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Stochastic Ordering Results on Implied Lifetime Distributions under a Specific Degradation Model

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Abstract: In this paper, a novel strategy is employed in which a degradation model affects the implied distribution of lifetimes differently compared to the traditional method. It is recognized that an existing link between the degradation measurements and failure time constructs the underlying time-to-failure model. We assume in this paper that the conditional survival function of a device under degradation is a piecewise linear function for a given level of degradation. The multiplicative degradation model is used as the underlying degradation model, which is often the case in many practical situations. It is found that the implied lifetime distribution is a classical mixture model. In this mixture model, the time to failure lies with some probabilities between two first passage times of the degradation process to reach two specified values. Stochastic comparisons in the model are investigated when the probabilities are changed. To illustrate the applicability of the results, several examples are given in cases when typical degradation models are candidates.

Keywords: multiplicative degradation model; time-to-failure model; hazard rate; majorization

MSC: 60E05; 62B10; 62N05; 94A17

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1. Introduction

The lack of observable failures often complicates reliability studies based on the time to failure. Accelerated life tests can accelerate product failure during test intervals by stressing the product beyond its typical use. Many tests supplement failure data with degradation data, which may include measurements of product wear at one or more points during the reliability test. The product life is defined as the time during which the degradation exceeds a predetermined threshold. Collecting degradation data has become necessary in many organizations, because extremely reliable equipment under test has few, if any, failures during the limited test period. A complete reference on degradation analysis for various life tests, including accelerated life tests, has shown that degradation analysis has the potential to significantly improve reliability analysis. However, degradation analysis can raise the possibility of inconsistencies in the experimenter's treatment of the data. The perceived relationship between the degradation measurements and the failure time is critical to the study.

When a stochastic model for degradation is assumed, the distribution of lifetimes is implied, as a consequence, and in many circumstances, these implied distributions of lifetimes are awkward and do not match the experimenter's expectations. The resulting estimate of the lifetime distribution usually must be solved numerically, with the uncertainty in the estimate calculated using simulations and large replicate samples such as bootstrap methods. The works related to the lifetime distributions have many applications, from technical sciences to gerontology. In the context of degradation models, lifetime prediction has many practical applications in various fields, including the following areas:

1. Automobile components: A time-to-failure model based on a degradation model can be used to predict the life of various automobile components such as engine components, brakes, and tires. To anticipate the time to failure of these parts, the model can take into account elements such as wear, corrosion, and mechanical stress.
2. Electronics: Time-to-failure models based on degradation models are extensively used in the electronics industry to predict the life of various electronic components such as capacitors, resistors, and transistors. To anticipate the time to failure of these components, these models can take into account elements such as temperature, humidity, and voltage stress.
3. Wind turbines: Based on degradation models, time-to-failure models can be used to predict the life of wind turbine components such as rotor blades, gearboxes, and generators. To anticipate the time to failure of these components, the models can take into account parameters such as wind speed, temperature, and mechanical load.
4. Aerospace: Time-to-failure models based on degradation models are extensively used in the aerospace industry to predict the life of various aircraft components such as engines, avionics systems, and landing gear. To anticipate the time to failure of these components, these models can take into account elements such as temperature, pressure, and mechanical stress.
5. Medical equipment: based on degradation models, time-to-failure models can be used to predict the life of various medical devices such as pacemakers, insulin pumps, and prosthetic joints. To anticipate the time to failure of these devices, the models can take into account elements such as wear, corrosion, and mechanical stress.

Reliability modeling and analysis of complex systems has always been an important topic in engineering. Degradation-based modeling of failure time as a fundamental process is a consistent method for analyzing the lifetime of complex systems in many practical situations (see, e.g., Nikulin et al. [1], Pham [2], Pelletier et al. [3], Chen et al. [4], and Wang et al. [5] for a monograph on this topic). The elements that deteriorate over time and have an observable process of deterioration can be considered by a stochastic deterioration model. In order to achieve and produce the high reliability of systems required by the majority of consumers, it is necessary to identify weaker systems. The relationship between the failure time and the degradation process may not be deterministic, and further investigation into the distribution of degradation levels and their impact on failure time is warranted.

The stochastic-process-based degradation model of Albabtain et al. [6] is used to model the lifetime of a system. The stochastic process is assumed to fluctuate around monotonic pattern paths. In the traditional definition, the failure of an object is assumed to correspond to the time when the degradation exceeds the given threshold $\mathfrak{D}_{\mathfrak{f}}$. Suppose the degradation process is $\{W(t), t \geq 0\}, W(0) = 0$ with a monotonically increasing sample path, as is often encountered in practice. The time to failure is denoted by T. Then, T is the time of the first pass to threshold $\mathfrak{D}_{\mathfrak{f}}$, i.e., $T = \inf\{t : W(t) > \mathfrak{D}_{\mathfrak{f}}\}$. The corresponding distribution function of the failures is denoted by F_T, and the implied survival function is denoted by $\bar{F}_T = 1 - F_T$. We also denote by $F_{W(t)}$ and $f_{W(t)}$ the distribution and density functions of $W(t)$, respectively. We have

$$\bar{F}_T(t) = P(T > t) = P(W(t) \leq \mathfrak{D}_{\mathfrak{f}}) = F_{W(t)}(\mathfrak{D}_{\mathfrak{f}}). \tag{1}$$

If $\{W(t), t \geq 0\}, W(0) = 0$ possesses a monotonically decreasing sample path, then the time to failure T is the first passage time to the threshold $\mathfrak{U}_{\mathfrak{f}}$ i.e., $T = \inf\{t : W(t) \leq \mathfrak{U}_{\mathfrak{f}}\}$. We obtain

$$\bar{F}_T(t) = P(T > t) = P(W(t) > \mathfrak{U}_{\mathfrak{f}}) = \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}}). \tag{2}$$

Degradation models differ significantly in the different areas of reliability modeling. In this section, we discuss the dynamic degradation-based model for analyzing failure time data, recently presented by Albabtain et al. [6]. The methodology underlying the model is applied to situations where a unit exhibits stochastic behavior over the time that the degradation occurs, and there is no specific value for the amount of degradation at which the unit fails. The flexible aspect of the dynamic degradation based failure time model is demonstrated when it is assumed that the failure of the unit follows a stochastic rule as part of the degradation process, as opposed to the traditional definition, where the failure of the unit is considered deterministic once the degradation amount reaches a predetermined threshold.

Suppose that the extent of the depletion at time t by $W(t)$ is denoted by pdf $f_{W(t)}(\cdot)$ and cdf $F_{W(t)}(\cdot)$. It is considered a postulate for increasing (decreasing) degradation paths that $W(t_1) \leq_{st} (\geq_{st}) W(t_2)$, for all $t_1 \leq t_2$. The previous literature has assumed that for a given threshold D_f, a system fails under degradation as soon as $W(t) > D_f$. This defines a termination rule for T that must be determined, such that $T \equiv \inf\{t \geq 0 \mid W(t) > D_f\}$. This definition of downtime was used by Albabtain et al. [6], so that an existing stochastic rule about the effect of degradation over time illustrates the process of item failure.

