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Existence and General Energy Decay of Solutions to a Coupled System of Quasi-Linear Viscoelastic Variable Coefficient Wave Equations with Nonlinear Source Terms

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Abstract: Viscoelastic damping phenomena are ubiquitous in diverse kinds of wave motions of nonlinear media. This arouses extensive interest in studying the existence, the finite time blow-up phenomenon and various large time behaviors of solutions to viscoelastic wave equations. In this paper, we are concerned with a class of variable coefficient coupled quasi-linear wave equations damped by viscoelasticity with a long-term memory fading at very general rates and possibly damped by friction but provoked by nonlinear interactions. We prove a local existence result for solutions to our concerned coupled model equations by applying the celebrated Faedo-Galerkin scheme. Based on the newly obtained local existence result, we prove that solutions would exist globally in time whenever their initial data satisfy certain conditions. In the end, we provide a criterion to guarantee that some of the global-in-time-existing solutions achieve energy decay at general rates uniquely determined by the fading rates of the memory. Compared with the existing results in the literature, our concerned model coupled wave equations are more general, and therefore our theoretical results have wider applicability. Modified energy functionals (can also be viewed as certain Lyapunov functionals) play key roles in proving our claimed general energy decay result in this paper.

Keywords: existence results; general energy decay; quasi-linear wave equations; variable coefficient wave equations; viscoelastic damping

MSC: 35L05; 35L15; 35L70

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1. Introduction

We are concerned, in this paper, with the initial boundary value problem (IBVP) for a coupled system of two quasi-linear space-variable coefficient wave equations whose energy is inhibited by viscoelastic dampings with long-term memories and possibly inhibited by frictional dampings, but provoked by nonlinear interactions. More precisely, we consider

$$\begin{cases} |\partial_t u|^{\rho_1}\partial_t^2 u - \mu_1 \operatorname{div}(A_1\nabla u) - \operatorname{div}(A_1\nabla\partial_t^2 u) \\ + \int_{-\infty}^t g_1(t-s)\operatorname{div}(A_1\nabla u)(s)ds \\ + a_{11}\partial_t u + a_{12}\partial_t v = f_1(u,v) & \text{in } \Omega\times(0,+\infty), \\ |\partial_t v|^{\rho_2}\partial_t^2 v - \mu_2 \operatorname{div}(A_2\nabla v) - \operatorname{div}(A_2\nabla\partial_t^2 v) \\ + \int_{-\infty}^t g_2(t-s)\operatorname{div}(A_2\nabla v)(s)ds \\ + a_{21}\partial_t u + a_{22}\partial_t v = f_2(u,v) & \text{in } \Omega\times(0,+\infty), \\ u = v = 0 & \text{on } \Gamma\times(0,+\infty), \\ u = u^0,\ v = v^0 & \text{in } \Omega\times(-\infty,0), \\ \partial_t u(0) = u^1,\ \partial_t v(0) = v^1 & \text{in } \Omega, \end{cases} \tag{1}$$

in which: $u = u(x,t)$ and $v = v(x,t)$, $(x,t) \in \Omega \times (0,+\infty)$, are the unknowns of IBVP (1); Ω is a nonempty bounded open subset of the N-dimensional Euclidean space $\mathbb{R}^N$, of which, the boundary, denoted by Γ (i.e., $\Gamma = \partial\Omega$), is smooth enough (say, is in the $\mathscr{C}^2$ class); N is a given positive integer; ρ_1 and ρ_2, as well as μ_1 and μ_2, are given positive constants; A_1 and A_2 are given $\mathbb{R}^{N\times N}$-valued functions depending merely on space variables; a_{11}, a_{12}, a_{21} and a_{22} are given functions which depend merely on space variables; g_1 and g_2, the so-called relaxation functions, are given functions mapping $\mathbb{R}_+$ (throughout this paper, $\mathbb{R}_+$ denotes the closed interval $[0,+\infty)$; see our notational conventions at the rear of this section) into itself; f_1 and f_2 are given real-valued functions defined in $\mathbb{R}^2$; the given functions u_0, v_0, u_1 and v_1 are initial data of the unknowns of IBVP (1); ∂_t denotes the partial differential operator $\frac{\partial}{\partial t}$; ∇ denotes, as usual, the gradient operator $(\partial_{x_1},\ldots,\partial_{x_N})^\top$ on the N-dimensional Euclidean space $\mathbb{R}^N$, with ∂_{x_k} denoting the partial differential operator $\frac{\partial}{\partial x_k}$, $k = 1,\ldots,N$; $\operatorname{div}\boldsymbol{\varphi}$ denotes formally the divergence of the vector field

$$\boldsymbol{\varphi} = (\varphi_1,\ldots,\varphi_N)^\top = (\varphi_1(x_1,\ldots,x_N),\ldots,\varphi_N(x_1,\ldots,x_N))^\top,$$

that is, $\operatorname{div}\boldsymbol{\varphi} = \sum_{k=1}^{N} \partial_{x_k}\varphi_k$. We shall explain later the sense in which the coupled quasi-linear viscoelastic variable coefficient wave Equations $(1)_1$ and $(1)_2$, as well as the homogeneous Dirichlet boundary condition '$u = v = 0$ on $\Gamma \times (0,+\infty)$' (i.e., $(1)_3$) are satisfied.

As with coupled parabolic equations (see [1,2] and the references cited therein), the coupled wave equations in IBVP (1) have important implications in Physics. The assumption that the coefficients A_1 and A_2 depend on space variables indicates that the underlying media/material is inhomogeneous. The assumption that the constants ρ_1 and ρ_2 are positive indicates that some of the structural properties of the concerned media/material are influenced significantly by the vibrating velocity. To include the terms

$$\int_{-\infty}^{t} g_1(t-s)\operatorname{div}(A_1\nabla u)(s)ds, \text{ and}$$
$$\int_{-\infty}^{t} g_2(t-s)\operatorname{div}(A_2\nabla v)(s)ds$$

in the model Equations $(1)_1$ and $(1)_2$, we stress that, in our concerned scenario in this paper, the wave motions of the concerned media/material are suppressed by its viscoelasticity (the kinetic energy is inhibited, the viscosity is influenced by the velocity, and the aftereffect or memory of $\operatorname{div}(A_1\nabla u)$ sustains for infinitely long time in the media/material); see Reference [3] for the description of viscoelasticity phenomenon and the explanation of the inducing mechanism of this phenomenon. The terms $a_{11}\partial_t u + a_{12}\partial_t v$, $a_{21}\partial_t u + a_{22}\partial_t v$, $f_1(u,v)$ and $f_2(u,v)$ are incorporated to emphasize that the waves u and v are 'strongly' coupled to a certain extent; we shall impose some suitable conditions on the coefficients a_{11}, a_{12}, a_{21} and a_{22} (see Assumption 5) to ensure that the term $a_{11}\partial_t u + a_{12}\partial_t v$, together with the term $a_{21}\partial_t u + a_{22}\partial_t v$, plays a role as frictional damping.

Partial differential equations describing the dynamics of viscoelastic materials have enormous implications to applications of these materials in engineering and scientific communities; the governing equations incorporate hereditary terms to stress that the aftereffect in the materials can not be neglected (see References [3–6]). In theoretical study or engineering applications, the aftereffect of some materials could be neglected for sufficiently large time, while the aftereffect of the other materials could last in infinitely long time periods. Let us point out again that, as indicated by the structure of the model equations in IBVP (1), the aftereffect of the material concerned in this paper could last in infinitely long time periods.

Since viscoelastic materials play important roles in diverse application areas (as alluded before), many experts in mathematical communities have been, in the last two decades, attracted into studying the dynamics of viscoelastic materials from mathematical perspectives. Muñoz Rivera [4] studied the large time behaviour of a class of viscoelastic equation, defined in bounded open subset of Euclidean spaces, in which the aftereffect in large time was neglected, and proved that the associated energy decays exponentially as time approaches infinity. Muñoz Rivera, Lapa and Barreto [5] established later some similar energy decaying results for plate equations. Aassila, Cavalcanti and Soriano [7] established exponentially decaying and polynomially decaying estimates for the energy of a constant-coefficient wave equation governing the vibration of materials occupying a domain whose boundary is of viscoelasticity under different conditions, respectively; in the meanwhile, they justified that the assertion that the energy approaches zero as time goes to infinity holds for all linear viscoelastic wave equations on bounded domains subject to homogeneous Dirichlet boundary condition. The idea in References [5,7] is strikingly illuminating for later study of problems for viscoelastic wave equations; see References [8–27] and the vast references cited therein. For example, Cavalcanti, Domingos Cavalcanti and Ferreira [8] considered the following initial boundary value problem

$$\begin{cases} |\partial_t u|^\rho \partial_t^2 u - \Delta u - \Delta \partial_t^2 u + \int_0^t g(t-s)\Delta u(s)ds - \gamma \partial_t u = 0 & \text{in } \Omega \times (0,+\infty), \\ u = v = 0 & \text{on } \Gamma \times (0,+\infty), \\ u = u^0,\ \partial_t u(0) = u^1 & \text{in } \Omega; \end{cases} \tag{2}$$

they proved, under certain conditions on the relaxation function g, that IBVP (2) admits global weak solutions in $H_0^1(\Omega;\mathbb{R}^2)$ whenever $\gamma \geqslant 0$, and that the energy $E(t)$ associated to the corresponding u decays exponentially whenever $\gamma > 0$, where

$$E(t) = \frac{1}{\rho+2}\int_\Omega |\partial_t u(t)|^{\rho+2}dx + \frac{1}{2}\int_\Omega |\nabla u(t)|^2 dx + \frac{1}{2}\int_\Omega |\nabla \partial_t u(t)|^2 dx, \quad t \in \mathbb{R}_+.$$

The (quasi-)linear wave equation (when $\rho = 0$, the model Equation $(2)_1$ is linear) in IBVP (2) includes two damping terms, namely, the viscoelastic damping $\int_0^t g(t-s)\Delta u(s)ds$ and the frictional damping $\partial_t u$. If $\rho = 0$, these two dampings seem to be equivalent in the sense that the energy of both IBVP (2) incorporating merely viscoelastic damping and IBVP (2) incorporating merely frictional damping decays exponentially. Hence it is interesting to compare the intensity of these two terms in inhibiting the energy of solutions IBVP (2) with $\rho > 0$. In this direction, Cavalcanti and Portillo Oquendo [9] obtained some interesting results. Berrimi and Messaoudi [10] studied viscoelastic equations including nonlinear source terms, and proved that the associated energy decays to zero as time goes to infinity whenever the initial values is sufficiently small. Cavalcanti, Domingos Cavalcanti and Martinez [11] extended in a certain sense the results in Reference [8] and proved that the energy of viscoelastic equations with general relaxation function g (which has a slow decaying rate compared to the one in Reference [8]) could also approach zero as time goes to infinity. The system of coupled viscoelastic wave equations has also been studied by several mathematicians in recent years. Han and Wang [12] studied the initial boundary value problem for a coupled system of viscoelastic wave equations with two nonlinear frictional damping terms, that is

$$\begin{cases} \partial_t^2 u - \Delta u + \int_0^t g_1(t-s)\Delta u(s)ds + |\partial_t u|^{m-1}\partial_t u = f_1(u,v) & \text{in } \Omega \times (0,+\infty), \\ \partial_t^2 v - \Delta v + \int_0^t g_2(t-s)\Delta v(s)ds + |\partial_t v|^{r-1}\partial_t v = f_2(u,v) & \text{in } \Omega \times (0,+\infty), \\ u = v = 0 & \text{on } \Gamma \times (0,+\infty), \\ u(0) = u^0,\ \partial_t u(0) = u^1,\ v(0) = v^0,\ \partial_t v(0) = v^1 & \text{in } \Omega; \end{cases} \tag{3}$$

they proved, under some additional conditions, that IBVP (3) is globally well-posed and provided a blow up criterion for IBVP (3) under some other conditions; as alluded above, the nonlinear terms $|\partial_t u|^{m-1}\partial_t u$ and $|\partial_t v|^{m-1}\partial_t v$, playing roles as frictional dampings, bring in dissipation mechanism in the energy of the system (3). Said-Houari, Messaoudi and Guesmia [13] and Mustafa [14] extended the results in References [8–11] to IBVP (3) with the nonlinear frictional dampings removed. As could be seen evidently from IBVP (3), the structural properties of the concerned materials do not depend on the velocity of the vibration of the materials. Liu [15] considered an initial boundary value problem which is more close to IBVP (1), our working model problem in this paper, namely

$$\begin{cases} |\partial_t u|^\rho \partial_t^2 u - \Delta u - \gamma_1 \Delta \partial_t^2 u + \int_0^t g_1(t-s)\Delta u(s)ds + f_1(u,v) = 0 & \text{in } \Omega\times(0,+\infty), \\ |\partial_t v|^\rho \partial_t^2 v - \Delta v - \gamma_2 \Delta \partial_t^2 v + \int_0^t g_2(t-s)\Delta v(s)ds + f_2(u,v) = 0 & \text{in } \Omega\times(0,+\infty), \\ u = v = 0 & \text{on } \Gamma\times(0,+\infty), \\ u(0) = u^0,\ \partial_t u(0) = u^1,\ v(0) = v^0,\ \partial_t v(0) = v^1 & \text{in } \Omega, \end{cases} \tag{4}$$

with $\rho > 0$; he established under certain additional conditions some uniform decaying estimates for the energy of IBVP (4). He [16] reported some uniform decaying results for the energy associated to IBVP (4) under some other conditions. The other more interesting existence and stability results concerning viscoelastic (quasi-)linear wave equations could be seen in References [17–29] and the references therein.

As can be infered from the above review: The model equations considered in the aforementioned references only reflect that the concerned materials are homogeneous in all directions and that the aftereffect is all neglected for large time. But it seems to be more realistic that the materials are inhomogeneous in directions and that the aftereffect could influence the materials all the time. This motivates us to study space-varying viscoelastic wave equations with infinitely long memory of which both improve the mathematical difficulty of the paper. As could be seen later, we shall not assume that $\rho_1 = \rho_2$; it is obvious that $\rho_1 \neq \rho_2$ has significant physical implications. Aside from these innovations, the nonlinearity in IBVP (1) seems to be more general than those studied in the existing references. Our goal in this paper is to prove under some conditions that solutions to IBVP (1) exist globally in time whenever their initial values are sufficiently small, and prove under some additional conditions that the energy associated to some of the global in time solutions approaches zero as time escapes to infinity.

Assumption 1. *For $i = 1,2$, $A_i \in \mathscr{C}^2(\bar{\Omega}; \mathbb{R}^{N\times N})$, the set of uniformly continuous functions of which partial derivatives whose orders not exceeding 2 are all uniformly continuous Ω. For every $x \in \Omega$, the matrix $A_i(x)$ is symmetric, $i = 1,2$. And ϑ_i is assumed to be strictly positive, where the constant ϑ_i is given by*

$$\vartheta_i = \inf_{\substack{x\in\Omega,\\ \xi\in\mathbb{R}^N\setminus\{\mathbf{0}\}}} \frac{\xi^\top A_i(x)\xi}{\xi^\top\xi}, \quad i = 1,2. \tag{5}$$

Assumption 2. *The relaxation function g_i is strictly monotonically decreasing, maps the closed interval $\mathbb{R}_+$ into itself and satisfies*

$$0 < \int_0^{+\infty} g_i(s)ds < \mu_i, \quad i = 1,2.$$

The derivative function g_i', of the relaxation function g_i, is locally Lebesgue integrable in $\mathbb{R}_+$(in other words, the function g_i is absolutely continuous in the interval $\mathbb{R}_+$), $i = 1,2$. There exists a nonincreasing absolutely continuous function ξ_i mapping $\mathbb{R}_+$ into itself and a function $K_i \in \mathscr{C}^2([0,r_i];\mathbb{R})$ (with r_i a given positive constant not less than $g_i(0)$), which is strictly increasing and strictly convex, and satisfies $K_i(0) = K_i'(0) = 0$, such that

$$K_1(t) \leqslant K_2(t) \text{ or } K_2(t) \leqslant K_1(t) \quad \text{for all } t \in [0, \min(r_1, r_2)], \tag{6}$$

that

$$g_i'(t) \leqslant -\xi_i(t) K_i(g_i(t)) \quad \text{for all } t \in \mathbb{R}_+,$$

and that there exists a positive constant $\mathfrak{k}_i$ *satisfying*

$$\lim_{\delta \to 0^+} \frac{K_i'^{-1}(\delta)}{\big(\max(K_1, K_2)\big)'^{-1}(\delta)} = \mathfrak{k}_i, \quad i = 1, 2,$$

with $\big(\max(K_1, K_2)\big)'^{-1}$ *denoting the inverse of the derivative of the function* $\max(K_1, K_2)$.

Assumption 3. *The constants* ρ_1 *and* ρ_2 *satisfy* $\min(\rho_1, \rho_2) > 0$ *and* $(N-2)\max(\rho_1, \rho_2) \leqslant 2$.

Assumption 4. f_i *is locally Lipschitz continuous in* $\mathbb{R}^2$*, and satisfies* $f_i(0,0) = 0$, $i = 1, 2$. *There exists a function* $F(u, v)$*, defined in the whole space* $\mathbb{R}^2$*, such that*

$$F(u, v) = \int_0^u f_1(\tilde{u}, 0) d\tilde{u} + \int_0^v f_2(u, \tilde{v}) d\tilde{v}, \quad u, v \in \mathbb{R}, \tag{7}$$

or equivalently $\partial_v f_1(u, v) = \partial_u f_2(u, v)$, u, $v \in \mathbb{R}$. f_i *satisfies the growth condition at infinity: There exist four absolute constants* $Ł_1$, $Ł_2$, p_1 *and* p_2 *satisfying* $\min(Ł_1, Ł_2, p_1, p_2) > 0$ *and* $(N-2)\max(p_1, p_2) \leqslant 2$*, such that, for every pair* $(u, v)^\top \in \mathbb{R}^2$*, it holds always that*

$$|f_i(u, v)| \leqslant Ł_i(|u|^{p_1+1} + |u|^{p_2+1} + |u|^{\rho_1+1} + |v|^{p_1+1} + |v|^{p_2+1} + |v|^{\rho_2+1}), \quad i = 1, 2. \tag{8}$$

Assumption 5. *For every* $x \in \Omega$: $a_{12}(x) = a_{21}(x)$*; and the matrix*

$$\begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}$$

is semi-positive definite. a_{ij} *belongs to the Banach space* $\mathscr{C}(\bar{\Omega})$*, the totality of uniformly continuous real-valued functions defined in* Ω, i, $j = 1, 2$.

For the sake of convenience of our later presentation, we write

$$\frac{1}{\zeta_i} = \inf_{\varphi \in H_0^1(\Omega) \setminus \{\mathbf{0}\}} \frac{1}{\|\nabla \varphi\|^2_{L^2(\Omega; \mathbb{R}^N)}} \int_\Omega \nabla^\top \varphi A_i \nabla \varphi dx, \quad i = 1, 2, \tag{9}$$

and write for every $p \geqslant 1$ satisfying $(N-2)p \leqslant 2N$:

$$\kappa_p = \sup_{\varphi \in H_0^1(\Omega) \setminus \{\mathbf{0}\}} \frac{\|\varphi\|_{L^p(\Omega)}}{\|\nabla \varphi\|_{L^2(\Omega; \mathbb{R}^N)}}. \tag{10}$$

For every given $\alpha \in \mathbb{R}_+$, we denote henceforth

$$\mho_\alpha = \sup_{(x,y)^\top \in \mathbb{R}^2_+ \setminus \{(0,0)^\top\}} \frac{x^\alpha + y^\alpha}{(x+y)^\alpha} = 2^{\max(1-\alpha, 0)}. \tag{11}$$

Hereafter, we associate to a_{ij} (i, $j = 1, 2$) the following constant

$$\varkappa_{a_{11}, a_{12}, a_{21}, a_{22}} = \sup_{\substack{(\varphi_1, \psi_1)^\top, (\varphi_2, \psi_2)^\top \in L^2(\Omega; \mathbb{R}^2), \\ \|(\varphi_1, \psi_1)^\top\|_{L^2(\Omega; \mathbb{R}^2)} = \|(\varphi_2, \psi_2)^\top\|_{L^2(\Omega; \mathbb{R}^2)} = 1}} \left| \int_\Omega \begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix}^\top \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \varphi_2 \\ \psi_2 \end{pmatrix} dx \right|. \tag{12}$$

To improve the readability, we would like to give some remarks on our list of five standing assumptions (see Assumptions 1–5 for the details) of this paper.

Remark 1. *By applying the celebrated Sobolev–Poincaré inequality (see [1], THEOREM 1, p. 292), we can conclude that* κ_p *given by* (10) *is a positive constant. In light of Assumption 1 on the coefficient matrix* $A_i(x)$*, the term* $\int_\Omega \nabla^\top \varphi A_i \nabla \varphi dx$ *is well-defined and* ζ_i *given by* (9) *is also a positive constant obeying*

$$\frac{1}{\sqrt{\| \operatorname{tr}\left((A_i)^2\right) \|_{L^\infty(\Omega)}}} = \frac{1}{\sqrt{\| \operatorname{tr}\left((A_i)^\top A_i\right) \|_{L^\infty(\Omega)}}} \leqslant \zeta_i \leqslant \frac{1}{\vartheta_i},$$

where ϑ_i *is given by* (5)*, and* tr *denotes the trace operator of square matrice,* $i = 1, 2$.

Remark 2. *By imposing the restriction* (6) *in Assumption 2, our principal aim in this paper is to guarantee the twice continuous differentiability of the function* $\max(K_1, K_2)$ *in the interval* $[0, \min(r_1, r_2)]$*. The restriction* (6) *can be probably removed via introducing the notion of subdifferential in convex analysis, or via utilizing Dini's derivatives, together with some complicated calculations techniques.*

Remark 3. *Let* $\alpha \in \mathbb{R}_+$*. We can prove, based on the notation* $\mho_\alpha$ *given by* (11)*, that*

$$x^\alpha + y^\alpha \leqslant \mho_\alpha (x+y)^\alpha, \quad (x, y) \in \mathbb{R}_+^2.$$

Based on Assumption 4 ((7) *and* (8)*, in particular), we could obtain the following lemma by some routine but tedious calculations.*

Lemma 1. *Let* $F(u, v)$*,* $f_1(u, v)$ *and* $f_2(u, v)$ *be three functions given as in Assumption 4. For every pair* $(u, v)^\top \in \mathbb{R}^2$*, it holds always that*

$$\begin{aligned} |F(u,v)| \leqslant & \frac{Ł_1 + Ł_2(p_1+1)}{p_1+2}|u|^{p_1+2} + \frac{Ł_1 + Ł_2(p_2+1)}{p_2+2}|u|^{p_2+2} \\ & + \frac{Ł_1 + Ł_2(\rho_1+1)}{\rho_1+2}|u|^{\rho_1+2} + \frac{2Ł_2|v|^{p_1+2}}{p_1+2} + \frac{2Ł_2|v|^{p_2+2}}{p_2+2} \\ & + \frac{Ł_2|v|^{\rho_1+2}}{\rho_1+2} + \frac{Ł_2|v|^{\rho_2+2}}{\rho_2+2}, \end{aligned}$$

where $Ł_i$ *is exactly the one given in* (8) *in Assumption 4,* $i = 1, 2$.

Remark 4. *By some routine calculations, we have immediately that*

$$\begin{aligned} (\varkappa_{a_{11},a_{12},a_{21},a_{22}})^2 & \leqslant \|(a_{11})^2 + (a_{12})^2 + (a_{21})^2 + (a_{22})^2\|_{L^\infty(\Omega)} \\ & \leqslant \|a_{11}\|^2_{L^\infty(\Omega)} + \|a_{12}\|^2_{L^\infty(\Omega)} + \|a_{21}\|^2_{L^\infty(\Omega)} + \|a_{22}\|^2_{L^\infty(\Omega)}. \end{aligned}$$

This, together with Assumption 5, implies that $\varkappa_{a_{11},a_{12},a_{21},a_{22}}$*, given by* (12)*, is indeed a non-negative constant.*

Notational Conventions. $\mathbb{R}$ is the field of real numbers; $\mathbb{R}_+ = [0, +\infty)$; $\mathbb{R}_- = (-\infty, 0]$. For $\varphi : \Omega \to \mathbb{R}$, we write formally $\nabla^\top \varphi = (\nabla \varphi)^\top$. For $1 \leqslant p \leqslant +\infty$, $L^p(\Omega)$ denotes, as usual, the classical Lebesgue space. The Hilbert space $H^1(\Omega)$, equipped with the inner product

$$(\varphi, \psi)^\top \mapsto \int_\Omega (\varphi\psi + \nabla^\top \varphi \nabla \psi) dx, \tag{13}$$

denotes the totality of square-integrable functions defined in Ω whose first order partial derivatives, in the distributional sense, are all square-integrable functions in Ω. $H_0^1(\Omega)$

denotes the totality of functions in $H^1(\Omega)$ having zero as their boundary values in the trace sense, or equivalently, $H_0^1(\Omega)$ is the completion of the totality $\mathscr{C}^\infty_{\mathrm{comp}}(\Omega)$ of infinitely differentiable functions defined in Ω having compact support in the Hilbert space $H^1(\Omega)$; inheriting the inner product (13) from $H^1(\Omega)$, $H_0^1(\Omega)$ is also a Hilbert space. We write $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_{H_0^1(\Omega)}$ for the norms induced by inner products of the Hilbert spaces $H^1(\Omega)$ and $H_0^1(\Omega)$. Let H be an inner product space, we write $\langle \dagger, \ddagger\rangle_H$ for the inner product of the space H. Let X be a Banach space with X' its topological dual, and J an interval; we write $\langle \dagger, \ddagger\rangle_{X',X}$ for the duality pairing (of the Banach space X and its dual X') and denote by $\mathscr{C}_w(J; X)$ the following space

$$\big\{\varphi : J \to X;\ J \ni t \to \langle \psi, \varphi(t)\rangle_{X',X} \in \mathbb{R} \text{ for every } \psi \in X'\big\}.$$

The rest of this paper is organized as follows. In Section 2, we prove that solutions to IBVP (1) exist globally in time whenever their initial data are sufficiently small. In Section 3, we provide a general decaying estimate on global-in-time solutions to IBVP (1); the estimate implies that global-in-time solutions to IBVP (1) decrease to zero as time goes to infinity, whenever their initial data satisfy some additional conditions. In Section 4, we provide several concluding remarks.

2. Global Existence Results Concerning Solutions to IBVP (1)

In this section, our main aim is to prove the global existence of solutions, whose initial data belonging to a certain function space, to IBVP (1). We shall first demonstrate the local existence of solutions to IBVP (1) via utilizing the Faedo-Galerkin method, and shall then prove the desired global existence by establishing *a priori* estimates and a standard continuation argument. For the sake of convenience of our later presentation, we write, in the sequel, for every $T \in (0, +\infty)$:

$$\begin{aligned}\mathcal{S}_{[0,T]} = \Big\{&(u,v)^\top; u,v : \Omega \times (-\infty, T] \to \mathbb{R},\\ &u|_{\Omega\times[0,T]}, v|_{\Omega\times[0,T]} \in \mathscr{C}_w([0,T]; H_0^1(\Omega)),\\ &\partial_t u|_{\Omega\times[0,T]}, \partial_t v|_{\Omega\times[0,T]} \in \mathscr{C}_w([0,T]; H_0^1(\Omega)),\\ &u|_{\Omega\times\mathbb{R}_-}, v|_{\Omega\times\mathbb{R}_-} \in L^\infty(\mathbb{R}_-; H_0^1(\Omega))\Big\},\end{aligned} \tag{14}$$

and for every $0 < T \leqslant +\infty$, we write similarly

$$\begin{aligned}\mathcal{S}_{[0,T)} = \Big\{&(u,v)^\top;\ u,v : \Omega \times (-\infty, T) \to \mathbb{R},\\ &(u,v)^\top|_{\Omega\times(-\infty,\tilde{T}]} \in \mathcal{S}_{[0,\tilde{T}]},\ \forall \tilde{T} \in (0,T)\Big\}.\end{aligned} \tag{15}$$

Definition 1. *Let $T \in (0, +\infty)$. The pair $(u, v)^\top \in \mathcal{S}_{[0,T]}$ is said to be a local weak solution, in the interval $[0, T]$, to IBVP (1) provided that the following two equalities hold for every pair $(\varphi, \psi)^\top \in H_0^1(\Omega; \mathbb{R}^2)$ of test functions:*

$$\begin{aligned}&\frac{1}{\rho_1+1}\int_\Omega |\partial_t u(t)|^{\rho_1}\partial_t u(t)\varphi dx + \int_\Omega \nabla^\top \varphi A_1 \nabla \partial_t u(t) dx\\ &- \frac{1}{\rho_1+1}\int_\Omega |u_1|^{\rho_1} u_1 \varphi dx - \int_\Omega \nabla^\top \varphi A_1 \nabla u_1 dx\\ &+ \int_0^t \int_\Omega \nabla^\top \varphi A_1 \Big(\nabla u(s) - \int_{-\infty}^s g_1(s-\tau)\nabla u(\tau) d\tau\Big) dx ds\\ =& \int_0^t \int_\Omega \varphi\big(f_1(u(s), v(s)) - a_{11}\partial_t u(s) - a_{12}\partial_t v(s)\big) dx ds, \quad t \in (0,T),\end{aligned} \tag{16}$$

and

$$\begin{aligned}
&\frac{1}{\rho_2+1}\int_\Omega |\partial_t v(t)|^{\rho_2}\partial_t v(t)\psi dx + \int_\Omega \nabla^\top \psi A_2 \nabla \partial_t v(t) dx \\
&- \frac{1}{\rho_2+1}\int_\Omega |v_1|^{\rho_2} v_1 \psi dx - \int_\Omega \nabla^\top \psi A_2 \nabla v_1 dx \\
&+ \int_0^t \int_\Omega \nabla^\top \psi A_2 (\nabla v(s) - \int_{-\infty}^s g_2(s-\tau)\nabla v(\tau) d\tau) dx ds \\
= &\int_0^t \int_\Omega \psi (f_2(u(s),v(s)) - a_{21}\partial_t u(s) - a_{22}\partial_t v(s)) dx ds, \quad t\in(0,T).
\end{aligned} \tag{17}$$

Definition 2. *Let $0 < T \leqslant +\infty$. $(u,v)^\top : \Omega\times(-\infty,T)\to\mathbb{R}^2$ is said to be a weak solution to IBVP (1), in the interval $[0,T)$, if for every $0 < T' < T$, $(u,v)^\top|_{\Omega\times(-\infty,T']}$ is a local weak solution, in the interval $[0,T']$, to IBVP (1). In the case that $T = +\infty$ (or equivalently, $(u,v)^\top \in \mathcal{S}_{\mathbb{R}_+}$; see (15) for the definition of $\mathcal{S}_{\mathbb{R}_+}$), $(u,v)^\top$ is called a global weak solution to IBVP (1); otherwise, $(u,v)^\top$ is still called a local weak solution to IBVP (1).*

To every solution pair $(u,v)^\top \in \mathcal{S}_{[0,T)}$ (see (15) for the definition of $\mathcal{S}_{[0,T)}$), with $0 < T \leqslant +\infty$, to IBVP (1), we associate the following functional (a certain Lyapunov functional candidate)

$$\begin{aligned}
E^{u,v}(t) = &\frac{1}{\rho_1+2}\|\partial_t u(t)\|^{\rho_1+2}_{L^{\rho_1+2}(\Omega)} + \frac{1}{\rho_2+2}\|\partial_t v(t)\|^{\rho_2+2}_{L^{\rho_2+2}(\Omega)} \\
&+ \frac{1}{2}(\mu_1 - \int_0^{+\infty} g_1(s)ds)\int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx \\
&+ \frac{1}{2}\int_\Omega \nabla^\top \partial_t u(t) A_1 \nabla \partial_t u(t) dx + \frac{1}{2}(g_1 \diamond^{A_1} \nabla u)(t) \\
&+ \frac{1}{2}(\mu_2 - \int_0^{+\infty} g_2(s)ds)\int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx \\
&+ \frac{1}{2}\int_\Omega \nabla^\top \partial_t v(t) A_2 \nabla \partial_t v(t) dx + \frac{1}{2}(g_2 \diamond^{A_2} \nabla v)(t) \\
&- \int_\Omega F(u(t),v(t)) dx, \quad t\in[0,T),
\end{aligned} \tag{18}$$

where F is given by (7), and the operation "$\diamond$", associated to two given functions $\boldsymbol{\Phi} \in L^\infty(\Omega;\mathbb{R}^{N\times N})$ and $g \in L^\infty(0,+\infty)$, is defined, in a formal way, as follows: For every $\boldsymbol{\psi} \in L^2_{\text{loc}}([-\infty,T);L^2(\Omega;\mathbb{R}^N))$,

$$(g \diamond^{\boldsymbol{\Phi}} \boldsymbol{\psi})(t) = \int_{-\infty}^t g(t-s)\int_\Omega (\boldsymbol{\psi}(t)-\boldsymbol{\psi}(s))^\top \boldsymbol{\Phi}(\boldsymbol{\psi}(t)-\boldsymbol{\psi}(s)) dx ds. \tag{19}$$

Lemma 2. *Let $\boldsymbol{\Phi} \in W^{1,\infty}(\Omega;\mathbb{R}^{N\times N})$, $g \in W^{1,\infty}(0,+\infty)$. For every*

$$\varphi \in L^2_{\text{loc}}([-\infty,T);H^1_0(\Omega))$$

satisfying $\varphi|_{\Omega\times[0,T)} \in H^1_{\text{loc}}([0,T);H^1_0(\Omega))$, it holds that

$$\begin{aligned}
&\frac{d}{dt}\Big((g \diamond^{\boldsymbol{\Phi}} \nabla\varphi)(t) - \int_0^{+\infty} g(s)ds \int_\Omega \nabla^\top \varphi(t)\boldsymbol{\Phi}\nabla\varphi(t)dx\Big) \\
= &(g' \diamond^{\boldsymbol{\Phi}} \nabla\varphi)(t) - 2\int_{-\infty}^t g(t-s)\int_\Omega \nabla^\top \partial_t\varphi(t)\boldsymbol{\Phi}\nabla\varphi(s) dx ds, \quad t\in(0,T).
\end{aligned}$$

Proof. Conduct some routine calculations, to obtain

$$\begin{aligned}
&\frac{d}{dt}\left((g\diamond^{\boldsymbol{\Phi}}\nabla\varphi)(t)-\int_0^{+\infty}g(s)ds\int_\Omega\nabla^\top\varphi(t)\boldsymbol{\Phi}\nabla\varphi(t)dx\right)\\
&=(g'\diamond^{\boldsymbol{\Phi}}\nabla\varphi)(t)-2\int_0^{+\infty}g(s)ds\int_\Omega\nabla^\top\partial_t\varphi(t)\boldsymbol{\Phi}\nabla\varphi(t)dx\\
&\quad+2\int_{-\infty}^t g(t-s)\int_\Omega\nabla^\top\partial_t\varphi(t)\boldsymbol{\Phi}(\nabla\varphi(t)-\nabla\varphi(s))dxds\\
&=(g'\diamond^{\Phi}\nabla\varphi)(t)-2\int_{-\infty}^t g(t-s)\int_\Omega\nabla^\top\partial_t\varphi(t)\boldsymbol{\Phi}\nabla\varphi(s)dxds,\quad t\in(0,T),
\end{aligned}$$

in which the "=" in the second line follows immediately from the very definition of the expression $(g\diamond^{\Phi}\nabla\varphi)(t)$ (see (19) for the detail). □

The differential identity in Lemma 2 is of great importance in our later calculations and will be used frequently in proving our main results in this paper. For example, we shall employ Lemma 2 as one of the main ingredients in the procedure of deducing an *a priori* inequality which plays a key role in proving the following local existence theorem.

