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Finsler Warped Product Metrics with Special Curvature Properties

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Abstract: The class of warped product metrics can often be interpreted as key space models for the general theory of relativity and theory of space–time. In this paper, we study several non-Riemannian quantities in Finsler geometry. These non-Riemannian quantities play an important role in understanding the geometric properties of Finsler metrics. In particular, we find differential equations of Finsler warped product metrics with vanishing χ-curvature or vanishing H-curvature. Furthermore, we show that, for Finsler warped product metrics, the χ-curvature vanishes if and only if the H-curvature vanishes.

Keywords: Finsler warped product metrics; χ-curvature; H-curvature

MSC: 53C30; 53C60

1. Introduction

There are several non-Riemannian quantities in Finsler geometry, such as the distortion, the (mean) Cartan torsion, the S-curvature, the (mean) Berwald curvature and the (mean) Landsberg curvature. We view the distortion and the (mean) Cartan torsion as non-Riemannian quantities of order zero, and the S-curvature, the (mean) Berwald curvature and the (mean) Landsberg curvature as non-Riemannian quantities of order one. Differentiating these quantities along geodesics, we obtain some non-Riemannian quantities of order two.

In this paper, we will consider two non-Riemannian quantities $\chi = \chi_i dx^i$ and $H = H_{ij} dx^i \otimes dx^j$ on the tangent bundle TM:

$$\chi_i := S_{.i|m} y^m - S_{|i},$$
$$H_{ij} := \frac{1}{2} S_{.i.j|m} y^m,$$

where S denotes the S-curvature of F and "|" and "." denote the horizontal and vertical covariant derivatives with respect to the Chern connection, respectively.

Shen [1] showed some relationships among the flag curvature, the S-curvature, the χ-curvature and the H-curvature. Cheng and Yuan [2] obtained a formula of χ-curvature for (α, β)-metrics. Based on this, they showed that the χ-curvature vanishes for a class of (α, β)-metrics. Shen [3] discussed several expressions for the χ-curvature of a spray. They showed that sprays, obtained by a projective deformation using the S-curvature, always have vanishing χ-curvature. They established a Beltrami theorem for sprays with vanishing χ-curvature.

The non-Riemannian quantity H was introduced by Zadeh [4] and developed by some other Finslerian geometers [5,6]. Xia [7] obtained some rigidity theorems of a compact Finsler manifold under some conditions related to H-curvature. They proved that the S-curvature for a Randers metric is almost isotropic if and only if the H-curvature almost vanishes. In particular, S-curvature is isotropic if and only if the H-curvature vanishes.

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Tang [8] showed that Randers metrics have almost isotropic S-curvature if and only if they have almost vanishing H-curvature. Furthermore, Randers metrics actually have zero S-curvature if and only if they have vanishing H-curvature. Mo [9] gave a characterization of spherically symmetric Finsler metrics with almost vanishing H-curvature. Zhu [10] showed that the χ-curvature vanishes if and only if the H-curvature vanishes for general (α, β)-metrics under some conditions. Sevim and Gabrani [11] showed that, on Finsler warped product manifolds, the χ-curvature vanishes if and only if the H-curvature vanishes.

The warped product metric was introduced by Bishop and O'Neill [12] to study Riemannian manifolds with negative curvature as a generalization of Riemannian product metrics. The notion of warped products was extended to the case of Finsler manifolds by Chen-Shen-Zhao [13] and Kozma-Peter-Varga [14], respectively. These metrics are called Finsler warped product metrics.

In [15], Shen and Marcal considered a new class of Finsler metrics using the warped product notion introduced by Chen-Shen-Zhao [13], with another "warping". This metric is consistent with static spacetime. They gave partial differential equations (PDEs) characterization for the proposed metrics to be Ricci-flat. Furthermore, they explicitly constructed two types of non-Riemannian examples.

In this paper, we obtain differential equations of such metrics with vanishing χ-curvature or vanishing H-curvature. Then, we obtain that the χ-curvature vanishes if and only if the H-curvature vanishes. The main results are as follows.

Theorem 1. *Let $F = \alpha\sqrt{\phi(z, \rho)}$ be a Finsler warped product metric on an $(n+1)$-dimensional manifold $M = \mathbb{R} \times \mathbb{R}^n$ $(n \geq 2)$, where $\alpha = |\bar{y}|$, $z = \frac{y^0}{|\bar{y}|}$, $\rho = |\bar{x}|$. Then, the χ-curvature vanishes if and only if the H-curvature vanishes.*

A Finsler metric F is said to be R-quadratic if its Riemann curvature R_v is quadratic in $v \in T_uM$ [16]. Najafi-Bidabad-Tayebi [17] and Mo [6] showed that all R-quadratic Finsler metrics have vanishing H-curvature, respectively. For a R-quadratic Finsler warped product metric, we have the following result.