The failure time T under this modified setting has the sf

$$\bar{F}_T(t) = \int_0^{\infty} S(w;t) f_{W(t)}(w)\, dw = E(S(W(t);t)), \tag{3}$$

where $S(w;t)$ is the limit of a conditional probability given, at the level of degradation w, by

$$S(w;t) = \lim_{\delta \to 0+} P(T > t \mid W(t) \in (w, w+\delta]).$$

To satisfy the degradation model, the bivariate function S must satisfy the following conditions for an increasing (or decreasing) degradation path:

(i) For all $w \geq 0$ and for all $t \geq 0$, $S(w;t) \in [0,1]$.
(ii) For any fixed $w \geq 0$, $S(w;t)$ is decreasing in $t \geq 0$.
(iii) For any fixed $t \geq 0$, $S(w;t)$ is decreasing (respectively, increasing) in $w \geq 0$.

Conditions (i)–(iii) guarantee that $\bar{F}_T$ in (3) is a valid survival function. The model (3) is a dynamic failure time model in that the construction of the model is modified depending on how the survival rate of the item undergoing degradation at a given time may be influenced by the extent of the degradation. This influence is accounted for by forming the function S.

The selection of S depends primarily on the knowledge of the engineer or operator who controls the performance of the system. For example, if a system hardly (strongly) degrades with time then $S(w;t) = \exp\{-\gamma(w)t\}$ may be an appropriate choice. For a less severe degradation process, $S(w;t) = \frac{1}{(1+t)^{\gamma(w)}}$ may be more appropriate. However, if there is no information about how the system degrades with time, then everything depends on the failure time data (observations at T), and a model selection strategy can be performed, i.e., some candidates are selected, and the best of them is chosen based on some possible model selection criteria in the literature.

It is assumed that data on $W(t)$ are not available for all $t \geq 0$, since the stochastic process $\{W(t), t \geq 0\}$ is usually partially observed with reference to the known sources of degradation models. To proceed along the line of serious statistical survival models, a common feature can be assumed for $S(\cdot;t)$, such that $S(w;t) = S_0^{\gamma(w)}(t)$ is the feature of the proportional hazard rate model if $S(w;t)$ is a survival function in t for each $w \geq 0$, in which $\gamma > 0$. The function γ may depend on some parameters. The initial probability (survival rate)

$$S_0(t) = \lim_{\delta \to 0+} P(T > t \mid W(t) \in (0, \delta])$$

measures the survival probability of the system at the time t when the extent of degradation is zero. As a corollary, we may need to assume that $S_0(t)$ is itself a survival function in $t \geq 0$. For $\lambda_0 > 0$, the exponential distribution may always be a good choice, so that $S_0(t) = \exp(-\lambda_0 t)$ describes an age-free behavior of the system under degradation. The Lomax distribution with the survival function given by $S_0(t) = \frac{1}{1+\lambda_0 t}$ is also a good choice for the base survival rate.

2. Stepwise Survival Rate at Interval Degradation Levels

In the literature, the correspondence between the randomness of the degradation and the randomness of the implied lifetime distribution is assumed to be strong and direct, such that failure occurs when the degradation level of the test object reaches a predetermined threshold (D_f). In such a case, the resulting lifetime distribution follows from (3) when $S(w;t) = 1$ for wD_f. However, Equation (3) holds as sf of the time to failure of an item under degradation when $0 < S(w;t) < 1$ at some time t and degradation w. The model (3) adds the possibility of undertaking situations in which the deterioration of an item is not due to degradation alone. In real-world problems, the item subject to degradation ages over time, and even if the extent of degradation does not change, it also ages. Therefore, the life of a device subject to degradation may decrease as the level of degradation increases. Therefore, at relatively high levels of degradation, the device will weaken, so that a given threshold for the level of degradation can readily be considered a deterministic rule for device failure. However, intervals for the degree of degradation can be specified to develop a more dynamic time-to-failure model.

Let us consider a degradation process with increasing degradation path and assume that $s_1 \geq s_2 \geq \ldots \geq s_k$, where $s_i \in [0,1]$ for $i = 1,2,\ldots,k$ are the survival rates of a unit subject to degradation when $W(t) = w$, respectively, as the value w takes, lies in

$$(\mathfrak{D}_{\mathfrak{f}(\mathfrak{o})}, \mathfrak{D}_{\mathfrak{f}(\mathfrak{1})}], (\mathfrak{D}_{\mathfrak{f}(\mathfrak{1})}, \mathfrak{D}_{\mathfrak{f}(\mathfrak{2})}], \cdots, (\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-\mathfrak{1})}, \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}], \cdots, (\mathfrak{D}_{\mathfrak{f}(\mathfrak{k}-\mathfrak{1})}, \mathfrak{D}_{\mathfrak{f}(\mathfrak{k})}],$$

where $\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-\mathfrak{1})} \leq \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}$ for every $i = 1,2,\ldots,k$, such that $\mathfrak{D}_{\mathfrak{f}(\mathfrak{o})} = -\infty$ and $\mathfrak{D}_{\mathfrak{f}(\mathfrak{k}+\mathfrak{1})} = +\infty$. Note that $\mathfrak{k} = k$ throughout the paper. The degradation points that are adjacent to each other may induce a same amount of probability of failure, in the way that the survival rate at degradation level $W(t) = w$ takes the form

$$S(w;t) = \sum_{i=1}^{k} s_i I[w \in J_i], \tag{4}$$

where $I[A]$ is the indicator function of the set A, and $J_i = (\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-\mathfrak{1})}, \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}]$. It is assumed that $s_i, i = 1,2,\ldots,k$ do not depend on w.

For example, in a multiplicative degradation model with an increasing mean degradation path, we assume that the probability of failure does not change for degradation amounts in given intervals, and when degradation exceeds the last point (the largest value) in each interval, the probability of failure increases. Example: For high reliability products, 100 percent survive before the degradation level reaches $\mathfrak{D}_{\mathfrak{f}(\mathfrak{1})}$, and when degradation reaches $\mathfrak{D}_{\mathfrak{f}(\mathfrak{1})}$, 10 percent of the products fail, and the remaining 90 percent survive before degradation reaches $\mathfrak{D}_{\mathfrak{f}(\mathfrak{2})}$, and all fail once degradation reaches $\mathfrak{D}_{\mathfrak{f}(\mathfrak{2})}$; the time to failure is then modeled by $s(w;t) = I[w \in J_1] + 0.9I[w \in J_2]$.