Theorem 1. *Suppose that Assumptions 1–4 hold true, and that the space-varying coefficient a_{ij} belongs to the Banach space $\mathscr{C}(\bar{\Omega})$, $i,j=1,2$. Then for every initial datum pair*

$$(u^0,v^0)^\top\in L^\infty(\mathbb{R}_-;H_0^1(\Omega;\mathbb{R}^2))$$

and every initial datum pair $(u^1,v^1)^\top\in H_0^1(\Omega;\mathbb{R}^2)$, IBVP (1) admits a local weak solution $(u,v)^\top\in\mathcal{S}_{[0,\tilde{T}]}$ (see (14) for the detailed definition of the notation $\mathcal{S}_{[0,\tilde{T}]}$), in the interval $[0,\tilde{T}]$, in which $\tilde{T}$ is a certain postive time instant depending merely on Ω, A_1, A_2, f_1, f_2, g_1, g_2, ρ_1, ρ_2, a_{11}, a_{12}, a_{21}, a_{22} , u^0, v^0, u^1 and v^1.

Please notice that we do not use Assumption 5 in Theorem 1 temporarily, instead, we used a weaker condition that 'a_{ij} belongs to the Banach space $\mathscr{C}(\bar{\Omega})$, $i,j=1,2$'. We shall prove Theorem 1 via the very standard Faedo-Galerkin procedure.

Proof. Thanks to Assumption 1, by recalling theory on elliptic partial differential equations, one can find: One orthonormal basis, designated by $\{e_{in}\}_{n=1}^\infty$, of the Hilbert space $L^2(\Omega)$ is composed of the solutions of the following eigenvalue problems

$$\begin{cases}\operatorname{div}(A_i\nabla e_i)=\lambda_i e_i & \text{in }\Omega,\\ e_i=0 & \text{on }\Gamma.\end{cases}$$

By using mainly the divergence theorem, we have

$$\begin{aligned}
\int_\Omega\nabla^\top e_{ik}A_i\nabla e_{i\ell}dx&=-\int_\Omega e_{ik}\operatorname{div}(A_i\nabla e_{i\ell})dx\\
&=\lambda_{i\ell}\int_\Omega e_{ik}A_ie_{i\ell}dx\\
&=\begin{cases}\lambda_{i\ell} & \text{if }k=\ell,\\ 0 & \text{if }k\neq\ell.\end{cases}
\end{aligned}$$

Since $A_i\in\mathscr{C}^2(\bar{\Omega};\mathbb{R}^{N\times N})$ (see Assumption 1), for any positive integer n, we have $e_{in}\in\mathscr{C}^2(\bar{\Omega})$, $i=1,2$. Let us introduce the following two sequences of approximate solutions

$$\left.\begin{aligned}
u_k(x,t)&=\sum_{j=1}^k e_{1j}\otimes\tilde{u}_{kj}(x,t)=\sum_{j=1}^k\tilde{u}_{kj}(t)e_{1j}(x),\\
v_k(x,t)&=\sum_{j=1}^k e_{2j}\otimes\tilde{v}_{kj}(x,t)=\sum_{j=1}^k\tilde{v}_{kj}(t)e_{2j}(x),
\end{aligned}\right\}\quad k\in\mathbb{N}. \tag{20}$$

As with (20), to construct sequences of approximate solutions is one of the main steps in applying the Faedo-Galerkin scheme to prove local existence result of evolution partial differential equations. After some calculations, we can find that $(u_k, v_k)^\top$ given by (20) is approximate solution pair to IBVP (1) if and only if

$$\begin{aligned} U_k(t) = (&\tilde{u}_{k1}(t), \dots, \tilde{u}_{kk}(t), \tilde{u}'_{k1}(t), \dots, \tilde{u}'_{kk}(t), \\ &\tilde{v}_{k1}(t), \dots, \tilde{v}_{kk}(t), \tilde{v}'_{k1}(t), \dots, \tilde{v}'_{kk}(t))^\top \end{aligned} \tag{21}$$

is the solution to the following Cauchy problem

$$\begin{cases} \int_\Omega |\sum_{j=1}^k \tilde{u}'_{kj}(t)e_{1j}|^{\rho_1} \sum_{j=1}^k \tilde{u}''_{kj}(t)e_{1j}e_{1\ell}dx \\ + \lambda_{1\ell}\tilde{u}_{k\ell}(t) + \lambda_{1\ell}\tilde{u}''_{k\ell}(t) - \lambda_{1\ell}\int_{-\infty}^t g_1(t-s)\tilde{u}_{k\ell}(s)ds \\ + \sum_{j=1}^k \tilde{u}'_{kj}(t)\int_\Omega a_{11}e_{1j}e_{1\ell}dx \\ + \sum_{j=1}^k \tilde{v}'_{kj}(t)\int_\Omega a_{12}e_{2j}e_{1\ell}dx \\ = \int_\Omega f_1(\sum_{j=1}^k \tilde{u}_{kj}(t)e_{1j}, \sum_{j=1}^k \tilde{v}_{kj}(t)e_{2j})e_{1\ell}dx, & t \in \mathbb{R}_+, \ell = 1, \dots, k, \\ \int_\Omega |\sum_{j=1}^k \tilde{v}'_{kj}(t)e_{2j}|^{\rho_2} \sum_{j=1}^k \tilde{v}''_{kj}(t)e_{2j}e_{2m}dx \\ + \lambda_{2m}\tilde{v}_{km}(t) + \lambda_{2m}\tilde{v}''_{km}(t) - \lambda_{2m}\int_{-\infty}^t g_2(t-s)\tilde{v}_{km}(s)ds \\ + \sum_{j=1}^k \tilde{v}'_{kj}(t)\int_\Omega a_{21}e_{2j}e_{2m}dx \\ + \sum_{j=1}^k \tilde{v}'_{kj}(t)\int_\Omega a_{22}e_{2j}e_{2m}dx \\ = \int_\Omega f_2(\sum_{j=1}^k \tilde{u}_{kj}(t)e_{1j}, \sum_{j=1}^k \tilde{v}_{kj}(t)e_{2j})e_{2m}dx, & t \in \mathbb{R}_+, m = 1, \dots, k, \\ \tilde{u}_{k\ell}(t) = \int_\Omega u^0(t)e_{1\ell}dx, \\ \tilde{v}_{k\ell}(t) = \int_\Omega v^0(t)e_{2\ell}dx, & t \in \mathbb{R}_-, \ell = 1, \dots, k, \\ \tilde{u}'_{k\ell}(0) = \int_\Omega u^1 e_{1\ell}dx, \\ \tilde{v}'_{k\ell}(0) = \int_\Omega v^1 e_{2\ell}dx, & \ell = 1, \dots, k. \end{cases} \tag{22}$$

In accordance with (21), we write

$$\begin{aligned} U_k^0(t) = (&\int_\Omega u^0(t)e_{11}dx, \dots, \int_\Omega u^0(t)e_{1k}dx, \int_\Omega u^1 e_{11}dx, \dots, \int_\Omega u^1 e_{1k}dx, \\ &\int_\Omega v^0(t)e_{21}dx, \dots, \int_\Omega v^0(t)e_{2k}dx, \int_\Omega v^1 e_{21}dx, \dots, \int_\Omega v^1 e_{2k}dx)^\top, \quad t \in \mathbb{R}_-. \end{aligned}$$

Then the Cauchy problem (22) can be recast into the following Cauchy problem for a functional differential equation

$$\begin{cases} U_k'(t) = \mathcal{F}_k(U_k(t), U_{kt}), & t \in \mathbb{R}_+, \\ U_k = U_k^0 & \text{in } \mathbb{R}_-, \end{cases} \tag{23}$$

where $U_{kt}(\tau) = U_k(t+\tau)$, $\tau \in \mathbb{R}_-$, and $\mathcal{F}_k : \mathbb{R}^{4k} \times L^\infty(\mathbb{R}_-; \mathbb{R}^{4k}) \to \mathbb{R}^{4k}$ is locally Lipschitz continuous. Then by a variant of the classical Cauchy–Lipschitz existence theory, the Cauchy problem (23), or equivalently, the Cauchy problem (22) is locally well-posed in Hadamard's sense. As a consequence, by applying a standard continuation argument, we can prove, based on the aforementioned local well-posedness result, that the Cauchy problem (22) (for system of ordinary differential equations) admits a unique solution

$$(\tilde{u}_{k1}(t), \ldots, \tilde{u}_{kk}(t), \tilde{v}_{k1}(t), \ldots, \tilde{v}_{kk}(t))^\top$$

in the classical sense in $[0, T^*)$, the maximal interval of existence. Thus the pair $(u_k, v_k)^\top$ in the form (20) is well-defined. Let us now introduce the following auxilliary functional

$$\begin{aligned} \Xi_k(t) = {} & E^{u_k, v_k}(t) + \int_\Omega F(u_k(t), v_k(t)) dx \\ = {} & \frac{1}{\rho_1 + 2} \|\partial_t u_k(t)\|_{L^{\rho_1+2}(\Omega)}^{\rho_1+2} + \frac{1}{\rho_2+2} \|\partial_t v_k(t)\|_{L^{\rho_2+2}(\Omega)}^{\rho_2+2} \\ & + \frac{1}{2}(\mu_1 - \int_0^{+\infty} g_1(s) ds) \int_\Omega \nabla^\top u_k(t) A_1 \nabla u_k(t) dx \\ & + \frac{1}{2} \int_\Omega \nabla^\top \partial_t u_k(t) A_1 \nabla \partial_t u_k(t) dx + \frac{1}{2}(g_1 \diamond^{A_1} \nabla u_k)(t) \\ & + \frac{1}{2}(\mu_2 - \int_0^{+\infty} g_2(s) ds) \int_\Omega \nabla^\top v_k(t) A_2 \nabla v_k(t) dx \\ & + \frac{1}{2} \int_\Omega \nabla^\top \partial_t v_k(t) A_2 \nabla \partial_t v_k(t) dx + \frac{1}{2}(g_2 \diamond^{A_2} \nabla v_k)(t), \quad t \in [0, T^*). \end{aligned} \tag{24}$$

Differentiate $\Xi_k(t)$ and simplify further the obtained result, to yield

$$\begin{aligned} \Xi_k'(t) = {} & \frac{1}{2}(g_1' \diamond^{A_1} \nabla u_k)(t) + \frac{1}{2}(g_2' \diamond^{A_2} \nabla v_k)(t) - \int_{-\infty}^t g_1(t-s) \int_\Omega \nabla^\top \partial_t u_k(t) A_1 \nabla u_k(s) dx ds \\ & - \int_{-\infty}^t g_2(t-s) \int_\Omega \nabla^\top \partial_t v_k(t) A_2 \nabla v_k(s) dx ds - \int_{-\infty}^t g_1(t-s) \int_\Omega \partial_t u_k(t) \operatorname{div}(A_1 \nabla u_k(s)) dx ds \\ & - \int_{-\infty}^t g_2(t-s) \int_\Omega \partial_t v_k(t) \operatorname{div}(A_2 \nabla v_k(s)) dx ds \\ & + \int_\Omega \partial_t u_k(t) f_1(u_k(t), v_k(t)) dx + \int_\Omega \partial_t v_k(t) f_2(u_k(t), v_k(t)) dx \\ & - \int_\Omega \begin{pmatrix} \partial_t u_k(t) \\ \partial_t v_k(t) \end{pmatrix}^\top \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \partial_t u_k(t) \\ \partial_t v_k(t) \end{pmatrix} dx \\ \leqslant {} & \int_\Omega \partial_t u_k(t) f_1(u_k(t), v_k(t)) dx + \int_\Omega \partial_t v_k(t) f_2(u_k(t), v_k(t)) dx \\ & + \varkappa_{a_{11}, a_{12}, a_{21}, a_{22}} (\|\partial_t u_k(t)\|_{L^2(\Omega)}^2 + \|\partial_t v_k(t)\|_{L^2(\Omega)}^2), \quad t \in [0, T^*). \end{aligned} \tag{25}$$

Before continuing our proof, it is worth noticing that the unique existence of solutions to the Cauchy problem (22) and solutions to the Cauchy problem (22) depending continuously on their initial data are both attributed to the local Lipschitz continuity of $\mathcal{F}_k$ in the right hand side of the partial differential equation in the Cauchy problem (22), and that the maximal existence time instant T^*, independent of k, depends on Ω, A_1, A_2, f_1, f_2, g_1, g_2, ρ_1, ρ_2, a_{11}, a_{12}, a_{21}, a_{22} , u^0, v^0, u^1 and v^1.

Thanks to

$$\frac{\rho_1}{\rho_1 + 2} + \frac{2}{\rho_1 + 2} = 1,$$

by the Fenchel–Young inequality, it holds that

$$|ab| \leqslant \frac{\rho_1}{\rho_1+2}|a|^{\frac{\rho_1+2}{\rho_1}} + \frac{2}{\rho_1+2}|b|^{\frac{\rho_1+2}{2}}, \quad a,b \in \mathbb{R},$$

from which it follows further that

$$\int_\Omega a_{11}|\partial_t u_k(t)|^2 dx \leqslant \frac{\rho_1}{\rho_1+2}\|a_{11}\|^{\frac{\rho_1+2}{\rho_1}}_{L^{\frac{\rho_1+2}{\rho_1}}(\Omega)} + \frac{2}{\rho_1+2}\|\partial_t u_k(t)\|^{\rho_1+2}_{L^{\rho_1+2}(\Omega)}, \quad t \in [0,T^*). \tag{26}$$

And similarly, we have

$$\int_\Omega a_{22}|\partial_t v_k(t)|^2 dx \leqslant \frac{\rho_2}{\rho_2+2}\|a_{22}\|^{\frac{\rho_2+2}{\rho_2}}_{L^{\frac{\rho_2+2}{\rho_2}}(\Omega)} + \frac{2}{\rho_2+2}\|\partial_t v_k(t)\|^{\rho_2+2}_{L^{\rho_2+2}(\Omega)}, \quad t \in [0,T^*). \tag{27}$$

In view of the Fenchel–Young inequality

$$|abc| \leqslant \frac{1}{\rho_1+2}|a|^{\rho_1+2} + \frac{1}{\rho_2+2}|b|^{\rho_2+2} + \frac{\rho_1\rho_2+\rho_1+\rho_2}{(\rho_1+2)(\rho_2+2)}|c|^{\frac{(\rho_1+2)(\rho_2+2)}{\rho_1\rho_2+\rho_1+\rho_2}}, \quad a,b,c \in \mathbb{R},$$

which follows directly from the identity

$$\frac{\rho_1\rho_2+\rho_1+\rho_2}{(\rho_1+2)(\rho_2+2)} + \frac{1}{\rho_1+2} + \frac{1}{\rho_2+2} = 1,$$

with the aid of the experience gained in the procedure of deriving (26) and (27), we have, after some routine but careful calculations, that

$$\Big|\int_\Omega a_{12}\partial_t u_k(t)\partial_t v_k(t)dx\Big| \leqslant \frac{1}{\rho_1+2}\|\partial_t u_k(t)\|^{\rho_1+2}_{L^{\rho_1+2}(\Omega)} + \frac{1}{\rho_2+2}\|\partial_t v_k(t)\|^{\rho_2+2}_{L^{\rho_2+2}(\Omega)} + \frac{\rho_1\rho_2+\rho_1+\rho_2}{(\rho_1+2)(\rho_2+2)}\|a_{12}\|^{\frac{(\rho_1+2)(\rho_2+2)}{\rho_1\rho_2+\rho_1+\rho_2}}_{L^{\frac{(\rho_1+2)(\rho_2+2)}{\rho_1\rho_2+\rho_1+\rho_2}}(\Omega)}, \quad t \in [0,T^*). \tag{28}$$

And analogously, we have

$$\Big|\int_\Omega a_{21}\partial_t u_k(t)\partial_t v_k(t)dx\Big| \leqslant \frac{1}{\rho_1+2}\|\partial_t u_k(t)\|^{\rho_1+2}_{L^{\rho_1+2}(\Omega)} + \frac{1}{\rho_2+2}\|\partial_t v_k(t)\|^{\rho_2+2}_{L^{\rho_2+2}(\Omega)} + \frac{\rho_1\rho_2+\rho_1+\rho_2}{(\rho_1+2)(\rho_2+2)}\|a_{21}\|^{\frac{(\rho_1+2)(\rho_2+2)}{\rho_1\rho_2+\rho_1+\rho_2}}_{L^{\frac{(\rho_1+2)(\rho_2+2)}{\rho_1\rho_2+\rho_1+\rho_2}}(\Omega)}, \quad t \in [0,T^*). \tag{29}$$

By utilizing the growth condition (8) upon the nonlinearity f_i $(i=1,2)$ in Assumption 4, we obtain immediately

$$\begin{aligned}\int_\Omega \partial_t u_k(t) f_1(u_k(t),v_k(t))dx \leqslant{}& Ł_1\int_\Omega |\partial_t u_k(t)||u_k(t)|^{p_1+1}dx + Ł_1\int_\Omega |\partial_t u_k(t)||u_k(t)|^{p_2+1}dx \\ &+ Ł_1\int_\Omega |\partial_t u_k(t)||u_k(t)|^{\rho_1+1}dx + Ł_1\int_\Omega |\partial_t u_k(t)||v_k(t)|^{p_1+1}dx \\ &+ Ł_1\int_\Omega |\partial_t u_k(t)||v_k(t)|^{p_2+1}dx + Ł_1\int_\Omega |\partial_t u_k(t)||v_k(t)|^{\rho_2+1}dx, \quad t \in [0,T^*).\end{aligned}$$

By virtue of observing

$$\frac{1}{\rho_1+2}+\frac{\rho_1+1}{\rho_1+2}=1, \tag{30}$$

inspired by the experience gained in deducing (26)–(29), we have, by applying also the Fenchel–Young inequality, immediately that

$$\begin{aligned}\int_\Omega |\partial_t u_k(t)||u_k(t)|^{\rho_1+1}dx \leqslant& \frac{1}{\rho_1+2}\int_\Omega |\partial_t u_k(t)|^{\rho_1+2}dx\\ &+\frac{\rho_1+1}{\rho_1+2}\int_\Omega |u_k(t)|^{\frac{(\rho_1+2)(\rho_1+2)}{\rho_1+1}}dx\\ =&\frac{1}{\rho_1+2}\|\partial_t u_k(t)\|^{\rho_1+2}_{L^{\rho_1+2}(\Omega)}\\ &+\frac{\rho_1+1}{\rho_1+2}\|u_k(t)\|^{\frac{(\rho_1+1)(\rho_1+2)}{\rho_1+1}}_{L^{\frac{(\rho_1+1)(\rho_1+2)}{\rho_1+1}}(\Omega)}, \quad t\in[0,T^*).\end{aligned} \tag{31}$$

In view of the coercivity condition on A_1 (see Assumption 1), we have by applying the Sobolev–Poincaré inequality (see Remark 1) that

$$\begin{aligned}\|u_k(t)\|^{\frac{(\rho_1+1)(\rho_1+2)}{\rho_1+1}}_{L^{\frac{(\rho_1+1)(\rho_1+2)}{\rho_1+1}}(\Omega)} &\leqslant (\kappa_{\frac{(\rho_1+1)(\rho_1+2)}{\rho_1+1}})^{\frac{(\rho_1+1)(\rho_1+2)}{\rho_1+1}}\|\nabla u_k(t)\|^{\frac{(\rho_1+1)(\rho_1+2)}{\rho_1+1}}_{L^2(\Omega;\mathbb{R}^N)}\\ &\leqslant (\kappa_{\frac{(\rho_1+1)(\rho_1+2)}{\rho_1+1}})^{\frac{(\rho_1+1)(\rho_1+2)}{\rho_1+1}}(\zeta_1)^{\frac{(\rho_1+1)(\rho_1+2)}{2\rho_1+2}}(\int_\Omega \nabla^\top u_k(t)A_1\nabla u_k(t)dx)^{\frac{(\rho_1+1)(\rho_1+2)}{2\rho_1+2}};\end{aligned} \tag{32}$$

see (9) and (10) for the detailed explanations on the notations ζ_1 and $\kappa_{\frac{(\rho_1+1)(\rho_1+2)}{\rho_1+1}}$, respectively. Combine (31) and (32), to arrive at

$$\begin{aligned}\int_\Omega |\partial_t u_k(t)||u_k(t)|^{\rho_1+1}dx \leqslant& \frac{1}{\rho_1+2}\|\partial_t u_k(t)\|^{\rho_1+2}_{L^{\rho_1+2}(\Omega)}\\ &+\mathscr{M}^1_{11}(\int_\Omega \nabla^\top u_k(t)A_1\nabla u_k(t)dx)^{\frac{(\rho_1+1)(\rho_1+2)}{2\rho_1+2}}, \quad t\in[0,T^*),\end{aligned} \tag{33}$$

in which the positive constant $\mathscr{M}^1_{11}$ is given by

$$\mathscr{M}^1_{11}=\frac{\rho_1+1}{\rho_1+2}(\kappa_{\frac{(\rho_1+1)(\rho_1+2)}{\rho_1+1}})^{\frac{(\rho_1+1)(\rho_1+2)}{\rho_1+1}}(\zeta_1)^{\frac{(\rho_1+1)(\rho_1+2)}{2\rho_1+2}}.$$

With the help of the experience of deriving (33), we conlude similarly that

$$\begin{aligned}\int_\Omega |\partial_t u_k(t)||v_k(t)|^{\rho_1+1}dx \leqslant& \frac{1}{\rho_1+2}\|\partial_t u_k(t)\|^{\rho_1+2}_{L^{\rho_1+2}(\Omega)}+\frac{\rho_1+1}{\rho_1+2}\|v_k(t)\|^{\frac{(\rho_1+1)(\rho_1+2)}{\rho_1+1}}_{L^{\frac{(\rho_1+1)(\rho_1+2)}{\rho_1+1}}(\Omega)}\\ \leqslant& \frac{1}{\rho_1+2}\|\partial_t u_k(t)\|^{\rho_1+2}_{L^{\rho_1+2}(\Omega)}\\ &+\mathscr{M}^1_{12}(\int_\Omega \nabla^\top v_k(t)A_2\nabla v_k(t)dx)^{\frac{(\rho_1+1)(\rho_1+2)}{2\rho_1+2}}, \quad t\in[0,T^*),\end{aligned} \tag{34}$$

in which the positive constant $\mathscr{M}^1_{12}$ is given by

$$\mathscr{M}^1_{12}=\frac{\rho_1+1}{\rho_1+2}(\kappa_{\frac{(\rho_1+1)(\rho_1+2)}{\rho_1+1}})^{\frac{(\rho_1+1)(\rho_1+2)}{\rho_1+1}}(\zeta_2)^{\frac{(\rho_1+1)(\rho_1+2)}{2\rho_1+2}}.$$

With the help of the experience of deriving (33) and (34), we arrive at

$$\int_\Omega |\partial_t u_k(t)||u_k(t)|^{p_2+1}dx \leqslant \frac{1}{\rho_1+2}\|\partial_t u_k(t)\|^{\rho_1+2}_{L^{\rho_1+2}(\Omega)} + \frac{\rho_1+1}{\rho_1+2}\|u_k(t)\|^{\frac{(p_2+1)(\rho_1+2)}{\rho_1+1}}_{L^{\frac{(p_2+1)(\rho_1+2)}{\rho_1+1}}(\Omega)}$$
$$\leqslant \frac{1}{\rho_1+2}\|\partial_t u_k(t)\|^{\rho_1+2}_{L^{\rho_1+2}(\Omega)}$$
$$+ \mathscr{M}^1_{13}(\int_\Omega \nabla^\top u_k(t)A_1\nabla u_k(t)dx)^{\frac{(p_2+1)(\rho_1+2)}{2\rho_1+2}}, \quad t\in[0,T^*), \tag{35}$$

in which the positive constant $\mathscr{M}^1_{13}$ is given by

$$\mathscr{M}^1_{13} = \frac{\rho_1+1}{\rho_1+2}(\kappa_{\frac{(p_2+1)(\rho_1+2)}{\rho_1+1}})^{\frac{(p_2+1)(\rho_1+2)}{\rho_1+1}}(\zeta_1)^{\frac{(p_2+1)(\rho_1+2)}{2\rho_1+2}}.$$

With the help of the experience of deriving (33)–(35), we conlude that

$$\int_\Omega |\partial_t u_k(t)||v_k(t)|^{p_2+1}dx \leqslant \frac{1}{\rho_1+2}\|\partial_t u_k(t)\|^{\rho_1+2}_{L^{\rho_1+2}(\Omega)} + \frac{\rho_1+1}{\rho_1+2}\|v_k(t)\|^{\frac{(p_2+1)(\rho_1+2)}{\rho_1+1}}_{L^{\frac{(p_2+1)(\rho_1+2)}{\rho_1+1}}(\Omega)}$$
$$\leqslant \frac{1}{\rho_1+2}\|\partial_t u_k(t)\|^{\rho_1+2}_{L^{\rho_1+2}(\Omega)}$$
$$+ \mathscr{M}^1_{14}(\int_\Omega \nabla^\top v_k(t)A_2\nabla v_k(t)dx)^{\frac{(p_2+1)(\rho_1+2)}{2\rho_1+2}}, \quad t\in[0,T^*), \tag{36}$$

in which the positive constant $\mathscr{M}^1_{14}$ is given by

$$\mathscr{M}^1_{14} = \frac{\rho_1+1}{\rho_1+2}(\kappa_{\frac{(p_2+1)(\rho_1+2)}{\rho_1+1}})^{\frac{(p_2+1)(\rho_1+2)}{\rho_1+1}}(\zeta_2)^{\frac{(p_2+1)(\rho_1+2)}{2\rho_1+2}}.$$

Based on the algebraic identity (30), we apply the Fenchel–Young inequality, to obtain

$$\int_\Omega |\partial_t u_k(t)||u_k(t)|^{\rho_1+1}dx \leqslant \frac{1}{\rho_1+2}\|\partial_t u_k(t)\|^{\rho_1+2}_{L^{\rho_1+2}(\Omega)}$$
$$+ \frac{\rho_1+1}{\rho_1+2}\|u_k(t)\|^{\rho_1+2}_{L^{\rho_1+2}(\Omega)}, \quad t\in[0,T^*). \tag{37}$$

Mimicking steps in deriving (32), having the notations in (9) and (10) at our disposal, and based on the coercivity condition on A_1 (see Assumption 1), we apply the Sobolev–Poincaré inequality (see Remark 1), to arrive at

$$\|u_k(t)\|^{\rho_1+2}_{L^{\rho_1+2}(\Omega)} \leqslant (\kappa_{\rho_1+2})^{\rho_1+2}\|\nabla u_k(t)\|^{\rho_1+2}_{L^2(\Omega;\mathbb{R}^N)}$$
$$\leqslant (\kappa_{\rho_1+2})^{\rho_1+2}(\zeta_1)^{\frac{\rho_1+2}{2}}(\int_\Omega \nabla^\top u_k(t)A_1\nabla u_k(t)dx)^{\frac{\rho_1+2}{2}},$$

which, together with (37), implies directly

$$\int_\Omega |\partial_t u_k(t)||u_k(t)|^{\rho_1+1}dx \leqslant \frac{1}{\rho_1+2}\|\partial_t u_k(t)\|^{\rho_1+2}_{L^{\rho_1+2}(\Omega)}$$
$$+ \mathscr{M}^1_{15}(\int_\Omega \nabla^\top u_k(t)A_1\nabla u_k(t)dx)^{\frac{\rho_1+2}{2}}, \quad t\in[0,T^*), \tag{38}$$

in which the positive constant $\mathscr{M}^1_{15}$ is given by

$$\mathscr{M}^1_{15} = \frac{\rho_1+1}{\rho_1+2}(\kappa_{\rho_1+2})^{\rho_1+2}(\zeta_1)^{\frac{\rho_1+2}{2}}.$$

With the aid of the experience of deriving (33)–(36), we conlude that

$$\int_\Omega |\partial_t u_k(t)||v_k(t)|^{\rho_2+1}dx \leqslant \frac{1}{\rho_1+2}\|\partial_t u_k(t)\|^{\rho_1+2}_{L^{\rho_1+2}(\Omega)} + \frac{\rho_1+1}{\rho_1+2}\|v_k(t)\|^{\frac{(\rho_2+1)(\rho_1+2)}{\rho_1+1}}_{L^{\frac{(\rho_2+1)(\rho_1+2)}{\rho_1+1}}(\Omega)}$$
$$\leqslant \frac{1}{\rho_1+2}\|\partial_t u_k(t)\|^{\rho_1+2}_{L^{\rho_1+2}(\Omega)} + \mathscr{M}^1_{16}\Big(\int_\Omega \nabla^\top v_k(t)A_2\nabla v_k(t)dx\Big)^{\frac{(\rho_2+1)(\rho_1+2)}{2\rho_1+2}}, \quad t\in[0,T^*), \tag{39}$$

in which the positive constant $\mathscr{M}^1_{16}$ is given by

$$\mathscr{M}^1_{16} = \frac{\rho_1+1}{\rho_1+2}\big(\kappa_{\frac{(\rho_2+1)(\rho_1+2)}{\rho_1+1}}\big)^{\frac{(\rho_2+1)(\rho_1+2)}{\rho_1+1}}(\zeta_2)^{\frac{(\rho_2+1)(\rho_1+2)}{2\rho_1+2}}.$$