Corollary 1. *Let $F = \alpha\sqrt{\phi(z, \rho)}$ be a Finsler warped product metric on an $(n+1)$-dimensional manifold $M = \mathbb{R} \times \mathbb{R}^n$ $(n \geq 2)$, where $\alpha = |\bar{y}|$, $z = \frac{y^0}{|\bar{y}|}$, $\rho = |\bar{x}|$. Suppose that F is R-quadratic, then the χ-curvature vanishes.*

2. Preliminaries

Set $M = \mathbb{R} \times \mathbb{R}^n$ with the following coordinates on TM:

$$x = \left(x^0, \bar{x}\right), \ \bar{x} = \left(x^1, \ldots, x^n\right),$$
$$y = \left(y^0, \bar{y}\right), \ \bar{y} = \left(y^1, \ldots, y^n\right).$$

Furthermore, consider a Finsler metric as follows:

$$F = \alpha\sqrt{\phi(z, \rho)},$$

where $\alpha = |\bar{y}|$, $z = \frac{y^0}{|\bar{y}|}$, $\rho = |\bar{x}|$ and ϕ is a suitable function on $\mathbb{R}^2$.

Throughout this paper, our index conventions are as follows:

$$0 \leq A, B, \ldots \leq n,$$
$$1 \leq i, j, \ldots \leq n.$$

For a Finsler warped product metric $F = \alpha\sqrt{\phi(z,\rho)}$, the fundamental form $g = g_{AB}dx^A \otimes dx^B$ is given by:

$$(g_{AB}) = \left(\begin{array}{c|c} \frac{1}{2}\phi_{zz} & \frac{1}{2}\Omega_z\frac{y^j}{\alpha} \\ \hline \frac{1}{2}\Omega_z\frac{y^i}{\alpha} & \frac{1}{2}\Omega\delta_{ij} - \frac{1}{2}z\Omega_z\frac{y^iy^j}{\alpha^2} \end{array}\right),$$

where $\Omega := 2\phi - z\phi_z$. Then:

$$\det(g_{AB}) = \frac{1}{2^{n+1}}\Omega^{n-1}\Lambda,$$

where $\Lambda := 2\phi\phi_{zz} - \phi_z^2$.

Henceforth, assume F is non-degenerate. In this case, the inverse of (g_{AB}) is:

$$\left(g^{AB}\right) = \left(\begin{array}{c|c} \frac{2}{\Lambda}(\Omega - z\Omega_z) & -\frac{2}{\Lambda}\Omega_z\frac{y^j}{\alpha} \\ \hline -\frac{2}{\Lambda}\Omega_z\frac{y^i}{\alpha} & \frac{2}{\Omega}\delta^{ij} + \frac{2\phi_z\Omega_z}{\Omega\Lambda}\frac{y^iy^j}{\alpha^2} \end{array}\right).$$

Proposition 1 ([15])**.** *$F = \alpha\sqrt{\phi(z,\rho)}$ is strongly convex if and only if $\Omega, \Lambda > 0$.*

The spray coefficients G^A are defined by:

$$G^A := \frac{1}{4}g^{AC}\left[\left(F^2\right)_{y^Cx^B}y^B - \left(F^2\right)_{x^C}\right].$$

The Riemann curvature of F is a family of endomorphisms:

$$R_y = R^A{}_B dx^B \otimes \frac{\partial}{\partial x^A} : T_xM \to T_xM,$$

defined by:

$$R^A{}_B := 2\left(G^A\right)_{x^B} - \left(G^A\right)_{x^Cy^B}y^C + 2G^C\left(G^A\right)_{y^Cy^B} - \left(G^A\right)_{y^C}\left(G^C\right)_{x^B}.$$

For the Riemannian curvature $R^A{}_B$ of the Finsler warped product metric $F = \alpha\sqrt{\phi(z,\rho)}$, we have [15]:

$$\begin{aligned} R^0{}_0 =& [\rho^2(U + zV)W_z - (2\rho^2W + 1)(U_z + V + zV_z)]\alpha^2 \\ &+ [2(V + W)(U_z + V + zV_z) - (V_z + W_z)(U + zV) \\ &+ 2U(U_{zz} + 2V_z + zV_{zz}) - \frac{1}{\rho}(U_{\rho z} + V_\rho + zV_{\rho z}) \\ &- (U_z + V + zV_z)^2 - (U - zU_z - z^2V_z)V_z]\langle\bar{x},\bar{y}\rangle^2, \end{aligned}$$

$$\begin{aligned} R^0{}_j =& z[(2\rho^2W + 1)(V + U_z + zV_z) - \rho^2W_z(U + zV)]\alpha y^j \\ &+ [z(U + zV)(V_z + W_z) - 2zU(U_{zz} + 2V_z + zV_{zz}) \\ &+ (U - zU_z - z^2V_z)(5W - U_z) - \frac{1}{\rho}(U_\rho - zU_{\rho z} - z^2V_{\rho z})]\langle\bar{x},\bar{y}\rangle^2\frac{y^j}{\alpha} \\ &+ [(U + zV)(U_z - V + zV_z - 2W) + (V - 3W)(U - zU_z - z^2V_z) \\ &+ \frac{1}{\rho}(U_\rho + zV_\rho)]\langle\bar{x},\bar{y}\rangle\alpha x^j, \end{aligned}$$