By using (3) and taking $s_0 = 1$ and $s_{k+1} = 0$, we obtain

$$\begin{aligned}\bar{F}_T(t) &= \sum_{i=1}^{k} s_i P[W(t) \in J_i] \\ &= \sum_{i=1}^{k} s_i (F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}) - F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-\mathfrak{1})})).\end{aligned} \tag{5}$$

Note that if $s_i = 1$ for every $i = 1,\ldots,k$ and $\mathfrak{D}_{\mathfrak{f}(\mathfrak{k})} = \mathfrak{D}_{\mathfrak{f}}$, where $\mathfrak{D}_{\mathfrak{f}}$ is the threshold for degradation in the standard model, then $\bar{F}_T(t) = \sum_{i=1}^{k} P[W(t) \in J_i] = F_{W(t)}(\mathfrak{D}_{\mathfrak{f}})$, i.e., (5) reduces to (1). The degradation process of a life unit does not always refer to products with high reliability, where gradual failure is foreseen. It also refers to situations where sudden failures are possible, with the probability of such failures increasing as the degree of degradation increases. The model (5) may contribute effectively in such situations. Let us suppose that $T_i := \inf\{t \geq 0 : W(t) > \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}\}$, $i = 0,1,\ldots,k+1$ is the first passage time of the stochastic process $\{W(t), t \geq 0\}$ to the value of $\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}$. By convention, $T_0 = 0$, and $T_{k+1} = +\infty$. If we denote by T the time to failure of the device degrading over time, then

$$p_i = s_i - s_{i+1} = P(T_i \leq T < T_{i+1}). \tag{6}$$

It is necessary that (5) and (8) have to be valid survival functions for the time to failure T. For example, $F_{W(+\infty)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}) = 0$, for all $i = 0, 1, \ldots k$, and further, when $F_{W(0)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}) = 1$, for every $i = 0, 1, \ldots k$, then (5) defines a valid SF.

We can also consider a degradation process with a decreasing degradation path and assume that $s_{k+1} \geq s_k \geq \ldots \geq s_1$, where $s_i \in [0, 1]$ for $i = 1, 2, \ldots, k+1$ are the survival rates of a unit subject to degradation when $W(t) = w$, respectively, as the value w lies in

$$(\mathfrak{U}_{\mathfrak{f}(\mathfrak{k})}, \mathfrak{U}_{\mathfrak{f}(\mathfrak{k}+1)}], (\mathfrak{U}_{\mathfrak{f}(\mathfrak{k}-1)}, \mathfrak{U}_{\mathfrak{f}(\mathfrak{k})}], \cdots, (\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-1)}, \mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}], \cdots, (\mathfrak{U}_{\mathfrak{f}(0)}, \mathfrak{U}_{\mathfrak{f}(1)}],$$

where $\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-1)} \leq \mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}$ for every $i = 1, 2, \ldots, k+1$, such that $\mathfrak{U}_{\mathfrak{f}(0)} = -\infty$, and $\mathfrak{U}_{\mathfrak{f}(\mathfrak{k}+1)} = +\infty$. The survival rate at degradation level $W(t) = w$ is

$$S(w; t) = \sum_{i=1}^{k+1} s_i I[w \in J_i^\star], \tag{7}$$

where $J_i^\star = (\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-1)}, \mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}]$. By appealing to (3) when $s_{k+1} = 1$ and $s_0 = 0$, we can obtain

$$\begin{aligned} \bar{F}_T(t) &= \sum_{i=1}^{k+1} s_i P[W(t) \in J_i^\star] \\ &= \sum_{i=1}^{k+1} s_i (\bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-1)}) - \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})})). \end{aligned} \tag{8}$$

In this case, if $s_1 = 0$ and $s_i = 1$ for every $i = 2, \ldots, k+1$, and $\mathfrak{U}_{\mathfrak{f}(1)} = \mathfrak{U}_{\mathfrak{f}}$, where $\mathfrak{U}_{\mathfrak{f}}$ is the threshold for degradation in the standard model, then $\bar{F}_T(t) = \sum_{i=2}^{k+1} P[W(t) \in J_i^\star] = \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}})$, i.e., (5) reduces to (2). Let us assume that $T_i^\star := \inf\{t \geq 0 : W(t) \leq \mathfrak{U}_{\mathfrak{f}(\mathfrak{k}+1-\mathfrak{i})}\}$, $i = 0, 1, \ldots, k+1$ is the first passage time of the stochastic process $\{W(t), t \geq 0\}$ to the value of $\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}$. By convention, $T_0^\star = 0$, and $T_{k+1}^\star = +\infty$. The time to failure of the device is the random variable T, and

$$\pi_i = s_{i+1} - s_i = P(T_{k-i}^\star \leq T < T_{k-i+1}^\star). \tag{9}$$

The following lemma is essential in deriving future results. It shows that the SF of T in the degradation model with an increasing degradation path is a convex transformation of $F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}), i = 0, 1, \ldots, k$, as $p_i \geq 0$ and $\sum_{i=0}^{k} p_i = 1$. Further, the SF of T in the degradation model with a decreasing degradation path is a convex transformation of $\bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}), i = 0, 1, \ldots, k$, as $\pi_i \geq 0$ and $\sum_{i=0}^{k} \pi_i = 1$.

Lemma 1. *Let $W(t)$, the degradation process, stochastically increase with t. Then, $\bar{F}_T(t) = \sum_{i=0}^{k} p_i F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})})$.*

Proof. From (5), we can write

$$\begin{aligned}
\bar{F}_T(t) &= \sum_{i=1}^{k}\Big(s_i F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}) - s_i F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-\mathfrak{1})})\Big) \\
&= \sum_{i=1}^{k} s_i F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}) - s_{i-1}F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-\mathfrak{1})}) + s_{i-1}F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-\mathfrak{1})}) - s_i F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-\mathfrak{1})}) \\
&= \sum_{i=1}^{k} s_i F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}) - \sum_{i=1}^{k} s_{i-1}F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-\mathfrak{1})}) + \sum_{i=1}^{k}(s_{i-1}-s_i)F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-\mathfrak{1})}) \\
&= s_k F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{k})}) - s_0 F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{o})}) + \sum_{i=0}^{k-1}(s_i - s_{i+1})F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}) \\
&= (s_k - s_{k+1})F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{k})}) + \sum_{i=0}^{k-1}(s_i - s_{i+1})F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}) \\
&= \sum_{i=0}^{k} p_i F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}),
\end{aligned}$$

where $s_0 = 1, s_{k+1} = 0$, $F_{W(t)}(\mathfrak{D}_{\mathfrak{f}(\mathfrak{o})}) = F_{W(t)}(-\infty) = 0$ and $p_i = s_i - s_{i+1}$. □

The following lemma parallels Lemma 1.

Lemma 2. *Let the degradation process $W(t)$ be stochastically decreasing in t. Then, $\bar{F}_T(t) = \sum_{i=0}^{k} \pi_i \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})})$.*

Proof. In the spirit of (8), one obtains

$$\begin{aligned}
\bar{F}_T(t) &= \sum_{i=1}^{k+1}\Big(s_i \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-\mathfrak{1})}) - s_i \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})})\Big) \\
&= \sum_{i=1}^{k+1} s_i \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-\mathfrak{1})}) - s_{i-1}\bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-\mathfrak{1})}) + s_{i-1}\bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-\mathfrak{1})}) - s_i \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}) \\
&= \sum_{i=1}^{k+1} s_{i-1}\bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-\mathfrak{1})}) - \sum_{i=1}^{k+1} s_i \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}) + \sum_{i=1}^{k+1}(s_i - s_{i-1})\bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-\mathfrak{1})}) \\
&= s_0 \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{o})}) - s_{k+1}\bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{k}+\mathfrak{1})}) + \sum_{i=1}^{k+1}(s_i - s_{i-1})\bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-\mathfrak{1})}) \\
&= \sum_{i=0}^{k} \pi_i \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}),
\end{aligned}$$

where $s_0 = 0, \bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{o})}) = \bar{F}_{W(t)}(-\infty) = 1$, $\bar{F}_{W(t)}(\mathfrak{U}_{\mathfrak{f}(\mathfrak{k}+\mathfrak{1})}) = \bar{F}_{W(t)}(+\infty) = 0$, and $\pi_i = s_{i+1} - s_i$. □

In the context of the standard families of degradation models studied by Bae et al. [7], we develop the failure-time model (3) under the multiplicative degradation model.