Combine (33)–(39), to arrive, after some simple calculations, at

$$\int_\Omega \partial_t u_k(t) f_1(u_k(t),v_k(t))dx \leqslant \frac{6Ł_1}{\rho_1+2}\|\partial_t u_k(t)\|^{\rho_1+2}_{L^{\rho_1+2}(\Omega)} + \mathscr{M}^1_{11}Ł_1\Big(\int_\Omega \nabla^\top u_k(t)A_1\nabla u_k(t)dx\Big)^{\frac{(\rho_1+1)(\rho_1+2)}{2\rho_1+2}}$$
$$+ \mathscr{M}^1_{12}Ł_1\Big(\int_\Omega \nabla^\top v_k(t)A_2\nabla v_k(t)dx\Big)^{\frac{(\rho_1+1)(\rho_1+2)}{2\rho_1+2}}$$
$$+ \mathscr{M}^1_{13}Ł_1\Big(\int_\Omega \nabla^\top u_k(t)A_1\nabla u_k(t)dx\Big)^{\frac{(\rho_2+1)(\rho_1+2)}{2\rho_1+2}}$$
$$+ \mathscr{M}^1_{14}Ł_1\Big(\int_\Omega \nabla^\top v_k(t)A_2\nabla v_k(t)dx\Big)^{\frac{(\rho_2+1)(\rho_1+2)}{2\rho_1+2}}$$
$$+ \mathscr{M}^1_{15}Ł_1\Big(\int_\Omega \nabla^\top u_k(t)A_1\nabla u_k(t)dx\Big)^{\frac{\rho_1+2}{2}}$$
$$+ \mathscr{M}^1_{16}Ł_1\Big(\int_\Omega \nabla^\top v_k(t)A_2\nabla v_k(t)dx\Big)^{\frac{(\rho_2+1)(\rho_1+2)}{2\rho_1+2}}, \quad t\in[0,T^*). \tag{40}$$

Mimicking the steps as in deducing (40) from (33)–(39), we could prove similarly that

$$\int_\Omega \partial_t v_k(t) f_2(u_k(t),v_k(t))dx \leqslant \frac{6Ł_2}{\rho_2+2}\|\partial_t u_k(t)\|^{\rho_2+2}_{L^{\rho_2+2}(\Omega)} + \mathscr{M}^1_{21}Ł_2\Big(\int_\Omega \nabla^\top u_k(t)A_1\nabla u_k(t)dx\Big)^{\frac{(\rho_1+1)(\rho_2+2)}{2\rho_2+2}}$$
$$+ \mathscr{M}^1_{22}Ł_2\Big(\int_\Omega \nabla^\top v_k(t)A_2\nabla v_k(t)dx\Big)^{\frac{(\rho_1+1)(\rho_2+2)}{2\rho_2+2}}$$
$$+ \mathscr{M}^1_{23}Ł_2\Big(\int_\Omega \nabla^\top u_k(t)A_1\nabla u_k(t)dx\Big)^{\frac{(\rho_2+1)(\rho_2+2)}{2\rho_2+2}}$$
$$+ \mathscr{M}^1_{24}Ł_2\Big(\int_\Omega \nabla^\top v_k(t)A_2\nabla v_k(t)dx\Big)^{\frac{(\rho_2+1)(\rho_2+2)}{2\rho_2+2}}$$
$$+ \mathscr{M}^1_{25}Ł_2\Big(\int_\Omega \nabla^\top u_k(t)A_1\nabla u_k(t)dx\Big)^{\frac{(\rho_1+1)(\rho_2+2)}{2\rho_2+2}}$$
$$+ \mathscr{M}^1_{26}Ł_2\Big(\int_\Omega \nabla^\top v_k(t)A_2\nabla v_k(t)dx\Big)^{\frac{\rho_2+2}{2}}, \quad t\in[0,T^*). \tag{41}$$

in which the positive constants $\mathscr{M}^1_{21}$, $\mathscr{M}^1_{22}$, $\mathscr{M}^1_{23}$, $\mathscr{M}^1_{24}$, $\mathscr{M}^1_{25}$ and $\mathscr{M}^1_{26}$ are given by

$$\mathscr{M}^1_{21} = \frac{\rho_2+1}{\rho_2+2}\big(\kappa_{\frac{(p_1+1)(\rho_2+2)}{\rho_2+1}}\big)^{\frac{(p_1+1)(\rho_2+2)}{\rho_2+1}}(\zeta_1)^{\frac{(p_1+1)(\rho_2+2)}{2\rho_2+2}},$$

$$\mathscr{M}^1_{22} = \frac{\rho_2+1}{\rho_2+2}\big(\kappa_{\frac{(p_1+1)(\rho_2+2)}{\rho_2+1}}\big)^{\frac{(p_1+1)(\rho_2+2)}{\rho_2+1}}(\zeta_2)^{\frac{(p_1+1)(\rho_2+2)}{2\rho_2+2}},$$

$$\mathscr{M}^1_{23} = \frac{\rho_2+1}{\rho_2+2}\big(\kappa_{\frac{(p_2+1)(\rho_2+2)}{\rho_2+1}}\big)^{\frac{(p_2+1)(\rho_2+2)}{\rho_2+1}}(\zeta_1)^{\frac{(p_2+1)(\rho_2+2)}{2\rho_2+2}},$$

$$\mathscr{M}^1_{24} = \frac{\rho_2+1}{\rho_2+2}\big(\kappa_{\frac{(p_2+1)(\rho_2+2)}{\rho_2+1}}\big)^{\frac{(p_2+1)(\rho_2+2)}{\rho_2+1}}(\zeta_2)^{\frac{(p_2+1)(\rho_2+2)}{2\rho_2+2}},$$

$$\mathscr{M}_{25}^1=\frac{\rho_2+1}{\rho_2+2}(\kappa_{\frac{(\rho_1+1)(\rho_2+2)}{\rho_2+1}})^{\frac{(\rho_1+1)(\rho_2+2)}{\rho_2+1}}(\zeta_1)^{\frac{(\rho_1+1)(\rho_2+2)}{2\rho_2+2}}$$

and

$$\mathscr{M}_{26}^1=\frac{\rho_2+1}{\rho_2+2}(\kappa_{\rho_2+2})^{\rho_2+2}(\zeta_2)^{\frac{\rho_2+2}{2}},$$

respectively. Plug (26), (27), (28), (29), (40) and (41) into (25) and simplify the obtained result further, to arrive at finally

$$\begin{aligned}\Xi_k'(t)\leqslant{}&\mathscr{M}_1^2+\frac{4+6Ł_1}{\rho_1+2}\|\partial_t u_k(t)\|_{L^{\rho_1+2}(\Omega)}^{\rho_1+2}+\frac{4+6Ł_2}{\rho_2+2}\|\partial_t v_k(t)\|_{L^{\rho_2+2}(\Omega)}^{\rho_2+2}\\
&+\mathscr{M}_{11}^1Ł_1(\int_\Omega\nabla^\top u_k(t)A_1\nabla u_k(t)dx)^{\frac{(\rho_1+1)(\rho_1+2)}{2\rho_1+2}}\\
&+\mathscr{M}_{12}^1Ł_1(\int_\Omega\nabla^\top v_k(t)A_2\nabla v_k(t)dx)^{\frac{(\rho_1+1)(\rho_1+2)}{2\rho_1+2}}\\
&+\mathscr{M}_{13}^1Ł_1(\int_\Omega\nabla^\top u_k(t)A_1\nabla u_k(t)dx)^{\frac{(\rho_2+1)(\rho_1+2)}{2\rho_1+2}}\\
&+\mathscr{M}_{14}^1Ł_1(\int_\Omega\nabla^\top v_k(t)A_2\nabla v_k(t)dx)^{\frac{(\rho_2+1)(\rho_1+2)}{2\rho_1+2}}\\
&+\mathscr{M}_{15}^1Ł_1(\int_\Omega\nabla^\top u_k(t)A_1\nabla u_k(t)dx)^{\frac{\rho_1+2}{2}}\\
&+\mathscr{M}_{16}^1Ł_1(\int_\Omega\nabla^\top v_k(t)A_2\nabla v_k(t)dx)^{\frac{(\rho_2+1)(\rho_1+2)}{2\rho_1+2}}\\
&+\mathscr{M}_{21}^1Ł_2(\int_\Omega\nabla^\top u_k(t)A_1\nabla u_k(t)dx)^{\frac{(\rho_1+1)(\rho_2+2)}{2\rho_2+2}}\\
&+\mathscr{M}_{22}^1Ł_2(\int_\Omega\nabla^\top v_k(t)A_2\nabla v_k(t)dx)^{\frac{(\rho_1+1)(\rho_2+2)}{2\rho_2+2}}\\
&+\mathscr{M}_{23}^1Ł_2(\int_\Omega\nabla^\top u_k(t)A_1\nabla u_k(t)dx)^{\frac{(\rho_2+1)(\rho_2+2)}{2\rho_2+2}}\\
&+\mathscr{M}_{24}^1Ł_2(\int_\Omega\nabla^\top v_k(t)A_2\nabla v_k(t)dx)^{\frac{(\rho_2+1)(\rho_2+2)}{2\rho_2+2}}\\
&+\mathscr{M}_{25}^1Ł_2(\int_\Omega\nabla^\top u_k(t)A_1\nabla u_k(t)dx)^{\frac{(\rho_1+1)(\rho_2+2)}{2\rho_2+2}}\\
&+\mathscr{M}_{26}^1Ł_2(\int_\Omega\nabla^\top v_k(t)A_2\nabla v_k(t)dx)^{\frac{\rho_2+2}{2}}\\
&+\varkappa_{a_{11},a_{12},a_{21},a_{22}}(\|\partial_t u_k(t)\|_{L^2(\Omega)}^2+\|\partial_t v_k(t)\|_{L^2(\Omega)}^2),\quad t\in[0,T^*),\end{aligned}\tag{42}$$

in which the positive constant $\mathscr{M}_1^2$ is given by

$$\begin{aligned}\mathscr{M}_1^2={}&\frac{\rho_1}{\rho_1+2}\|a_{11}\|_{L^{\frac{\rho_1+2}{\rho_1}}(\Omega)}^{\frac{\rho_1+2}{\rho_1}}+\frac{\rho_1\rho_2+\rho_1+\rho_2}{(\rho_1+2)(\rho_2+2)}\|a_{12}\|_{L^{\frac{(\rho_1+2)(\rho_2+2)}{\rho_1\rho_2+\rho_1+\rho_2}}(\Omega)}^{\frac{(\rho_1+2)(\rho_2+2)}{\rho_1\rho_2+\rho_1+\rho_2}}\\
&+\frac{\rho_2}{\rho_2+2}\|a_{22}\|_{L^{\frac{\rho_2+2}{\rho_2}}(\Omega)}^{\frac{\rho_2+2}{\rho_2}}+\frac{\rho_1\rho_2+\rho_1+\rho_2}{(\rho_1+2)(\rho_2+2)}\|a_{21}\|_{L^{\frac{(\rho_1+2)(\rho_2+2)}{\rho_1\rho_2+\rho_1+\rho_2}}(\Omega)}^{\frac{(\rho_1+2)(\rho_2+2)}{\rho_1\rho_2+\rho_1+\rho_2}}.\end{aligned}$$

With the aid of the definition $\Xi_k(t)$ (see (24) for the details), by some routine but careful calculations, we arrive at

$$
\begin{aligned}
&\mathscr{M}_{11}^{1}Ł_1(\int_\Omega \nabla^\top u_k(t)A_1\nabla u_k(t)dx)^{\frac{(p_1+1)(\rho_1+2)}{2\rho_1+2}} \\
&+\mathscr{M}_{12}^{1}Ł_1(\int_\Omega \nabla^\top v_k(t)A_2\nabla v_k(t)dx)^{\frac{(p_1+1)(\rho_1+2)}{2\rho_1+2}} \\
\leqslant\, &\mathscr{M}_{11}^{3}(\frac{1}{2}(\mu_1-\int_0^{+\infty}g_1(s)ds)\int_\Omega \nabla^\top u_k(t)A_1\nabla u_k(t)dx)^{\frac{(p_1+1)(\rho_1+2)}{2\rho_1+2}} \\
&+\mathscr{M}_{11}^{3}(\frac{1}{2}(\mu_2-\int_0^{+\infty}g_2(s)ds)\int_\Omega \nabla^\top v_k(t)A_2\nabla v_k(t)dx)^{\frac{(p_1+1)(\rho_1+2)}{2\rho_1+2}} \\
\leqslant\, &\mathscr{M}_{11}^{3}(\frac{1}{2}(\mu_1-\int_0^{+\infty}g_1(s)ds)\int_\Omega \nabla^\top u_k(t)A_1\nabla u_k(t)dx \\
&\qquad+\frac{1}{2}(\mu_2-\int_0^{+\infty}g_2(s)ds)\int_\Omega \nabla^\top v_k(t)A_2\nabla v_k(t)dx)^{\frac{(p_1+1)(\rho_1+2)}{2\rho_1+2}} \\
\leqslant\, &\mathscr{M}_{11}^{3}\big(\Xi_k(t)\big)^{\frac{(p_1+1)(\rho_1+2)}{2\rho_1+2}},
\end{aligned} \tag{43}
$$

in which the positive constant $\mathscr{M}_{11}^{3}$ is given by

$$
\mathscr{M}_{11}^{3}=Ł_1\mho_{\frac{(p_1+1)(\rho_1+2)}{2\rho_1+2}}\max(\mathscr{M}_{11}^{1}(\frac{1}{2}(\mu_1-\int_0^{+\infty}g_1(s)ds))^{-\frac{(p_1+1)(\rho_1+2)}{2\rho_1+2}},\mathscr{M}_{12}^{1}(\frac{1}{2}(\mu_2-\int_0^{+\infty}g_2(s)ds))^{-\frac{(p_1+1)(\rho_1+2)}{2\rho_1+2}});
$$

see (11) for the detailed explanation on the notation $\mho_{\frac{(p_1+1)(\rho_1+2)}{2\rho_1+2}}$ and see Remark 3 for its applications. Take similar steps as in deriving (43), to get

$$
\begin{aligned}
&\mathscr{M}_{13}^{1}Ł_1(\int_\Omega \nabla^\top u_k(t)A_1\nabla u_k(t)dx)^{\frac{(p_2+1)(\rho_1+2)}{2\rho_1+2}} \\
&+\mathscr{M}_{14}^{1}Ł_1(\int_\Omega \nabla^\top v_k(t)A_2\nabla v_k(t)dx)^{\frac{(p_2+1)(\rho_1+2)}{2\rho_1+2}} \\
\leqslant\, &\mathscr{M}_{12}^{3}\big(\Xi_k(t)\big)^{\frac{(p_2+1)(\rho_1+2)}{2\rho_1+2}},
\end{aligned} \tag{44}
$$

in which the positive constant $\mathscr{M}_{12}^{3}$ is given by

$$
\begin{aligned}
\mathscr{M}_{12}^{3}=Ł_1\mho_{\frac{(p_2+1)(\rho_1+2)}{2\rho_1+2}}\max(&\mathscr{M}_{13}^{1}(\frac{1}{2}(\mu_1-\int_0^{+\infty}g_1(s)ds))^{-\frac{(p_2+1)(\rho_1+2)}{2\rho_1+2}}, \\
&\mathscr{M}_{14}^{1}(\frac{1}{2}(\mu_2-\int_0^{+\infty}g_2(s)ds))^{-\frac{(p_2+1)(\rho_1+2)}{2\rho_1+2}});
\end{aligned}
$$

see (11) for the detailed explanation on the notation $\mho_{\frac{(p_2+1)(\rho_1+2)}{2\rho_1+2}}$ and see Remark 3 for its applications. Take similar steps as in deriving (43) and (44), to get

$$
\begin{aligned}
&\mathscr{M}_{21}^{1}Ł_2(\int_\Omega \nabla^\top u_k(t)A_1\nabla u_k(t)dx)^{\frac{(p_1+1)(\rho_2+2)}{2\rho_2+2}} \\
&+\mathscr{M}_{22}^{1}Ł_2(\int_\Omega \nabla^\top v_k(t)A_2\nabla v_k(t)dx)^{\frac{(p_1+1)(\rho_2+2)}{2\rho_2+2}} \\
\leqslant\, &\mathscr{M}_{21}^{3}\big(\Xi_k(t)\big)^{\frac{(p_1+1)(\rho_2+2)}{2\rho_2+2}},
\end{aligned} \tag{45}
$$

in which the positive constant $\mathscr{M}_{21}^{3}$ is given by

$$
\mathscr{M}_{21}^{3}=Ł_2\mho_{\frac{(p_1+1)(\rho_2+2)}{2\rho_2+2}}\max(\mathscr{M}_{21}^{1}(\frac{1}{2}(\mu_1-\int_0^{+\infty}g_1(s)ds))^{-\frac{(p_1+1)(\rho_2+2)}{2\rho_2+2}},\mathscr{M}_{22}^{1}(\frac{1}{2}(\mu_2-\int_0^{+\infty}g_2(s)ds))^{-\frac{(p_1+1)(\rho_2+2)}{2\rho_2+2}});
$$

see (11) for the detailed explanation on the notation $\mho_{\frac{(p_1+1)(\rho_2+2)}{2\rho_2+2}}$ and see Remark 3 for its applications. Take similar steps as in deriving (43), (44) and (45), to get

$$\mathscr{M}_{23}^{1}Ł_2\Big(\int_{\Omega}\nabla^{\top}u_k(t)A_1\nabla u_k(t)dx\Big)^{\frac{(p_2+1)(\rho_2+2)}{2\rho_2+2}}$$
$$+\mathscr{M}_{24}^{1}Ł_2\Big(\int_{\Omega}\nabla^{\top}v_k(t)A_2\nabla v_k(t)dx\Big)^{\frac{(p_2+1)(\rho_2+2)}{2\rho_2+2}}$$
$$\leqslant\mathscr{M}_{22}^{3}\big(\Xi_k(t)\big)^{\frac{(p_2+1)(\rho_2+2)}{2\rho_2+2}}, \tag{46}$$

in which the positive constant $\mathscr{M}_{22}^{3}$ is given by

$$\mathscr{M}_{22}^{3}=Ł_2\mho_{\frac{(p_2+1)(\rho_2+2)}{2\rho_2+2}}\max\Big(\mathscr{M}_{23}^{1}\big(\frac{1}{2}(\mu_1-\int_0^{+\infty}g_1(s)ds)\big)^{-\frac{(p_2+1)(\rho_2+2)}{2\rho_2+2}},$$
$$\mathscr{M}_{24}^{1}\big(\frac{1}{2}(\mu_2-\int_0^{+\infty}g_2(s)ds)\big)^{-\frac{(p_2+1)(\rho_2+2)}{2\rho_2+2}}\Big);$$

see (11) for the detailed explanation on the notation $\mho_{\frac{(p_2+1)(\rho_2+2)}{2\rho_2+2}}$ and see Remark 3 for its applications. By the Fenchel–Young inequality, we have

$$\|\partial_t u_k(t)\|_{L^2(\Omega)}^2+\|\partial_t v_k(t)\|_{L^2(\Omega)}^2\leqslant\Big(\frac{\rho_1}{\rho_1+2}+\frac{\rho_2}{\rho_2+2}\Big)\operatorname{meas}\Omega$$
$$+\frac{2}{\rho_1+2}\|\partial_t u_k(t)\|_{L^{\rho_1+2}(\Omega)}^{\rho_1+2}$$
$$+\frac{2}{\rho_2+2}\|\partial_t v_k(t)\|_{L^{\rho_2+2}(\Omega)}^{\rho_2+2}. \tag{47}$$

Plug (43), (44), (45), (46) and (47) into (42) and perform some simple computations, to arrive at finally the semi-linear differential inequality

$$\Xi_k'(t)\leqslant\mathscr{M}_1^4+\mathscr{M}_2^4\Xi_k(t)$$
$$+\mathscr{M}_3^4(\Xi_k(t))^{\frac{\rho_1+2}{2}}+\mathscr{M}_4^4(\Xi_k(t))^{\frac{\rho_2+2}{2}}$$
$$+\mathscr{M}_5^4(\Xi_k(t))^{\frac{(\rho_2+1)(\rho_1+2)}{2\rho_1+2}}+\mathscr{M}_6^4(\Xi_k(t))^{\frac{(\rho_1+1)(\rho_2+2)}{2\rho_2+2}}$$
$$+\mathscr{M}_{11}^{3}\big(\Xi_k(t)\big)^{\frac{(p_1+1)(\rho_1+2)}{2\rho_1+2}}+\mathscr{M}_{12}^{3}\big(\Xi_k(t)\big)^{\frac{(p_2+1)(\rho_1+2)}{2\rho_1+2}}$$
$$+\mathscr{M}_{21}^{3}\big(\Xi_k(t)\big)^{\frac{(p_1+1)(\rho_2+2)}{2\rho_2+2}}+\mathscr{M}_{22}^{3}\big(\Xi_k(t)\big)^{\frac{(p_2+1)(\rho_2+2)}{2\rho_2+2}},\quad t\in[0,T^*), \tag{48}$$

in which $\mathscr{M}_1^4$, $\mathscr{M}_2^4$, $\mathscr{M}_3^4$, $\mathscr{M}_4^4$, $\mathscr{M}_5^4$ and $\mathscr{M}_6^4$ are given by

$$\mathscr{M}_1^4=\mathscr{M}_1^2+\varkappa_{a_{11},a_{12},a_{21},a_{22}}\Big(\frac{\rho_1}{\rho_1+2}+\frac{\rho_2}{\rho_2+2}\Big)\operatorname{meas}\Omega,$$

$$\mathscr{M}_2^4=4+6\max(Ł_1,Ł_2)+2\varkappa_{a_{11},a_{12},a_{21},a_{22}},$$
$$\mathscr{M}_3^4=\mathscr{M}_{15}^{1}Ł_1\Big(\frac{1}{2}(\mu_1-\int_0^{+\infty}g_1(s)ds)\Big)^{-\frac{\rho_1+2}{2}},$$
$$\mathscr{M}_4^4=\mathscr{M}_{26}^{1}Ł_2\Big(\frac{1}{2}(\mu_2-\int_0^{+\infty}g_2(s)ds)\Big)^{-\frac{\rho_2+2}{2}},$$
$$\mathscr{M}_5^4=\mathscr{M}_{16}^{1}Ł_1\Big(\frac{1}{2}(\mu_2-\int_0^{+\infty}g_2(s)ds)\Big)^{-\frac{(\rho_2+1)(\rho_1+2)}{2\rho_1+2}}$$

and

$$\mathscr{M}_6^4=\mathscr{M}_{25}^{1}Ł_2\Big(\frac{1}{2}(\mu_1-\int_0^{+\infty}g_1(s)ds)\Big)^{-\frac{(\rho_2+1)(\rho_1+2)}{2\rho_2+2}},$$

respectively. By careful calculations, we have

$$
\begin{aligned}
\Xi_k(0) = {} & \frac{1}{\rho_1+2}\|u_k^0\|_{L^{\rho_1+2}(\Omega)}^{\rho_1+2} + \frac{1}{\rho_2+2}\|v_k^0\|_{L^{\rho_2+2}(\Omega)}^{\rho_2+2} \\
& + \frac{1}{2}(\mu_1 - \int_0^{+\infty} g_1(s)ds)\int_\Omega \nabla^\top u_k^0 A_1 \nabla u_k^0 dx \\
& + \frac{1}{2}\int_\Omega \nabla^\top u_k^1 A_1 \nabla u_k^1 dx + \frac{1}{2}(g_1 \diamond^{A_1} \nabla u_k^0)(0) \\
& + \frac{1}{2}(\mu_2 - \int_0^{+\infty} g_2(s)ds)\int_\Omega \nabla^\top v_k^0 A_2 \nabla v_k^0 dx \\
& + \frac{1}{2}\int_\Omega \nabla^\top v_k^1 A_2 \nabla v_k^1 dx + \frac{1}{2}(g_2 \diamond^{A_2} \nabla v_k^0)(0) \\
\leqslant {} & \frac{1}{\rho_1+2}(\kappa_{\rho_1+2})^{\rho_1+2}(\zeta_1)^{\frac{\rho_1+2}{2}}(\int_\Omega \nabla^\top u_k^0 A_1 \nabla u_k^0 dx)^{\frac{\rho_1+2}{2}} \\
& + \frac{1}{\rho_2+2}(\kappa_{\rho_2+2})^{\rho_2+2}(\zeta_2)^{\frac{\rho_2+2}{2}}(\int_\Omega \nabla^\top v_k^0 A_2 \nabla v_k^0 dx)^{\frac{\rho_2+2}{2}} \\
& + \frac{1}{2}(\mu_1 - \int_0^{+\infty} g_1(s)ds)\int_\Omega \nabla^\top u_k^0 A_1 \nabla u_k^0 dx \\
& + \frac{1}{2}\int_\Omega \nabla^\top u_k^1 A_1 \nabla u_k^1 dx + \frac{1}{2}(g_1 \diamond^{A_1} \nabla u_k^0)(0) \\
& + \frac{1}{2}(\mu_2 - \int_0^{+\infty} g_2(s)ds)\int_\Omega \nabla^\top v_k^0 A_2 \nabla v_k^0 dx \\
& + \frac{1}{2}\int_\Omega \nabla^\top v_k^1 A_2 \nabla v_k^1 dx + \frac{1}{2}(g_2 \diamond^{A_2} \nabla v_k^0)(0) \\
\leqslant {} & \frac{1}{\rho_1+2}(\kappa_{\rho_1+2})^{\rho_1+2}(\zeta_1)^{\frac{\rho_1+2}{2}}(\int_\Omega \nabla^\top u^0 A_1 \nabla u^0 dx)^{\frac{\rho_1+2}{2}} \\
& + \frac{1}{\rho_2+2}(\kappa_{\rho_2+2})^{\rho_2+2}(\zeta_2)^{\frac{\rho_2+2}{2}}(\int_\Omega \nabla^\top v^0 A_2 \nabla v^0 dx)^{\frac{\rho_2+2}{2}} \\
& + \frac{1}{2}(\mu_1 - \int_0^{+\infty} g_1(s)ds)\int_\Omega \nabla^\top u^0 A_1 \nabla u^0 dx \\
& + \frac{1}{2}\int_\Omega \nabla^\top u^1 A_1 \nabla u^1 dx + \frac{1}{2}(g_1 \diamond^{A_1} \nabla u^0)(0) \\
& + \frac{1}{2}(\mu_2 - \int_0^{+\infty} g_2(s)ds)\int_\Omega \nabla^\top v^0 A_2 \nabla v^0 dx \\
& + \frac{1}{2}\int_\Omega \nabla^\top v^1 A_2 \nabla v^1 dx + \frac{1}{2}(g_2 \diamond^{A_2} \nabla v^0)(0) \leqslant \mathscr{G}_0,
\end{aligned}
$$

where the nonnegative constant $\mathscr{G}_0$ is given by

$$
\begin{aligned}
\mathscr{G}_0 = {} & \frac{1}{\rho_1+2}(\kappa_{\rho_1+2})^{\rho_1+2}(\zeta_1\|\sqrt{\mathrm{tr}((A_1)^2)}\|_{L^\infty(\Omega)})^{\frac{\rho_1+2}{2}}\|\nabla u^0\|_{L^2(\Omega;\mathbb{R}^N)}^{\rho_1+2} \\
& + \frac{1}{\rho_2+2}(\kappa_{\rho_2+2})^{\rho_2+2}(\zeta_2\|\sqrt{\mathrm{tr}((A_2)^2)}\|_{L^\infty(\Omega)})^{\frac{\rho_2+2}{2}}\|\nabla v^0\|_{L^2(\Omega;\mathbb{R}^N)}^{\rho_2+2} \\
& + \frac{1}{2}(\mu_1 - \int_0^{+\infty} g_1(s)ds)\|\sqrt{\mathrm{tr}((A_1)^2)}\|_{L^\infty(\Omega)}\|\nabla u^0\|_{L^2(\Omega;\mathbb{R}^N)}^2 \\
& + \frac{1}{2}\|\sqrt{\mathrm{tr}((A_1)^2)}\|_{L^\infty(\Omega)}\big(\|\nabla u^1\|_{L^2(\Omega;\mathbb{R}^N)}^2 + \int_0^{+\infty} g_1(s)ds\|\nabla u^0\|_{L^\infty(\mathbb{R}_-;L^2(\Omega;\mathbb{R}^N))}^2\big) \\
& + \frac{1}{2}(\mu_2 - \int_0^{+\infty} g_2(s)ds)\|\sqrt{\mathrm{tr}((A_2)^2)}\|_{L^\infty(\Omega)}\|\nabla v^0\|_{L^2(\Omega;\mathbb{R}^N)}^2 \\
& + \frac{1}{2}\|\sqrt{\mathrm{tr}((A_2)^2)}\|_{L^\infty(\Omega)}\big(\|\nabla v^1\|_{L^2(\Omega;\mathbb{R}^N)}^2 + \int_0^{+\infty} g_2(s)ds\|\nabla v^0\|_{L^\infty(\mathbb{R}_-;L^2(\Omega;\mathbb{R}^N))}^2\big).
\end{aligned}
$$

Let us introduce an auxiliary Cauchy problem

$$\begin{cases} \mathscr{G}' = \mathscr{M}_1^4 + \mathscr{M}_2^4\mathscr{G} + \mathscr{M}_3^4\mathscr{G}^{\frac{\rho_1+2}{2}} + \mathscr{M}_4^4\mathscr{G}^{\frac{\rho_2+2}{2}} \\ \quad + \mathscr{M}_5^4\mathscr{G}^{\frac{(\rho_2+1)(\rho_1+2)}{2\rho_1+2}} + \mathscr{M}_6^4\mathscr{G}^{\frac{(\rho_1+1)(\rho_2+2)}{2\rho_2+2}} \\ \quad + \mathscr{M}_{11}^3\mathscr{G}^{\frac{(p_1+1)(\rho_1+2)}{2\rho_1+2}} + \mathscr{M}_{12}^3\mathscr{G}^{\frac{(p_2+1)(\rho_1+2)}{2\rho_1+2}} \\ \quad + \mathscr{M}_{21}^3\mathscr{G}^{\frac{(p_1+1)(\rho_2+2)}{2\rho_2+2}} + \mathscr{M}_{22}^3\mathscr{G}^{\frac{(p_2+1)(\rho_2+2)}{2\rho_2+2}} \quad \text{for } t \in [0, T^*), \\ \mathscr{G}(0) = \mathscr{G}_0. \end{cases} \tag{49}$$

Since the right hand side of the differential equation in the problem (49) is smooth (and hence, is locally Lipschitz continuous), by the classical Cauchy–Lipschitz existence theory of ordinary differential equations, the Cauchy problem (49) admits a unique solution, denoted by $\mathscr{G}(t)$, in an interval $[0, T^{**})$, where T^{**} is a certain constant fulfilling $T^{**} \in (0, T^*]$. It is not difficult to conclude: Both $\mathscr{G}(t)$ and T^{**} are uniquely determined by Ω, A_1, A_2, f_1, f_2, g_1, g_2, ρ_1, ρ_2, a_{11}, a_{12}, a_{21}, a_{22} , u^0, v^0, u^1 and v^1; $\mathscr{G}(t)$ is strictly increasing in the interval $[0, T^{**})$; and T^{**} can be chosen arbitrarily in the interval $(0, \check{T}]$ with $\check{T}$ given by

$$\check{T} = \min(T^*, \frac{1}{\text{II}(\mathscr{G}_0)^{\alpha-1}(\alpha-1)})$$

in which $\text{II} = \max(\varsigma, \ni)$ with ς given by

$$\begin{aligned} \varsigma = {} & \mathscr{M}_1^4 + \mathscr{M}_2^4(1-\frac{1}{\alpha}) + \mathscr{M}_3^4(1-\frac{\rho_1+2}{2\alpha}) + \mathscr{M}_4^4(1-\frac{\rho_2+2}{2\alpha}) \\ & + \mathscr{M}_5^4(1-\frac{(\rho_2+1)(\rho_1+2)}{2\alpha(\rho_1+1)}) + \mathscr{M}_6^4(1-\frac{(\rho_1+1)(\rho_2+2)}{2\alpha(\rho_2+1)}) \\ & + \mathscr{M}_{11}^3(1-\frac{(p_1+1)(\rho_1+2)}{2\alpha(\rho_1+1)}) + \mathscr{M}_{12}^3(1-\frac{(p_2+1)(\rho_1+2)}{2\alpha(\rho_1+1)}) \\ & + \mathscr{M}_{21}^3(1-\frac{(p_1+1)(\rho_2+2)}{2\alpha(\rho_2+1)}) + \mathscr{M}_{22}^3(1-\frac{(p_2+1)(\rho_2+2)}{2\alpha(\rho_2+1)}), \end{aligned}$$

$\ni$ given by

$$\begin{aligned} \ni = {} & \frac{\mathscr{M}_2^4}{\alpha} + \frac{\mathscr{M}_3^4(\rho_1+2)}{2\alpha} + \frac{\mathscr{M}_4^4(\rho_2+2)}{2\alpha} \\ & + \frac{\mathscr{M}_5^4(\rho_2+1)(\rho_1+2)}{2\alpha(\rho_1+1)} + \frac{\mathscr{M}_6^4(\rho_1+1)(\rho_2+2)}{2\alpha(\rho_2+1)} \\ & + \frac{\mathscr{M}_{11}^3(p_1+1)(\rho_1+2)}{2\alpha(\rho_1+1)} + \frac{\mathscr{M}_{12}^3(p_2+1)(\rho_1+2)}{2\alpha(\rho_1+1)} \\ & + \frac{\mathscr{M}_{21}^3(p_1+1)(\rho_2+2)}{2\alpha(\rho_2+1)} + \frac{\mathscr{M}_{22}^3(p_2+1)(\rho_2+2)}{2\alpha(\rho_2+1)}, \end{aligned}$$

and α given by

$$\begin{aligned} \alpha = \max\Big(& \frac{\rho_1+2}{2}, \frac{\rho_2+2}{2}, \\ & \frac{(\rho_2+1)(\rho_1+2)}{2\rho_1+2}, \frac{(\rho_1+1)(\rho_2+2)}{2\rho_2+2}, \\ & \frac{(p_1+1)(\rho_1+2)}{2\rho_1+2}, \frac{(p_2+1)(\rho_1+2)}{2\rho_1+2}, \\ & \frac{(p_1+1)(\rho_2+2)}{2\rho_2+2}, \frac{(p_2+1)(\rho_2+2)}{2\rho_2+2}\Big). \end{aligned}$$

Thanks to $0 \leqslant \Xi_k(0) \leqslant \mathscr{G}_0$, by standard comparison theory of ordinary differential equations, we deduce from (48) and (49) that

$$\Xi_k(t) \leqslant \mathscr{G}(t) \quad \text{for } t \in [0, \check{T}). \tag{50}$$

Now let us put $\tilde{T} = \frac{1}{2}\min(2, \check{T})$. In light of (50), we have by recalling that the function $\mathscr{G}(t)$ is strictly increasing in the interval $[0, \check{T}) \supset [0, \tilde{T}]$

$$\Xi_k(t) \leqslant \mathscr{G}(t) \leqslant \mathscr{G}(\tilde{T}) \quad \text{for } t \in [0, \tilde{T}],$$

and have further by recalling (24) and performing some more elementary calculations

$$\max(\|u_k(t)\|^2_{H^1_0(\Omega)}, \|\partial_t u_k(t)\|^2_{H^1_0(\Omega)}, \|v_k(t)\|^2_{H^1_0(\Omega)}, \|\partial_t v_k(t)\|^2_{H^1_0(\Omega)}) \leqslant \frac{2(1+(\kappa_2)^2)}{\mathscr{M}_1^5}\Xi_k(t) \leqslant \mathscr{G}(\tilde{T}) \quad \text{for } t \in [0, \tilde{T}], \tag{51}$$

in which: the positive constant $\mathscr{M}_1^5$ is given by

$$\mathscr{M}_1^5 = \min(\frac{1}{\zeta_1}(\mu_1 - \int_0^{+\infty} g_1(s)ds), \frac{1}{\zeta_1}, \frac{1}{\zeta_2}(\mu_2 - \int_0^{+\infty} g_2(s)ds), \frac{1}{\zeta_2}), \tag{52}$$

and $\mathscr{G}(\tilde{T})$, a constant in the sense that it is independent of k and t, depends on Ω, A_1, A_2, f_1, f_2, g_1, g_2, ρ_1, ρ_2, a_{11}, a_{12}, a_{21}, a_{22} , u^0, v^0, u^1 and v^1.