$$
\begin{aligned}
R^i{}_0 =&[\rho^2 W_z(V-W)-(2\rho^2W+1)V_z]\alpha y^i+[(2W-V-U_z)(V_z+W_z)\\
&+2U(V_{zz}+W_{zz})-\frac{1}{\rho}(V_{\rho z}+W_{\rho z})]\langle\bar{x},\bar{y}\rangle^2\frac{y^i}{\alpha}\\
&+[(U_z-W)W_z-2UW_{zz}+\frac{1}{\rho}W_{\rho z}]\langle\bar{x},\bar{y}\rangle\alpha x^i,
\end{aligned}
$$

$$
\begin{aligned}
R^i{}_j =&-[2W+(2\rho^2W+1)(V+W)]\alpha^2\delta^i_j\\
&+[(V+W)^2+2U(V_z+W_z)-\frac{1}{\rho}(V_\rho+W_\rho)]\langle\bar{x},\bar{y}\rangle^2\delta^i_j\\
&+[2W(2W-zW_z)+W_z(U-zW)-\frac{2}{\rho}W_\rho]\alpha^2x^ix^j+[(V+W)\\
&+z(V_z+W_z)(2\rho^2W+1)+(\rho^2(V+W)+1)(2W-zW_z)]y^iy^j\\
&-[2zU(V_{zz}+W_{zz})+(3U-zU_z-zV+5zW)(V_z+W_z).\\
&-\frac{z}{\rho}(V_{\rho z}+W_{\rho z})]\langle\bar{x},\bar{y}\rangle^2\frac{y^iy^j}{\alpha^2}+[-(2W-zW_z)^2-2U(W_z-zW_{zz})\\
&+\frac{1}{\rho}(2W_\rho-zW_{\rho z})+W_z(U-zU_z-z^2W_z)]\langle\bar{x},\bar{y}\rangle x^iy^j\\
&+[-(V+W)^2+(V_z+W_z)(U+3zW)+\frac{1}{\rho}(V_\rho+W_\rho)]\langle\bar{x},\bar{y}\rangle x^jy^i,
\end{aligned}
$$

where $\langle\bar{x},\bar{y}\rangle := \sum_{k=1}^n x^ky^k$,

$$
U := \frac{1}{2\rho\Lambda}\Big(2\phi\phi_{\rho z}-\phi_\rho\phi_z\Big), \quad V := \frac{1}{2\rho\Lambda}\Big(\phi_\rho\phi_{zz}-\phi_z\phi_{\rho z}\Big), \quad W := \frac{1}{2\rho\Omega}\phi_\rho. \tag{1}
$$

Thus, the Ricci curvature of $F=\alpha\sqrt{\phi(z,\rho)}$ is [15]:

$$
\begin{aligned}
\mathrm{Ric} :=& R^A{}_A\\
=&[-(2\rho^2W+1)(U_z+nV+(n-3)W)-2(nW+\rho W_\rho-\rho^2W_z(U-zW))]\alpha^2\\
&+[2U(U_{zz}+nV_z+(n-2)W_z)-\frac{1}{\rho}(U_{\rho z}+nV_\rho+(n-3)W_\rho)\\
&+nV(V+2W)+W((n-5)W+2zW_z)+U_z(2W-U_z)]\langle\bar{x},\bar{y}\rangle^2.
\end{aligned}
$$

3. χ-Curvature

In this section, we first derive the expression for the χ-curvature of a Finsler warped product metric $F=\alpha\sqrt{\phi(z,\rho)}$. Then, we obtain differential equations of such metrics with vanishing χ-curvature.

Lemma 1 ([15])**.** *For a Finsler warped product metric $F=\alpha\sqrt{\phi(z,\rho)}$, the χ-curvature of F is given by:*

$$
\chi_0 = (\frac{1}{\rho}\Psi_{\rho z}-2U\Psi_{zz}-2W\Psi_z)\frac{\langle\bar{x},\bar{y}\rangle^2}{\alpha}+(2\rho^2W+1)\Psi_z\alpha,
$$

$$
\begin{aligned}
\chi_i =& [2zU\Psi_{zz}-\frac{z}{\rho}\Psi_{\rho z}+2(U+2zW)\Psi_z]\frac{\langle\bar{x},\bar{y}\rangle^2}{\alpha^2}y^i\\
&-z(2\rho^2W+1)\Psi_zy^i-2(U+zW)\Psi_z\langle\bar{x},\bar{y}\rangle x^i,
\end{aligned}
$$

where

$$
\Psi := U_z+(n+2)V+(n-1)W.
$$

Lemma 2 ([15]). *For $n \geq 2$, $A\alpha^2 + B\langle \bar{x}, \bar{y}\rangle^2 = 0$ if and only if $A = 0$ and $B = 0$, where A, B are functions of z and ρ.*

Lemma 3. *For $n \geq 2$:*

$$A\alpha^2 y^i + B\langle \bar{x}, \bar{y}\rangle \alpha^2 x^i + C\langle \bar{x}, \bar{y}\rangle^2 y^i = 0 \tag{2}$$

if and only if $A = 0$, $B = 0$ and $C = 0$, where A, B, C are functions of z and ρ.

Proof. "Necessity". Suppose that (2) holds. Contracting (2) with y^i, we have:

$$A\alpha^2 + (B + C)\langle \bar{x}, \bar{y}\rangle^2 = 0.$$

By Lemma 2, we obtain $A = 0, B + C = 0$.