The general multiplicative degradation model is stated as

$$W(t) = \eta(t)X, \tag{10}$$

where η is the mean degradation path, and X is the random variation around $\eta(t)$ having PDF f_X, CDF F_X, and SF $\bar{F}_X$. If the mean degradation path is considered as a monotonically increasing function, then we develop $\bar{F}_T$ under the multiplicative degradation model (10). Note that $F_{W(t)}(w) = F_X\left(\frac{w}{\eta(t)}\right)$; thus, it is deduced from Lemma 1 that

$$\begin{aligned}\bar{F}_T(t) &= E(S(X\eta(t);t)) \\ &= E\left[\sum_{i=1}^{k} s_i I(X\eta(t) \in (\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-1)}, \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}])\right] \\ &= \sum_{i=0}^{k} p_i F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right). \end{aligned} \tag{11}$$

The PDF of T, the time to failure under the degradation model 10 when $\eta(t)$ is increasing in $t \geq 0$ ($\eta'(t) \geq 0$, for all $t \geq 0$), having SF (11), is obtained as follows:

$$f_T(t) = \frac{\eta'(t)}{\eta^2(t)} \sum_{i=0}^{k} p_i \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right). \tag{12}$$

The failure rate associated with the SF given in (11) is then derived as

$$r_T(t) = \frac{\eta'(t) \sum_{i=0}^{k} p_i \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})} f_X(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)})}{\eta^2(t) \sum_{i=0}^{k} p_i F_X(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)})}. \tag{13}$$

If the mean degradation path $\eta(t)$ is a monotonically decreasing function, then the time to failure is denoted by T_1 with SF $\bar{F}_{T_1}$. This SF can be obtained in the setting of the multiplicative degradation model (10). We see that $\bar{F}_{W(t)}(w) = \bar{F}_X\left(\frac{w}{\eta(t)}\right)$. Therefore, using Lemma 2, we obtain

$$\begin{aligned}\bar{F}_{T_1}(t) &= E\left[\sum_{i=1}^{k+1} s_i I(X\eta(t) \in (\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}-1)}, \mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}])\right] \\ &= \sum_{i=0}^{k} \pi_i \bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right). \end{aligned} \tag{14}$$

The PDF of T_1, the time to failure under the degradation model (10) when $\eta(t)$ is decreasing in $t \geq 0$ ($\eta'(t) \leq 0$, for all $t \geq 0$), having SF (14), is revealed to be:

$$f_{T_1}(t) = \frac{-\eta'(t)}{\eta^2(t)} \sum_{i=0}^{k} \pi_i \mathfrak{U}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right). \tag{15}$$

The failure rate of T with the SF given in (14) is

$$r_{T_1}(t) = \frac{-\eta'(t) \sum_{i=0}^{k} \pi_i \mathfrak{U}_{\mathfrak{f}(\mathfrak{i})} f_X(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)})}{\eta^2(t) \sum_{i=0}^{k} \pi_i \bar{F}_X(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)})}. \tag{16}$$

3. Stochastic Ordering Results

In this section, we study some stochastic ordering properties of the time-to-failure distributions of two devices under the multiplicative degradation model. In industrial science, it is well known that products can have different qualities, some of which are more reliable, while others fail earlier. The extent to which each subject resists not failing under degradation can be evaluated by p_i's and π_i's in the models (5) and (8), respectively (see, e.g., Lemma 1). Let $P = (p_0, \ldots, p_k)$ and $P^\star = (p_0^\star, \ldots, p_n^\star)$ denote two probability vectors assigned to a couple of devices working under a multiplicative degradation model with an increasing mean degradation path. We suppose that P and $P^\star$ are associated with

with random lifetimes T and $T^{\star}$, respectively, such that $p_i = P(T_i \le T < T_{i+1})$ and also $p_i^{\star} = P(T_i \le T^{\star} < T_{i+1})$, where $T_i := \inf\{t \ge 0 \mid W(t) > \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}\}$ for $i = 0, 1, \ldots, k+1$. In a similar manner, let $\Pi = (\pi_0, \ldots, \pi_k)$ and $\Pi^{\star} = (\pi_0^{\star}, \ldots, \pi_k^{\star})$ denote other probability vectors related to a pair of devices working under a multiplicative degradation model with a decreasing mean degradation path. It is assumed that Π and $\Pi^{\star}$ are associated with random lifetimes T_1 and $T_1^{\star}$, respectively, such that $\pi_i = P(T_i^{\star} \le T_1 < T_{i+1}^{\star})$ and also $\pi_i^{\star} = P(T_i^{\star} \le T_1^{\star} < T_{i+1}^{\star})$, where $T_i^{\star} := \inf\{t \ge 0 | W(t) \le \mathfrak{U}_{\mathfrak{f}(\mathfrak{k}-\mathfrak{i}+\mathfrak{1})}\}$ for $i = 0, 1, \ldots, k+1$. Suppose that $W(t) = \eta(t)X$ is the underlying degradation model. We impose a partial order condition among P and $P^{\star}$ or/and conditions on the distribution of X (random variation around $\eta(t)$), such that some stochastic orders between T and $T^{\star}$ are procured. Further, we find some conditions on Π and $\Pi^{\star}$ and other conditions on the distribution of X, such that several stochastic orders between T_1 and $T_1^{\star}$ are fulfilled.

There are some concepts in applied probability that we need to introduce before we develop our stochastic comparison results. The following definition can be found in Joag-dev et al. [8].

Definition 1. *The function w, as a transformation on (x, y), is said to be totally positive of order 2, TP_2, [reverse regular of order 2, RR_2] in $(x, y) \in \mathfrak{A} \times \mathfrak{B}$, if $w(x, y) \ge 0$ and*

$$\begin{vmatrix} w(x_1, y_1) & w(x_1, y_2) \\ w(x_2, y_1) & w(x_2, y_2) \end{vmatrix} \ge [\le] 0,$$

for all $x_1 \le x_2 \in \mathfrak{A}$ and for all $y_1 \le y_2 \in \mathfrak{B}$, where $\mathfrak{A}$ and $\mathfrak{B}$ are two subsets of $\mathbb{R}$.

It is plain to verify that the TP_2 $[RR_2]$ property of w, as a transformation on (i, k), is equivalent to $\frac{w(i,k_2)}{w(i,k_1)}$ being nondecreasing [nonincreasing] in i whenever $k_1 \le k_2$ by considering the conventions $\frac{a}{0} = +\infty$ when $a > 0$ and $\frac{a}{0} = 0$, if $a = 0$.

The following lemma from Joag-dev et al. [8] known as the general composition theorem (or basic composition formula) is frequently used in this paper.

Lemma 3.

