\pi/m} f^*_{nm}(t,z)F_n(t)F_m(z)dtdz \\
& - \int_0^{\pi/n}\int_0^{\pi/m} \overline{f}^*_{nm}(t,z)F_n(t)F_m(z)dtdz \geq 0,
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
& \int_0^{\pi/n}\int_{i\pi/m}^{(i+1)\pi/m} f^*_{nm}(t,z)F_n(t)F_m(z)dtdz \\
& - \int_0^{\pi/n}\int_{i\pi/m}^{(i+1)\pi/m} \overline{f}^*_{nm}(t,z)F_n(t)F_m(z)dtdz \geq 0,
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
& \int_{k\pi/n}^{(k+1)\pi/n}\int_0^{\pi/m} f^*_{nm}(t,z)F_n(t)F_m(z)dtdz \\
& - \int_{k\pi/n}^{(k+1)\pi/n}\int_0^{\pi/m} \overline{f}^*_{nm}(t,z)F_n(t)F_m(z)dtdz \geq 0,
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
& \int_{k\pi/n}^{(k+1)\pi/n}\int_{i\pi/m}^{(i+1)\pi/m} f^*_{nm}(t,z)F_n(t)F_m(z)dtdz \\
& - \int_{k\pi/n}^{(k+1)\pi/n}\int_{i\pi/m}^{(i+1)\pi/m} \overline{f}^*_{nm}(t,z)F_n(t)F_m(z)dtdz \geq 0.
\end{aligned} \tag{50}
$$

It follows from (46) that inequalities (47)–(50) must contain the equal sign. In (47) there is the equal sign only if

$$\overline{f}^*_{nm}(t,z) = f^*_{nm}(t,z)$$

on $[0,\pi/n] \times [0,\pi/m]$. The equal sign in (48), according to Corollary 1, is possible if and only if

$$\overline{f}^*_{nm}(t,z) = \varphi(z) + f_i(t)$$

on $[0,\pi/n] \times [i\pi/m,(i+1)\pi/m]$. Similarly, in (49) the equal sign is possible if and only if

$$\overline{f}^*_{nm}(t,z) = \psi(t) + g_k(z)$$

on $[k\pi/n,(k+1)\pi/n] \times [0,\pi/m]$. The equal sign in (50) is possible if and only if $\overline{f}^*_{nm}(t,z)$ is the extremal function of Lemma S for the class $W^{1,1}_{P_{k,i}}$ on each rectangle $P_{k,i}$. For $0 \le t \le \frac{\pi}{n}$

$$\overline{f}^*_{nm}(t,\frac{\pi}{m}) = f^*_{nm}(t,\frac{\pi}{m}) = \frac{\pi}{m} + t,$$

but, on the other hand, $\overline{f}^*_{nm}(t,\frac{\pi}{m}) = \frac{\pi}{m} + f_1(t)$, because $\overline{f}^*_{nm}(t,z) = \varphi(z) + f_1(t)$ on $[0,\pi/n] \times [\pi/m, 2\pi/m]$. As a result of the continuity of the function $\overline{f}^*_{nm}(t,z)$ we have $f_1(t) = t$.

We prove similarly that $f_i(t) = t, ß = \overline{2, m-1}$. Therefore, on $[0,\pi/n] \times [0,\pi]$ we obtain

$$\overline{f}^*_{nm}(t,z) = f^*_{nm}(t,z). \tag{51}$$

We prove similarly that on $[0,\pi] \times [0,\pi/m]$

$$\overline{f}^*_{nm}(t,z) = f^*_{nm}(t,z). \tag{52}$$

Since $f^*_{nm}(t,z)$ and $\overline{f}^*_{nm}(t,z)$ are the extremal functions of Lemma S for the class $W^{1,1}_{P_{1,i}}$ on each rectangle $P_{1,i}$ and coincide on the larger side $\{(\frac{\pi}{n}, z) : i\frac{\pi}{m} \le z \le (i+1)\frac{\pi}{m}\}$ of the rectangle, then according to Lemma 2 they coincide on all rectangles $P_{1,i}$. We prove similarly that

$$\overline{f}^*_{nm}(t,z) = f^*_{nm}(t,z)$$

on $P_{2,i}, P_{3,i}, \ldots, P_{k,i}, \ldots, P_{n-1,i}$. So, on $[\pi/n,\pi] \times [\pi/m,\pi]$ we have

$$\overline{f}^*_{nm}(t,z) = f^*_{nm}(t,z). \tag{53}$$

From (51)–(53), taking into account the parity and 2π-periodicity in both variables of functions $f^*_{nm}(x,y)$ and $\overline{f}^*_{nm}(x,y)$ we get that $\overline{f}^*_{nm}(x,y) = f^*_{nm}(x,y)$ on the whole plane XOY. Thus, our assumption is wrong. Therefore, $f^*_{nm}(x,y)$ is the unique extremal function from the class H_0. Since any extremal function $\gamma^*_{nm}(x,y)$ has the form $\gamma^*_{nm}(x,y) = \pm\varphi^*_{nm}(x - x_0, y - y_0) + K$, and $\varphi^*_{nm}(x,y) = f^*_{nm}(x,y)$, then

$$\gamma^*_{nm}(x,y) = \pm f^*_{nm}(x - x_0, y - y_0 + K).$$

The lemma has been proved. □

Proof of Theorem 1. Let us prove that there exists the function $\widehat{f}_{nm}(x,y) \in H^{1,1}$, realizing the exact upper bound of the best approximation on the class $H^{1,1}$, i.e., $E_{n,m}(\widehat{f}_{nm}) = E_{n,m}(H^{1,1})$. Since $E_{n,m}(f) = E_{n,m}(f - f(0,0))$, then $E_{n,m}(H^{1,1}) = E_{n,m}(H^{1,1}_0)$, where $H^{1,1}_0$ is the subset of functions from the class $H^{1,1}$ that are equal to 0 at the origin. Let us prove that $H^{1,1}_0$ is the compact set in the metric space of 2π-periodic functions in each of the variables. If $f(x,y) \in H^{1,1}_0$ then $|f(x,y) - f(0,0)| = |f(x,y)| \le |x| + |y| \le 2\pi$. This implies that the set $H^{1,1}_0$ is bounded and (see, for example, [13] (pp. 123–125)) compact. The best approximation functional $E_{n,m}(f)$ is known to be continuous (see, for example, [2]