Thus, (2) can be simplified as $B\left(\alpha^2 x^i - \langle \bar{x}, \bar{y}\rangle y^i\right) = 0$. Contracting it with x^i yields:

$$B\left(\rho^2\alpha^2 - \langle \bar{x}, \bar{y}\rangle^2\right) = 0.$$

We obtain $B = 0$. Thus, $A = 0, B = 0$ and $C = 0$.

"Sufficiency". It is obviously true. □

Theorem 2. *Let $F = \alpha\sqrt{\phi(z, \rho)}$ be a Finsler warped product metric on an $(n+1)$-dimensional manifold $M = \mathbb{R} \times \mathbb{R}^n$ $(n \geq 2)$, where $\alpha = |\bar{y}|, z = \frac{y^0}{|\bar{y}|}, \rho = |\bar{x}|$. Then F has vanishing χ-curvature if and only if ϕ satisfies $\Psi_z = 0$.*

Proof. "Necessity". Suppose that F has vanishing χ-curvature, i.e., $\chi_0 = 0, \chi_i = 0$. For $\chi_0 = 0$, by Lemma 1 and Lemma 2, we obtain that:

$$\begin{cases} 0 = -2\rho U\Psi_{zz} + \Psi_{\rho z} - 2\rho W\Psi_z, & (3) \\ 0 = \left(2\rho^2 W + 1\right)\Psi_z. & (4) \end{cases}$$

Since $\chi_i = 0$, by Lemma 1 and Lemma 3, we obtain that:

$$\begin{cases} 0 = 2\rho z U\Psi_{zz} - z\Psi_{\rho z} + 2\rho(U + 2zW)\Psi_z, & (5) \\ 0 = (U + zW)\Psi_z. & (6) \end{cases}$$

Since (5) $- 2\rho \times$ (6) $= -z \times$ (3), F has vanishing χ-curvature if and only if:

$$\begin{cases} 0 = \left(2\rho^2 W + 1\right)\Psi_z, \\ 0 = 2\rho z U\Psi_{zz} - z\Psi_{\rho z} + 2\rho(U + 2zW)\Psi_z, \\ 0 = (U + zW)\Psi_z. \end{cases} \tag{7}$$

We divide the problem into two cases:

Case(i) $\Psi_z = 0$. It is easy to verify that (7) holds.

Case(ii) $\Psi_z \neq 0$. We see that (7) is equivalent to:

$$\begin{cases} 0 = 2\rho^2 W + 1, & (8) \\ 0 = U + zW, & (9) \\ 0 = 2\rho U\Psi_{zz} - \Psi_{\rho z} + 2\rho W\Psi_z. & (10) \end{cases}$$

By (8), we have:

$$W = -\frac{1}{2\rho^2}.$$

Substituting it into $W := \frac{1}{2\rho\Omega}\phi_\rho$ yields:

$$2\phi - z\phi_z + \rho\phi_\rho = 0. \tag{11}$$

Plugging $W = -\frac{1}{2\rho^2}$ into (9), we obtain:

$$U = \frac{z}{2\rho^2}.$$

By (11) and $V := \frac{1}{2\rho\Lambda}\left(\phi_\rho\phi_{zz} - \phi_z\phi_{\rho z}\right)$, we have:

$$V = -\frac{1}{2\rho^2}.$$

Finally, we obtain $\Psi := U_z + (n+2)V + (n-1)W = -\frac{n}{\rho^2}$. Hence, $\Psi_z = 0$, which is a contradiction to our assumption.

"Sufficiency". It is obvious by Lemma 1.

This completes the proof of Theorem 2. □

4. *H*-Curvature

In this section, we derive the expression for the H-curvature of Finsler warped product metric $F = \alpha\sqrt{\phi(z,\rho)}$. Then, we obtain differential equations of such metrics with vanishing H-curvature.

The H-curvature can be expressed in terms of χ-curvature [8], that is:

$$H_{ij} = \frac{1}{4}\left(\chi_{i.j} + \chi_{j.i}\right). \tag{12}$$

Lemma 4. *For a Finsler warped product metric $F = \alpha\sqrt{\phi(z,\rho)}$, the H-curvature is given by:*