(i) (discrete case): Let g be TP_2 in $(j, i) \in \{1, 2\} \times \mathfrak{A}_k$ and also let w be TP_2 (respectively, RR_2) in $(i, t) \in \mathfrak{A}_k \times \mathfrak{B}$, where $\mathfrak{A}_k = \{0, 1, \ldots, k\}$. Then, the function $w^{\star}$, given by

$$w^{\star}(j, t) := \sum_{i=0}^{k} g(j, i) w(i, t), \text{ is } TP_2 \text{ (respectively, } RR_2) \text{ in } (j, t) \in \{1, 2\} \times \mathfrak{B}.$$

(ii) (continuous case): Let $g(j, y)$ be TP_2 in $(j, y) \in \{1, 2\} \times \mathfrak{Y}$, and let $w(y, x)$ be TP_2 (respectively, RR_2) in $(y, x) \in \mathfrak{Y} \times \mathfrak{B}$, where $\mathfrak{Y}$ and $\mathfrak{B}$ are two subsets of $[0, +\infty)$. Then,

$$w^{\star}(j, x) := \int_0^{+\infty} g(j, y) w(y, x)\, dy \text{ is } TP_2 \text{ (respectively, } RR_2) \text{ in } (j, x) \in \{1, 2\} \times \mathfrak{B}.$$

The following definition proposes some class of functions.

Definition 2. *Suppose that w, as a transformation of nonnegative values, is a nonnegative function. Then, w is said to have*

(i) One-sided scaled-ratio increasing (decreasing), OSSRI (OSSRD), property, if $\frac{w(tx)}{w(x)}$ is increasing (decreasing) in $x \ge 0$ for every $t \ge 1$.

(ii) Two-sided scaled-ratio increasing (decreasing), TSSRI (TSSRD), property, if $\frac{w(t_2x) - w(t_1x)}{w(s_2x) - w(s_1x)}$ is increasing (decreasing) in $x \ge 0$ for every $t_i \ge s_i \ge 0, i = 1, 2$, with $t_1 \le t_2$ and $s_1 \le s_2$

From Definition 2, it is apparent that if $t_1 = s_1 = 0$ and also $w(0) = 0$, then from assertion (ii) the ratio $\frac{w(t_2x)}{w(s_2x)}$ is increasing (decreasing) in x for every $t_2 > s_2 \ge 0$. Equivalently, this realizes that $\frac{w(tx)}{w(x)}$ is increasing (decreasing) in x for all $t \geq 1$. Therefore, every w with $w(0) = 0$ having the TSSRI (TSSRD) property will also fulfill the OSSRI (OSSRD) property.

Remark 1. *The properties in Definition 2(i) can be applied to generate reliability classes of lifetime distributions. One can state that X has the increasing proportional probability (IPLR) property, if and only if f_X has the OSSRD property, and X has the decreasing proportional likelihood ratio (DPLR) property, if and only if f_X has the OSSRI property (see Romero and Díaz (2001) for definitions of IPLR and DPLR). One can also see that X has the increasing proportional hazard rate (IPHR) property, if and only if $\bar{F}_X$ has the OSSRD property, and in parallel, X has the decreasing proportional hazard rate (DPHR) property, if and only if $\bar{F}_X$ has the OSSRI property (see Belzunce et al. [9] for IPHR and DPHR properties). It can also be shown that X has the decreasing proportional reversed failure rate (DPRFR) property, if and only if F_X has the OSSRD property, and also X has the increasing proportional failure rate (IPRFR) property, if and only if F_X has the OSSRI property (see Oliveira and Torrado [10] for the DPRFR and IPRFR classes).*

In applied probability theory, stochastic orderings between random variables are a useful approach for comparing the reliability of systems (see, e.g., Müller and Stoyan [11], Osaki [12], Shaked and Shanthikumar [13], and Belzunce et al. [14]). Stochastic orderings are considered a fundamental tool for decision making under uncertainty (see, e.g., Mosler [15] and Li and Li [16]).

Let us assume that T and $T^\star$ are random variables with absolutely continuous CDFs F_T and $F_{T^\star}$, SFs $\bar{F}_T$ and $\bar{F}_{T^\star}$, and PDFs f_T and $f_{T^\star}$, respectively. We suppose that T and $T^\star$ have hazard rate functions h_T and $h_{T^\star}$ and reversed hazard rate functions $\tilde{h}_T$ and $\tilde{h}_{T^\star}$, respectively. Then:

Definition 3. *We say that T is smaller than or equal to $T^\star$ in the*

(i) *Likelihood ratio order (denoted as $T \leq_{lr} T^\star$), if $\frac{f_{T^\star}(t)}{f_T(t)}$ is increasing in $t \geq 0$.*
(ii) *Hazard rate order (denoted as $T \leq_{hr} T^\star$), if $\frac{\bar{F}_{T^\star}(t)}{\bar{F}_T(t)}$ is increasing in $t \geq 0$, or equivalently, $h_T(t) \geq h_{T^\star}(t)$, for all $t \geq 0$.*
(iii) *Reversed hazard rate order (denoted as $T \leq_{rhr} T^\star$), if $\frac{F_{T^\star}(t)}{F_T(t)}$ is increasing in $t \geq 0$, or equivalently, $\tilde{h}_T(t) \leq \tilde{h}_{T^\star}(t)$, for all $t > 0$.*
(iv) *Usual stochastic order (denoted as $T \leq_{st} T^\star$) if $\bar{F}_T(t) \leq \bar{F}_{T^\star}(t)$, for all $t \geq 0$.*

As given in Shaked and Shanthikumar [13], we have:

$$T \leq_{lr} T^\star \Rightarrow T \leq_{hr} T^\star \Rightarrow T \leq_{st} T^\star.$$

It is, furthermore, well known that

$$T \leq_{lr} T^\star \Rightarrow T \leq_{rhr} T^\star \Rightarrow T \leq_{st} T^\star.$$

To compare T and $T^\star$ according to the usual stochastic ordering, a sufficient condition is the well-known majorization ordering as given in the next definition. Majorization is a partial order relation of two probability vectors with the same dimension that causes the elements in one vector to be less far apart or more equal than the elements in another vector. The majorization order provides an elegant framework for comparing two probability vectors (see, e.g., Marshall et al. [17]).

We take $\mathbb{X} = (x_0, \ldots, x_k)$ and $\mathbb{Y} = (y_0, \ldots, y_k)$ as two vectors of real numbers, such that $x_{(0)} \leq \ldots \leq x_{(k)}$ and $y_{(0)} \leq \ldots \leq y_{(k)}$ denote the increasing arrangement of the values of $\mathbb{X}$ and values of $\mathbb{Y}$, respectively, where $x_{(i)}$ is the ith smallest value among $x_0, \ldots, x_k$, and $y_{(i)}$ is the ith smallest value among $y_0, \ldots, y_k$, for $i = 1, \ldots, k$.

Definition 4. *It is said that* $\mathbb{X}$ *is majorized by* $\mathbb{Y}$*, written as* $\mathbb{X} \preceq \mathbb{Y}$*, whenever* $\sum_{i=0}^{k} x_i = \sum_{i=0}^{k} y_i$*, and* $\sum_{i=0}^{j} x_{(k-i)} \leq \sum_{i=0}^{j} y_{(k-i)}$*, for every* $j = 0, \ldots, k-1$.