Since $\rho_i > 0$ and $p_i > 0$ (see Assumptions 3 and 4), we have

$$\left.\begin{aligned} \rho_i + 2 < 2(\rho_i + 1), \\ p_i + 2 < 2(p_i + 1), \end{aligned}\right\} \quad i = 1, 2.$$

This, together with Assumptions 3 and 4, and the Rellich–Kondrachov theorem (see [1], Theorem 1, p. 286), implies that the Sobolev embeddings

$$H^1_0(\Omega) \hookrightarrow L^{\rho_i+2}(\Omega), \text{ and}$$
$$H^1_0(\Omega) \hookrightarrow L^{p_i+2}(\Omega)$$

are both compact, $i = 1, 2$. By applying the Banach–Alaoglu theorem (see [30], Theorem 3.16, p. 66), we can prove, via utilizing (51) and by applying the aforementioned compact embeddings $H^1_0(\Omega) \hookrightarrow\hookrightarrow L^{\rho_i+2}(\Omega)$ and $H^1_0(\Omega) \hookrightarrow\hookrightarrow L^{p_i+2}(\Omega)$, $i = 1, 2$, that there exists a pair $(u, v)^\top$ (whose restriction to $\Omega \times [0, \tilde{T}]$ could be proved to be weakly continuous with respect to time t in the Hilbert space $H^1_0(\Omega; \mathbb{R}^2)$), and a subsequence $\{i_k\}$ of $\{k\}$ (that is, a strictly increasing sequence in $\mathbb{N}$) such that

$$u_{i_k} \rightharpoonup u \text{ weakly}^* \text{ in } L^\infty(0, \tilde{T}; H^1_0(\Omega)) \text{ as } k \to \infty,$$
$$v_{i_k} \rightharpoonup v \text{ weakly}^* \text{ in } L^\infty(0, \tilde{T}; H^1_0(\Omega)) \text{ as } k \to \infty,$$
$$\partial_t u_{i_k} \rightharpoonup \partial_t u \text{ weakly}^* \text{ in } L^\infty(0, \tilde{T}; H^1_0(\Omega)) \text{ as } k \to \infty,$$
$$\partial_t v_{i_k} \rightharpoonup \partial_t v \text{ weakly}^* \text{ in } L^\infty(0, \tilde{T}; H^1_0(\Omega)) \text{ as } k \to \infty,$$

and such that

$$|\partial_t u_{i_k}(t)|^{\rho_1}\partial_t u_{i_k}(t) \to |\partial_t u(t)|^{\rho_1}\partial_t u(t) \text{ in } L^{\frac{\rho_1+2}{\rho_1+1}}(\Omega) \text{ as } k \to \infty,$$
$$|\partial_t v_{i_k}(t)|^{\rho_2}\partial_t v_{i_k}(t) \to |\partial_t v(t)|^{\rho_2}\partial_t v(t) \text{ as } k \to \infty,$$
$$f_1(u_{i_k}(t), v_{i_k}(t)) \to f_1(u(t), v(t)) \text{ in } L^{\min(\frac{\rho_1+2}{\rho_1+1}, \frac{\rho_2+2}{\rho_2+1}, \frac{p_1+2}{p_1+1}, \frac{p_2+2}{p_2+1})}(\Omega) \text{ as } k \to \infty,$$
$$f_2(u_{i_k}(t), v_{i_k}(t)) \to f_2(u(t), v(t)) \text{ in } L^{\min(\frac{\rho_1+2}{\rho_1+1}, \frac{\rho_2+2}{\rho_2+1}, \frac{p_1+2}{p_1+1}, \frac{p_2+2}{p_2+1})}(\Omega) \text{ as } k \to \infty.$$

In addition, for every pair $(\varphi,\psi)^\top \in H_0^1(\Omega;\mathbb{R}^2)$, it holds that

$$
\begin{aligned}
&\frac{1}{\rho_1+1}\int_\Omega |\partial_t u_{i_k}(t)|^{\rho_1}\partial_t u_{i_k}(t)\varphi dx + \int_\Omega \nabla^\top\varphi A_1\nabla\partial_t u_{i_k}(t)dx \\
&-\frac{1}{\rho_1+1}\int_\Omega |\sum_{j=1}^{i_k}\int_\Omega u^1 e_{1j}dx e_{1j}|^{\rho_1}\sum_{j=1}^{i_k}\int_\Omega u^1 e_{1j}dx e_{1j}\varphi dx \\
&-\int_\Omega \nabla^\top\varphi A_1\nabla(\sum_{j=1}^{i_k}\int_\Omega u^1 e_{1j}dx e_{1j})dx \\
&+\int_0^t\int_\Omega \nabla^\top\varphi A_1(\nabla u_{i_k}(s)-\int_{-\infty}^s g_1(s-\tau)\nabla u_{i_k}(\tau)d\tau)dxds \\
=&\int_0^t\int_\Omega \varphi(f_1(u_{i_k}(s),v_{i_k}(s))-a_{11}\partial_t u_{i_k}(s)-a_{12}\partial_t v_{i_k}(s))dxds,\quad t\in[0,\tilde{T}],
\end{aligned}
\tag{53}
$$

and

$$
\begin{aligned}
&\frac{1}{\rho_2+1}\int_\Omega |\partial_t v_{i_k}(t)|^{\rho_2}\partial_t v_{i_k}(t)\psi dx + \int_\Omega \nabla^\top\psi A_2\nabla\partial_t v_{i_k}(t)dx \\
&-\frac{1}{\rho_2+1}\int_\Omega |\sum_{j=1}^{i_k}\int_\Omega v^1 e_{2j}dx e_{2j}|^{\rho_2}\sum_{j=1}^{i_k}\int_\Omega v^1 e_{2j}dx e_{2j}\psi dx \\
&-\int_\Omega \nabla^\top\psi A_2\nabla(\sum_{j=1}^{i_k}\int_\Omega v^1 e_{2j}dx e_{2j})dx \\
&+\int_0^t\int_\Omega \nabla^\top\psi A_2(\nabla v_{i_k}(s)-\int_{-\infty}^s g_2(s-\tau)\nabla v_{i_k}(\tau)d\tau)dxds \\
=&\int_0^t\int_\Omega \psi(f_2(u_{i_k}(s),v_{i_k}(s))-a_{21}\partial_t u_{i_k}(s)-a_{22}\partial_t v_{i_k}(s))dxds,\quad t\in[0,\tilde{T}].
\end{aligned}
\tag{54}
$$

By recalling that

$$
\begin{aligned}
&\sum_{j=1}^{i_k}\int_\Omega u^0(t)e_{1j}dx e_{1j}\to u^0 \text{ in } L^\infty(\mathbb{R}_-;H_0^1(\Omega)) \text{ as } k\to\infty,\\
&\sum_{j=1}^{i_k}\int_\Omega v^0(t)e_{2j}dx e_{2j}\to v^0 \text{ in } L^\infty(\mathbb{R}_-;H_0^1(\Omega)) \text{ as } k\to\infty,\\
&\sum_{j=1}^{i_k}\int_\Omega u^1 e_{1j}dx e_{1j}\to u^1 \text{ in } H_0^1(\Omega) \text{ as } k\to\infty,\\
&\sum_{j=1}^{i_k}\int_\Omega v^1 e_{2j}dx e_{2j}\to v^1 \text{ in } H_0^1(\Omega) \text{ as } k\to\infty,
\end{aligned}
$$

we could conclude, based on the idea of passing to the limit of (53) and (54), that $(u,v)^\top$, the limit of $(u_{i_k},v_{i_k})^\top$, satisfies (16), (17), $(1)_3$ and $(1)_4$.

Lastly, we can mimick steps in [9,23], to show that $(u,v)^\top\in S_{[0,\tilde{T}]}$. To summarize, $(u,v)^\top$, the limit of $(u_{i_k},v_{i_k})^\top$, is indeed a local solution, in the interval $[0,\tilde{T}]$, to IBVP (1) in the sense of Definition 1. The proof is complete. □

Remark 5. *Illuminated by the integral identity*

$$
\begin{aligned}
\int_0^t\int_\Omega \nabla^\top\varphi A_1\nabla\partial_t^2 u_{i_k}(s)dxds = &\int_\Omega \nabla^\top\varphi A_1\nabla\partial_t u_{i_k}(t)dx \\
&-\int_\Omega \nabla^\top\varphi A_1\nabla(\sum_{j=1}^{i_k}\int_\Omega u^1 e_{1j}dx e_{1j})dx,\quad t\in[0,\tilde{T}],\quad \forall\varphi\in H_0^1(\Omega),
\end{aligned}
$$

and by the integral identity

$$\int_0^t \int_\Omega \nabla^\top \psi A_2 \nabla \partial_t^2 v_{i_k}(s) dx ds = \int_\Omega \nabla^\top \psi A_2 \nabla \partial_t v_{i_k}(t) dx - \int_\Omega \nabla^\top \psi A_2 \nabla (\sum_{j=1}^{i_k} \int_\Omega v^1 e_{2j} dx e_{2j}) dx, \quad t \in [0, \tilde{T}], \quad \forall \psi \in H_0^1(\Omega),$$

we conclude that

$$\partial_t^2 u_{i_k} \to \partial_t^2 u \text{ weakly}^* \text{ in } \mathcal{D}'(0, \tilde{T}; H_0^1(\Omega)), \text{ as } k \to \infty, \text{ and}$$
$$\partial_t^2 v_{i_k} \to \partial_t^2 v \text{ weakly}^* \text{ in } \mathcal{D}'(0, \tilde{T}; H_0^1(\Omega)), \text{ as } k \to \infty.$$

Besides, enlightened by the integral identity

$$\int_0^t \int_\Omega |\partial_t u_{i_k}(s)|^{\rho_1} \partial_t^2 u_{i_k}(s) \varphi dx ds = \frac{1}{\rho_1 + 1} \int_\Omega |\partial_t u_{i_k}(t)|^{\rho_1} \partial_t u_{i_k}(t) \varphi dx - \frac{1}{\rho_1 + 1} \int_\Omega |\sum_{j=1}^{i_k} \int_\Omega u^1 e_{1j} dx e_{1j}|^{\rho_1} \sum_{j=1}^{i_k} \int_\Omega u^1 e_{1j} dx e_{1j} \varphi dx, \quad t \in [0, \tilde{T}], \quad \forall \varphi \in L^2(\Omega),$$

and by the integral identity

$$\int_0^t \int_\Omega |\partial_t v_{i_k}(s)|^{\rho_2} \partial_t^2 v_{i_k}(s) \psi dx ds = \frac{1}{\rho_2 + 1} \int_\Omega |\partial_t v_{i_k}(t)|^{\rho_2} \partial_t v_{i_k}(t) \psi dx - \frac{1}{\rho_2 + 1} \int_\Omega |\sum_{j=1}^{i_k} \int_\Omega v^1 e_{2j} dx e_{2j}|^{\rho_2} \sum_{j=1}^{i_k} \int_\Omega v^1 e_{2j} dx e_{2j} \psi dx, \quad t \in [0, \tilde{T}], \quad \forall \psi \in L^2(\Omega),$$

we conclude that

$$|\partial_t u_{i_k}|^{\rho_1} \partial_t^2 u_{i_k} \to |\partial_t u|^{\rho_1} \partial_t^2 u \text{ weakly}^* \text{ in } \mathcal{D}'(0, \tilde{T}; L^2(\Omega)), \text{ as } k \to \infty, \text{ and}$$
$$|\partial_t v_{i_k}|^{\rho_2} \partial_t^2 v_{i_k} \to |\partial_t v|^{\rho_2} \partial_t^2 v \text{ weakly}^* \text{ in } \mathcal{D}'(0, \tilde{T}; L^2(\Omega)), \text{ as } k \to \infty.$$

To sum up, the solution $(u, v)^\top \in \mathcal{S}_{[0,\tilde{T}]}$ (see (14) for the definition of $\mathcal{S}_{[0,\tilde{T}]}$) in the sense of Definition 1 whose existence justified by Theorem 1 satisfies automatically

$$\partial_t^2 u, \ \partial_t^2 v \in \mathcal{D}'(0, \tilde{T}; H_0^1(\Omega)), \text{ and } |\partial_t u|^{\rho_1} \partial_t^2 u, \ |\partial_t v|^{\rho_2} \partial_t^2 v \in \mathcal{D}'(0, \tilde{T}; L^2(\Omega)).$$

Remark 6. *By re-checking the proof of Theorem 1, we may find that the restriction on the symmetric matrices-valued A_i function, defined in the domain Ω, (see Assumption 1 for the details) could be weakened to: $A_i \in W^{1,\infty}(\Omega)$ satisfies*

$$\operatorname*{ess\,inf}_{\substack{x \in \Omega, \\ \xi \in \mathbb{R}^N \setminus \{\mathbf{0}\}}} \frac{\xi^\top A_i(x) \xi}{\xi^\top \xi} > 0, \quad i = 1, 2,$$

and may find that the restriction that $a_{ij} \in \mathscr{C}(\bar{\Omega})$ on the coefficient a_{ij} in Assumption 5 could be weakened to $a_{ij} \in L^\infty(\Omega)$, $i, j = 1, 2$. For the sake of convenience of our calculations, unless stated otherwise, we abide by Assumptions 1 and 5 in the rest of the paper.

Theorem 2. *Suppose that Assumptions 1–5 hold true. For every pair*

$$(u^0, v^0)^\top \in L^\infty(\mathbb{R}_-; H_0^1(\Omega; \mathbb{R}^2))$$

and every pair $(u^1, v^1)^\top \in H_0^1(\Omega;\mathbb{R}^2)$, IBVP (1) admits a solution $(u,v)^\top \in S_{[0,T)}$ in the sense of Definition 2, in which $[0,T)$, is the maximal existence time interval of the solution pair $(u,v)^\top$ with the maximal existence time instant $0 < T \leqslant +\infty$ independent of $(u,v)^\top$ but depending on Ω, A_1, A_2, f_1, f_2, g_1, g_2, ρ_1, ρ_2, a_{11}, a_{12}, a_{21}, a_{22} , u^0, v^0, u^1 and v^1.

Proof. Theorem 2 can be proved by a standard continuation procedure. And therefore we choose to leave out the detailed steps in this paper. □

Lemma 3. *Suppose that Assumptions 1–5 hold true. $E^{u,v}(t)$ defined as in (18), associated to every pair $(u,v)^\top$ of solution, in the interval $[0,T)$, to IBVP (1) in the sense of Definition 2 (see Theorem 2 for the existence of $(u,v)^\top$), is non-increasing in $[0,T)$.*

Proof. Mimicking steps conducted in (25), we have

$$\begin{aligned}\frac{d}{dt}E^{u,v}(t) =&\frac{1}{2}(g_1' \diamond^{A_1} \nabla u)(t) + \frac{1}{2}(g_2' \diamond^{A_2} \nabla v)(t)\\ &- \int_\Omega \begin{pmatrix}\partial_t u(t)\\ \partial_t v(t)\end{pmatrix}^\top \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}\partial_t u(t)\\ \partial_t v(t)\end{pmatrix}dx, \quad t \in [0,T),\end{aligned} \tag{55}$$

which, together with Assumptions 2 and 5, implies immediately that $E^{u,v}(t)$ is indeed non-increasing in the interval $[0,T)$. The proof is complete. □

Theorem 2, having Theorem 1 as its basis, states that to every initial-datum pair

$$(u^0, v^0)^\top \in L^\infty(\mathbb{R}_-; H_0^1(\Omega;\mathbb{R}^2))$$

and every initial-datum pair $(u^1, v^1)^\top \in H_0^1(\Omega;\mathbb{R}^2)$, there corresponds a weak solution pair $(u,v)^\top \in S_{[0,T)}$ in the sense of Definition 2 to IBVP (1), in which $[0,T)$ is the maximal existence time interval of the solution pair $(u,v)^\top$. Now it is natural to start to investigate the global existence of solutions to IBVP (1).

To make it convenient to present our other results in the rest of this paper, let us introduce two auxilliary functionals

$$\begin{aligned}I(t) =&\frac{1}{4}(\mu_1 - \int_0^{+\infty} g_1(s)ds)\int_\Omega \nabla^\top u(t)A_1\nabla u(t)dx\\ &+ \frac{1}{4}\int_\Omega \nabla^\top\partial_t u(t)A_1\nabla\partial_t u(t)dx + \frac{1}{4}(g_1 \diamond^{A_1} \nabla u)(t)\\ &+ \frac{1}{4}(\mu_2 - \int_0^{+\infty} g_2(s)ds)\int_\Omega \nabla^\top v(t)A_2\nabla v(t)dx\\ &+ \frac{1}{4}\int_\Omega \nabla^\top\partial_t v(t)A_2\nabla\partial_t v(t)dx + \frac{1}{4}(g_2 \diamond^{A_2} \nabla v)(t) - \int_\Omega F(u(t),v(t))dx, \quad t \in [0,T)\end{aligned} \tag{56}$$

and

$$\begin{aligned}J(t) =&\frac{1}{2}(\mu_1 - \int_0^{+\infty} g_1(s)ds)\int_\Omega \nabla^\top u(t)A_1\nabla u(t)dx\\ &+ \frac{1}{2}\int_\Omega \nabla^\top\partial_t u(t)A_1\nabla\partial_t u(t)dx + \frac{1}{2}(g_1 \diamond^{A_1} \nabla u)(t)\\ &+ \frac{1}{2}(\mu_1 - \int_0^{+\infty} g_2(s)ds)\int_\Omega \nabla^\top v(t)A_2\nabla v(t)dx\\ &+ \frac{1}{2}\int_\Omega \nabla^\top\partial_t v(t)A_2\nabla\partial_t v(t)dx + \frac{1}{2}(g_2 \diamond^{A_2} \nabla v)(t) - \int_\Omega F(u(t),v(t))dx, \quad t \in [0,T).\end{aligned} \tag{57}$$

We shall see below that our global existence and general energy decay results hold true only for solutions to IBVP (1) having small initial data. To measure such smallness, we need introduce the following two constants

$$\beta_1 = \frac{4\mathscr{M}_{11}^6(Ł_1 + Ł_2(p_1+1))}{p_1+2} + \frac{4\mathscr{M}_{12}^6(Ł_1 + Ł_2(p_2+1))}{p_2+2} + \frac{4\mathscr{M}_{13}^6(Ł_1 + Ł_2(\rho_1+1))}{\rho_1+2}, \tag{58}$$

and

$$\beta_2 = \frac{8Ł_2\mathscr{M}_{21}^6}{p_1+2} + \frac{8Ł_2\mathscr{M}_{22}^6}{p_2+2} + \frac{4Ł_2\mathscr{M}_{23}^6}{\rho_1+2} + \frac{4Ł_2\mathscr{M}_{24}^6}{\rho_2+2}, \tag{59}$$

in which the constants $\mathscr{M}_{11}^6$, $\mathscr{M}_{12}^6$, $\mathscr{M}_{13}^6$, $\mathscr{M}_{14}^6$, $\mathscr{M}_{21}^6$, $\mathscr{M}_{22}^6$, $\mathscr{M}_{23}^6$ and $\mathscr{M}_{24}^6$ are given by

$$\mathscr{M}_{11}^6 = (\frac{\zeta_1(\kappa_{p_1+2})^2}{\mu_1 - \int_0^{+\infty} g_1(s)ds})^{\frac{p_1+2}{2}}(4E^{u,v}(0))^{\frac{p_1}{2}}, \tag{60}$$

$$\mathscr{M}_{12}^6 = (\frac{\zeta_1(\kappa_{p_2+2})^2}{\mu_1 - \int_0^{+\infty} g_1(s)ds})^{\frac{p_2+2}{2}}(4E^{u,v}(0))^{\frac{p_2}{2}}, \tag{61}$$

$$\mathscr{M}_{13}^6 = (\frac{\zeta_1(\kappa_{\rho_1+2})^2}{\mu_1 - \int_0^{+\infty} g_1(s)ds})^{\frac{\rho_1+2}{2}}(4E^{u,v}(0))^{\frac{\rho_1}{2}}, \tag{62}$$

$$\mathscr{M}_{14}^6 = (\frac{\zeta_1(\kappa_{\rho_2+2})^2}{\mu_1 - \int_0^{+\infty} g_1(s)ds})^{\frac{\rho_2+2}{2}}(4E^{u,v}(0))^{\frac{\rho_2}{2}}, \tag{63}$$

$$\mathscr{M}_{21}^6 = (\frac{\zeta_2(\kappa_{p_1+2})^2}{\mu_2 - \int_0^{+\infty} g_2(s)ds})^{\frac{p_1+2}{2}}(4E^{u,v}(0))^{\frac{p_1}{2}}, \tag{64}$$

$$\mathscr{M}_{22}^6 = (\frac{\zeta_2(\kappa_{p_2+2})^2}{\mu_2 - \int_0^{+\infty} g_2(s)ds})^{\frac{p_2+2}{2}}(4E^{u,v}(0))^{\frac{p_2}{2}}, \tag{65}$$

$$\mathscr{M}_{23}^6 = (\frac{\zeta_2(\kappa_{\rho_1+2})^2}{\mu_2 - \int_0^{+\infty} g_2(s)ds})^{\frac{\rho_1+2}{2}}(4E^{u,v}(0))^{\frac{\rho_1}{2}}, \tag{66}$$

and

$$\mathscr{M}_{24}^6 = (\frac{\zeta_2(\kappa_{\rho_2+2})^2}{\mu_2 - \int_0^{+\infty} g_2(s)ds})^{\frac{\rho_2+2}{2}}(4E^{u,v}(0))^{\frac{\rho_2}{2}}, \tag{67}$$

respectively, where the constant ζ_i is given as in (9), $i = 1, 2$, and $E^{u,v}(0) = E^{u,v}(t)|_{t=0}$ can be formulated explicitly as

$$\begin{aligned} E^{u,v}(0) = &\frac{1}{\rho_1+2}\|u^1\|_{L^{\rho_1+2}(\Omega)}^{\rho_1+2} + \frac{1}{\rho_2+2}\|v^1\|_{L^{\rho_2+2}(\Omega)}^{\rho_2+2} \\ &+ \frac{1}{2}(\mu_1 - \int_0^{+\infty} g_1(s)ds)\int_\Omega \nabla^\top u^0(0)A_1\nabla u^0(0)dx \\ &+ \frac{1}{2}\int_\Omega \nabla^\top u^1 A_1 \nabla u^1 dx + \frac{1}{2}(g_1 \diamond^{A_1} \nabla u^0)(0) \\ &+ \frac{1}{2}(\mu_2 - \int_0^{+\infty} g_2(s)ds)\int_\Omega \nabla^\top v^0(0)A_2\nabla v^0(0)dx \\ &+ \frac{1}{2}\int_\Omega \nabla^\top v^1 A_2 \nabla v^1 dx + \frac{1}{2}(g_2 \diamond^{A_2} \nabla v^0)(0) - \int_\Omega F(u^0, v^0)dx. \end{aligned}$$

Please consult (18) for the detailed expression of $E^{u,v}(t)$.

Lemma 4. *Suppose that Assumptions 1–5 hold true. For every weak solution pair* $(u, v)^\top \in S_{[0,T)}$ *to IBVP* (1) *with* $[0, T)$ *the maximal existence interval of* $(u, v)^\top$*, if the associated functional* $I(t)$ *given by* (56) *satisfies* $I(0) > 0$ *and the associated constants* β_1 *and* β_2*, given by* (58) *and* (59)*, respectively, satisfy* $\max(\beta_1, \beta_2) < 1$*, then* $I(t) > 0$ *holds for all* $t \in [0, T)$*.*

Proof. Thanks to the assumption that $I(0) > 0$ and to the continuity of the function $I(t)$, there exists a time $T_1 \in (0, T)$, such that

$$I(t) \geqslant 0, \quad t \in [0, T_1]. \tag{68}$$

By the very definition of $I(t)$ (see (56) for the details), we have

$$\begin{aligned} J(t) = I(t) &+ \frac{1}{4}(\mu_1 - \int_0^{+\infty} g_1(s)ds) \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx \\ &+ \frac{1}{4}\int_\Omega \nabla^\top \partial_t u(t) A_1 \nabla \partial_t u(t) dx + \frac{1}{4}(g_1 \diamond^{A_1} \nabla u)(t) \\ &+ \frac{1}{4}(\mu_2 - \int_0^{+\infty} g_2(s)ds) \int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx \\ &+ \frac{1}{4}\int_\Omega \nabla^\top \partial_t v(t) A_2 \nabla \partial_t v(t) dx + \frac{1}{4}(g_2 \diamond^{A_2} \nabla v)(t), \quad t \in [0, T_1], \end{aligned} \tag{69}$$

where the functional $J(t)$, associated to the solution $(u, v)^\top$ to IBVP (1), is defined as in (57). Combine (68) and (69), to arrive at directly

$$\begin{aligned} &\frac{1}{4}(\mu_1 - \int_0^{+\infty} g_1(s)ds) \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx \\ &+ \frac{1}{4}(\mu_2 - \int_0^{+\infty} g_2(s)ds) \int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx \leqslant J(t), \quad t \in [0, T_1]. \end{aligned} \tag{70}$$

In light of the definition (57) of the functional $J(t)$, we have

$$J(t) \leqslant E^{u,v}(t) \leqslant E^{u,v}(0), \quad t \in [0, T_1], \tag{71}$$

where the second '$\leqslant$' follows from the non-increasing monotonicity (see Lemma 3 for the details). Substitute (71) into (70), to arrive at immediately

$$\begin{aligned} &(\mu_1 - \int_0^{+\infty} g_1(s)ds) \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx \\ &+ (\mu_2 - \int_0^{+\infty} g_2(s)ds) \int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx \leqslant 4E^{u,v}(0), \quad t \in [0, T_1]. \end{aligned} \tag{72}$$

By Lemma 1, we have

$$\begin{aligned} |\int_\Omega F(u(t), v(t)) dx| \leqslant & \frac{Ł_1 + Ł_2(p_1+1)}{p_1+2} \|u(t)\|_{L^{p_1+2}(\Omega)}^{p_1+2} + \frac{2Ł_2}{p_1+2} \|v(t)\|_{L^{p_1+2}(\Omega)}^{p_1+2} \\ &+ \frac{Ł_1 + Ł_2(p_2+1)}{p_2+2} \|u(t)\|_{L^{p_2+2}(\Omega)}^{p_2+2} + \frac{2Ł_2}{p_2+2} \|v(t)\|_{L^{p_2+2}(\Omega)}^{p_2+2} \\ &+ \frac{Ł_1 + Ł_2(\rho_1+1)}{\rho_1+2} \|u(t)\|_{L^{\rho_1+2}(\Omega)}^{\rho_1+2} + \frac{Ł_2}{\rho_1+2} \|v(t)\|_{L^{\rho_1+2}(\Omega)}^{\rho_1+2} \\ &+ \frac{Ł_2}{\rho_2+2} \|v(t)\|_{L^{\rho_2+2}(\Omega)}^{\rho_2+2}, \quad t \in [0, T_1]. \end{aligned} \tag{73}$$

Having the notations in (9) and (10) at our disposal and based on the coercivity condition on A_1 (see Assumption 1), mimicking steps in deducing (32), we apply the Sobolev–Poincaré inequality (see Remark 1), to conclude

$$\|u(t)\|_{L^{p_1+2}(\Omega)}^{p_1+2} \leqslant \mathscr{M}_{11}^6 (\mu_1 - \int_0^{+\infty} g_1(s)ds) \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx. \tag{74}$$

And similarly, we have

$$\|u(t)\|_{L^{p_2+2}(\Omega)}^{p_2+2} \leqslant \mathscr{M}_{12}^6(\mu_1 - \int_0^{+\infty} g_1(s)ds) \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx, \tag{75}$$

$$\|u(t)\|_{L^{\rho_1+2}(\Omega)}^{\rho_1+2} \leqslant \mathscr{M}_{13}^6(\mu_1 - \int_0^{+\infty} g_1(s)ds) \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx, \tag{76}$$

and

$$\|u(t)\|_{L^{\rho_2+2}(\Omega)}^{\rho_2+2} \leqslant \mathscr{M}_{14}^6(\mu_1 - \int_0^{+\infty} g_1(s)ds) \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx. \tag{77}$$

Having the notations in (9) and (10) at our disposal and based on the coercivity condition on A_2 (see Assumption 1), mimicking steps in deducing (32) and (74)–(77), we apply the Sobolev–Poincaré inequality (see Remark 1), to obtain

$$\|v(t)\|_{L^{p_1+2}(\Omega)}^{p_1+2} \leqslant \mathscr{M}_{21}^6(\mu_2 - \int_0^{+\infty} g_2(s)ds) \int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx. \tag{78}$$

And analogously, we have

$$\|v(t)\|_{L^{p_2+2}(\Omega)}^{p_2+2} \leqslant \mathscr{M}_{22}^6(\mu_2 - \int_0^{+\infty} g_2(s)ds) \int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx, \tag{79}$$

$$\|v(t)\|_{L^{\rho_1+2}(\Omega)}^{\rho_1+2} \leqslant \mathscr{M}_{23}^6(\mu_2 - \int_0^{+\infty} g_2(s)ds) \int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx, \tag{80}$$

and

$$\|v(t)\|_{L^{\rho_2+2}(\Omega)}^{\rho_2+2} \leqslant \mathscr{M}_{24}^6(\mu_2 - \int_0^{+\infty} g_2(s)ds) \int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx. \tag{81}$$

Plug (74)–(81) into (73), to obtain

$$\begin{aligned} |\int_\Omega F(u(t),v(t))dx| \leqslant & \frac{\beta_1}{4}(\mu_1 - \int_0^{+\infty} g_1(s)ds) \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx \\ & + \frac{\beta_2}{4}(\mu_2 - \int_0^{+\infty} g_2(s)ds) \int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx, \quad t \in [0, T_1], \end{aligned}$$

where the constants β_1 and β_2 are defined as in (58) and (59), respectively. Owing to the assumption that $\max(\beta_1, \beta_2) < 1$, it follows that

$$\begin{aligned} |\int_\Omega F(u(t),v(t))dx| < & \frac{1}{4}(\mu_1 - \int_0^{+\infty} g_1(s)ds) \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx \\ & + \frac{1}{4}(\mu_2 - \int_0^{+\infty} g_2(s)ds) \int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx, \quad t \in [0, T_1]. \end{aligned} \tag{82}$$