$$\begin{aligned}
H_{00} =& [\frac{1}{2\rho}\Psi_{\rho zz} - (U_z + W)\Psi_{zz} - U\Psi_{zzz} - W_z\Psi_z]\frac{\langle\bar{x},\bar{y}\rangle^2}{\alpha^2} + \frac{1}{2}(2\rho^2 W + 1)\Psi_{zz} + \rho^2 W_z\Psi_z,\\
H_{0i} =& \frac{1}{2}[-\frac{z}{\rho}\Psi_{\rho zz} + 2zU\Psi_{zzz} - \frac{1}{\rho}\Psi_{\rho z} + (3U + 3zW + 2zU_z)\Psi_{zz} + (3zW_z + 3W + U_z)\Psi_z]\frac{\langle\bar{x},\bar{y}\rangle^2}{\alpha^3}y^i\\
&+ \frac{1}{2}[\frac{1}{\rho}\Psi_{\rho z} - (3U + zW)\Psi_{zz} - (3W + U_z + zW_z)\Psi_z]\frac{\langle\bar{x},\bar{y}\rangle}{\alpha}x^i\\
&- \frac{z}{2}[2\rho^2 W_z\Psi_z + (2\rho^2 W + 1)\Psi_{zz}]\frac{y^i}{\alpha},\\
H_{ij} =& \frac{1}{2}[-2z(4U + zU_z + 2zW)\Psi_{zz} - 2z^2 U\Psi_{zzz} + \frac{3z}{\rho}\Psi_{\rho z} + \frac{z^2}{\rho}\Psi_{\rho zz}\\
&- 2(6zW + zU_z + 2z^2 W_z + 2U)\Psi_z]\frac{\langle\bar{x},\bar{y}\rangle^2}{\alpha^4}y^i y^j\\
&+ \frac{z}{2}[(2\rho^2 W + 1 + 2z\rho^2 W_z)\Psi_z + z(2\rho^2 W + 1)\Psi_{zz}]\frac{y^i y^j}{\alpha^2}\\
&+ \frac{1}{2}[(zU_z + 5zW + z^2 W_z + 2U)\Psi_z + z(3U + zW)\Psi_{zz} - \frac{z}{\rho}\Psi_{\rho z}]\frac{\langle\bar{x},\bar{y}\rangle}{\alpha^2}(x^i y^j + x^j y^i)\\
&+ \frac{1}{2}[2zU\Psi_{zz} - \frac{z}{\rho}\Psi_{\rho z} + 2(U + 2zW)\Psi_z]\frac{\langle\bar{x},\bar{y}\rangle^2}{\alpha^2}\delta_{ij}\\
&- \frac{z}{2}(2\rho^2 W + 1)\Psi_z\delta_{ij} - (U + zW)\Psi_z x^i x^j.
\end{aligned}$$

Proof. For a Finsler warped product metric $F = \alpha\sqrt{\phi(z,\rho)}$:

$$\begin{aligned} H =& H_{AB}dx^A \otimes dx^B \\ =& H_{00}dx^0 \otimes dx^0 + H_{0j}dx^0 \otimes dx^j + H_{i0}dx^i \otimes dx^0 + H_{ij}dx^i \otimes dx^j, \end{aligned}$$

where $H_{00} = \frac{1}{4}(\chi_{0.0} + \chi_{0.0}), H_{0j} = \frac{1}{4}(\chi_{0.j} + \chi_{j.0}), H_{i0} = \frac{1}{4}(\chi_{i.0} + \chi_{0.i})$ and $H_{ij} = \frac{1}{4}(\chi_{i.j} + \chi_{j.i})$. Differentiating χ_A with respect to y, we obtain:

$$\begin{aligned}
\chi_{0.0} =& [\frac{1}{\rho}\Psi_{\rho zz} - 2(U_z + W)\Psi_{zz} - 2U\Psi_{zzz} - 2W_z\Psi_z]\frac{\langle\bar{x},\bar{y}\rangle^2}{\alpha^2} + (2\rho^2 W + 1)\Psi_{zz} + 2\rho^2 W_z\Psi_z, \\
\chi_{0.i} =& [-\frac{z}{\rho}\Psi_{\rho zz} + 2zU\Psi_{zzz} - \frac{1}{\rho}\Psi_{\rho z} + 2(zU_z + zW + U)\Psi_{zz} + 2(zW_z + W)\Psi_z]\frac{\langle\bar{x},\bar{y}\rangle^2}{\alpha^3}y^i \\
&+ 2(\frac{1}{\rho}\Psi_{\rho z} - 2U\Psi_{zz} - 2W\Psi_z)\frac{\langle\bar{x},\bar{y}\rangle}{\alpha}x^i \\
&+ [(2\rho^2 W + 1 - 2z\rho^2 W_z)\Psi_z - z(2\rho^2 W + 1)\Psi_{zz}]\frac{y^i}{\alpha}, \\
\chi_{i.0} =& [-\frac{z}{\rho}\Psi_{\rho zz} + 2zU\Psi_{zzz} - \frac{1}{\rho}\Psi_{\rho z} + 2(zU_z + 2zW + 2U)\Psi_{zz} + 2(U_z + 2zW_z + 2W)\Psi_z]\frac{\langle\bar{x},\bar{y}\rangle^2}{\alpha^3}y^i \\
&- 2[(U + zW)\Psi_{zz} + (U_z + zW_z + W)\Psi_z]\frac{\langle\bar{x},\bar{y}\rangle}{\alpha}x^i \\
&- [z(2\rho^2 W + 1)\Psi_{zz} + (2\rho^2 W + 1 + 2z\rho^2 W_z)\Psi_z]\frac{y^i}{\alpha}, \\
\chi_{i.j} =& [-2z^2 U\Psi_{zzz} + \frac{z^2}{\rho}\Psi_{\rho zz} - 2z(4U + zU_z + 2zW)\Psi_{zz} + \frac{3z}{\rho}\Psi_{\rho z} \\
&- 2(6zW + zU_z + 2z^2 W_z + 2U)\Psi_z]\frac{\langle\bar{x},\bar{y}\rangle^2}{\alpha^4}y^i y^j \\
&+ z[(2\rho^2 W + 1 + 2z\rho^2 W_z)\Psi_z + z(2\rho^2 W + 1)\Psi_{zz}]\frac{y^i y^j}{\alpha^2} \\
&+ 2z[(U_z + W + zW_z)\Psi_z + (U + zW)\Psi_{zz}]\frac{\langle\bar{x},\bar{y}\rangle}{\alpha^2}x^i y^j \\
&+ 2[2(U + 2zW)\Psi_z + 2zU\Psi_{zz} - \frac{z}{\rho}\Psi_{\rho z}]\frac{\langle\bar{x},\bar{y}\rangle}{\alpha^2}x^j y^i \\
&+ [2zU\Psi_{zz} - \frac{z}{\rho}\Psi_{\rho z} + 2(U + 2zW)\Psi_z]\frac{\langle\bar{x},\bar{y}\rangle^2}{\alpha^2}\delta_{ij} - z(2\rho^2 W + 1)\Psi_z\delta_{ij} - 2(U + zW)\Psi_z x^i x^j.
\end{aligned}$$