In this part of the paper, we assume that T and $T^\star$ are two random variables denoting the time to failure under the dynamic multiplicative degradation model $W(t) = X\eta(t)$, where η is an increasing function with SFs

$$\bar{F}_T(t) = \sum_{i=0}^{k} p_i F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) \text{ and } \bar{F}_{T^\star}(t) = \sum_{i=0}^{k} p_i^\star F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right).$$

The corresponding PDFs are derived as

$$f_T(t) = \frac{\eta'(t)}{\eta^2(t)} \sum_{i=0}^{k} p_i \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) \text{ and } f_{T^\star}(t) = \sum_{i=0}^{k} p_i^\star \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right).$$

We also suppose that T_1 and $T_1^\star$ are two random variables denoting the time to failure under the multiplicative degradation model $W(t) = X\eta(t)$, where η is a decreasing function with SFs

$$\bar{F}_{T_1}(t) = \sum_{i=0}^{k} \pi_i \bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) \text{ and } \bar{F}_{T_1^\star}(t) = \sum_{i=0}^{k} \pi_i^\star \bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right).$$

The associated PDFs are obtained as

$$f_{T_1}(t) = \frac{-\eta'(t)}{\eta^2(t)} \sum_{i=0}^{k} \pi_i \mathfrak{U}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) \text{ and } f_{T_1^\star}(t) = \frac{-\eta'(t)}{\eta^2(t)} \sum_{i=0}^{k} \pi_i^\star \mathfrak{U}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right).$$

We utilize the following technical lemma.

Lemma 4.

(i) *Let* $w_0, w_1, \ldots, w_k$ *be a set of real numbers satisfying* $\sum_{i=0}^{k} w_i = 0$*. If* $h(i)$ *is nondecreasing in* $i = 0, 1, \ldots, k$*, then*

$$W_j = \sum_{i=j}^{k} w_i \geq 0, \text{ for all } j = 1, 2, \ldots, k \text{ implies that } \sum_{i=0}^{k} h(i) w_i \geq 0.$$

(ii) *Let* $w_0, w_1, \ldots, w_k$ *be real numbers. If* $h(i) \geq 0$ *is nonincreasing for* $i = 0, 1, \ldots, k$*, then*

$$W_j = \sum_{i=0}^{j} w_i \geq 0, \text{ for all } j = 0, 1, \ldots, k \text{ implies that } \sum_{i=0}^{k} h(i) w_i \geq 0.$$

The next result discusses the sufficient conditions for the stochastic comparison of T and $T^\star$ and also the stochastic ordering of T_1 and $T_1^\star$ according to the usual stochastic order.

Theorem 1.

(i) *Let* $P = (p_0, \ldots, p_k)$ *and* $P^\star = (p_0^\star, \ldots, p_k^\star)$ *be two probability vectors satisfying* $p_0 \leq \ldots \leq p_k$ *and* $p_0^\star \leq \ldots \leq p_k^\star$*, such that* $P \preceq P^\star$*. Then,* $T \leq_{st} T^\star$.

(ii) *Let* $\Pi = (\pi_0, \ldots, \pi_k)$ *and* $\Pi^\star = (\pi_0^\star, \ldots, \pi_k^\star)$ *be two probability vectors with* $\pi_0 \geq \ldots \geq \pi_k$ *and* $\pi_1^\star \geq \ldots \geq \pi_k^\star$*, such that* $\Pi^\star \preceq \Pi$*. Then,* $T_1 \leq_{st} T_1^\star$.

Proof. Firstly, we prove assertion (i). Note that for any $t \geq 0$,

$$F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(0)}}{\eta(t)}\right) \leq F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(1)}}{\eta(t)}\right) \leq \cdots \leq F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{k})}}{\eta(t)}\right). \tag{17}$$

By appealing to Equation (11) and since $p_i \leq p_j$, for every $i < j$ and also from (17), $F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) \leq F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{j})}}{\eta(t)}\right)$ for every $i < j$, as $i, j = 0, 1, \ldots, k$, thus, by rearranging the elements in sigma in Equation (11), we conclude the following: (It is straightforward that if $a_0 \leq a_1 \leq \cdots \leq a_k$ and also $b_0 \leq b_1 \leq \cdots \leq b_k$, then $\sum_{i=0}^{k} a_i b_i = \sum_{i=0}^{k} a_{k-i} b_{k-i} = \sum_{i=0}^{k} a_{(k-i)} b_{k-i}$ in which $a_{(0)} \leq a_{(1)} \leq \cdots \leq a_{(k)}$ denote the ordered values of $a_0, a_1, \ldots, a_k$.)

$$\bar{F}_T(t) = \sum_{i=0}^{k} p_i F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) = \sum_{i=0}^{k} p_{(k-i)} F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{k}-\mathfrak{i})}}{\eta(t)}\right).$$

Similarly,

$$\bar{F}_{T^\star}(t) = \sum_{i=0}^{k} p_i^\star F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) = \sum_{i=0}^{k} p^\star_{(k-i)} F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{k}-\mathfrak{i})}}{\eta(t)}\right).$$

Let us take $h(i) = F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{k}-\mathfrak{i})}}{\eta(t)}\right)$, which by (17), is a nonincreasing function in $i = 0, 1, \ldots, k$. Since $P \preceq P^\star$, thus $\sum_{i=0}^{j}(p^\star_{(k-i)} - p_{(k-i)}) \geq 0$, for all $j = 0, 1, \ldots, k$. Therefore, from Lemma 4(ii),

$$\bar{F}_{T^\star}(t) - \bar{F}_T(t) = \sum_{i=0}^{k}(p^\star_{(k-i)} - p_{(k-i)}) F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{k}-\mathfrak{i})}}{\eta(t)}\right)$$

is nonnegative, which means that $T \leq_{st} T^\star$. We now prove assertion (ii). For each fixed $t \geq 0$, we have:

$$\bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{o})}}{\eta(t)}\right) \geq \bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{1})}}{\eta(t)}\right) \geq \cdots \geq \bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{k})}}{\eta(t)}\right). \quad (18)$$

By applying Equation (14) and since $\pi_i \geq \pi_j$, for every $i < j$ and also from (18), $\bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) \geq \bar{F}_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{j})}}{\eta(t)}\right)$ for every $i < j$, when $i, j = 0, 1, \ldots, k$, thus, by rearranging the elements of sigma in Equation (14), we can obtain the following: (It is plain to see if $a_0 \geq a_1 \geq \cdots \geq a_k$ and also $b_0 \geq b_1 \leq \cdots \geq b_k$, then $\sum_{i=0}^{k} a_i b_i = \sum_{i=0}^{k} a_{k-i} b_{k-i} = \sum_{i=0}^{k} a_{(k-i)} b_{k-i}$.)

$$\bar{F}_{T_1}(t) = \sum_{i=0}^{k} \pi_i \bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) = \sum_{i=0}^{k} \pi_{(k-i)} \bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{k}-\mathfrak{i})}}{\eta(t)}\right).$$

In parallel,

$$\bar{F}_{T_1^\star}(t) = \sum_{i=0}^{k} \pi_i^\star \bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) = \sum_{i=0}^{k} \pi^\star_{(k-i)} \bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{k}-\mathfrak{i})}}{\eta(t)}\right).$$