This, together with the definition (56) of $I(t)$, implies that

$$I(t) > 0 \quad \text{for } t \in [0, T_1]. \tag{83}$$

Lastly, to finish the proof, we introduce

$$T_2 = \sup\{0 < T' \leqslant T;\, I(t) > 0,\ t \in [0, T')\}.$$

Obviously, T_2 does not exceed T. By a contradiction argument, we shall show that T_2 coincides actually with T. We assume to the contrary that $T_2 < T$, then by the definition of T_2 as well as the continuity of $I(t)$, we have immediately that $I(T_2) \geqslant 0$. Enlightened by the procedure of deriving (83) from (68), we have therefore $I(T_2) > 0$. Thanks to the continuity of $I(t)$, there exists a $T_2 < T_3 < T$ such that $I(t) > 0$ holds for all $t \in [0, T_3]$. This contradicts the definition of T_2. This implies indeed that T_2 coincides actually with T. In other words, $I(t) > 0$ holds for all $t \in [0, T)$. □

Theorem 3. *Suppose that Assumptions 1–5 hold true. For every weak solution pair $(u,v)^\top \in S_{[0,T)}$ to IBVP (1) with $[0,T)$ the maximal existence interval of $(u,v)^\top$, if the associated functional $I(t)$ given by (56) satisfies $I(0) > 0$ and the associated constants β_1 and β_2, given by (58) and (59), respectively, satisfy $\max(\beta_1,\beta_2) < 1$, then $T = +\infty$. In other words, weak solutions to IBVP (1) exist globally in time whenever their initial data satisfy $I(0) > 0$ and $\max(\beta_1,\beta_2) < 1$.*

Proof. We shall prove, by a contradiction argument, that T is exactly $+\infty$. Let us assume that $T < +\infty$. Following the idea used to obtain (72) in the proof of Lemma 4, we have

$$
\begin{aligned}
&(\mu_1 - \int_0^{+\infty} g_1(s)ds) \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx \\
&+ \int_\Omega \nabla^\top \partial_t u(t) A_1 \nabla \partial_t u(t) dx \\
&+ (\mu_2 - \int_0^{+\infty} g_2(s)ds) \int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx \\
&+ \int_\Omega \nabla^\top \partial_t v(t) A_2 \nabla \partial_t v(t) dx \\
\leqslant\, & 4I(t) + (\mu_1 - \int_0^{+\infty} g_1(s)ds) \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx \\
&+ \int_\Omega \nabla^\top \partial_t u(t) A_1 \nabla \partial_t u(t) dx + (g_1 \diamond^{A_1} \nabla u)(t) \\
&+ (\mu_2 - \int_0^{+\infty} g_2(s)ds) \int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx \\
&+ \int_\Omega \nabla^\top \partial_t v(t) A_2 \nabla \partial_t v(t) dx + (g_2 \diamond^{A_2} \nabla v)(t) \\
=\, & 4J(t) \leqslant 4E^{u,v}(t) \leqslant 4E^{u,v}(0), \quad t \in [0,T).
\end{aligned}
$$

This, together with (9) (see Remark 1 for the details), implies that

$$
\begin{aligned}
&\max(\|\nabla u(t)\|^2_{L^2(\Omega;\mathbb{R}^N)}, \|\nabla \partial_t u(t)\|^2_{L^2(\Omega;\mathbb{R}^N)}, \\
&\qquad \|\nabla v(t)\|^2_{L^2(\Omega;\mathbb{R}^N)}, \|\nabla \partial_t v(t)\|^2_{L^2(\Omega;\mathbb{R}^N)}) \leqslant \frac{4}{\mathscr{M}_1^5} E^{u,v}(0), \quad t \in [0,T),
\end{aligned} \tag{84}
$$

which, together with (10) (see also Remark 1 for the details), implies further

$$
\begin{aligned}
&\max(\|u(t)\|^2_{H_0^1(\Omega)}, \|\partial_t u(t)\|^2_{H_0^1(\Omega)}, \\
&\qquad \|v(t)\|^2_{H_0^1(\Omega)}, \|\partial_t v(t)\|^2_{H_0^1(\Omega)}) \leqslant \frac{4(1+(\kappa_2)^2)}{\mathscr{M}_1^5} E^{u,v}(0), \quad t \in [0,T),
\end{aligned}
$$

where the positive constant $\mathscr{M}_1^5$, given by (52), is independent of the time variable t. Now, by a standard continution procedure, we could obtain a time $T_4 > T$ such that IBVP (1) admits a solution in $[0,T_4)$. This contradicts the assumption that $[0,T)$ is the maximal existence interval of IBVP (1). This implies immediately that $[0,T)$, the maximal existence time interval of the solution $(u,v)^\top$, coincides actually with $\mathbb{R}_+$. In other words, the solution $(u,v)^\top$ exists globally in time. The proof is complete. □

3. General Energy Decay Results Concerning Solutions to IBVP (1)

Let us now associate to every solution $(u,v)^\top$ to IBVP (1) the functional

$$
\begin{aligned}
L_1(t) =\, & \frac{1}{\rho_1+1} \int_\Omega u(t) |\partial_t u(t)|^{\rho_1} \partial_t u(t) dx + \int_\Omega \nabla^\top u(t) A_1 \nabla \partial_t u(t) dx \\
&+ \frac{1}{\rho_2+1} \int_\Omega v(t) |\partial_t v(t)|^{\rho_2} \partial_t v(t) dx + \int_\Omega \nabla^\top v(t) A_2 \nabla \partial_t v(t) dx \\
&+ \frac{1}{2} \int_\Omega \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}^\top \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} dx.
\end{aligned} \tag{85}
$$

As we shall see, $L_1(t)$ plays a role of the energy perturbation, and one differential inequality concerning the functional $L_1(t)$ plays an important role in proving our main results in this paper. This inequality can only be established for those solutions whose initial data are small. To give a precise sense by which we mean the smallness, we need to introduce two useful constants $\hbar_1$ and $\hbar_2$:

$$\begin{aligned}\hbar_1 = &(4E^{u,v}(0))^{\frac{p_1}{2}}(Ł_1 + \frac{Ł_1}{p_1+2} + \frac{Ł_2(p_1+1)}{p_1+2})(\frac{\zeta_1(\kappa_{p_1+2})^2}{\mu_1 - \int_0^{+\infty} g_1(s)ds})^{\frac{p_1+2}{2}}\\ &+ (4E^{u,v}(0))^{\frac{p_2}{2}}(Ł_1 + \frac{Ł_1}{p_2+2} + \frac{Ł_2(p_2+1)}{p_2+2})(\frac{\zeta_1(\kappa_{p_2+2})^2}{\mu_1 - \int_0^{+\infty} g_1(s)ds})^{\frac{p_2+2}{2}}\\ &+ (4E^{u,v}(0))^{\frac{\rho_1}{2}}(Ł_1 + \frac{Ł_2(\rho_1+1)}{\rho_1+2})(\frac{\zeta_1(\kappa_{\rho_1+2})^2}{\mu_1 - \int_0^{+\infty} g_1(s)ds})^{\frac{\rho_1+2}{2}}\\ &+ \frac{Ł_1(\rho_2+1)(4E^{u,v}(0))^{\frac{\rho_2}{2}}}{\rho_2+2}(\frac{\zeta_1(\kappa_{\rho_2+2})^2}{\mu_1 - \int_0^{+\infty} g_1(s)ds})^{\frac{\rho_2+2}{2}},\end{aligned} \tag{86}$$

$$\begin{aligned}\hbar_2 = &(4E^{u,v}(0))^{\frac{p_1}{2}}(Ł_2 + \frac{Ł_2}{p_1+2} + \frac{Ł_1(p_1+1)}{p_1+2})(\frac{\zeta_2(\kappa_{p_1+2})^2}{\mu_2 - \int_0^{+\infty} g_2(s)ds})^{\frac{p_1+2}{2}}\\ &+ (4E^{u,v}(0))^{\frac{p_2}{2}}(Ł_2 + \frac{Ł_2}{p_2+2} + \frac{Ł_1(p_2+1)}{p_2+2})(\frac{\zeta_2(\kappa_{p_2+2})^2}{\mu_2 - \int_0^{+\infty} g_2(s)ds})^{\frac{p_2+2}{2}}\\ &+ \frac{Ł_2(4E^{u,v}(0))^{\frac{\rho_1}{2}}}{\rho_1+2}(\frac{\zeta_2(\kappa_{\rho_1+2})^2}{\mu_2 - \int_0^{+\infty} g_2(s)ds})^{\frac{\rho_1+2}{2}}\\ &+ (4E^{u,v}(0))^{\frac{\rho_2}{2}}(Ł_2 + \frac{Ł_1(\rho_2+1)}{\rho_2+2})(\frac{\zeta_2(\kappa_{\rho_2+2})^2}{\mu_2 - \int_0^{+\infty} g_2(s)ds})^{\frac{\rho_2+2}{2}}.\end{aligned} \tag{87}$$

Lemma 5. *Suppose that Assumptions 1–5 hold true. If the associated functional $I(t)$ given by (56) satisfies $I(0) > 0$, the associated constants β_1 and β_2, given by (58) and (59), respectively, satisfy $\max(\beta_1, \beta_2) < 1$, and the associated constants $\hbar_1$ and $\hbar_2$, given by (86) and (87), respectively, satisfy $\max(\hbar_1, \hbar_2) < \frac{1}{4}$, then weak solutions to IBVP (1) exist globally in time, and render the associated functional $L_1(t)$ given by (85) to satisfy*

$$\begin{aligned}L_1'(t) \leqslant &\frac{1}{\rho_1+1}\int_\Omega |\partial_t u(t)|^{\rho_1+2}dx + \frac{1}{\rho_2+1}\int_\Omega |\partial_t v(t)|^{\rho_2+2}dx\\ &+ \int_\Omega \nabla^\top \partial_t u(t) A_1 \nabla \partial_t u(t)dx + \int_\Omega \nabla^\top \partial_t v(t) A_2 \nabla \partial_t v(t)dx\\ &- \frac{1}{2}(\mu_1 - \int_0^{+\infty} g_1(s)ds)\int_\Omega \nabla^\top u(t) A_1 \nabla u(t)dx - \frac{1}{2}(\mu_2 - \int_0^{+\infty} g_2(s)ds)\int_\Omega \nabla^\top v(t) A_2 \nabla v(t)dx\\ &+ \frac{1}{\mu_1 - \int_0^{+\infty} g_1(s)ds}\int_0^{+\infty}\frac{(g_1(s))^2}{g_1(s) - Gg_1'(s)}ds((g_1 - Gg_1') \diamond^{A_1} \nabla u)(t)\\ &+ \frac{1}{\mu_2 - \int_0^{+\infty} g_2(s)ds}\int_0^{+\infty}\frac{(g_2(s))^2}{g_2(s) - Gg_2'(s)}ds((g_2 - Gg_2') \diamond^{A_2} \nabla v)(t), \quad t \in \mathbb{R}_+,\end{aligned} \tag{88}$$

where G is a positive constant yet to be determined later.

Proof. As mentioned in Lemma 5, thanks to the assumptions $I(0) > 0$ and $\max(\beta_1, \beta_2) < 1$, it follows from Theorem 3 that $L_1(t)$ exists globally in time. Recalling (85), we differentiate $L_1(t)$, to arrive at

$$
\begin{aligned}
L_1'(t) = & \frac{1}{\rho_1+1}\int_\Omega |\partial_t u(t)|^{\rho_1+2}dx + \frac{1}{\rho_2+1}\int_\Omega |\partial_t v(t)|^{\rho_2+2}dx \\
& + \int_\Omega \nabla^\top \partial_t u(t) A_1 \nabla \partial_t u(t)dx + \int_\Omega \nabla^\top \partial_t v(t) A_2 \nabla \partial_t v(t)dx \\
& - \mu_1 \int_\Omega \nabla^\top u(t) A_1 \nabla u(t)dx - \mu_2 \int_\Omega \nabla^\top v(t) A_2 \nabla v(t)dx \\
& + \int_{-\infty}^t g_1(t-s)\int_\Omega \nabla^\top u(t) A_1 \nabla u(s)dxds \\
& + \int_{-\infty}^t g_2(t-s)\int_\Omega \nabla^\top v(t) A_2 \nabla v(s)dxds \\
& + \int_\Omega (u(t)f_1(u(t),v(t)) + v(t)f_2(u(t),v(t)))dx, \quad t \in \mathbb{R}_+.
\end{aligned} \tag{89}
$$

By some routine calculations, we have

$$
\begin{aligned}
& \int_{-\infty}^t g_1(t-s)\int_\Omega \nabla^\top u(t) A_1 \nabla u(s)dxds - \mu_1 \int_\Omega \nabla^\top u(t) A_1 \nabla u(t)dx \\
= & -(\mu_1 - \int_0^{+\infty} g_1(s)ds)\int_\Omega \nabla^\top u(t) A_1 \nabla u(t)dx \\
& - \int_{-\infty}^t g_1(t-s)\int_\Omega \nabla^\top u(t) A_1 (\nabla u(t) - \nabla u(s))dxds, \quad t \in \mathbb{R}_+.
\end{aligned} \tag{90}
$$

Since A_1 satisfies the coercivity condition that $\vartheta_1 > 0$ with ϑ_1 given by (5) (see Assumption 1 for the details), from the Cauchy–Schwarz inequality it follows that

$$
\begin{aligned}
& - \int_{-\infty}^t g_1(t-s)\int_\Omega \nabla^\top u(t) A_1 (\nabla u(t) - \nabla u(s))dxds \\
\leqslant & \Big(\int_{-\infty}^t \int_\Omega \frac{(g_1(t-s))^2}{g_1(t-s) - Gg_1'(t-s)} \nabla^\top u(t) A_1 \nabla u(t)dxds\Big)^{\frac{1}{2}} \\
& \cdot \Big(\int_{-\infty}^t \int_\Omega (g_1(t-s) - Gg_1'(t-s))(\nabla^\top u(t) - \nabla^\top u(s)) A_1 (\nabla u(t) - \nabla u(s))dxds\Big)^{\frac{1}{2}} \\
= & \Big(\int_\Omega \nabla^\top u(t) A_1 \nabla u(t)dx\Big)^{\frac{1}{2}} \Big(\int_0^{+\infty} \frac{(g_1(s))^2}{g_1(s) - Gg_1'(s)}ds((g_1 - Gg_1') \diamond^{A_1} \nabla u)(t)\Big)^{\frac{1}{2}} \\
\leqslant & \frac{\mu_1 - \int_0^{+\infty} g_1(s)ds}{4}\int_\Omega \nabla^\top u(t) A_1 \nabla u(t)dx \\
& + \frac{1}{\mu_1 - \int_0^{+\infty} g_1(s)ds}\int_0^{+\infty} \frac{(g_1(s))^2}{g_1(s) - Gg_1'(s)}ds((g_1 - Gg_1') \diamond^{A_1} \nabla u)(t), \quad t \in \mathbb{R}_+,
\end{aligned}
$$

which, together with (90), implies

$$
\begin{aligned}
& \int_{-\infty}^t g_1(t-s)\int_\Omega \nabla^\top u(t) A_1 \nabla u(s)dxds - \mu_1 \int_\Omega \nabla^\top u(t) A_1 \nabla u(t)dx \\
\leqslant & -\frac{3}{4}(\mu_1 - \int_0^{+\infty} g_1(s)ds)\int_\Omega \nabla^\top u(t) A_1 \nabla u(t)dx \\
& + \frac{1}{\mu_1 - \int_0^{+\infty} g_1(s)ds}\int_0^{+\infty} \frac{(g_1(s))^2}{g_1(s) - Gg_1'(s)}ds((g_1 - Gg_1') \diamond^{A_1} \nabla u)(t), \quad t \in \mathbb{R}_+,
\end{aligned} \tag{91}
$$

where G is a sufficiently large positive constant. Since A_2 satisfies the coercivity condition that $\vartheta_2 > 0$ with ϑ_2 given by (5) (see Assumption 1 for the details), it follows by mimicking steps in deducing (91) that

$$
\begin{aligned}
&\int_{-\infty}^{t} g_2(t-s)\int_{\Omega}\nabla^{\top}v(t)A_2\nabla v(s)dxds-\mu_2\int_{\Omega}\nabla^{\top}v(t)A_2\nabla v(t)dx\\
\leqslant&-\frac{3}{4}(\mu_2-\int_0^{+\infty}g_2(s)ds)\int_{\Omega}\nabla^{\top}v(t)A_2\nabla v(t)dx\\
&+\frac{1}{\mu_2-\int_0^{+\infty}g_2(s)ds}\int_0^{+\infty}\frac{(g_2(s))^2}{g_2(s)-Gg_2'(s)}ds((g_2-Gg_2')\diamond^{A_2}\nabla v)(t),\quad t\in\mathbb{R}_+.
\end{aligned}
\tag{92}
$$

By recalling Assumption 4 (especially (8)), by recalling the experience of deriving (31), (33)–(36), and by utilizing mainly the Fenchel–Young inequality, we have

$$
\begin{aligned}
&\int_{\Omega}(u(t)f_1(u(t),v(t))+v(t)f_2(u(t),v(t)))dx\\
\leqslant&Ł_1\int_{\Omega}(|u(t)|^{p_1+2}+|u(t)|^{p_2+2}+|u(t)|^{\rho_1+2})dx\\
&+Ł_1\int_{\Omega}|u(t)||v(t)|^{p_1+1}dx+Ł_1\int_{\Omega}|u(t)||v(t)|^{p_2+1}dx+Ł_1\int_{\Omega}|u(t)||v(t)|^{\rho_2+1}dx\\
&+Ł_2\int_{\Omega}(|v(t)|^{p_1+2}+|v(t)|^{p_2+2}+|v(t)|^{\rho_2+2})dx\\
&+Ł_2\int_{\Omega}|v(t)||u(t)|^{p_1+1}dx+Ł_2\int_{\Omega}|v(t)||u(t)|^{p_2+1}dx+Ł_2\int_{\Omega}|v(t)||u(t)|^{\rho_1+1}dx\\
\leqslant&(Ł_1+\frac{Ł_1}{p_1+2}+\frac{Ł_2(p_1+1)}{p_1+2})\|u(t)\|^{p_1+2}_{L^{p_1+2}(\Omega)}\\
&+(Ł_1+\frac{Ł_1}{p_2+2}+\frac{Ł_2(p_2+1)}{p_2+2})\|u(t)\|^{p_2+2}_{L^{p_2+2}(\Omega)}\\
&+(Ł_1+\frac{Ł_2(\rho_1+1)}{\rho_1+2})\|u(t)\|^{\rho_1+2}_{L^{\rho_1+2}(\Omega)}+\frac{Ł_1(\rho_2+1)}{\rho_2+2}\|u(t)\|^{\rho_2+2}_{L^{\rho_2+2}(\Omega)}\\
&+(Ł_2+\frac{Ł_2}{p_1+2}+\frac{Ł_1(p_1+1)}{p_1+2})\|v(t)\|^{p_1+2}_{L^{p_1+2}(\Omega)}\\
&+(Ł_2+\frac{Ł_2}{p_2+2}+\frac{Ł_1(p_2+1)}{p_2+2})\|v(t)\|^{p_2+2}_{L^{p_2+2}(\Omega)}\\
&+\frac{Ł_2}{\rho_1+2}\|v(t)\|^{\rho_1+2}_{L^{\rho_1+2}(\Omega)}+(Ł_2+\frac{Ł_1(\rho_2+1)}{\rho_2+2})\|v(t)\|^{\rho_2+2}_{L^{\rho_2+2}(\Omega)},\quad t\in\mathbb{R}_+.
\end{aligned}
\tag{93}
$$

With the aid of (74), (75), (76), (77), (78), (79), (80) and (81), we combine (89), (91), (92) and (93), to obtain

$$
\begin{aligned}
L_1'(t)\leqslant&\frac{1}{\rho_1+1}\int_{\Omega}|\partial_t u(t)|^{\rho_1+2}dx+\frac{1}{\rho_2+1}\int_{\Omega}|\partial_t v(t)|^{\rho_2+2}dx\\
&+\int_{\Omega}\nabla^{\top}\partial_t u(t)A_1\nabla\partial_t u(t)dx+\int_{\Omega}\nabla^{\top}\partial_t v(t)A_2\nabla\partial_t v(t)dx\\
&+(\hbar_1-\frac{3}{4})(\mu_1-\int_0^{+\infty}g_1(s)ds)\int_{\Omega}\nabla^{\top}u(t)A_1\nabla u(t)dx\\
&+(\hbar_2-\frac{3}{4})(\mu_2-\int_0^{+\infty}g_2(s)ds)\int_{\Omega}\nabla^{\top}v(t)A_2\nabla v(t)dx\\
&+\frac{1}{\mu_1-\int_0^{+\infty}g_1(s)ds}\int_0^{+\infty}\frac{(g_1(s))^2}{g_1(s)-Gg_1'(s)}ds((g_1-Gg_1')\diamond^{A_1}\nabla u)(t)\\
&+\frac{1}{\mu_2-\int_0^{+\infty}g_2(s)ds}\int_0^{+\infty}\frac{(g_2(s))^2}{g_2(s)-Gg_2'(s)}ds((g_2-Gg_2')\diamond^{A_2}\nabla v)(t),\quad t\in\mathbb{R}_+.
\end{aligned}
$$

This, together with the assumption

$$
\max(\hbar_1,\hbar_2)<\frac{1}{4},
$$

implies that the proof of Lemma 5 is complete. □

As with $L_1(t)$ defined by (85), the following energy perturbation functional associated to each solution pair $(u, v)^\top$ to IBVP (1) is of great importance:

$$L_2(t) = \int_\Omega \big(\operatorname{div}(A_1\nabla\partial_t u(t)) - \frac{|\partial_t u(t)|^{\rho_1}\partial_t u(t)}{\rho_1+1}\big) \int_{-\infty}^{t} g_1(t-s)\big(u(t)-u(s)\big)dsdx \\ + \int_\Omega \big(\operatorname{div}(A_2\nabla\partial_t v(t)) - \frac{|\partial_t v(t)|^{\rho_2}\partial_t v(t)}{\rho_2+1}\big) \int_{-\infty}^{t} g_2(t-s)\big(v(t)-v(s)\big)dsdx. \quad (94)$$

As with what we did in Lemma 5 for $L_1(t)$, we shall establish a useful differential inequality for $L_2(t)$ in the following lemma.

Lemma 6. *Suppose that Assumptions 1–5 hold true. If the associated functional $I(t)$ given by (56) satisfies $I(0) > 0$, the associated constants β_1 and β_2, given by (58) and (59), respectively, satisfy $\max(\beta_1, \beta_2) < 1$, then weak solutions to IBVP (1) exist globally in time, and render the associated functional $L_2(t)$ given by (94) to satisfy*

$$\begin{aligned} L_2'(t) \leqslant{} & \delta \int_0^{+\infty} g_1(s)ds \int_\Omega \nabla^\top u(t) A_1 \nabla u(t)dx \\ & + \delta \int_0^{+\infty} g_2(s)ds \int_\Omega \nabla^\top v(t) A_2 \nabla v(t)dx \\ & - \Big(\int_0^{+\infty} g_1(s)ds - \delta g_1(0)\Big) \int_\Omega \nabla^\top \partial_t u(t) A_1 \nabla \partial_t u(t)dx \\ & - \Big(\int_0^{+\infty} g_2(s)ds - \delta g_2(0)\Big) \int_\Omega \nabla^\top \partial_t v(t) A_2 \nabla \partial_t v(t)dx \\ & - \frac{1}{\rho_1+1} \int_0^{+\infty} g_1(s)ds \int_\Omega |\partial_t u(t)|^{\rho_1+2}dx \\ & - \frac{1}{\rho_2+1} \int_0^{+\infty} g_2(s)ds \int_\Omega |\partial_t v(t)|^{\rho_2+2}dx \\ & + \int_\Omega \begin{pmatrix} \partial_t u(t) \\ \partial_t v(t) \end{pmatrix}^\top \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \partial_t u(t) \\ \partial_t v(t) \end{pmatrix} dx \\ & + \mathscr{W}_{11}(g_1 \diamond^{A_1} \nabla u)(t) + \mathscr{W}_{21}(g_2 \diamond^{A_2} \nabla v)(t) \\ & - \mathscr{W}_{12}(g_1' \diamond^{A_1} \nabla u)(t) - \mathscr{W}_{22}(g_2' \diamond^{A_2} \nabla v)(t), \quad t \in \mathbb{R}_+, \end{aligned} \quad (95)$$

where the positive constant δ is given arbitrarily, the positive constants $\mathscr{W}_{11}$, $\mathscr{W}_{12}$, $\mathscr{W}_{21}$ and $\mathscr{W}_{22}$ are given respectively by

$$\begin{aligned} \mathscr{W}_{11} ={} & \frac{1}{2\delta}\Big(\mu_1 - \int_0^{+\infty} g_1(s)ds\Big)^2 + \int_0^{+\infty} g_1(s)ds + \frac{(\kappa_2)^2 \varkappa_{a_{11},a_{12},a_{21},a_{22}}\zeta_1}{4} \int_0^{+\infty} g_1(s)ds \\ & + \frac{3(\kappa_2 Ł_1)^2\zeta_1}{\delta} \Bigg((\kappa_{2p_1+2})^{2p_1+2}(\zeta_1)^{p_1+1}\Big(\frac{4E^{u,v}(0)}{\mu_1 - \int_0^{+\infty} g_1(s)ds}\Big)^{p_1} + (\kappa_{2p_2+2})^{2p_2+2}(\zeta_1)^{p_2+1}\Big(\frac{4E^{u,v}(0)}{\mu_1 - \int_0^{+\infty} g_1(s)ds}\Big)^{p_2} \\ & + (\kappa_{2\rho_1+2})^{2\rho_1+2}(\zeta_1)^{\rho_1+1}\Big(\frac{4E^{u,v}(0)}{\mu_1 - \int_0^{+\infty} g_1(s)ds}\Big)^{\rho_1} \Bigg), \end{aligned} \quad (96)$$

$$\mathscr{W}_{12} = \frac{1}{2\delta} + \frac{(\kappa_2)^2(\kappa_{2\rho_1+2})^{2\rho_1+2}(\zeta_1)^{\rho_1+2}(4E^{u,v}(0))^{\rho_1}}{2\delta(\rho_1+1)^2}, \quad (97)$$

$$
\begin{aligned}
\mathscr{W}_{21} = & \frac{1}{2\delta}(\mu_2 - \int_0^{+\infty} g_2(s)ds)^2 + \int_0^{+\infty} g_2(s)ds + \frac{(\kappa_2)^2 \varkappa_{a_{11},a_{12},a_{21},a_{22}}\zeta_2}{4}\int_0^{+\infty} g_2(s)ds \\
& + \frac{3(\kappa_2 Ł_2)^2\zeta_2}{\delta}\Bigg((\kappa_{2p_1+2})^{2p_1+2}(\zeta_2)^{p_1+1}(\frac{4E^{u,v}(0)}{\mu_2 - \int_0^{+\infty} g_2(s)ds})^{p_1} + (\kappa_{2p_2+2})^{2p_2+2}(\zeta_2)^{p_2+1}(\frac{4E^{u,v}(0)}{\mu_2 - \int_0^{+\infty} g_2(s)ds})^{p_2} \\
& + (\kappa_{2\rho_2+2})^{2\rho_2+2}(\zeta_2)^{\rho_2+1}(\frac{4E^{u,v}(0)}{\mu_2 - \int_0^{+\infty} g_2(s)ds})^{\rho_2}\Bigg)
\end{aligned} \tag{98}
$$

and

$$
\mathscr{W}_{22} = \frac{1}{2\delta} + \frac{(\kappa_2)^2(\kappa_{2\rho_2+2})^{2\rho_2+2}(\zeta_2)^{\rho_2+2}(4E^{u,v}(0))^{\rho_2}}{2\delta(\rho_2+1)^2}. \tag{99}
$$

Proof. As pointed in Lemma 6, in view of the assumptions $I(0) > 0$ and $\max(\beta_1, \beta_2) < 1$, we conclude by Theorem 3 that $L_2(t)$ exists globally in time. Differentiate both sides of the Equation (94), to yield

$$
\begin{aligned}
L_2'(t) = & \mu_1 \int_{-\infty}^t g_1(t-s)\int_\Omega \nabla^\top u(t) A_1 \nabla(u(t)-u(s))dxds \\
& - \int_\Omega \int_{-\infty}^t g_1(t-s)\nabla^\top u(s)ds A_1 \int_{-\infty}^t g_1(t-s)\nabla(u(t)-u(s))dsdx \\
& - \int_{-\infty}^t g_1'(t-s)\int_\Omega \nabla^\top \partial_t u(t) A_1 \nabla(u(t)-u(s))dxds \\
& - \int_{-\infty}^t g_1'(t-s)\int_\Omega \frac{|\partial_t u(t)|^{\rho_1}\partial_t u(t)}{\rho_1+1}(u(t)-u(s))dxds - \int_{-\infty}^t g_1(t-s)\int_\Omega \nabla^\top \partial_t u(t) A_1 \nabla \partial_t u(t) dxds \\
& - \frac{1}{\rho_1+1}\int_0^{+\infty} g_1(s)ds \int_\Omega |\partial_t u(t)|^{\rho_1+2}dx + \mu_2 \int_{-\infty}^t g_2(t-s)\int_\Omega \nabla^\top v(t) A_2 \nabla(v(t)-v(s))dxds \\
& - \int_\Omega \int_{-\infty}^t g_2(t-s)\nabla^\top v(s)ds A_2 \int_{-\infty}^t g_2(t-s)\nabla(v(t)-v(s))dsdx \\
& - \int_{-\infty}^t g_2'(t-s)\int_\Omega \nabla^\top \partial_t v(t) A_2 \nabla(v(t)-v(s))dxds - \int_{-\infty}^t g_2'(t-s)\int_\Omega \frac{|\partial_t v(t)|^{\rho_2}\partial_t v(t)}{\rho_2+1}(v(t)-v(s))dxds \\
& - \int_{-\infty}^t g_2(t-s)\int_\Omega \nabla^\top \partial_t v(t) A_2 \nabla \partial_t v(t) dxds - \frac{1}{\rho_2+1}\int_0^{+\infty} g_2(s)ds \int_\Omega |\partial_t v(t)|^{\rho_2+2}dx \\
& - \int_\Omega f_1(u(t),v(t))\int_{-\infty}^t g_1(t-s)(u(t)-u(s))dsdx - \int_\Omega f_2(u(t),v(t))\int_{-\infty}^t g_2(t-s)(v(t)-v(s))dsdx \\
& + \int_\Omega (a_{11}\partial_t u(t) + a_{12}\partial_t v(t))\int_{-\infty}^t g_1(t-s)(u(t)-u(s))dsdx \\
& + \int_\Omega (a_{21}\partial_t u(t) + a_{22}\partial_t v(t))\int_{-\infty}^t g_2(t-s)(v(t)-v(s))dsdx, \quad t \in \mathbb{R}_+.
\end{aligned} \tag{100}
$$

We shall split (100) into several parts, and we shall treat each part separately. By some routine calculations, it is not difficult to find that

$$
\begin{aligned}
& \mu_1 \int_{-\infty}^t g_1(t-s)\int_\Omega \nabla^\top u(t) A_1 \nabla(u(t)-u(s))dxds \\
& - \int_\Omega \int_{-\infty}^t g_1(t-s)\nabla^\top u(s)ds A_1 \int_{-\infty}^t g_1(t-s)\nabla(u(t)-u(s))dsdx \\
= & \int_\Omega \int_{-\infty}^t g_1(t-s)\nabla^\top (u(t)-u(s))ds A_1 \int_{-\infty}^t g_1(t-s)\nabla(u(t)-u(s))dsdx \\
& + (\mu_1 - \int_0^{+\infty} g_1(s)ds)\int_{-\infty}^t g_1(t-s)\int_\Omega \nabla^\top u(t) A_1 \nabla(u(t)-u(s))dxds, \quad t \in \mathbb{R}_+.
\end{aligned} \tag{101}
$$

But by exploiting the positive definiteness of A_1 (see Assumption 1, especially (5), for the details) and by mainly utilizing the Cauchy–Schwarz inequality, we have

$$\varphi^\top A_1(x)\psi \leqslant \frac{\delta}{2(\mu_1 - \int_0^{+\infty} g_1(s)ds)}\varphi^\top A_1(x)\varphi + \frac{1}{2\delta}(\mu_1 - \int_0^{+\infty} g_1(s)ds)\psi^\top A_1(x)\psi, \quad \varphi, \psi \in \mathbb{R}^N,\ x \in \Omega.$$

This implies immediately

$$\begin{aligned}\int_{-\infty}^t g_1(t-s)\int_\Omega \nabla^\top u(t)A_1\nabla(u(t)-u(s))dxds \leqslant& \frac{\delta\int_0^{+\infty} g_1(s)ds}{2(\mu_1 - \int_0^{+\infty} g_1(s)ds)}\int_\Omega \nabla^\top u(t)A_1\nabla u(t)dx \\ &+ \frac{1}{2\delta}(\mu_1 - \int_0^{+\infty} g_1(s)ds)(g_1 \diamond^{A_1} \nabla u)(t), \quad t \in \mathbb{R}_+. \end{aligned} \tag{102}$$