By simple calculations, we obtain the expression of H_{AB}. □

Lemma 5. *For $n \geq 2$:*

$$A\alpha^4 x^i + B\langle\bar{x},\bar{y}\rangle^2\alpha^2 x^i + C\langle\bar{x},\bar{y}\rangle\alpha^2 y^i + D\langle\bar{x},\bar{y}\rangle^3 y^i = 0 \tag{13}$$

if and only if $A = 0, B = 0, C = 0$ and $D = 0$, where A, B, C, D are functions of z and ρ.

Proof. "Necessity". Suppose that (13) holds. Contracting (13) with y^i yields:

$$A\langle\bar{x},\bar{y}\rangle\alpha^4 + B\langle\bar{x},\bar{y}\rangle^3\alpha^2 + C\langle\bar{x},\bar{y}\rangle\alpha^4 + D\langle\bar{x},\bar{y}\rangle^3\alpha^2 = 0,$$

that is:

$$(A + C)\alpha^2 + (B + D)\langle\bar{x},\bar{y}\rangle^2 = 0.$$

By Lemma 2, we obtain $C = -A, D = -B$.

Thus, (13) can be simplified as $A\alpha^4 x^i + B\langle\bar{x},\bar{y}\rangle^2\alpha^2 x^i - A\langle\bar{x},\bar{y}\rangle\alpha^2 y^i - B\langle\bar{x},\bar{y}\rangle^3 y^i = 0$. Contracting it with x^i yields:

$$\Big(\rho^2\alpha^2 - \langle\bar{x},\bar{y}\rangle^2\Big)\Big[A\alpha^2 + B\langle\bar{x},\bar{y}\rangle^2\Big] = 0.$$

By Lemma 2, we obtain $A = 0$ and $B = 0$. Thus, $A = 0, B = 0, C = 0$ and $D = 0$.

"Sufficiency". It is obviously true. □

Lemma 6. *For $n \geq 2$:*

$$A\langle\bar{x},\bar{y}\rangle^2 y^i y^j + B\alpha^2 y^i y^j + C\alpha^4 x^i x^j + D\langle\bar{x},\bar{y}\rangle\alpha^2\Big(x^i y^j + x^j y^i\Big) + E\langle\bar{x},\bar{y}\rangle^2\alpha^2\delta_{ij} + F\alpha^4\delta_{ij} = 0 \tag{14}$$

if and only if $A = 0, B = -F = C\rho^2$ and $E = C = -D$, where A, B, C, D, E, F are functions of z and ρ. In particular, for $n > 2$, if (14) holds, then $A = B = C = D = E = F = 0$.

Proof. "Necessity". Suppose that (14) holds. Contracting (14) with y^j, we have:

$$(B + F)\alpha^2 y^i + (C + D)\langle\bar{x},\bar{y}\rangle\alpha^2 x^i + (A + D + E)\langle\bar{x},\bar{y}\rangle^2 y^i = 0.$$

By Lemma 3, we obtain $F = -B, D = -C$ and $E = -A + C$.

Thus, (14) can be simplified as $A\langle\bar{x},\bar{y}\rangle^2 y^i y^j + B\alpha^2 y^i y^j + C\alpha^4 x^i x^j - C\langle\bar{x},\bar{y}\rangle\alpha^2\Big(x^i y^j + x^j y^i\Big) + (-A + C)\langle\bar{x},\bar{y}\rangle^2\alpha^2\delta_{ij} - B\alpha^4\delta_{ij} = 0$. Contracting it with x^j yields:

$$\Big(C\rho^2 - B\Big)\alpha^4 x^i - A\langle\bar{x},\bar{y}\rangle^2\alpha^2 x^i + \Big(B - C\rho^2\Big)\langle\bar{x},\bar{y}\rangle\alpha^2 y^i + A\langle\bar{x},\bar{y}\rangle^3 y^i = 0.$$

By Lemma 5, we obtain $C\rho^2 - B = 0$ and $A = 0$. Thus, $A = 0, B = -F = C\rho^2$ and $E = C = -D$.