We set $h(i) = \bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{k}-\mathfrak{i})}}{\eta(t)}\right)$, which by (18), is a nondecreasing function in $i = 0, 1, \ldots, k$. Since $\Pi^\star \preceq \Pi$, thus $\sum_{i=j}^{k}(\pi^\star_{(k-i)} - \pi_{(k-i)}) \geq 0$, for all $j = 1, \ldots, k$ and $\sum_{i=0}^{k}(\pi^\star_{(k-i)} - \pi_{(k-i)}) = 0$. Hence, an application of Lemma 4(i) yields

$$\bar{F}_{T_1^\star}(t) - \bar{F}_{T_1}(t) = \sum_{i=0}^{k}(\pi^\star_{(k-i)} - \pi_{(k-i)}) \bar{F}_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{k}-\mathfrak{i})}}{\eta(t)}\right),$$

which is nonnegative, which means that $T_1 \leq_{st} T_1^\star$. The proof is complete. □

Remark 2. *The result of Theorem 1 shows that the usual stochastic ordering between T and $T^\star$ and also that of T_1 and $T_1^\star$ do not depend on the distribution of the random variation X. The conditions imposed on $T \leq_{st} T^\star$ in Theorem 1(i) consist of an order relation between the p_i's* *(i.e., $p_0 \leq \ldots \leq p_k$) and the same order relation between the $p_i^\star$'s (i.e., $p_0^\star \leq \ldots \leq p_k^\star$) and a majorization order condition of P and $P^\star$. The probability vector ($P^\star$), which majorizes the other probability vector (P) leads to a more reliable product under a multiplicative degradation model with increasing $\eta(t)$. The order relations $p_0 \leq \ldots \leq p_k$ and $p_0^\star \leq \ldots \leq p_k^\star$ are valid assumptions in practical works. This is because in a multiplicative degradation model, as $\eta(t)$ increases with elapsed time t, the amount of degradation $W(t)$ increases, and thus the probability of failure increases accordingly. Note that the first elements of P and $P^\star$ are associated with smaller amounts of $W(t)$. The conditions necessary to obtain $T \leq_{st} T^\star$ in Theorem 1(ii) are, first, an order relation of π_i's (i.e., $\pi_0 \geq \ldots \geq \pi_k$) and an analogous order relation of $\pi_i^\star$'s (i.e., $\pi_0^\star \geq \ldots \geq \pi_k^\star$) and, second, the majorization order of $\Pi^\star$ and Π. The probability vector (Π), which majorizes the other probability vector ($\Pi^\star$) will lead to a less reliable product under the multiplicative degradation model with decreasing $\eta(t)$. The ordering constraints $\pi_0 \geq \ldots \geq \pi_k$ and $\pi_0^\star \geq \ldots \geq \pi_k^\star$ are also valid assumptions in practice. This is because in a multiplicative degradation model with decreasing $\eta(t)$ with time t, the factor $W(t)$ for degradation decreases; therefore, the probability of failure of the product increases accordingly. Note that the first elements of Π and $\Pi^\star$ are associated with smaller amounts $W(t)$.*

The following theorems impose some conditions to explain the order $\leq_{lr}$ between time-to-failure random variables in the dynamic multiplicative degradation model with increasing mean degradation path $\eta(t)$ (Theorem 2(i)) and the dynamic multiplicative degradation model with decreasing mean degradation path $\eta(t)$ (Theorem 2(ii)).

Theorem 2.

(i) Let $P = (p_0, \ldots, p_k)$ and $P^\star = (p_0^\star, \ldots, p_k^\star)$ be two probability vectors so that $\frac{p_i^\star}{p_i}$ is nondecreasing in $i = 0, 1, \ldots, k$. If f_X is OSSRD (OSSRI), then $T \leq_{lr} (\geq_{lr}) T^\star$.

(ii) Let $\Pi = (\pi_0, \ldots, \pi_k)$ and $\Pi^\star = (\pi_0^\star, \ldots, \pi_k^\star)$ be two probability vectors so that $\frac{\pi_i^\star}{\pi_i}$ is nondecreasing in $i = 0, 1, \ldots, k$. If f_X is OSSRI (OSSRD), then $T_1 \leq_{lr} (\geq_{lr}) T_1^\star$.

Proof. To prove (i) it suffices to demonstrate that

$$\frac{f_{T^\star}(t)}{f_T(t)} = \frac{\sum_{i=0}^k p_i^\star \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right)}{\sum_{i=0}^k p_i \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right)}$$

is nondecreasing (nonincreasing) in $t > 0$. Set $g(j,i) = p_i$, for $j = 1$ and $g(j,i) = p_i^\star$, for $j = 2$ and also $w(i,t) = \mathfrak{D}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right)$. Therefore, $T \leq_{lr} (\geq_{lr}) T^\star$, if and only if $w^\star(j,t) := \sum_{i=0}^k g(j,i) w(i,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0,+\infty)$. Note that, by assumption, $\frac{p_i^\star}{p_i}$ is nondecreasing in $i = 0,1,\ldots,k$; hence, $g(j,i)$ is TP_2 in $(j,i) \in \{1,2\} \times \{0,1,\ldots,k\}$, and also since f_X is OSSRD (OSSRI), and $\eta(t)$ is nondecreasing in $t \geq 0$, thus, for every $i_1 < i_2 \in \{0,1,\ldots,k\}$,

$$\frac{w(i_2,t)}{w(i_1,t)} = \frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}_2)} f_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}_2)}}{\eta(t)}\right)}{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}_1)} f_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}_1)}}{\eta(t)}\right)}$$

is nondecreasing (nonincreasing) in $t \geq 0$. This means $w(i,t)$ is TP_2 (RR_2) in $(i,t) \in \{0,1,\ldots,k\} \times \{1,2\}$. By Lemma 3(i), $w^\star(j,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0,+\infty)$, and this completes the proof of (i). To prove (ii) one needs to show that

$$\frac{f_{T_1^\star}(t)}{f_{T_1}(t)} = \frac{\sum_{i=0}^k \pi_i^\star \mathfrak{U}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right)}{\sum_{i=0}^k \pi_i \mathfrak{U}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right)}$$

is nondecreasing (nonincreasing) in $t > 0$. We take $g^\star(j,i) = \pi_i$, for $j = 1$ and $g^\star(j,i) = \pi_i^\star$, for $j = 2$ and also set $w_1(i,t) = \mathfrak{U}_{\mathfrak{f}(\mathfrak{i})} f_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right)$. Thus, $T_1 \leq_{lr} (\geq_{lr}) T_1^\star$, if and only if $w_2(j,t) := \sum_{i=0}^k g^\star(j,i) w_1(i,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0,+\infty)$. From this assumption, $\frac{\pi_i^\star}{\pi_i}$ is nondecreasing in $i = 0,1,\ldots,k$; hence, $g^\star(j,i)$ is TP_2 in $(j,i) \in \{1,2\} \times \{0,1,\ldots,k\}$, and also since f_X is OSSRI (OSSRD), and $\eta(t)$ is nonincreasing in $t \geq 0$, thus, for every $i_1 < i_2 \in \{0,1,\ldots,k\}$,

$$\frac{w_1(i_2,t)}{w_1(i_1,t)} = \frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}_2)} f_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}_2)}}{\eta(t)}\right)}{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}_1)} f_X\left(\frac{\mathfrak{U}_{\mathfrak{f}(\mathfrak{i}_1)}}{\eta(t)}\right)}$$

is nondecreasing (nonincreasing) in $t \geq 0$. This means $w_1(i,t)$ is TP_2 (RR_2) in $(i,t) \in \{0,1,\ldots,k\} \times \{1,2\}$. By Lemma 3(i), $w_2(j,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0,+\infty)$, which validates the proof of (ii). □

The following theorem establishes the conditions for ordering $\leq_{hr}$ between the time-to-failure random variables in the dynamic multiplicative degradation model with increasing mean degradation path $\eta(t)$.