(p. 17)). Since $E_{n,m}(f)$ is the continuous functional and the set $H_0^{1,1}$ is compact, then there exists the function $\widehat{f}(x,y) \in H_0^{1,1}$ on which the functional $E_{n,m}(f)$ reaches its exact upper bound, i.e., $E_{n,m}(H^{1,1}) = E_{n,m}(H_0^{1,1}) = E_{n,m}(\widehat{f}_{nm})$. Let us assume that $E_{n,m}(H^{1,1}) = \mathcal{E}_{n,m}$. Since

$$\mathcal{E}_{n,m} = E_{n,m}(H^{1,1}) = E_{n,m}(\widehat{f}_{nm}) = \left\| \widehat{f}_{nm}(x,y) - T^*_{n-1,m-1}(\widehat{f},x,y) \right\|_C$$
$$\leq \left\| \widehat{f}_{nm}(x,y) - F_{n,m}(\widehat{f}_{nm},x,y) \right\|_C \leq \mathcal{E}_{n,m},$$

then

$$\left\| \widehat{f}_{nm}(x,y) - F_{n,m}(\widehat{f}_{nm},x,y) \right\|_C = \mathcal{E}_{n,m}. \quad (54)$$

Here, $T^*_{n-1,m-1}(\widehat{f},x,y)$ is the polynomial of the best approximation of the function $\widehat{f}_{nm}(x,y)$ of the degree $(n-1)$ in the variable x and the degree $(m-1)$ in the variable y in the uniform metric. It follows from relation (54) that the function $\widehat{f}_{nm}(x,y)$ belongs to the set of extremal functions for the Favard method on the class $H^{1,1}$, i.e.,

$$\widehat{f}_{nm}(x,y) = \pm f^*_{n,m}(x-x_0, y-y_0) + K. \quad (55)$$

Since $K_{f^*_{n,m}} = \pi/n + \pi/m$, from relation (55) we get $K_{\widehat{f}_{nm}} = pi/n + \pi/m$. Since $E_{n,m}(\widehat{f}_{nm}) \leq K_{\widehat{f}_{nm}}/2 = \pi/2n + \pi/2m$, and as a result (2) $\mathcal{E}_{n,m} > \pi/2n + \pi/2m$, then our assumption is wrong. Hence, the statement of Theorem 1 is true. □

Let us denote by $H^{1,1}_{u+v} := \{f(x,y) \in H^{1,1} : f(x,y) = u(x) + v(y)\}$ as the subset of the functions from the class $H^{1,1}$ that can be represented as a sum of two functions, each of which depends on only one variable. It follows from the definition of the class $H^{1,1}$ that

$$u(x) \in H^1, \quad v(x) \in H^1. \quad (56)$$

Theorem 1 (see, for example, [14]) implies the following statement.

Lemma 4. *If the functions $u(x)$ and $v(y)$ are continuous 2π-periodic in the variables x and y, and $T^*_{n-1}(u,x)$, $T^*_{m-1}(v,y)$ are the polynomials of the best approximation of these functions, then $E_{n,m}(u+v) = E_n(u) + E_m(v)$, and $T^*_{n-1}(u,x) + T^*_{m-1}(v,y)$ is the unique polynomial of the best approximation for the function $f(x,y) = u(x) + v(y) \in H^1$.*

Using Lemmas 4 and (56), we prove the relation

$$E_{n,m}(H^{1,1}_{u+v}) = E_n(H^1) + E_m(H^1) = \frac{\pi}{2n} + \frac{\pi}{2m}.$$

From the last relation and the equality

$$\sup_{f \in H^{1,1}_{u+v}} \|f(x,y) - F_{n,m}(f,x,y)\|_C = \sup_{u \in H^1} \|u(x) - F_n(u,x)\|_C$$
$$+ \sup_{v \in H^1} \|v(y) - F_m(v,y)\|_C = \frac{\pi}{2n} + \frac{\pi}{2m}$$

the following statement follows.

Theorem 5. *For any natural numbers n and m*

$$\sup_{f \in H^{1,1}_{u+v}} \|f(x,y) - F_{n,m}(f,x,y)\|_C = \frac{\pi}{2n} + \frac{\pi}{2m} = E_{n,m}(H^{1,1}_{u+v}),$$

that is, the Favard method implements the exact upper bound of the best approximations on the class $H^{1,1}_{u+v}$.

3. Conclusions

In this paper, we proved that the approximation of the class $H^{1,1}$ by Favard method is greater than the value of the best approximation of this class by trigonometric polynomials, the exact value of which being unknown. We have also managed to build classes for which these values are equal.

The question of Theorem 1 validity for Hölder classes of functions of $n \geq 3$ variables being 2π-periodic in each variable, still remains open. To solve it, we have to establish analogues of equality (1) and Lemmas 2 and 3 for these classes of functions.

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