In this case, (14) can be rewritten as:

$$\alpha^2 C\Big[\rho^2 y^i y^j + \alpha^2 x^i x^j - \langle\bar{x},\bar{y}\rangle(x^i y^j + x^j y^i) + \langle\bar{x},\bar{y}\rangle^2\delta^{ij} - \rho^2\alpha^2\delta^{ij}\Big] = 0. \tag{15}$$

Now putting $i = j$ and taking summation over i, we obtain:

$$(n-2)C\alpha^2\Big(\langle\bar{x},\bar{y}\rangle^2 - \rho^2\alpha^2\Big) = 0.$$

Thus, when $n = 2$, the above equation always holds; when $n > 2$, we obtain $C = 0$. Thus, $A = B = C = D = E = F = 0$.

"Sufficiency". When $n > 2$, it is obviously true. When $n = 2$, we see that the right side of (14) is reduced to the left side of (15). Furthermore, we have that (15) holds for any i and j $(1 \leq i, j \leq 2)$. Thus, (14) holds. This completes the proof of Lemma 6. □

Theorem 3. *Let $F = \alpha\sqrt{\phi(z,\rho)}$ be a Finsler warped product metric on an $(n+1)$-dimensional manifold $M = \mathbb{R}\times\mathbb{R}^n$ $(n \geq 2)$, where $\alpha = |\bar{y}|, z = \frac{y^0}{|\bar{y}|}, \rho = |\bar{x}|$. Then, F has vanishing H-curvature if and only if ϕ satisfies $\Psi_z = 0$.*

Proof. "Necessity". Suppose that F has vanishing H-curvature, i.e, $H_{00} = 0, H_{0i} = 0$, $H_{ij} = 0$. Since $H_{00} = 0$, by Lemma 4 and Lemma 2, we obtain that:

$$\begin{cases} 0 = \Big[2\rho U\Psi_{zz} - \Psi_{\rho z} + 2\rho W\Psi_z\Big]_z, & (16)\\ 0 = \Big[\Big(2\rho^2 W + 1\Big)\Psi_z\Big]_z. & (17)\end{cases}$$

For $H_{0i} = 0$, by Lemma 4 and Lemma 3, we obtain that:

$$\begin{cases} 0 = 2\rho z U \Psi_{zzz} - z\Psi_{\rho zz} + \rho(3U + 3zW + 2zU_z)\Psi_{zz} - \Psi_{\rho z} \\ \quad +\rho(3zW_z + 3W + U_z)\Psi_z, & (18) \\ 0 = \rho(3U + zW)\Psi_{zz} - \Psi_{\rho z} + \rho(3W + U_z + zW_z)\Psi_z. & (19) \end{cases}$$

Since $H_{ij} = 0$, by Lemma 4 and Lemma 6, we have that:

$$\begin{cases} 0 = 2\rho z^2 U \Psi_{zzz} - z^2\Psi_{\rho zz} + 2\rho z(4U + zU_z + 2zW)\Psi_{zz} - 3z\Psi_{\rho z} \\ \quad +2\rho\left(6zW + zU_z + 2z^2W_z + 2U\right)\Psi_z, & (20) \\ 0 = z\left(2\rho^2 W + 1\right)\Psi_{zz} + \left[(1-z)\left(2\rho^2 W + 1\right) + 2\rho^2 z W_z\right]\Psi_z, & (21) \\ 0 = \left[z\left(2\rho^2 W + 1\right) + 2\rho^2(U + zW)\right]\Psi_z, & (22) \\ 0 = 2\rho^2 z U \Psi_{zz} - \rho z \Psi_{\rho z} + \left[2\rho^2(U + 2zW) - z\left(2\rho^2 W + 1\right)\right]\Psi_z, & (23) \\ 0 = \rho z(5U + zW)\Psi_{zz} - 2z\Psi_{\rho z} + \rho\left(zU_z + 9zW + z^2W_z + 4U\right)\Psi_z. & (24) \end{cases}$$

(21) $- z \times$ (17) yields:

$$(1-z)\left(2\rho^2 W + 1\right)\Psi_z = 0.$$

(22) + (23) yields:

$$4\rho^2(U + zW)\Psi_z + \rho z\left(2\rho U \Psi_{zz} - \Psi_{\rho z} + 2\rho W \Psi_z\right) = 0.$$

(24) $- 2z \times$ (19) yields:

$$-z(U + zW)\Psi_{zz} + \left[-z^2 W_z + 4(U + zW) - z(W + U_z)\right]\Psi_z = 0.$$

Since (18) = (19) $+ z \times$ (16) and (20) $= z^2 \times$ (16) $+ z \times$ (19) + (24), F has vanishing H-curvature if and only if:

$$\begin{cases} 0 = \left[2\rho U \Psi_{zz} - \Psi_{\rho z} + 2\rho W \Psi_z\right]_z, & (25) \\ 0 = \left[\left(2\rho^2 W + 1\right)\Psi_z\right]_z, & (26) \\ 0 = \rho(3U + zW)\Psi_{zz} - \Psi_{\rho z} + \rho(3W + U_z + zW_z)\Psi_z, & (27) \\ 0 = (1-z)\left(2\rho^2 W + 1\right)\Psi_z, & (28) \\ 0 = \left[z\left(2\rho^2 W + 1\right) + 2\rho^2(U + zW)\right]\Psi_z, & (29) \\ 0 = 4\rho^2(U + zW)\Psi_z + \rho z\left(2\rho U \Psi_{zz} - \Psi_{\rho z} + 2\rho W \Psi_z\right), & (30) \\ 0 = -z(U + zW)\Psi_{zz} + \left[-z^2 W_z + 4(U + zW) - z(W + U_z)\right]\Psi_z. & (31) \end{cases}$$

We divide the problem into two cases:

Case(i) $\Psi_z = 0$. It is easy to verify that (25)–(31) hold.