Besides, by using again the positive definiteness of A_1, as in (102), and by employing Jensen's inequality, we have

$$\begin{aligned}&\int_{-\infty}^t g_1(t-s)(\boldsymbol{U}(s))^\top ds A_1 \int_{-\infty}^t g_1(t-s)\boldsymbol{U}(s)ds \\ \leqslant& \int_{-\infty}^t g_1(t-s)ds \int_{-\infty}^t g_1(t-s)(\boldsymbol{U}(s))^\top A_1 \boldsymbol{U}(s)ds, \quad \boldsymbol{U} \in L^2(-\infty, t; \mathbb{R}^N), \quad t \in \mathbb{R}_+,\end{aligned}$$

which implies directly

$$\begin{aligned}&\int_\Omega \int_{-\infty}^t g_1(t-s)\nabla^\top(u(t)-u(s))ds A_1 \int_{-\infty}^t g_1(t-s)\nabla(u(t)-u(s))dsdx \\ \leqslant& \int_\Omega \int_{-\infty}^t g_1(t-s)ds \int_{-\infty}^t g_1(t-s)\nabla^\top(u(t)-u(s))A_1\nabla(u(t)-u(s))dsdx \\ =& \int_0^{+\infty} g_1(s)ds(g_1 \diamond^{A_1} \nabla u)(t), \quad t \in \mathbb{R}_+.\end{aligned} \tag{103}$$

Plug (102) and (103) into (101), to obtain

$$\begin{aligned}&\mu_1 \int_{-\infty}^t g_1(t-s)\int_\Omega \nabla^\top u(t)A_1\nabla(u(t)-u(s))dxds \\ &- \int_\Omega \int_{-\infty}^t g_1(t-s)\nabla^\top u(s)ds A_1 \int_{-\infty}^t g_1(t-s)\nabla(u(t)-u(s))dsdx \\ \leqslant& \frac{\delta}{2}\int_0^{+\infty} g_1(s)ds \int_\Omega \nabla^\top u(t)A_1\nabla u(t)dx + \mathscr{M}_{11}^7(g_1 \diamond^{A_1} \nabla u)(t), \quad t \in \mathbb{R}_+,\end{aligned} \tag{104}$$

with the positive constant $\mathscr{M}_{11}^7$ given by

$$\mathscr{M}_{11}^7 = \frac{1}{2\delta}(\mu_1 - \int_0^{+\infty} g_1(s)ds)^2 + \int_0^{+\infty} g_1(s)ds. \tag{105}$$

And similarly, we have also

$$\begin{aligned}&\mu_2 \int_{-\infty}^t g_2(t-s)\int_\Omega \nabla^\top v(t)A_2\nabla(v(t)-v(s))dxds \\ &- \int_\Omega \int_{-\infty}^t g_2(t-s)\nabla^\top v(s)ds A_2 \int_{-\infty}^t g_2(t-s)\nabla(v(t)-v(s))dsdx \\ \leqslant& \frac{\delta}{2}\int_0^{+\infty} g_2(s)ds \int_\Omega \nabla^\top v(t)A_2\nabla v(t)dx + \mathscr{M}_{21}^7(g_2 \diamond^{A_2} \nabla v)(t), \quad t \in \mathbb{R}_+,\end{aligned} \tag{106}$$

where the positive constant $\mathscr{M}_{21}^7$ is given by

$$\mathscr{M}_{21}^7 = \frac{1}{2\delta}(\mu_2 - \int_0^{+\infty} g_2(s)ds)^2 + \int_0^{+\infty} g_2(s)ds. \tag{107}$$

Thanks to the positive definiteness of A_1 (see Assumption 1, especially (5)), taking steps similar to those in the derivation of (102), we can prove

$$\begin{aligned} & -\int_{-\infty}^{t} g_1'(t-s)\int_{\Omega}\nabla^{\top}\partial_t u(t)A_1\nabla(u(t)-u(s))dxds \\ \leqslant & \frac{\delta g_1(0)}{2}\int_{\Omega}\nabla^{\top}\partial_t u(t)A_1\nabla\partial_t u(t)dx-\frac{1}{2\delta}(g_1'\diamond^{A_1}\nabla u)(t), \quad t\in\mathbb{R}_+, \end{aligned} \tag{108}$$

and

$$\begin{aligned} & -\int_{-\infty}^{t} g_2'(t-s)\int_{\Omega}\nabla^{\top}\partial_t v(t)A_2\nabla(v(t)-v(s))dxds \\ \leqslant & \frac{\delta g_2(0)}{2}\int_{\Omega}\nabla^{\top}\partial_t v(t)A_2\nabla\partial_t v(t)dx-\frac{1}{2\delta}(g_2'\diamond^{A_2}\nabla v)(t), \quad t\in\mathbb{R}_+. \end{aligned} \tag{109}$$

Apply the Fenchel–Young inequality, use the notations introduced in (9) and (10), and conduct some routine calculations, to obtain

$$\begin{aligned} & -\int_{-\infty}^{t} g_1'(t-s)\int_{\Omega}\frac{|\partial_t u(t)|^{\rho_1}\partial_t u(t)}{\rho_1+1}(u(t)-u(s))dxds \\ \leqslant & -\int_{-\infty}^{t} g_1'(t-s)\int_{\Omega}\Bigg(\frac{\delta}{2(\kappa_{2\rho_1+2})^{2\rho_1+2}(\zeta_1)^{\rho_1+1}(4E^{u,v}(0))^{\rho_1}}|\partial_t u(t)|^{2\rho_1+2} \\ & \qquad +\frac{(\kappa_{2\rho_1+2})^{2\rho_1+2}(\zeta_1)^{\rho_1+1}(4E^{u,v}(0))^{\rho_1}}{2\delta(\rho_1+1)^2}|u(t)-u(s)|^2\Bigg)dxds \\ = & \frac{\delta g_1(0)}{2(\kappa_{2\rho_1+2})^{2\rho_1+2}(\zeta_1)^{\rho_1+1}(4E^{u,v}(0))^{\rho_1}}\|\partial_t u(t)\|^{2\rho_1+2}_{L^{2\rho_1+2}(\Omega)} \\ & -\frac{(\kappa_{2\rho_1+2})^{2\rho_1+2}(\zeta_1)^{\rho_1+1}(4E^{u,v}(0))^{\rho_1}}{2\delta(\rho_1+1)^2}\int_{-\infty}^{t} g_1'(t-s)\int_{\Omega}|u(t)-u(s)|^2dxds \\ \leqslant & \frac{\delta g_1(0)}{2(\kappa_{2\rho_1+2})^{2\rho_1+2}(\zeta_1)^{\rho_1+1}(4E^{u,v}(0))^{\rho_1}}\|\partial_t u(t)\|^{2\rho_1+2}_{L^{2\rho_1+2}(\Omega)}-\mathscr{M}_{12}^{7}(g_1'\diamond^{A_1}\nabla u)(t), \quad t\in\mathbb{R}_+, \end{aligned} \tag{110}$$

with the positive constant $\mathscr{M}_{12}^{7}$ given by

$$\mathscr{M}_{12}^{7}=\frac{(\kappa_2)^2(\kappa_{2\rho_1+2})^{2\rho_1+2}(\zeta_1)^{\rho_1+2}(4E^{u,v}(0))^{\rho_1}}{2\delta(\rho_1+1)^2}. \tag{111}$$

To proceed further, we need the help of the following inequality

$$\begin{aligned} & \int_{\Omega}\nabla^{\top}\partial_t u(t)A_1\nabla\partial_t u(t)dx+\int_{\Omega}\nabla^{\top}\partial_t v(t)A_2\nabla\partial_t v(t)dx \\ \leqslant & 4I(t)+(\mu_1-\int_0^{+\infty}g_1(s)ds)\int_{\Omega}\nabla^{\top}u(t)A_1\nabla u(t)dx+\int_{\Omega}\nabla^{\top}\partial_t u(t)A_1\nabla\partial_t u(t)dx+(g_1\diamond^{A_1}\nabla u)(t) \\ & +(\mu_2-\int_0^{+\infty}g_2(s)ds)\int_{\Omega}\nabla^{\top}v(t)A_2\nabla v(t)dx+\int_{\Omega}\nabla^{\top}\partial_t v(t)A_2\nabla\partial_t v(t)dx+(g_2\diamond^{A_2}\nabla v)(t) \\ = & 4J(t)\leqslant 4E^{u,v}(0), \quad t\in\mathbb{R}_+, \end{aligned} \tag{112}$$

which can be proved via applying Lemma 4 and using (56) as well as (57). With the help of (112), we can prove easily

$$\begin{aligned} \|\partial_t u(t)\|^{2\rho_1+2}_{L^{2\rho_1+2}(\Omega)} & \leqslant (\kappa_{2\rho_1+2})^{2\rho_1+2}(\zeta_1)^{\rho_1+1}(\int_{\Omega}\nabla^{\top}\partial_t u(t)A_1\nabla\partial_t u(t)dx)^{\rho_1+1} \\ & \leqslant (\kappa_{2\rho_1+2})^{2\rho_1+2}(\zeta_1)^{\rho_1+1}(4E^{u,v}(0))^{\rho_1}\int_{\Omega}\nabla^{\top}\partial_t u(t)A_1\nabla\partial_t u(t)dx. \end{aligned} \tag{113}$$

This, together with (110), implies

$$
\begin{aligned}
&-\int_{-\infty}^{t} g_1'(t-s)\int_{\Omega}\frac{|\partial_t u(t)|^{\rho_1}\partial_t u(t)}{\rho_1+1}(u(t)-u(s))dxds\\
&\leqslant \frac{\delta g_1(0)}{2}\int_{\Omega}\nabla^{\top}\partial_t u(t)A_1\nabla\partial_t u(t)dx-\mathscr{M}_{12}^{7}(g_1'\diamond^{A_1}\nabla u)(t),\quad t\in\mathbb{R}_+.
\end{aligned}
\tag{114}
$$

By applying (112), we take steps similar to those used to obtain (113), to arrive at

$$
\|\partial_t v(t)\|_{L^{2\rho_2+2}(\Omega)}^{2\rho_2+2}\leqslant(\kappa_{2\rho_2+2})^{2\rho_2+2}(\zeta_2)^{\rho_2+1}(4E^{u,v}(0))^{\rho_2}\int_{\Omega}\nabla^{\top}\partial_t v(t)A_2\nabla\partial_t v(t)dx.
$$

With this at our hand, we can use the idea similar to the one utilized to establish the inequality (114), to prove successfully that

$$
\begin{aligned}
&-\int_{-\infty}^{t} g_2'(t-s)\int_{\Omega}\frac{|\partial_t v(t)|^{\rho_2}\partial_t v(t)}{\rho_2+1}(v(t)-v(s))dxds\\
&\leqslant\frac{\delta g_2(0)}{2}\int_{\Omega}\nabla^{\top}\partial_t v(t)A_2\nabla\partial_t v(t)dx-\mathscr{M}_{22}^{7}(g_2'\diamond^{A_2}\nabla v)(t),\quad t\in\mathbb{R}_+,
\end{aligned}
\tag{115}
$$

where the positive constant $\mathscr{M}_{22}^{7}$ is given by

$$
\mathscr{M}_{22}^{7}=\frac{(\kappa_2)^2(\kappa_{2\rho_2+2})^{2\rho_2+2}(\zeta_2)^{\rho_2+2}(4E^{u,v}(0))^{\rho_2}}{2\delta(\rho_2+1)^2}.
\tag{116}
$$

With the help of Assumption 5 and the notations in (9), (10) and (12), by mainly exploiting the Cauchy–Schwarz inequality and Jensen's inequality, we have

$$
\begin{aligned}
&\int_{\Omega}(a_{11}\partial_t u(t)+a_{12}\partial_t v(t))\int_{-\infty}^{t} g_1(t-s)(u(t)-u(s))dsdx\\
&+\int_{\Omega}(a_{21}\partial_t u(t)+a_{22}\partial_t v(t))\int_{-\infty}^{t} g_2(t-s)(v(t)-v(s))dsdx\\
=&\int_{\Omega}\begin{pmatrix}\partial_t u(t)\\ \partial_t v(t)\end{pmatrix}^{\top}\begin{pmatrix}a_{11}&a_{12}\\ a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}\int_{-\infty}^{t} g_1(t-s)(u(t)-u(s))ds\\ \int_{-\infty}^{t} g_2(t-s)(v(t)-v(s))ds\end{pmatrix}dx\\
\leqslant&\int_{\Omega}\begin{pmatrix}\partial_t u(t)\\ \partial_t v(t)\end{pmatrix}^{\top}\begin{pmatrix}a_{11}&a_{12}\\ a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}\partial_t u(t)\\ \partial_t v(t)\end{pmatrix}dx\\
&+\frac{1}{4}\int_{\Omega}\begin{pmatrix}\int_{-\infty}^{t} g_1(t-s)(u(t)-u(s))ds\\ \int_{-\infty}^{t} g_2(t-s)(v(t)-v(s))ds\end{pmatrix}^{\top}\begin{pmatrix}a_{11}&a_{12}\\ a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}\int_{-\infty}^{t} g_1(t-s)(u(t)-u(s))ds\\ \int_{-\infty}^{t} g_2(t-s)(v(t)-v(s))ds\end{pmatrix}dx\\
\leqslant&\int_{\Omega}\begin{pmatrix}\partial_t u(t)\\ \partial_t v(t)\end{pmatrix}^{\top}\begin{pmatrix}a_{11}&a_{12}\\ a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}\partial_t u(t)\\ \partial_t v(t)\end{pmatrix}dx\\
&+\frac{\varkappa_{a_{11},a_{12},a_{21},a_{22}}}{4}\int_{\Omega}\Big[\Big(\int_{-\infty}^{t} g_1(t-s)(u(t)-u(s))ds\Big)^2+\Big(\int_{-\infty}^{t} g_2(t-s)(v(t)-v(s))ds\Big)^2\Big]dx\\
\leqslant&\int_{\Omega}\begin{pmatrix}\partial_t u(t)\\ \partial_t v(t)\end{pmatrix}^{\top}\begin{pmatrix}a_{11}&a_{12}\\ a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}\partial_t u(t)\\ \partial_t v(t)\end{pmatrix}dx\\
&+\frac{\varkappa_{a_{11},a_{12},a_{21},a_{22}}}{4}\int_{\Omega}\Big[\int_{-\infty}^{t} g_1(t-s)ds\int_{-\infty}^{t} g_1(t-s)|u(t)-u(s)|^2ds\\
&\quad+\int_{-\infty}^{t} g_2(t-s)ds\int_{-\infty}^{t} g_2(t-s)|v(t)-v(s)|^2ds\Big]dx\\
\leqslant&\int_{\Omega}\begin{pmatrix}\partial_t u(t)\\ \partial_t v(t)\end{pmatrix}^{\top}\begin{pmatrix}a_{11}&a_{12}\\ a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}\partial_t u(t)\\ \partial_t v(t)\end{pmatrix}dx+\mathscr{M}_{13}^{7}(g_1\diamond^{A_1}\nabla u)(t)+\mathscr{M}_{23}^{7}(g_2\diamond^{A_2}\nabla v)(t),\quad t\in\mathbb{R}_+,
\end{aligned}
\tag{117}
$$

where the positive constants $\mathscr{M}_{13}^{7}$ and $\mathscr{M}_{23}^{7}$ are given respectively by

$$\mathscr{M}_{13}^{7} = \frac{(\kappa_2)^2 \varkappa_{a_{11},a_{12},a_{21},a_{22}} \zeta_1}{4} \int_0^{+\infty} g_1(s)ds, \tag{118}$$

and

$$\mathscr{M}_{23}^{7} = \frac{(\kappa_2)^2 \varkappa_{a_{11},a_{12},a_{21},a_{22}} \zeta_2}{4} \int_0^{+\infty} g_2(s)ds. \tag{119}$$

With the growth condition (8) in Assumption 4 at our hand, we can prove via applying the Cauchy–Schwarz inequality and using the notations in (9) and (10) that

$$\begin{aligned}
&-\int_\Omega f_1(u(t),v(t)) \int_{-\infty}^t g_1(t-s)(u(t)-u(s))dsdx \\
&-\int_\Omega f_2(u(t),v(t)) \int_{-\infty}^t g_2(t-s)(v(t)-v(s))dsdx \\
\leqslant{}& \frac{\delta_1}{2} \int_{-\infty}^t g_1(t-s) \int_\Omega |f_1(u(t),v(t))|^2 dxds + \frac{1}{2\delta_1} \int_{-\infty}^t g_1(t-s) \int_\Omega |u(t)-u(s)|^2 dxds \\
&+ \frac{\delta_2}{2} \int_{-\infty}^t g_2(t-s) \int_\Omega |f_2(u(t),v(t))|^2 dxds + \frac{1}{2\delta_2} \int_{-\infty}^t g_2(t-s) \int_\Omega |v(t)-v(s)|^2 dxds \\
\leqslant{}& 3\delta_1 (Ł_1)^2 \int_{-\infty}^t g_1(t-s) \Big(\|u(t)\|_{L^{2p_1+2}}^{2p_1+2} + \|u(t)\|_{L^{2p_2+2}}^{2p_2+2} + \|u(t)\|_{L^{2\rho_1+2}}^{2\rho_1+2} \\
&+ \|v(t)\|_{L^{2p_1+2}}^{2p_1+2} + \|v(t)\|_{L^{2p_2+2}}^{2p_2+2} + \|v(t)\|_{L^{2\rho_2+2}}^{2\rho_2+2} \Big) ds + \frac{(\kappa_2)^2 \zeta_1}{2\delta_1} (g_1 \diamond^{A_1} \nabla u)(t) \\
&+ 3\delta_2 (Ł_2)^2 \int_{-\infty}^t g_2(t-s) \Big(\|u(t)\|_{L^{2p_1+2}}^{2p_1+2} + \|u(t)\|_{L^{2p_2+2}}^{2p_2+2} + \|u(t)\|_{L^{2\rho_1+2}}^{2\rho_1+2} \\
&+ \|v(t)\|_{L^{2p_1+2}}^{2p_1+2} + \|v(t)\|_{L^{2p_2+2}}^{2p_2+2} + \|v(t)\|_{L^{2\rho_2+2}}^{2\rho_2+2} \Big) ds + \frac{(\kappa_2)^2 \zeta_2}{2\delta_2} (g_2 \diamond^{A_2} \nabla u)(t) \\
={}& 3\delta_1 (Ł_1)^2 \int_0^{+\infty} g_1(s) ds \Big(\|u(t)\|_{L^{2p_1+2}}^{2p_1+2} + \|u(t)\|_{L^{2p_2+2}}^{2p_2+2} + \|u(t)\|_{L^{2\rho_1+2}}^{2\rho_1+2} \\
&+ \|v(t)\|_{L^{2p_1+2}}^{2p_1+2} + \|v(t)\|_{L^{2p_2+2}}^{2p_2+2} + \|v(t)\|_{L^{2\rho_2+2}}^{2\rho_2+2} \Big) + \frac{(\kappa_2)^2 \zeta_1}{2\delta_1} (g_1 \diamond^{A_1} \nabla u)(t) \\
&+ 3\delta_2 (Ł_2)^2 \int_0^{+\infty} g_2(s) ds \Big(\|u(t)\|_{L^{2p_1+2}}^{2p_1+2} + \|u(t)\|_{L^{2p_2+2}}^{2p_2+2} + \|u(t)\|_{L^{2\rho_1+2}}^{2\rho_1+2} \\
&+ \|v(t)\|_{L^{2p_1+2}}^{2p_1+2} + \|v(t)\|_{L^{2p_2+2}}^{2p_2+2} + \|v(t)\|_{L^{2\rho_2+2}}^{2\rho_2+2} \Big) + \frac{(\kappa_2)^2 \zeta_2}{2\delta_2} (g_2 \diamond^{A_2} \nabla u)(t), \quad t \in \mathbb{R}_+.
\end{aligned} \tag{120}$$

In view of the assumptions $I(0) > 0$ and $\max(\beta_1, \beta_2) < 1$, we conclude by Theorem 3 the inequality (72) holds true in $\mathbb{R}_+$. This, together with the notations in (9) and (10), implies

$$\begin{aligned}
\|u(t)\|_{L^{2p_1+2}}^{2p_1+2} &\leqslant (\kappa_{2p_1+2})^{2p_1+2} (\zeta_1)^{p_1+1} \Big(\int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx \Big)^{p_1+1} \\
&\leqslant (\kappa_{2p_1+2})^{2p_1+2} (\zeta_1)^{p_1+1} \Big(\frac{4E^{u,v}(0)}{\mu_1 - \int_0^{+\infty} g_1(s)ds} \Big)^{p_1} \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx,
\end{aligned} \tag{121}$$

$$\|u(t)\|_{L^{2p_2+2}}^{2p_2+2} \leqslant (\kappa_{2p_2+2})^{2p_2+2} (\zeta_1)^{p_2+1} \Big(\frac{4E^{u,v}(0)}{\mu_1 - \int_0^{+\infty} g_1(s)ds} \Big)^{p_2} \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx, \tag{122}$$

$$\|u(t)\|_{L^{2\rho_1+2}}^{2\rho_1+2} \leqslant (\kappa_{2\rho_1+2})^{2\rho_1+2} (\zeta_1)^{\rho_1+1} \Big(\frac{4E^{u,v}(0)}{\mu_1 - \int_0^{+\infty} g_1(s)ds} \Big)^{\rho_1} \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx, \tag{123}$$

$$\|v(t)\|_{L^{2p_1+2}}^{2p_1+2} \leqslant (\kappa_{2p_1+2})^{2p_1+2} (\zeta_2)^{p_1+1} \Big(\frac{4E^{u,v}(0)}{\mu_2 - \int_0^{+\infty} g_2(s)ds} \Big)^{p_1} \int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx, \tag{124}$$

$$\|v(t)\|_{L^{2p_2+2}}^{2p_2+2} \leqslant (\kappa_{2p_2+2})^{2p_2+2}(\zeta_2)^{p_2+1}(\frac{4E^{u,v}(0)}{\mu_2 - \int_0^{+\infty} g_2(s)ds})^{p_2} \int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx, \quad (125)$$

and

$$\|v(t)\|_{L^{2\rho_2+2}}^{2\rho_2+2} \leqslant (\kappa_{2\rho_2+2})^{2\rho_2+2}(\zeta_2)^{\rho_2+1}(\frac{4E^{u,v}(0)}{\mu_2 - \int_0^{+\infty} g_2(s)ds})^{\rho_2} \int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx. \quad (126)$$

Plug (121), (122), (123), (124), (125) and (126) into (120), to get

$$\begin{aligned}
&- \int_\Omega f_1(u(t),v(t)) \int_{-\infty}^t g_1(t-s)(u(t)-u(s))dsdx \\
&- \int_\Omega f_2(u(t),v(t)) \int_{-\infty}^t g_2(t-s)(v(t)-v(s))dsdx \\
=& 3\delta_1(\text{Ł}_1)^2 \mathscr{M}_{14}^7 \int_0^{+\infty} g_1(s)ds \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx + \frac{(\kappa_2)^2\zeta_1}{2\delta_1}(g_1 \diamond^{A_1} \nabla u)(t) \\
&+ 3\delta_2(\text{Ł}_2)^2 \mathscr{M}_{24}^7 \int_0^{+\infty} g_2(s)ds \int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx \\
&+ \frac{(\kappa_2)^2\zeta_2}{2\delta_2}(g_2 \diamond^{A_2} \nabla u)(t), \quad t \in \mathbb{R}_+,
\end{aligned}$$

where the positive constant δ_i is given in an arbitrary way, the positive constants $\mathscr{M}_{14}^7$ and $\mathscr{M}_{24}^7$ are given respectively by

$$\begin{aligned}
\mathscr{M}_{14}^7 =& (\kappa_{2p_1+2})^{2p_1+2}(\zeta_1)^{p_1+1}(\frac{4E^{u,v}(0)}{\mu_1 - \int_0^{+\infty} g_1(s)ds})^{p_1} \\
&+ (\kappa_{2p_2+2})^{2p_2+2}(\zeta_1)^{p_2+1}(\frac{4E^{u,v}(0)}{\mu_1 - \int_0^{+\infty} g_1(s)ds})^{p_2} \\
&+ (\kappa_{2\rho_1+2})^{2\rho_1+2}(\zeta_1)^{\rho_1+1}(\frac{4E^{u,v}(0)}{\mu_1 - \int_0^{+\infty} g_1(s)ds})^{\rho_1}, \quad (127)
\end{aligned}$$

and

$$\begin{aligned}
\mathscr{M}_{24}^7 =& (\kappa_{2p_1+2})^{2p_1+2}(\zeta_2)^{p_1+1}(\frac{4E^{u,v}(0)}{\mu_2 - \int_0^{+\infty} g_2(s)ds})^{p_1} \\
&+ (\kappa_{2p_2+2})^{2p_2+2}(\zeta_2)^{p_2+1}(\frac{4E^{u,v}(0)}{\mu_2 - \int_0^{+\infty} g_2(s)ds})^{p_2} \\
&+ (\kappa_{2\rho_2+2})^{2\rho_2+2}(\zeta_2)^{\rho_2+1}(\frac{4E^{u,v}(0)}{\mu_2 - \int_0^{+\infty} g_2(s)ds})^{\rho_2}. \quad (128)
\end{aligned}$$

Pick δ_1 and δ_2 so that

$$3\delta_1(\text{Ł}_1)^2 \mathscr{M}_{14}^7 = 3\delta_2(\text{Ł}_2)^2 \mathscr{M}_{24}^7 = \frac{\delta}{2},$$

to yield

$$\begin{aligned}
&- \int_\Omega f_1(u(t),v(t)) \int_{-\infty}^t g_1(t-s)(u(t)-u(s))dsdx \\
&- \int_\Omega f_2(u(t),v(t)) \int_{-\infty}^t g_2(t-s)(v(t)-v(s))dsdx \\
\leqslant& \frac{\delta}{2} \int_0^{+\infty} g_1(s)ds \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx + \frac{\delta}{2} \int_0^{+\infty} g_2(s)ds \int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx \\
&+ \frac{3(\kappa_2\text{Ł}_1)^2\zeta_1\mathscr{M}_{14}^7}{\delta}(g_1 \diamond^{A_1} \nabla u)(t) + \frac{3(\kappa_2\text{Ł}_2)^2\zeta_2\mathscr{M}_{24}^7}{\delta}(g_2 \diamond^{A_2} \nabla v)(t), \quad t \in \mathbb{R}_+. \quad (129)
\end{aligned}$$

Plug (104), (106), (108), (109), (114), (115), (117) and (129) into (100) and perform some routine but tedious calculations, to finish the proof of Lemma 6. □

Remark 7. *In some occasions, we are inclined to establish an inequality similar to (117) without the term*

$$\int_\Omega \begin{pmatrix} \partial_t u(t) \\ \partial_t v(t) \end{pmatrix}^\top \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \partial_t u(t) \\ \partial_t v(t) \end{pmatrix} dx. \tag{130}$$

To this end, we actually have several different approaches. For example, by using the Cauchy–Schwarz inequalities (in several different forms), we have

$$\begin{aligned}
&\int_\Omega (a_{11}\partial_t u(t) + a_{12}\partial_t v(t)) \int_{-\infty}^t g_1(t-s)(u(t)-u(s))dsdx \\
&\quad + \int_\Omega (a_{21}\partial_t u(t) + a_{22}\partial_t v(t)) \int_{-\infty}^t g_2(t-s)(v(t)-v(s))dsdx \\
&= \int_\Omega \begin{pmatrix} \partial_t u(t) \\ \partial_t v(t) \end{pmatrix}^\top \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \int_{-\infty}^t g_1(t-s)(u(t)-u(s))ds \\ \int_{-\infty}^t g_2(t-s)(v(t)-v(s))ds \end{pmatrix} dx \\
&\leqslant \frac{1}{2(\kappa_2)^2} \int_\Omega \begin{pmatrix} \partial_t u(t) \\ \partial_t v(t) \end{pmatrix}^\top \begin{pmatrix} \frac{\int_0^{+\infty} g_1(s)ds - \delta g_1(0)}{\zeta_1} & 0 \\ 0 & \frac{\int_0^{+\infty} g_2(s)ds - \delta g_2(0)}{\zeta_2} \end{pmatrix} \begin{pmatrix} \partial_t u(t) \\ \partial_t v(t) \end{pmatrix} dx \\
&\quad + \frac{(\kappa_2)^2}{2} \int_\Omega \begin{pmatrix} \int_{-\infty}^t g_1(t-s)(u(t)-u(s))ds \\ \int_{-\infty}^t g_2(t-s)(v(t)-v(s))ds \end{pmatrix}^\top \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^\top \\
&\quad \cdot \begin{pmatrix} \frac{\zeta_1}{\int_0^{+\infty} g_1(s)ds - \delta g_1(0)} & 0 \\ 0 & \frac{\zeta_2}{\int_0^{+\infty} g_2(s)ds - \delta g_2(0)} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \int_{-\infty}^t g_1(t-s)(u(t)-u(s))ds \\ \int_{-\infty}^t g_2(t-s)(v(t)-v(s))ds \end{pmatrix} dx \\
&\leqslant \frac{1}{2}\Big(\int_0^{+\infty} g_1(s)ds - \delta g_1(0)\Big) \int_\Omega \nabla^\top \partial_t u(t) A_1 \nabla \partial_t u(t) dx \\
&\quad + \frac{1}{2}\Big(\int_0^{+\infty} g_2(s)ds - \delta g_2(0)\Big) \int_\Omega \nabla^\top \partial_t v(t) A_2 \nabla \partial_t v(t) dx \\
&\quad + \frac{(\kappa_2)^2 \varkappa_{a_{11},a_{12},a_{21},a_{22}}}{8} \max\Big(\frac{\zeta_1}{\int_0^{+\infty} g_1(s)ds - \delta g_1(0)}, \frac{\zeta_2}{\int_0^{+\infty} g_2(s)ds - \delta g_2(0)}\Big) \\
&\quad \cdot \int_\Omega \Big[\Big(\int_{-\infty}^t g_1(t-s)(u(t)-u(s))ds\Big)^2 + \Big(\int_{-\infty}^t g_2(t-s)(v(t)-v(s))ds\Big)^2 \Big] dx \\
&\leqslant \frac{1}{2}\Big(\int_0^{+\infty} g_1(s)ds - \delta g_1(0)\Big) \int_\Omega \nabla^\top \partial_t u(t) A_1 \nabla \partial_t u(t) dx \\
&\quad + \frac{1}{2}\Big(\int_0^{+\infty} g_2(s)ds - \delta g_2(0)\Big) \int_\Omega \nabla^\top \partial_t v(t) A_2 \nabla \partial_t v(t) dx \\
&\quad + \frac{(\kappa_2)^2 \varkappa_{a_{11},a_{12},a_{21},a_{22}}}{8} \max\Big(\frac{\zeta_1}{\int_0^{+\infty} g_1(s)ds - \delta g_1(0)}, \frac{\zeta_2}{\int_0^{+\infty} g_2(s)ds - \delta g_2(0)}\Big) \\
&\quad \int_\Omega \Big[\int_{-\infty}^t g_1(t-s)ds \int_{-\infty}^t g_1(t-s)|u(t)-u(s)|^2 ds \\
&\qquad + \int_{-\infty}^t g_2(t-s)ds \int_{-\infty}^t g_2(t-s)|v(t)-v(s)|^2 ds \Big] dx \\
&\leqslant \frac{1}{2}\Big(\int_0^{+\infty} g_1(s)ds - \delta g_1(0)\Big) \int_\Omega \nabla^\top \partial_t u(t) A_1 \nabla \partial_t u(t) dx \\
&\quad + \frac{1}{2}\Big(\int_0^{+\infty} g_2(s)ds - \delta g_2(0)\Big) \int_\Omega \nabla^\top \partial_t v(t) A_2 \nabla \partial_t v(t) dx \\
&\quad + \tilde{\mathscr{M}}_{13}^7 (g_1 \diamond^{A_1} \nabla u)(t) + \tilde{\mathscr{M}}_{23}^7 (g_2 \diamond^{A_2} \nabla v)(t), \quad t \in \mathbb{R}_+,
\end{aligned} \tag{131}$$

where the positive constants $\tilde{\mathscr{M}}^7_{13}$ *and* $\tilde{\mathscr{M}}^7_{23}$*, slightly different from* $\mathscr{M}^7_{13}$ *and* $\mathscr{M}^7_{23}$ *in* (117) *(see* (118) *and* (119) *for the details), are given by*

$$\tilde{\mathscr{M}}^7_{13} = \frac{(\kappa_2)^4 \varkappa_{a_{11},a_{12},a_{21},a_{22}} \zeta_1}{8} \max\Big(\frac{\zeta_1}{\int_0^{+\infty} g_1(s)ds - \delta g_1(0)}, \frac{\zeta_2}{\int_0^{+\infty} g_2(s)ds - \delta g_2(0)}\Big) \int_0^{+\infty} g_1(s)ds, \tag{132}$$

and

$$\tilde{\mathscr{M}}^7_{23} = \frac{(\kappa_2)^4 \varkappa_{a_{11},a_{12},a_{21},a_{22}} \zeta_2}{8} \max\Big(\frac{\zeta_1}{\int_0^{+\infty} g_1(s)ds - \delta g_1(0)}, \frac{\zeta_2}{\int_0^{+\infty} g_2(s)ds - \delta g_2(0)}\Big) \int_0^{+\infty} g_2(s)ds. \tag{133}$$