Case(ii) $\Psi_z \neq 0$. From (28), we can see $2\rho^2 W + 1 = 0$. Thus, $W_z = 0$. Plugging $2\rho^2 W + 1 = 0$ into (29) yields:

$$U + zW = 0.$$

Differentiating it with respect to z and combining this with $W_z = 0$, we have $U_z + W = 0$.

Substituting $2\rho^2 W + 1 = 0$ and $U + zW = 0$ into (27) yields:

$$2\rho U \Psi_{zz} - \Psi_{\rho z} + 2\rho W \Psi_z = 0.$$

It is easy to verify that (25), (26), (30) and (31) hold. Now F has vanishing H-curvature if and only if the following hold:

$$\begin{cases} 0 = 2\rho^2 W + 1, \\ 0 = U + zW, \\ 0 = 2\rho U \Psi_{zz} - \Psi_{\rho z} + 2\rho W \Psi_z. \end{cases}$$

The result is the same as in Case(ii) of Theorem 2. This is a contradiction.

"Sufficiency". It is obvious by Lemma 4.

This completes the proof of Theorem 3. □

Proof of Theorem 1. By Theorem 2 and Theorem 3, the result is obvious. □

Example 1. *(Minkowski metrics). Let $\phi = \phi(z, \rho)$ be a function defined by:*

$$\phi(z, \rho) = e^{2z},$$

where $|z| < 1$. We have the Finsler warped product metric:

$$F = \alpha\sqrt{\phi} = \alpha e^z.$$

Since $\Omega = 2(1-z)e^{2z} > 0$, $\Lambda = 4e^{4z} > 0$, by Proposition 1, we obtain that $F = \alpha\sqrt{\phi}$ gives a positive-definite metric. We know that $\Psi_z = 0$. By Theorems 2 and 3, we have that its χ-curvature and H-curvature vanish.

Example 2. *(Randers metrics). Let $\phi = \phi(z, \rho)$ be a function defined by:*

$$\phi(z, \rho) = \left(f(\rho)z + \sqrt{f^2(\rho)z^2 + g(\rho)}\right)^2,$$

where $g(\rho) > 0$ and $f(\rho)z + \sqrt{f^2(\rho)z^2 + g(\rho)} > 0$. We have the Finsler warped product metric:

$$F = \alpha\sqrt{\phi} = \alpha(f(\rho)z + \sqrt{red f^2(\rho)z^2 + g(\rho)}).$$

Since $\Omega = \frac{2g(\rho)\phi^{\frac{1}{2}}}{\sqrt{f^2(\rho)z^2+g(\rho)}} > 0$, $\Lambda = \frac{4f^2(\rho)g(\rho)\phi^{\frac{3}{2}}}{(f^2(\rho)z^2+g(\rho))^{\frac{3}{2}}} > 0$, by Proposition 1, we obtain that $F = \alpha\sqrt{\phi}$ gives a positive-definite metric. We know that $\Psi_z = 0$. By Theorems 2 and 3, we have that its χ-curvature and H-curvature vanish.

Example 3. *(Quadratic polynomial). Let $\phi = \phi(z, \rho)$ be a function defined by:*

$$\phi(z, \rho) = c_2 z^2 + c_1(\rho)z + c_0 c_1^2(\rho),$$

where c_0, c_2 are constants, $|z| < 1$, $4c_0c_2 > 1$ and $2c_0c_1(\rho) > 1$. We have the Finsler warped product metric:

$$F = \alpha\sqrt{\phi} = \alpha\sqrt{c_2 z^2 + c_1(\rho)z + c_0 c_1^2(\rho)}.$$

Since $\Omega = c_1(\rho)(z + 2c_0c_1(\rho)) > 0$, $\Lambda = c_1^2(\rho)(4c_0c_2 - 1) > 0$, by Proposition 1, we obtain that $F = \alpha\sqrt{\phi}$ gives a positive-definite metric. We know that $\Psi_z = 0$. By Theorems 2 and 3, we have that its χ-curvature and H-curvature vanish.

5. Conclusions

Non-Riemannian quantities play an important role in Finsler geometry. In this paper, we firstly obtain differential equations of Finsler warped product metrics with vanishing χ-curvature or vanishing H-curvature. Based on these, we obtain that the χ-curvature vanishes if and only if the H-curvature vanishes. Since the solution of $\Psi_z = 0$ is unknown, the classification of such metrics is to be continued.

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