As mentioned above, compared to the estimate (117)*, the last three lines of the sequence* (131) *of inequalities do not include the term* (130)*. As will be seen, this could be beneficial to relatively wide choice of portions of the energy functional* $E^{u,v}(t)$ *(see* (18)*), in the procedure of constructing modified energy functionals.*

As with $L_1(t)$ and $L_2(t)$ (see (85) and (94), respectively), the following energy perturbation functional will be useful in our later presentation: To every pair $(u,v)^\top \in \mathcal{S}_{\mathbb{R}_+}$ (see (15) for the definition of $\mathcal{S}_{\mathbb{R}_+}$), we associate the functional

$$L_3(t) = \frac{\mu_1 - \int_0^{+\infty} g_1(s)ds}{\int_0^{+\infty} g_1(s)ds} \int_0^{+\infty} g_1(s) \int_{t-s}^t \int_\Omega \nabla^\top u(r) A_1 \nabla u(r) dxdrds + \frac{\mu_2 - \int_0^{+\infty} g_2(s)ds}{\int_0^{+\infty} g_2(s)ds} \int_0^{+\infty} g_2(s) \int_{t-s}^t \int_\Omega \nabla^\top v(r) A_2 \nabla v(r) dxdrds, \quad t \in \mathbb{R}_+. \tag{134}$$

Lemma 7. *Suppose that Assumptions 1–5 hold true. If the associated functional* $I(t)$ *given by* (56) *satisfies* $I(0) > 0$*, the associated constants* β_1 *and* β_2*, given by* (58) *and* (59)*, respectively, satisfy* $\max(\beta_1, \beta_2) < 1$*, then weak solutions to IBVP* (1) *exist globally in time, and render the associated functional* $L_3(t)$ *given by* (134) *to satisfy*

$$\begin{aligned} L_3'(t) \leqslant\; & 2\Big(\mu_1 - \int_0^{+\infty} g_1(s)ds\Big) \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx \\ & + 2\Big(\mu_2 - \int_0^{+\infty} g_2(s)ds\Big) \int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx \\ & - \frac{\mu_1 - \int_0^{+\infty} g_1(s)ds}{2\int_0^{+\infty} g_1(s)ds} (g_1 \diamond^{A_1} \nabla u)(t) \\ & - \frac{\mu_2 - \int_0^{+\infty} g_2(s)ds}{2\int_0^{+\infty} g_2(s)ds} (g_2 \diamond^{A_2} \nabla v)(t), \quad t \in \mathbb{R}_+. \end{aligned} \tag{135}$$

Proof. Differentiate directly $L_3(t)$ (see (134)) with respect to t, to arrive at

$$
\begin{aligned}
L_3'(t) = {} & \frac{\mu_1 - \int_0^{+\infty} g_1(s)ds}{\int_0^{+\infty} g_1(s)ds} \int_0^{+\infty} g_1(s)ds \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx \\
& - \frac{\mu_1 - \int_0^{+\infty} g_1(s)ds}{\int_0^{+\infty} g_1(s)ds} \int_0^{+\infty} g_1(s) \int_\Omega \nabla^\top u(t-s) A_1 \nabla u(t-s) dxds \\
& + \frac{\mu_2 - \int_0^{+\infty} g_2(s)ds}{\int_0^{+\infty} g_2(s)ds} \int_0^{+\infty} g_2(s)ds \int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx \\
& - \frac{\mu_2 - \int_0^{+\infty} g_2(s)ds}{\int_0^{+\infty} g_2(s)ds} \int_0^{+\infty} g_2(s) \int_\Omega \nabla^\top v(t-s) A_2 \nabla v(t-s) dxds \\
= {} & (\mu_1 - \int_0^{+\infty} g_1(s)ds) \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx \\
& - \frac{\mu_1 - \int_0^{+\infty} g_1(s)ds}{\int_0^{+\infty} g_1(s)ds} \int_{-\infty}^t g_1(t-s) \int_\Omega \nabla^\top u(s) A_1 \nabla u(s) dxds \\
& + (\mu_2 - \int_0^{+\infty} g_2(s)ds) \int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx \\
& - \frac{\mu_2 - \int_0^{+\infty} g_2(s)ds}{\int_0^{+\infty} g_2(s)ds} \int_{-\infty}^t g_2(t-s) \int_\Omega \nabla^\top v(s) A_2 \nabla v(s) dxds, \quad t \in \mathbb{R}_+. \qquad (136)
\end{aligned}
$$

By direct calculations, we have

$$
\begin{aligned}
\int_{-\infty}^t g_1(t-s) \int_\Omega \nabla^\top u(s) A_1 \nabla u(s) dxds = {} & \frac{1}{2} \int_{-\infty}^t g_1(t-s) \int_\Omega (\nabla^\top u(t) - \nabla^\top u(s)) A_1 (\nabla u(t) - \nabla u(s)) dxds \\
& + \frac{1}{2} \int_{-\infty}^t g_1(t-s) \int_\Omega (\nabla^\top u(t) + \nabla^\top u(s)) A_1 (\nabla u(t) + \nabla u(s)) dxds \\
& - \int_{-\infty}^t g_1(t-s) \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dxds \\
= {} & \frac{1}{2} \int_{-\infty}^t g_1(t-s) \int_\Omega (\nabla^\top u(t) + \nabla^\top u(s)) A_1 (\nabla u(t) + \nabla u(s)) dxds \\
& + \frac{1}{2} (g_1 \diamond^{A_1} \nabla u)(t) - \int_0^{+\infty} g_1(s)ds \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx, \quad t \in \mathbb{R}_+. \qquad (137)
\end{aligned}
$$

And we have analogously

$$
\begin{aligned}
\int_{-\infty}^t g_2(t-s) \int_\Omega \nabla^\top v(s) A_2 \nabla v(s) dxds = {} & \frac{1}{2} \int_{-\infty}^t g_2(t-s) \int_\Omega (\nabla^\top v(t) + \nabla^\top v(s)) A_2 (\nabla v(t) + \nabla v(s)) dxds \\
& + \frac{1}{2} (g_2 \diamond^{A_2} \nabla v)(t) - \int_0^{+\infty} g_2(s)ds \int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx, \quad t \in \mathbb{R}_+.
\end{aligned}
$$

This, together with (136) and (137), implies immediately

$$
\begin{aligned}
L_3'(t) = {} & 2(\mu_1 - \int_0^{+\infty} g_1(s)ds) \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx + 2(\mu_2 - \int_0^{+\infty} g_2(s)ds) \int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx \\
& - \frac{\mu_1 - \int_0^{+\infty} g_1(s)ds}{2 \int_0^{+\infty} g_1(s)ds} \int_{-\infty}^t g_1(t-s) \int_\Omega (\nabla^\top u(t) + \nabla^\top u(s)) A_1 (\nabla u(t) + \nabla u(s)) dxds \\
& - \frac{\mu_2 - \int_0^{+\infty} g_2(s)ds}{2 \int_0^{+\infty} g_2(s)ds} \int_{-\infty}^t g_2(t-s) \int_\Omega (\nabla^\top v(t) + \nabla^\top v(s)) A_2 (\nabla v(t) + \nabla v(s)) dxds \\
& - \frac{\mu_1 - \int_0^{+\infty} g_1(s)ds}{2 \int_0^{+\infty} g_1(s)ds} (g_1 \diamond^{A_1} \nabla u)(t) - \frac{\mu_2 - \int_0^{+\infty} g_2(s)ds}{2 \int_0^{+\infty} g_2(s)ds} (g_2 \diamond^{A_2} \nabla v)(t), \quad t \in \mathbb{R}_+. \qquad (138)
\end{aligned}
$$

Thanks to Assumptions 1 and 2, it holds that

$$\left.\begin{aligned}&\int_{-\infty}^{t} g_1(t-s)\int_{\Omega}(\nabla^{\top}u(t)+\nabla^{\top}u(s))A_1(\nabla u(t)+\nabla u(s))dxds \geqslant 0,\\&\int_{-\infty}^{t} g_2(t-s)\int_{\Omega}(\nabla^{\top}v(t)+\nabla^{\top}v(s))A_2(\nabla v(t)+\nabla v(s))dxds \geqslant 0,\end{aligned}\right\}\quad t\in\mathbb{R}_+,$$

which, together with (138), implies that the proof of Lemma 7 is complete. □

To obtain our claimed general energy decay result, we need to design various modified (perturbed) energy functionals by adding energy perturbation functionals to the conventional energy functional $E^{u,v}(t)$ (see (18) for the precise definition). Here, we are in a position to introduce the modified energy functional

$$\daleth(t) = \mathfrak{m}_0 E^{u,v}(t) + \mathfrak{m}_1 L_1(t) + \mathfrak{m}_2 L_2(t), \quad t \in \mathbb{R}_+, \tag{139}$$

where $\mathfrak{m}_k$ is a positive constant yet to be determined later, $k = 0, 1, 2$.

Lemma 8. *Suppose that Assumptions 1–5 hold true. Then there exists a triple*

$$(\mathfrak{m}_0, \mathfrak{m}_1, \mathfrak{m}_2)^{\top} \in (0, +\infty)^3,$$

such that each solution $(u, v)^{\top}$ to IBVP (1) makes the following differential inequality hold:

$$\begin{aligned}\daleth'(t) \leqslant & -\frac{1}{\rho_1+1}\int_{\Omega}|\partial_t u(t)|^{\rho_1+2}dx - \frac{1}{\rho_2+1}\int_{\Omega}|\partial_t v(t)|^{\rho_2+2}dx\\& -\frac{3}{2}(\mu_1 - \int_0^{+\infty} g_1(s)ds)\int_{\Omega}\nabla^{\top}u(t)A_1\nabla u(t)dx\\& -\frac{3}{2}(\mu_2 - \int_0^{+\infty} g_2(s)ds)\int_{\Omega}\nabla^{\top}v(t)A_2\nabla v(t)dx\\& -\int_{\Omega}\nabla^{\top}\partial_t u(t)A_1\nabla\partial_t u(t)dx - \int_{\Omega}\nabla^{\top}\partial_t v(t)A_2\nabla\partial_t v(t)dx\\& +\frac{\mu_1 - \int_0^{+\infty} g_1(s)ds}{3\int_0^{+\infty} g_1(s)ds}(g_1 \diamond^{A_1} \nabla u)(t) + \frac{\mu_2 - \int_0^{+\infty} g_2(s)ds}{3\int_0^{+\infty} g_2(s)ds}(g_2 \diamond^{A_2} \nabla v)(t) \quad \textit{for a.e. } t \in \mathbb{R}_+,\end{aligned} \tag{140}$$

whenever its initial datum $(u^0, v^0, u^1, v^1)^{\top} \in L^{\infty}(\mathbb{R}_-; H_0^1(\Omega;\mathbb{R}^2)) \times H_0^1(\Omega;\mathbb{R}^2)$ render the associated functional $I(t)$ given by (56) to satisfy $I(0) > 0$, the associated constants β_1 and β_2, given by (58) and (59), respectively, to satisfy $\max(\beta_1, \beta_2) < 1$, the associated constants $\hbar_1$ and $\hbar_2$, given by (86) and (87), respectively, satisfy $\max(\hbar_1, \hbar_2) < \frac{1}{4}$, and the constants $\mathscr{W}_{11}$ and $\mathscr{W}_{21}$, given by (96) and (98), respectively, to satisfy

$$\left.\begin{aligned}\mathscr{W}_{11} &\leqslant \frac{\mu_1 - \int_0^{+\infty} g_1(s)ds}{6\int_0^{+\infty} g_1(s)ds \max(\frac{10}{\int_0^{+\infty} g_1(s)ds}, \frac{10}{\int_0^{+\infty} g_2(s)ds})},\\\mathscr{W}_{21} &\leqslant \frac{\mu_2 - \int_0^{+\infty} g_2(s)ds}{6\int_0^{+\infty} g_2(s)ds \max(\frac{10}{\int_0^{+\infty} g_1(s)ds}, \frac{10}{\int_0^{+\infty} g_2(s)ds})}.\end{aligned}\right\} \tag{141}$$

Proof. Thanks to (139), it follows that

$$\daleth'(t) = \mathfrak{m}_0 \frac{d}{dt}E^{u,v}(t) + \mathfrak{m}_1 L_1'(t) + \mathfrak{m}_2 L_2'(t) \quad \text{for a.e. } t \in \mathbb{R}_+,$$

where $(\mathfrak{m}_0, \mathfrak{m}_1, \mathfrak{m}_2)^{\top} \in (0, +\infty)^3$, as in (139), is yet to be determined later. By Lemmas 3, 5 and 6, this, together with (55), (88) and (95), implies

$$
\begin{aligned}
\beth'(t) \leqslant & \frac{1}{\rho_1+1}\big(\mathfrak{m}_1 - \mathfrak{m}_2\int_0^{+\infty} g_1(s)ds\big)\int_\Omega |\partial_t u(t)|^{\rho_1+2}dx + \frac{1}{\rho_2+1}\big(\mathfrak{m}_1 - \mathfrak{m}_2\int_0^{+\infty} g_2(s)ds\big)\int_\Omega |\partial_t v(t)|^{\rho_2+2}dx \\
& + \big(\delta\mathfrak{m}_2\int_0^{+\infty} g_1(s)ds - \frac{\mathfrak{m}_1}{2}(\mu_1 - \int_0^{+\infty} g_1(s)ds)\big)\int_\Omega \nabla^\top u(t)A_1\nabla u(t)dx \\
& + \big(\delta\mathfrak{m}_2\int_0^{+\infty} g_2(s)ds - \frac{\mathfrak{m}_1}{2}(\mu_2 - \int_0^{+\infty} g_2(s)ds)\big)\int_\Omega \nabla^\top v(t)A_2\nabla v(t)dx \\
& + \big(\mathfrak{m}_1 - \mathfrak{m}_2(\int_0^{+\infty} g_1(s)ds - \delta g_1(0))\big)\int_\Omega \nabla^\top\partial_t u(t)A_1\nabla\partial_t u(t)dx \\
& + \big(\mathfrak{m}_1 - \mathfrak{m}_2(\int_0^{+\infty} g_2(s)ds - \delta g_2(0))\big)\int_\Omega \nabla^\top\partial_t v(t)A_2\nabla\partial_t v(t)dx + \mathfrak{m}_2\mathscr{W}_{11}(g_1\diamond^{A_1}\nabla u)(t) \\
& + \frac{\mathfrak{m}_1}{\mu_1 - \int_0^{+\infty} g_1(s)ds}\int_0^{+\infty}\frac{(g_1(s))^2}{g_1(s)-Gg_1'(s)}ds((g_1 - Gg_1')\diamond^{A_1}\nabla u)(t) + \mathfrak{m}_2\mathscr{W}_{21}(g_2\diamond^{A_2}\nabla v)(t) \\
& + \frac{\mathfrak{m}_1}{\mu_2 - \int_0^{+\infty} g_2(s)ds}\int_0^{+\infty}\frac{(g_2(s))^2}{g_2(s)-Gg_2'(s)}ds((g_2 - Gg_2')\diamond^{A_2}\nabla v)(t) + \big(\frac{\mathfrak{m}_0}{2} - \mathfrak{m}_2\mathscr{W}_{12}\big)(g_1'\diamond^{A_1}\nabla u)(t) \\
& + \big(\frac{\mathfrak{m}_0}{2} - \mathfrak{m}_2\mathscr{W}_{22}\big)(g_2'\diamond^{A_2}\nabla v)(t) + (\mathfrak{m}_2 - \mathfrak{m}_0)\int_\Omega \begin{pmatrix}\partial_t u(t)\\ \partial_t v(t)\end{pmatrix}^\top \begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}\partial_t u(t)\\ \partial_t v(t)\end{pmatrix}dx \quad \text{for a.e. } t\in\mathbb{R}_+.
\end{aligned} \tag{142}
$$

Since $g_i : \mathbb{R}_+ \to (0, +\infty)$ is strictly decreasing, g_i is of bounded variation and is therefore Lebesgue measurable, $i = 1, 2$. For any $G > 0$, it is easy to show that

$$
0 < \frac{(g_i(s))^2}{g_i(s) - Gg_i'(s)} < g_i(s) \leqslant g_i(0), \quad s \in \mathbb{R}_+,\ i = 1, 2. \tag{143}
$$

It is also obvious to see that

$$
\lim_{G\to+\infty}\frac{(g_i(s))^2}{g_i(s) - Gg_i'(s)} = 0, \quad s \in \mathbb{R}_+,\ i = 1, 2. \tag{144}
$$

By Lebesgue's dominated convergence theorem, we combine (143) and (144), to obtain

$$
\lim_{G\to+\infty}\int_0^{+\infty}\frac{(g_i(s))^2}{g_i(s) - Gg_i'(s)}ds = 0, \quad i = 1, 2. \tag{145}
$$

By recalling the limit theory of one-variable functions and in view of (145), we conclude that there exists a $G_0 > 0$ such that

$$
\int_0^{+\infty}\frac{(g_i(s))^2}{g_i(s) - Gg_i'(s)}ds < \frac{(\mu_i - \int_0^{+\infty} g_i(s)ds)^2}{24\int_0^{+\infty} g_i(s)ds}, \quad G \geqslant G_0,\ i = 1, 2. \tag{146}
$$

With (146) as one of the main tools, we have here by some routine computations

$$
\begin{aligned}
& \frac{4}{\mu_1 - \int_0^{+\infty} g_1(s)ds}\int_0^{+\infty}\frac{(g_1(s))^2}{g_1(s) - G_0g_1'(s)}ds((g_1 - G_0g_1')\diamond^{A_1}\nabla u)(t) \\
= & \frac{4}{\mu_1 - \int_0^{+\infty} g_1(s)ds}\left(\int_0^{+\infty}\frac{(g_1(s))^2}{g_1(s) - G_0g_1'(s)}ds - \frac{(\mu_1 - \int_0^{+\infty} g_1(s)ds)^2}{24\int_0^{+\infty} g_1(s)ds}\right)((g_1 - G_0g_1')\diamond^{A_1}\nabla u)(t) \\
& + \frac{\mu_1 - \int_0^{+\infty} g_1(s)ds}{6\int_0^{+\infty} g_1(s)ds}(g_1\diamond^{A_1}\nabla u)(t) - \frac{G_0(\mu_1 - \int_0^{+\infty} g_1(s)ds)}{6\int_0^{+\infty} g_1(s)ds}(g_1'\diamond^{A_1}\nabla u)(t) \\
\leqslant & \frac{\mu_1 - \int_0^{+\infty} g_1(s)ds}{6\int_0^{+\infty} g_1(s)ds}(g_1\diamond^{A_1}\nabla u)(t) - \frac{G_0(\mu_1 - \int_0^{+\infty} g_1(s)ds)}{6\int_0^{+\infty} g_1(s)ds}(g_1'\diamond^{A_1}\nabla u)(t) \quad \text{for a.e. } t\in\mathbb{R}_+.
\end{aligned} \tag{147}
$$

With the aid of (146), we take several steps similar to (147), to arrive at

$$\begin{aligned}&\frac{4}{\mu_2-\int_0^{+\infty}g_2(s)ds}\int_0^{+\infty}\frac{(g_2(s))^2}{g_2(s)-G_0g_2'(s)}ds((g_2-G_0g_2')\diamond^{A_2}\nabla v)(t)\\ \leqslant&\frac{\mu_2-\int_0^{+\infty}g_2(s)ds}{6\int_0^{+\infty}g_2(s)ds}(g_2\diamond^{A_2}\nabla v)(t)\\ &-\frac{G_0(\mu_2-\int_0^{+\infty}g_2(s)ds)}{6\int_0^{+\infty}g_2(s)ds}(g_2'\diamond^{A_2}\nabla v)(t)\quad \text{for a.e. } t\in\mathbb{R}_+. \end{aligned} \tag{148}$$

Now we are ready to choose appropriate values of the parameters δ and $\mathfrak{m}_i$, $i=0,1,2$. Actually, there are many ways to accomplish this goal. For instance, we can pick

$$\mathfrak{m}_1=4,\quad \mathfrak{m}_2=\max\Big(\frac{10}{\int_0^{+\infty}g_1(s)ds},\frac{10}{\int_0^{+\infty}g_2(s)ds}\Big), \tag{149}$$

$$\begin{aligned}\delta=\min\Big(&\frac{\int_0^{+\infty}g_1(s)ds}{2g_1(0)},\frac{\mu_1-\int_0^{+\infty}g_1(s)ds}{2\int_0^{+\infty}g_1(s)ds\max(\frac{10}{\int_0^{+\infty}g_1(s)ds},\frac{10}{\int_0^{+\infty}g_2(s)ds})},\\ &\frac{\int_0^{+\infty}g_2(s)ds}{2g_2(0)},\frac{\mu_2-\int_0^{+\infty}g_2(s)ds}{2\int_0^{+\infty}g_2(s)ds\max(\frac{10}{\int_0^{+\infty}g_1(s)ds},\frac{10}{\int_0^{+\infty}g_2(s)ds})}\Big),\end{aligned} \tag{150}$$

and

$$\begin{aligned}\mathfrak{m}_0=\max\Big(&\max(\frac{10}{\int_0^{+\infty}g_1(s)ds},\frac{10}{\int_0^{+\infty}g_2(s)ds}),\,2\sup\Big\{\max(\mathscr{W}_{12}\max(\frac{10}{\int_0^{+\infty}g_1(s)ds},\frac{10}{\int_0^{+\infty}g_2(s)ds})\\ &+\frac{G_0(\mu_2-\int_0^{+\infty}g_2(s)ds)}{6\int_0^{+\infty}g_2(s)ds},\mathscr{W}_{22}\max(\frac{10}{\int_0^{+\infty}g_1(s)ds},\frac{10}{\int_0^{+\infty}g_2(s)ds})+\frac{G_0(\mu_2-\int_0^{+\infty}g_2(s)ds)}{6\int_0^{+\infty}g_2(s)ds});\\ &\max(\beta_1,\beta_2)<1,\,\max(\hbar_1,\hbar_2)<\frac{1}{4}\text{ and (141) is satisfied}\Big\}\Big).\end{aligned} \tag{151}$$

It is not difficult to find that when the parameters δ and $\mathfrak{m}_i$, $i=0,1,2$, take the values shown above, it holds that

$$\left.\begin{aligned}&\mathfrak{m}_1-\mathfrak{m}_2\int_0^{+\infty}g_1(s)ds\leqslant-1,\quad \mathfrak{m}_1-\mathfrak{m}_2\int_0^{+\infty}g_2(s)ds\leqslant-1,\\ &\delta\mathfrak{m}_2\int_0^{+\infty}g_1(s)ds-\frac{\mathfrak{m}_1}{2}(\mu_1-\int_0^{+\infty}g_1(s)ds)\leqslant-\frac{3}{2}(\mu_1-\int_0^{+\infty}g_1(s)ds),\\ &\delta\mathfrak{m}_2\int_0^{+\infty}g_2(s)ds-\frac{\mathfrak{m}_1}{2}(\mu_2-\int_0^{+\infty}g_2(s)ds)\leqslant-\frac{3}{2}(\mu_2-\int_0^{+\infty}g_2(s)ds),\\ &\mathfrak{m}_1-\mathfrak{m}_2(\int_0^{+\infty}g_1(s)ds-\delta g_1(0))\leqslant 4-\frac{10}{\int_0^{+\infty}g_1(s)ds}(\int_0^{+\infty}g_1(s)ds-\frac{\int_0^{+\infty}g_1(s)ds}{2g_1(0)}\cdot g_1(0))\leqslant-1,\\ &\mathfrak{m}_1-\mathfrak{m}_2(\int_0^{+\infty}g_2(s)ds-\delta g_2(0))\leqslant 4-\frac{10}{\int_0^{+\infty}g_2(s)ds}(\int_0^{+\infty}g_2(s)ds-\frac{\int_0^{+\infty}g_2(s)ds}{2g_2(0)}\cdot g_2(0))\leqslant-1,\\ &\frac{\mathfrak{m}_0}{2}-\mathfrak{m}_2\mathscr{W}_{12}-\frac{G_0(\mu_1-\int_0^{+\infty}g_1(s)ds)}{6\int_0^{+\infty}g_1(s)ds}\geqslant 0,\quad \frac{\mathfrak{m}_0}{2}-\mathfrak{m}_2\mathscr{W}_{22}-\frac{G_0(\mu_2-\int_0^{+\infty}g_2(s)ds)}{6\int_0^{+\infty}g_2(s)ds}\geqslant 0,\\ &\mathfrak{m}_2-\mathfrak{m}_0\leqslant 0,\quad \mathfrak{m}_2\mathscr{W}_{11}\leqslant\frac{\mu_1-\int_0^{+\infty}g_1(s)ds}{6\int_0^{+\infty}g_1(s)ds},\quad \mathfrak{m}_2\mathscr{W}_{21}\leqslant\frac{\mu_2-\int_0^{+\infty}g_2(s)ds}{6\int_0^{+\infty}g_2(s)ds}.\end{aligned}\right\}$$

In view of the fact that the term (130) in (142) is non-negative, by some routine calculations, we conclude immediately: This, together with (142), (147) and (148), implies that the proof of Lemma 8 is complete. □

Theorem 4. *Suppose that Assumptions 1–5 hold true. For every quadruple*

$$(u^0, v^0, u^1, v^1)^\top \in L^\infty(\mathbb{R}_-; H_0^1(\Omega;\mathbb{R}^2)) \times H_0^1(\Omega;\mathbb{R}^2)$$

of initial datum, if it renders the associated functional $I(t)$ given by (56) to satisfy $I(0) > 0$, the associated constants β_1 and β_2, given respectively by (58) and (59), to satisfy $\max(\beta_1, \beta_2) < 1$, the associated constants $\hbar_1$ and $\hbar_2$, given respectively by (86) and (87), to satisfy $\max(\hbar_1, \hbar_2) < \frac{1}{4}$, and the associated constants $\mathscr{W}_{11}$ and $\mathscr{W}_{21}$, given respectively by (96) and (98), to satisfy (141), then the corresponding global in time solution $(u, v)^\top \in \mathcal{S}_{\mathbb{R}_+}$ (see (15) for the definition of $\mathcal{S}_{\mathbb{R}_+}$) to IBVP (1) makes the associated energy $E^{u,v}(t)$, given by (18), satisfy

$$E^{u,v}(t) \leqslant \varpi_1 \mathscr{K}^{-1}\Big(\varpi_2 \int_0^t \min(\xi_1(s), \xi_2(s))ds\Big) \quad \text{for all } t \in \mathbb{R}_+, \tag{152}$$

where the positive constants ϖ_1 and ϖ_2, depending on $(u^0, v^0, u^1, v^1)^\top$, is independent of the time variable t, and $\mathscr{K}^{-1}$ is the inverse function of $\mathscr{K}(t)$ which is given by

$$\mathscr{K}(t) = \int_t^{\min(r_1,r_2)} \frac{ds}{sK'(s)} \quad \text{for all } t \in (0, \min(r_1, r_2)]. \tag{153}$$

Remark 8. *By Assumption 2 (in particular, the restriction (6)), we conclude that the $\mathscr{K}(t)$, given by (153), is well-defined as a continuous function, that it takes non-negative real numbers as its values, and that it obeys*

$$\mathscr{K}'(t) = -\frac{1}{sK'(s)} \quad \text{for all } t \in (0, \min(r_1, r_2)].$$

This implies, in particular, that the function $\mathscr{K}(t)$ is strictly monotonically decreasing in the interval $(0, \min(r_1, r_2)]$. We have directly by applying the definition (153) of $\mathscr{K}(t)$

$$\lim_{t \to \min(r_1,r_2)^-} \mathscr{K}(t) = \mathscr{K}(\min(r_1, r_2)) = 0.$$

We apply Assumption 2 again and the definition (153) of $\mathscr{K}(t)$, to get $\lim\limits_{t\to 0^+} \mathscr{K}(t) = +\infty$. In conclusion, $\mathscr{K}(t)$, given by (153), is a strictly decreasing continuous function mapping the interval $(0, \min(r_1, r_2)]$ onto the closed interval $\mathbb{R}_+$. This implies, among other things, the inverse function $\mathscr{K}^{-1}$ of $\mathscr{K}$, appearing in (152), is well-defined, and $\mathscr{K}^{-1}$ actually maps the closed interval $\mathbb{R}_+$ onto the interval $(0, \min(r_1, r_2)]$.

Proof of Theorem 4. In view of (18), (82) and (140), we have

$$\begin{aligned}
\beth'(t) \leqslant & -\frac{(\rho_1+2)(\rho_2+2)}{(\rho_1+1)(\rho_2+2)+(\rho_1+2)(\rho_2+1)} E^{u,v}(t) \\
& - \Big(\frac{1}{\rho_1+1} - \frac{\rho_2+2}{(\rho_1+1)(\rho_2+2)+(\rho_1+2)(\rho_2+1)}\Big) \int_\Omega |\partial_t u(t)|^{\rho_1+2} dx \\
& - \Big(\frac{1}{\rho_2+1} - \frac{\rho_1+2}{(\rho_1+1)(\rho_2+2)+(\rho_1+2)(\rho_2+1)}\Big) \int_\Omega |\partial_t v(t)|^{\rho_2+2} dx \\
& - \frac{1}{2}\Big(3 - \frac{(\rho_1+2)(\rho_2+2)}{(\rho_1+1)(\rho_2+2)+(\rho_1+2)(\rho_2+1)}\Big)\Big(\mu_1 - \int_0^{+\infty} g_1(s)ds\Big) \int_\Omega \nabla^\top u(t) A_1 \nabla u(t) dx \\
& - \frac{1}{2}\Big(3 - \frac{(\rho_1+2)(\rho_2+2)}{(\rho_1+1)(\rho_2+2)+(\rho_1+2)(\rho_2+1)}\Big)\Big(\mu_2 - \int_0^{+\infty} g_2(s)ds\Big) \int_\Omega \nabla^\top v(t) A_2 \nabla v(t) dx
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\Big(2-\frac{(\rho_1+2)(\rho_2+2)}{(\rho_1+1)(\rho_2+2)+(\rho_1+2)(\rho_2+1)}\Big)\int_\Omega \nabla^\top\partial_t u(t)A_1\nabla\partial_t u(t)dx\\
&-\frac{1}{2}\Big(2-\frac{(\rho_1+2)(\rho_2+2)}{(\rho_1+1)(\rho_2+2)+(\rho_1+2)(\rho_2+1)}\Big)\int_\Omega \nabla^\top\partial_t v(t)A_2\nabla\partial_t v(t)dx\\
&-\frac{(\rho_1+2)(\rho_2+2)}{(\rho_1+1)(\rho_2+2)+(\rho_1+2)(\rho_2+1)}\int_\Omega F(u(t),v(t))dx\\
&+\mathscr{M}_{11}^8(g_1\diamond^{A_1}\nabla u)(t)+\mathscr{M}_{12}^8(g_2\diamond^{A_2}\nabla v)(t)\\
\leqslant&-\frac{(\rho_1+2)(\rho_2+2)}{(\rho_1+1)(\rho_2+2)+(\rho_1+2)(\rho_2+1)}E^{u,v}(t)\\
&-\frac{1}{4}\Big(6-\frac{3(\rho_1+2)(\rho_2+2)}{(\rho_1+1)(\rho_2+2)+(\rho_1+2)(\rho_2+1)}\Big)(\mu_1-\int_0^{+\infty}g_1(s)ds)\int_\Omega \nabla^\top u(t)A_1\nabla u(t)dx\\
&-\frac{1}{4}\Big(6-\frac{3(\rho_1+2)(\rho_2+2)}{(\rho_1+1)(\rho_2+2)+(\rho_1+2)(\rho_2+1)}\Big)(\mu_2-\int_0^{+\infty}g_2(s)ds)\int_\Omega \nabla^\top v(t)A_2\nabla v(t)dx\\
&+\mathscr{M}_{11}^8(g_1\diamond^{A_1}\nabla u)(t)+\mathscr{M}_{12}^8(g_2\diamond^{A_2}\nabla v)(t)\\
\leqslant&-\mathscr{M}_2^8E^{u,v}(t)+\mathscr{M}_{11}^8(g_1\diamond^{A_1}\nabla u)(t)+\mathscr{M}_{12}^8(g_2\diamond^{A_2}\nabla v)(t)\quad\text{for a.e. } t\in\mathbb{R}_+, \qquad (154)
\end{aligned}
$$

in which the constants $\mathscr{M}_{11}^8$, $\mathscr{M}_{12}^8$ and $\mathscr{M}_2^8$ are given respectively by

$$
\mathscr{M}_{11}^8=\frac{\mu_1-\int_0^{+\infty}g_1(s)ds}{3\int_0^{+\infty}g_1(s)ds}+\frac{(\rho_1+2)(\rho_2+2)}{2(\rho_1+1)(\rho_2+2)+2(\rho_1+2)(\rho_2+1)}, \qquad (155)
$$

$$
\mathscr{M}_{12}^8=\frac{\mu_2-\int_0^{+\infty}g_2(s)ds}{3\int_0^{+\infty}g_2(s)ds}+\frac{(\rho_1+2)(\rho_2+2)}{2(\rho_1+1)(\rho_2+2)+2(\rho_1+2)(\rho_2+1)}, \qquad (156)
$$

and

$$
\mathscr{M}_2^8=\frac{(\rho_1+2)(\rho_2+2)}{(\rho_1+1)(\rho_2+2)+(\rho_1+2)(\rho_2+1)}. \qquad (157)
$$

Case I. K_1 and K_2 are both linear.

By some routine but tedious calculations, we have

$$
\begin{aligned}
\beth'(t)\min(\xi_1(t),\xi_2(t))\leqslant&-\mathscr{M}_2^8E^{u,v}(t)\min(\xi_1(t),\xi_2(t))+\mathscr{M}_{11}^8\min(\xi_1(t),\xi_2(t))(g_1\diamond^{A_1}\nabla u)(t)\\
&+\mathscr{M}_{12}^8\min(\xi_1(t),\xi_2(t))(g_2\diamond^{A_2}\nabla v)(t),\\
\leqslant&-\mathscr{M}_2^8E^{u,v}(t)\min(\xi_1(t),\xi_2(t))+\mathscr{M}_{11}^8\xi_1(t)(g_1\diamond^{A_1}\nabla u)(t)+\mathscr{M}_{12}^8\xi_2(t)(g_2\diamond^{A_2}\nabla v)(t),\\
\leqslant&-\mathscr{M}_2^8E^{u,v}(t)\min(\xi_1(t),\xi_2(t))-\mathscr{M}_{31}^8(g_1'\diamond^{A_1}\nabla u)(t)-\mathscr{M}_{32}^8(g_2'\diamond^{A_2}\nabla v)(t),\\
\leqslant&-\mathscr{M}_2^8E^{u,v}(t)\min(\xi_1(t),\xi_2(t))-\mathscr{M}_4^8\frac{d}{dt}E^{u,v}(t)\quad\text{for a.e. } t\in\mathbb{R}_+,
\end{aligned}
$$

or equivalently, we have

$$
\begin{aligned}
&\frac{d}{dt}\big(\mathscr{M}_4^8E^{u,v}(t)+\beth(t)\min(\xi_1(t),\xi_2(t))\big)\\
=&\mathscr{M}_4^8\frac{d}{dt}E^{u,v}(t)+\beth'(t)\min(\xi_1(t),\xi_2(t))+\beth(t)\frac{d}{dt}\min(\xi_1(t),\xi_2(t))\\
\leqslant&-\mathscr{M}_2^8E^{u,v}(t)\min(\xi_1(t),\xi_2(t))\quad\text{for a.e. } t\in\mathbb{R}_+. \qquad (158)
\end{aligned}
$$

$\mathscr{M}_4^8E^{u,v}(t)+\beth(t)\min(\xi_1(t),\xi_2(t))\sim E^{u,v}(t)$, more precisely, there exist two positive constants $\mathscr{M}_5^8$ and $\mathscr{M}_6^8$, such that

$$
\frac{1}{\mathscr{M}_6^8}E^{u,v}(t)\leqslant\mathscr{M}_4^8E^{u,v}(t)+\beth(t)\min(\xi_1(t),\xi_2(t))\leqslant\frac{1}{\mathscr{M}_5^8}E^{u,v}(t),\quad t\in\mathbb{R}_+. \qquad (159)
$$

Therefore, we deduce from (158) immediately that

$$\frac{d}{dt}\big(\mathscr{M}_4^8 E^{u,v}(t)+\beth(t)\min(\xi_1(t),\xi_2(t))\big)\leqslant -\mathscr{M}_2^8\mathscr{M}_5^8\min(\xi_1(t),\xi_2(t))\big(\mathscr{M}_4^8 E^{u,v}(t)+\beth(t)\min(\xi_1(t),\xi_2(t))\big).$$

This, together with Gronwall's Lemma, implies

$$\big(\mathscr{M}_4^8 E^{u,v}(t)+\beth(t)\min(\xi_1(t),\xi_2(t))\big)\leqslant \big(\mathscr{M}_4^8 E^{u,v}(0)+\beth(0)\min(\xi_1(0),\xi_2(0))\big)e^{-\mathscr{M}_2^8\mathscr{M}_5^8\int_0^t \min(\xi_1(s),\xi_2(s))ds},\quad t\in\mathbb{R}_+.$$

By recalling the afore-mentioned equivalence (159), we realize that this implies

$$E^{u,v}(t)\leqslant \mathscr{M}_7^8 e^{-\mathscr{M}_2^8\mathscr{M}_5^8\int_0^t \min(\xi_1(s),\xi_2(s))ds},\quad t\in\mathbb{R}_+, \tag{160}$$

where the positive constant $\mathscr{M}_7^8$ is given by

$$\mathscr{M}_7^8=\mathscr{M}_6^8\big(\mathscr{M}_4^8 E^{u,v}(0)+\beth(0)\min(\xi_1(0),\xi_2(0))\big). \tag{161}$$

Case II. K_1 or K_2 is non-linear.

We combine $\beth(t)$ (with δ and $\mathfrak{m}_i$, $i=0,1,2$ given as in (149), (150) and (151), respectively) and $L_3(t)$, to associate with each solution $(u,v)^\top$ to IBVP (1) a new functional

$$F(t)=\beth(t)+\frac{17}{24}L_3(t),\quad t\in\mathbb{R}_+. \tag{162}$$

Obviously, $F(t)\geqslant 0$ for all $t\in\mathbb{R}_+$. Besides, by Lemmas 7 and 8, there exists a positive constant $\mathscr{M}_1^9$ such that

$$F'(t)\leqslant -\frac{E^{u,v}(t)}{\mathscr{M}_1^9}\quad \text{for a.e. } t\in\mathbb{R}_+. \tag{163}$$

For example, we could put $\mathscr{M}_1^9=\frac{1}{\mathscr{M}_2^9}$ with

$$\mathscr{M}_2^9=\min\Big(\frac{1}{9},\frac{(\rho_1+2)(\rho_2+2)}{(\rho_1+1)(\rho_2+2)+(\rho_1+2)(\rho_2+1)},\frac{(\mu_1-\int_0^{+\infty}g_1(s)ds)(\mu_2-\int_0^{+\infty}g_2(s)ds)}{24\int_0^{+\infty}g_2(s)ds(\mu_1-\int_0^{+\infty}g_1(s)ds)+24\int_0^{+\infty}g_1(s)ds(\mu_2-\int_0^{+\infty}g_2(s)ds)}\Big).$$

In this situation, we have

$$\begin{aligned}
F'(t)=&\beth'(t)+\frac{17}{24}L_3'(t)\\
\leqslant&-\frac{E^{u,v}(t)}{\mathscr{M}_1^9}-\Big(\frac{1}{\rho_1+1}-\frac{\rho_2+2}{(\rho_1+1)(\rho_2+2)+(\rho_1+2)(\rho_2+1)}\Big)\|\partial_t u(t)\|_{L^{\rho_1+2}(\Omega)}^{\rho_1+2}\\
&-\Big(\frac{1}{\rho_2+1}-\frac{\rho_1+2}{(\rho_1+1)(\rho_2+2)+(\rho_1+2)(\rho_2+1)}\Big)\|\partial_t v(t)\|_{L^{\rho_2+2}(\Omega)}^{\rho_2+2}\\
&-\frac{17}{18}\int_\Omega \nabla^\top\partial_t u(t)A_1\nabla\partial_t u(t)dx-\frac{17}{18}\int_\Omega \nabla^\top\partial_t v(t)A_2\nabla\partial_t v(t)dx-\frac{1}{48}(g_1\diamond^{A_1}\nabla u)(t)\Big(\frac{\mu_1-\int_0^{+\infty}g_1(s)ds}{\int_0^{+\infty}g_1(s)ds}\\
&-\frac{(\mu_1-\int_0^{+\infty}g_1(s)ds)(\mu_2-\int_0^{+\infty}g_2(s)ds)}{\int_0^{+\infty}g_2(s)ds(\mu_1-\int_0^{+\infty}g_1(s)ds)+\int_0^{+\infty}g_1(s)ds(\mu_2-\int_0^{+\infty}g_2(s)ds)}\Big)-\frac{1}{48}(g_2\diamond^{A_2}\nabla v)(t)\Big(\frac{\mu_2-\int_0^{+\infty}g_2(s)ds}{\int_0^{+\infty}g_2(s)ds}\\
&-\frac{(\mu_1-\int_0^{+\infty}g_1(s)ds)(\mu_2-\int_0^{+\infty}g_2(s)ds)}{\int_0^{+\infty}g_2(s)ds(\mu_1-\int_0^{+\infty}g_1(s)ds)+\int_0^{+\infty}g_1(s)ds(\mu_2-\int_0^{+\infty}g_2(s)ds)}\Big)\leqslant-\frac{E^{u,v}(t)}{\mathscr{M}_1^9}\quad\text{for a.e. } t\in\mathbb{R}_+,
\end{aligned}$$

which implies directly

$$\int_0^t E^{u,v}(s)ds \leqslant -\mathscr{M}_1^9 \int_0^t \digamma'(s)ds = \mathscr{M}_1^9(\digamma(0) - \digamma(t)) \leqslant \mathscr{M}_1^9 \digamma(0) \quad \text{for all } t \in \mathbb{R}_+.$$

This, in turn, implies

$$\int_0^{+\infty} E^{u,v}(t)dt = \lim_{t\to+\infty} \int_0^t E^{u,v}(s)ds \leqslant \limsup_{t\to+\infty} \int_0^t E^{u,v}(s)ds \leqslant \mathscr{M}_1^9 \digamma(0). \tag{164}$$

Fix provisionally t (sufficiently large if necessary) in the interval $(0, +\infty)$, write

$$\begin{cases} \mathscr{U}(t) = \int_{-\infty}^t \int_\Omega (\nabla^\top u(t) - \nabla^\top u(s)) A_1 (\nabla u(t) - \nabla u(s)) dxds, \\ \mathscr{V}(t) = \int_{-\infty}^t \int_\Omega (\nabla^\top v(t) - \nabla^\top v(s)) A_2 (\nabla v(t) - \nabla v(s)) dxds, \end{cases}$$

and pick a $\mathfrak{q} \in (0, +\infty)$ such that

$$\left.\begin{aligned} \mathscr{U}(t) &< \frac{\min(1, r_1, r_2)}{\mathfrak{q} g_1(0)}, \\ \mathscr{V}(t) &< \frac{\min(1, r_1, r_2)}{\mathfrak{q} g_2(0)}, \end{aligned}\right\} \quad t \in (0, +\infty).$$

We assume first that $\mathscr{U}(t) \equiv 0$ and $\mathscr{V}(t) \equiv 0$. In this case, we deduce from (154) that

$$\begin{aligned} \beth'(t) &\leqslant -\mathscr{M}_2^8 E^{u,v}(t) + \mathscr{M}_{11}^8 (g_1 \diamond^{A_1} \nabla u)(t) + \mathscr{M}_{12}^8 (g_2 \diamond^{A_2} \nabla v)(t) \\ &\leqslant -\mathscr{M}_2^8 E^{u,v}(t) + g_1(0) \mathscr{M}_{11}^8 \mathscr{U}(t) + g_2(0) \mathscr{M}_{12}^8 \mathscr{V}(t) \\ &= -\mathscr{M}_2^8 E^{u,v}(t) \quad \text{for a.e. } t \in \mathbb{R}_+. \end{aligned}$$

This, together with the equivalence $\beth(t) \sim E^{u,v}(t)$, implies that the energy $E^{u,v}(t)$ associated to the system (1) the decays exponentially as time t escapes to infinity. Let us now assume that $\mathscr{U}(t) > 0$ and $\mathscr{V}(t) > 0$ when t is sufficiently large. Without loss of generality, we assume that $\mathscr{U}(t) > 0$ and $\mathscr{V}(t) > 0$ for all $t \in \mathbb{R}_+$. With Jensen's inequality as one of our main tools, by some routine but tedious computations, we arrive at

$$\begin{aligned} K_1\big(\mathfrak{q}(g_1 \diamond^{A_1} \nabla u)(t)\big) &= K_1\left(\frac{\int_{-\infty}^t (\mathfrak{q}\mathscr{A}(t) g_1(t-s)) \int_\Omega (\nabla^\top u(t) - \nabla^\top u(s)) A_1 (\nabla u(t) - \nabla u(s)) dxds}{\int_{-\infty}^t \int_\Omega (\nabla^\top u(t) - \nabla^\top u(s)) A_1 (\nabla u(t) - \nabla u(s)) dxds} \right) \\ &\leqslant \frac{\int_{-\infty}^t K_1\big((\mathfrak{q}\mathscr{A}(t) g_1(t-s))\big) \int_\Omega (\nabla^\top u(t) - \nabla^\top u(s)) A_1 (\nabla u(t) - \nabla u(s)) dxds}{\int_{-\infty}^t \int_\Omega (\nabla^\top u(t) - \nabla^\top u(s)) A_1 (\nabla u(t) - \nabla u(s)) dxds} \\ &\leqslant \frac{\mathfrak{q} \int_{-\infty}^t \xi_1(t-s) K_1\big(g_1(t-s)\big) \int_\Omega (\nabla^\top u(t) - \nabla^\top u(s)) A_1 (\nabla u(t) - \nabla u(s)) dxds}{\xi_1(t)} \\ &\leqslant \frac{\mathfrak{q} \int_{-\infty}^t (-g_1'(t-s)) \int_\Omega (\nabla^\top u(t) - \nabla^\top u(s)) A_1 (\nabla u(t) - \nabla u(s)) dxds}{\xi_1(t)} = -\frac{\mathfrak{q}(g_1' \diamond^{A_1} \nabla u)(t)}{\xi_1(t)} \quad \text{for a.e. } t \in \mathbb{R}_+. \end{aligned} \tag{165}$$

Since K_1 is strictly increasing (see Assumption 2 for the details), it follows immediately

$$(g_1 \diamond^{A_1} \nabla u)(t) \leqslant \frac{1}{\mathfrak{q}} K_1^{-1}\Big(-\frac{\mathfrak{q}(g_1' \diamond^{A_1} \nabla u)(t)}{\xi_1(t)}\Big) \quad \text{for a.e. } t \in \mathbb{R}_+. \tag{166}$$

And analogously, we can prove

$$(g_2 \diamond^{A_2} \nabla v)(t) \leqslant \frac{1}{\mathfrak{q}} K_2^{-1}\Big(-\frac{\mathfrak{q}(g_2' \diamond^{A_2} \nabla v)(t)}{\xi_2(t)}\Big) \quad \text{for a.e. } t \in \mathbb{R}_+. \tag{167}$$

Combine (154), (166) and (167), to obtain

$$\begin{aligned}\beth'(t) \leqslant& -\mathscr{M}_2^8 E^{u,v}(t) + \mathscr{M}_{11}^8 (g_1 \diamond^{A_1} \nabla u)(t) + \mathscr{M}_{12}^8 (g_2 \diamond^{A_2} \nabla v)(t) \\ \leqslant& -\mathscr{M}_2^8 E^{u,v}(t) + \frac{\mathscr{M}_{11}^8}{\mathfrak{q}} K_1^{-1}(-\frac{\mathfrak{q}(g_1' \diamond^{A_1} \nabla u)(t)}{\xi_1(t)}) + \frac{\mathscr{M}_{12}^8}{\mathfrak{q}} K_2^{-1}(-\frac{\mathfrak{q}(g_2' \diamond^{A_2} \nabla v)(t)}{\xi_2(t)}) \quad \text{for a.e. } t \in \mathbb{R}_+. \end{aligned} \tag{168}$$

Let us define a new function $K(t)$ by giving

$$K(t) = \max(K_1(t), K_2(t)) \quad \text{for all } t \in (0, \min(r_1, r_2)].$$

It is not difficult to check that $K(t)$ is strictly increasing and strictly convex. Put

$$\natural(t) = K'(\varepsilon_0 \frac{E^{u,v}(t)}{E^{u,v}(0)}) \beth(t) + E^{u,v}(t) \quad \text{for a.e. } t \in \mathbb{R}_+, \tag{169}$$

where the positive constant ε_0 is suitably chosen so that

$$\varepsilon_0 \frac{E^{u,v}(t)}{E^{u,v}(0)} \leqslant \min(r_1, r_2) \quad \text{for all } t \in \mathbb{R}_+.$$

By direct computations, we deduce from (168) and (169) that

$$\begin{aligned}\natural'(t) =& \frac{\varepsilon_0 \beth(t)}{E^{u,v}(0)} K''(\varepsilon_0 \frac{E^{u,v}(t)}{E^{u,v}(0)}) \frac{d}{dt} E^{u,v}(t) + K'(\varepsilon_0 \frac{E^{u,v}(t)}{E^{u,v}(0)}) \beth'(t) + \frac{d}{dt} E^{u,v}(t) \\ \leqslant& \frac{d}{dt} E^{u,v}(t) - \mathscr{M}_2^8 E^{u,v}(t) K'(\varepsilon_0 \frac{E^{u,v}(t)}{E^{u,v}(0)}) + \frac{\mathscr{M}_{11}^8}{\mathfrak{q}} K'(\varepsilon_0 \frac{E^{u,v}(t)}{E^{u,v}(0)}) K_1^{-1}(-\frac{\mathfrak{q}(g_1' \diamond^{A_1} \nabla u)(t)}{\xi_1(t)}) \\ &+ \frac{\mathscr{M}_{12}^8}{\mathfrak{q}} K'(\varepsilon_0 \frac{E^{u,v}(t)}{E^{u,v}(0)}) K_2^{-1}(-\frac{\mathfrak{q}(g_2' \diamond^{A_2} \nabla v)(t)}{\xi_2(t)}) \quad \text{for a.e. } t \in \mathbb{R}_+. \end{aligned} \tag{170}$$

By applying Assumption 2, we have

$$\lim_{\delta \to 0^+} \frac{K_i'^{-1}(\delta)}{K'^{-1}(\delta)} = \mathfrak{k}_i, \quad i = 1, 2.$$

By applying the method of change variable, we have furthermore

$$\lim_{\delta \to 0^+} \frac{K_i'^{-1}(K'(\delta))}{\delta} = \mathfrak{k}_i, \quad i = 1, 2.$$

By recalling the limit theory of one-variable functions, we conclude that there exists a positive constant $\wp \leqslant \min(r_1, r_2)$, such that

$$K_i'^{-1}(K'(\delta)) < 2\mathfrak{k}_i \delta, \quad i = 1, 2. \tag{171}$$

whenever the positive variable δ does not exceed $\wp$. With (171) as one of our main tools, by applying mainly the Fenchel–Young inequality, we have

$$\begin{aligned}K'(\varepsilon_0 \frac{E^{u,v}(t)}{E^{u,v}(0)}) K_1^{-1}(-\frac{\mathfrak{q}(g_1' \diamond^{A_1} \nabla u)(t)}{\xi_1(t)}) \leqslant& K_1^*(K'(\varepsilon_0 \frac{E^{u,v}(t)}{E^{u,v}(0)})) + K_1(K_1^{-1}(-\frac{\mathfrak{q}(g_1' \diamond^{A_1} \nabla u)(t)}{\xi_1(t)})) \\ \leqslant& K'(\varepsilon_0 \frac{E^{u,v}(t)}{E^{u,v}(0)}) K_1'^{-1}(K'(\varepsilon_0 \frac{E^{u,v}(t)}{E^{u,v}(0)})) - K_1(K_1'^{-1}(K'(\varepsilon_0 \frac{E^{u,v}(t)}{E^{u,v}(0)}))) \\ &- \frac{\mathfrak{q}(g_1' \diamond^{A_1} \nabla u)(t)}{\xi_1(t)} \\ \leqslant& 2\mathfrak{k}_1 \varepsilon_0 \frac{E^{u,v}(t)}{E^{u,v}(0)} K'(\varepsilon_0 \frac{E^{u,v}(t)}{E^{u,v}(0)}) - \frac{\mathfrak{q}(g_1' \diamond^{A_1} \nabla u)(t)}{\xi_1(t)} \quad \text{for a.e. } t \in \mathbb{R}_+. \end{aligned} \tag{172}$$

By analogy to (172), we have

$$K'(\varepsilon_0\frac{E^{u,v}(t)}{E^{u,v}(0)})K_2^{-1}(-\frac{\mathfrak{q}(g_2'\diamond^{A_2}\nabla v)(t)}{\xi_2(t)}) \leqslant 2\mathfrak{k}_2\varepsilon_0\frac{E^{u,v}(t)}{E^{u,v}(0)}K'(\varepsilon_0\frac{E^{u,v}(t)}{E^{u,v}(0)}) - \frac{\mathfrak{q}(g_2'\diamond^{A_2}\nabla v)(t)}{\xi_2(t)} \quad \text{for a.e. } t\in\mathbb{R}_+. \tag{173}$$

We substitute (172) and (173) into (170), to conclude

$$\begin{aligned}\natural'(t) &\leqslant \frac{d}{dt}E^{u,v}(t) - \mathscr{M}_2^8E^{u,v}(t)K'(\varepsilon_0\frac{E^{u,v}(t)}{E^{u,v}(0)})\\
&\quad+\frac{2\mathfrak{k}_1\varepsilon_0\mathscr{M}_{11}^8}{\mathfrak{q}}\frac{E^{u,v}(t)}{E^{u,v}(0)}K'(\varepsilon_0\frac{E^{u,v}(t)}{E^{u,v}(0)}) - \frac{\mathscr{M}_{11}^8(g_1'\diamond^{A_1}\nabla u)(t)}{\xi_1(t)}\\
&\quad+\frac{2\mathfrak{k}_2\varepsilon_0\mathscr{M}_{12}^8}{\mathfrak{q}}\frac{E^{u,v}(t)}{E^{u,v}(0)}K'(\varepsilon_0\frac{E^{u,v}(t)}{E^{u,v}(0)}) - \frac{\mathscr{M}_{12}^8(g_2'\diamond^{A_2}\nabla v)(t)}{\xi_2(t)}\\
&\leqslant \frac{2\varepsilon_0\mathfrak{k}_1\mathscr{M}_{11}^8}{\mathfrak{q}}\frac{E^{u,v}(t)}{E^{u,v}(0)}K'(\varepsilon_0\frac{E^{u,v}(t)}{E^{u,v}(0)}) - \frac{\mathscr{M}_{11}^8(g_1'\diamond^{A_1}\nabla u)(t)}{\xi_1(t)}\\
&\quad+\frac{2\varepsilon_0\mathfrak{k}_2\mathscr{M}_{12}^8}{\mathfrak{q}}\frac{E^{u,v}(t)}{E^{u,v}(0)}K'(\varepsilon_0\frac{E^{u,v}(t)}{E^{u,v}(0)}) - \frac{\mathscr{M}_{12}^8(g_2'\diamond^{A_2}\nabla v)(t)}{\xi_2(t)}\\
&\quad- \mathscr{M}_2^8E^{u,v}(t)K'(\varepsilon_0\frac{E^{u,v}(t)}{E^{u,v}(0)}) \quad \text{for a.e. } t\in\mathbb{R}_+.\end{aligned} \tag{174}$$

With Lemma 3 (in particular, (55)) and (174) as our main tools, we perform some routine computations, to arrive at

$$\begin{aligned}\natural'(t)\min(\xi_1(t),\xi_2(t)) &\leqslant \frac{2\varepsilon_0\mathfrak{k}_1\mathscr{M}_{11}^8}{\mathfrak{q}}\frac{E^{u,v}(t)}{E^{u,v}(0)}K'(\varepsilon_0\frac{E^{u,v}(t)}{E^{u,v}(0)})\min(\xi_1(t),\xi_2(t))\\
&\quad-\frac{\mathscr{M}_{11}^8(g_1'\diamond^{A_1}\nabla u)(t)}{\xi_1(t)}\min(\xi_1(t),\xi_2(t))\\
&\quad+\frac{2\varepsilon_0\mathfrak{k}_2\mathscr{M}_{12}^8}{\mathfrak{q}}\frac{E^{u,v}(t)}{E^{u,v}(0)}K'(\varepsilon_0\frac{E^{u,v}(t)}{E^{u,v}(0)})\min(\xi_1(t),\xi_2(t))\\
&\quad-\frac{\mathscr{M}_{12}^8(g_2'\diamond^{A_2}\nabla v)(t)}{\xi_2(t)}\min(\xi_1(t),\xi_2(t))\\
&\quad-\mathscr{M}_2^8E^{u,v}(t)K'(\varepsilon_0\frac{E^{u,v}(t)}{E^{u,v}(0)})\min(\xi_1(t),\xi_2(t))\\
&\leqslant -\mathscr{M}_1^{10}\mathcal{K}(\varepsilon_0\frac{E^{u,v}(t)}{E^{u,v}(0)})\min(\xi_1(t),\xi_2(t)) - \mathscr{M}_2^{10}\frac{d}{dt}E^{u,v}(t) \quad \text{for a.e. } t\in\mathbb{R}_+,\end{aligned} \tag{175}$$

in which, $\mathscr{M}_1^{10}$, $\mathscr{M}_2^{10}$ and $\mathcal{K}(t)$ are given respectively by

$$\mathscr{M}_1^{10} = \frac{\mathscr{M}_2^8E^{u,v}(0)}{\varepsilon_0} - \frac{2}{\mathfrak{q}}(\mathfrak{k}_1\mathscr{M}_{11}^8 + \mathfrak{k}_2\mathscr{M}_{12}^8),$$

$$\mathscr{M}_2^{10} = 2\max(\mathscr{M}_{11}^8, \mathscr{M}_{12}^8),$$

and

$$\mathcal{K}(t) = tK'(t) \quad \text{for all } t\in(0,\min(r_1,r_2)]. \tag{176}$$

The strict increasing monotonicity of $K'(t)$, the strict convexity of $K(t)$ and the continuity of $\mathcal{K}(t)$ imply that the function $\mathcal{K}(t)$ is strictly increasing. For the sake our later presentation, we introduce the auxiliary functional

$$\tilde{\natural}(t) = \natural(t)\min(\xi_1(t),\xi_2(t)) + \mathscr{M}_2^{10}E^{u,v}(t) \quad \text{for all } t\in\mathbb{R}_+. \tag{177}$$

Since $\mathscr{M}_2^{10} > 0$, $\tilde{\natural}(t) \sim E^{u,v}(t)$. More exactly, there exists, as with (159), positive constants $\mathscr{M}_1^{11}$ and $\mathscr{M}_2^{11}$, such that

$$\mathscr{M}_1^{11}\tilde{\natural}(t) \leqslant E^{u,v}(t) \leqslant \mathscr{M}_2^{11}\tilde{\natural}(t) \quad \text{for all } t \in \mathbb{R}_+. \tag{178}$$

With the help of (175) and the definition (177) of $\tilde{\natural}(t)$, we have

$$\begin{aligned}
\tilde{\natural}'(t) &= \natural'(t)\min(\xi_1(t),\xi_2(t)) + \natural(t)\frac{d}{dt}\min(\xi_1(t),\xi_2(t)) + \mathscr{M}_2^{10}\frac{d}{dt}E^{u,v}(t) \\
&\leqslant -\mathscr{M}_1^{10}\mathcal{K}(\varepsilon_0\frac{E^{u,v}(t)}{E^{u,v}(0)})\min(\xi_1(t),\xi_2(t)) \quad \text{for a.e. } t \in \mathbb{R}_+,
\end{aligned}$$

which, together with the equivalence (178), implies

$$\tilde{\natural}'(t) \leqslant -\mathscr{M}_1^{10}\mathcal{K}(\varepsilon_0\frac{\mathscr{M}_1^{11}\tilde{\natural}(t)}{E^{u,v}(0)})\min(\xi_1(t),\xi_2(t)) \quad \text{for a.e. } t \in \mathbb{R}_+, \tag{179}$$

where the positive constant ε_0 is appropriately picked so that

$$\varepsilon_0\frac{\mathscr{M}_1^{11}\tilde{\natural}(t)}{E^{u,v}(0)} \leqslant \min(r_1,r_2) \quad \text{for all } t \in \mathbb{R}_+.$$

Based on the differential inequality (179), we perform some routine calculations, to yield

$$\begin{aligned}
\mathscr{M}_1^{10}\int_0^t \min(\xi_1(s),\xi_2(s))ds &\leqslant \int_0^t \frac{-\tilde{\natural}'(s)}{\mathcal{K}(\varepsilon_0\frac{\mathscr{M}_1^{11}\tilde{\natural}(s)}{E^{u,v}(0)})}ds \\
&= \frac{E^{u,v}(0)}{\varepsilon_0\mathscr{M}_1^{11}}\int_{\frac{\varepsilon_0\mathscr{M}_1^{11}\tilde{\natural}(t)}{E^{u,v}(0)}}^{\frac{\varepsilon_0\mathscr{M}_1^{11}\tilde{\natural}(0)}{E^{u,v}(0)}} \frac{ds}{\mathcal{K}(s)} \\
&\leqslant \frac{E^{u,v}(0)}{\varepsilon_0\mathscr{M}_1^{11}}\int_{\frac{\varepsilon_0\mathscr{M}_1^{11}\tilde{\natural}(t)}{E^{u,v}(0)}}^{\min(r_1,r_2)} \frac{ds}{\mathcal{K}(s)} \\
&= \frac{E^{u,v}(0)}{\varepsilon_0\mathscr{M}_1^{11}}\mathscr{K}(\frac{\varepsilon_0\mathscr{M}_1^{11}\tilde{\natural}(t)}{E^{u,v}(0)}) \quad \text{for all } t \in \mathbb{R}_+,
\end{aligned} \tag{180}$$

where the function $\mathscr{K}(t)$ is defined as in (153). It is not difficult to observe that

$$\mathscr{K}(t) = \int_t^{\min(r_1,r_2)} \frac{ds}{\mathcal{K}(s)} \quad \text{for all } t \in (0,\min(r_1,r_2)],$$

where the function $\mathcal{K}(t)$ is given as in (176). By recalling that the function $\mathscr{K}(t)$ is strictly decreasing (see Remark 8 for the detailed explanation), we deduce from (180) that

$$\frac{\varepsilon_0\mathscr{M}_1^{11}\tilde{\natural}(t)}{E^{u,v}(0)} \leqslant \mathscr{K}^{-1}\Big(\frac{\varepsilon_0\mathscr{M}_1^{10}\mathscr{M}_1^{11}}{E^{u,v}(0)}\int_0^t \min(\xi_1(s),\xi_2(s))ds\Big) \quad \text{for all } t \in \mathbb{R}_+,$$

which, together with the equivalence (178), implies directly

$$E^{u,v}(t) \leqslant \frac{\mathscr{M}_2^{11}E^{u,v}(0)}{\varepsilon_0\mathscr{M}_1^{11}}\mathscr{K}^{-1}\Big(\frac{\varepsilon_0\mathscr{M}_1^{10}\mathscr{M}_1^{11}}{E^{u,v}(0)}\int_0^t \min(\xi_1(s),\xi_2(s))ds\Big) \text{ for all } t \in \mathbb{R}_+.$$

This, together with (160), implies that the proof of Theorem 4 is complete. □

4. Conclusions

In this paper, we studied the initial boundary value problem (that is, IBVP (1)) for a coupled system of two quasi-linear viscoelastic space-variable coefficient wave equations. We proved (see Theorems 1 and 2 for the details), under some seemingly natural conditions on A_1, A_2, μ_1, μ_2, ρ_1, ρ_2, f_1, f_2, g_1, g_2, a_{11}, a_{12}, a_{21} and a_{22}, via the celebrated Faedo–Galerkin method, that IBVP (1) admits a local solution in the sense of Definition 1 and 2 for every initial datum in the space

$$L^\infty(\mathbb{R}_-; H_0^1(\Omega;\mathbb{R}^2)) \times H_0^1(\Omega;\mathbb{R}^2).$$

Based on our new obtained local existence results, we proved, via establishing a priori inequalities, a global existence result for solutions, having small initial data, to IBVP (1) (see Theorem 3 for the details). Based on our new established global existence result, we proved via constructing various modified energy functionals (functionals, equivalent to the energy functional $E^{u,v}(t)$, defined by (18), of IBVP (1) and can be seen as Lyapunov functional from other perspectives), that if the initial data satisfy some additional conditions, then global in time solutions would decrease to zero, at the optimal decaying rate in a sense given by Remark 2.3 in Reference [22], as time escapes to infinity; see Theorem 4.

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