Theorem 3. *Let $P = (p_0,\ldots,p_k)$ and $P^\star = (p_0^\star,\ldots,p_k^\star)$ be two probability vectors, such that*

(i) $\frac{p_i^\star}{p_i}$ *is nondecreasing in $i = 0,1,\ldots,k$. If F_X is OSSRD (OSSRI), then we have $T \leq_{hr} (\geq_{hr}) T^\star$.*

(ii) $\frac{s_i^\star}{s_i}$ *is nondecreasing in $i = 1,2,\ldots,k$. If F_X is TSSRD (TSSRI), then we have $T \leq_{hr} (\geq_{hr}) T^\star$.*

Proof. For assertion (i) to be proved, it is enough to show that

$$\frac{\bar{F}_{T^\star}(t)}{\bar{F}_T(t)} = \frac{\sum_{i=0}^k p_i^\star F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right)}{\sum_{i=0}^k p_i F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right)}$$

is nondecreasing (nonincreasing) in $t \geq 0$. Let us take $g(j,i) = p_i$, for $j = 1$, and $g(j,i) = p_i^\star$, for $j = 2$, and also $w(i,t) = F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right)$. Thus, $T \leq_{hr} (\geq_{hr}) T^\star$, if and only if $w^\star(j,t) := \sum_{i=0}^k g(j,i) w(i,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0,+\infty)$. By assumption, $\frac{p_i^\star}{p_i}$ is nondecreasing in i; hence, $g(j,i)$ is TP_2 in (j,i), and further, since F_X is OSSRD (OSSRI), and $\eta(t)$ is nondecreasing in $t \geq 0$, thus, for every $i_1 < i_2$, in the domain of i,

$$\frac{w(i_2,t)}{w(i_1,t)} = \frac{F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}_2)}}{\eta(t)}\right)}{F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}_1)}}{\eta(t)}\right)}$$

is nondecreasing (nonincreasing) in $t \geq 0$. This is equivalent to saying that $w(i,t)$ is TP_2 (RR_2) in (i,t). By Lemma 3(i), $w^\star(j,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0,+\infty)$, and this ends the proof of (i). For the proof of assertion (ii), one needs to prove that

$$\frac{\bar{F}_{T^\star}(t)}{\bar{F}_T(t)} = \frac{\sum_{i=1}^k s_i^\star \left(F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) - F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-1)}}{\eta(t)}\right)\right)}{\sum_{i=1}^k s_i \left(F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) - F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-1)}}{\eta(t)}\right)\right)}$$

is nondecreasing (nonincreasing) in $t \geq 0$. We can set $g(j,i) = s_i$, for $j = 1$ and $g(j,i) = s_i^\star$, for $j = 2$ and also take $w(i,t) = F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})}}{\eta(t)}\right) - F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-1)}}{\eta(t)}\right)$, which is nonnegative since $\mathfrak{D}_{\mathfrak{f}(\mathfrak{i})} \geq \mathfrak{D}_{\mathfrak{f}(\mathfrak{i}-1)}$. By these notations, $T \leq_{hr} (\geq_{hr}) T^\star$, if and only if $w^\star(j,t) := \sum_{i=1}^k g(j,i) w(i,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0,+\infty)$. From assumption, $\frac{s_i^\star}{s_i}$ is nondecreasing in i; hence, $g(j,i)$ is TP_2 in (j,i), and moreover, since F_X is TSSRD (TSSRI), and $\eta(t)$ is nondecreasing in $t \geq 0$, thus, for every $i_1 < i_2$,

$$\frac{w(i_2,t)}{w(i_1,t)} = \frac{F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(i_2)}}{\eta(t)}\right) - F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(i_2-1)}}{\eta(t)}\right)}{F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(i_1)}}{\eta(t)}\right) - F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(i_1-1)}}{\eta(t)}\right)}$$

is nondecreasing (nonincreasing) in $t \geq 0$. This is equivalent to $w(i,t)$ being TP_2 (RR_2) in (i,t). On applying Lemma 3(i), $w^\star(j,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0,+\infty)$, and this gives the required result in assertion (ii). □

In the context of Theorem 3, if $\frac{p_i^\star}{p_i}$ is nondecreasing in $i = 0,1,\ldots,k$, then $\frac{s_i^\star}{s_i}$ is also nondecreasing in $i = 1,2,\ldots,k$. We can use Lemma 3(i) to prove it. Let us take $g(j,i) = p_i^\star$, for $j = 2$ and $g(j,i) = p_i$, for $j = 1$ when $i = 0,1,\ldots,k$. Set $w(i,t) = I[i \geq t]$, where $t = 1,2,\ldots,k$ and $i = 0,1,\ldots,k$. Since $\frac{p_i^\star}{p_i}$ is nondecreasing in $i = 0,1,\ldots,k$, thus $g(j,i)$ is TP_2 in (j,i), and it is also straightforward to show that $w(i,t) = I[i \geq t]$ is TP_2 in (i,t). Hence, $w^\star(j,i) = \sum_{i=0}^{k} g(j,i)w(i,t)$ is TP_2 in (j,t), i.e., $\frac{s_i^\star}{s_i}$ is nondecreasing in $i = 1,2,\ldots,k$. Therefore, the condition on probabilities in Theorem 3(ii) is weaker than the condition imposed on probabilities in Theorem 3(i). It is also plain to show that if F_X is TSSRD (TSSRI) then F_X is OSSRD (OSSRI). Therefore, the condition on the random effect distribution in Theorem 3(ii) is stronger than the condition on the random effect distribution in Theorem 3(i).

The theorem below presents conditions to make the order $\leq_{hr}$ between the time-to-failure random variables in the dynamic multiplicative degradation model with decreasing mean degradation path $\eta(t)$. The proof being similar to the proof of Theorem 3 has been omitted.

Theorem 4. *Let $\Pi = (\pi_0,\ldots,\pi_k)$ and $\Pi^\star = (\pi_0^\star,\ldots,\pi_k^\star)$ be two probability vectors such that*

(i) $\frac{\pi_i^\star}{\pi_i}$ *is nondecreasing in $i = 0,1,\ldots,k$. If $\bar{F}_X$ is OSSRI (OSSRD), then we have $T_1 \leq_{hr} (\geq_{hr}) T_1^\star$.*

(ii) $\frac{s_i^\star}{s_i}$ *is nondecreasing in $i=1,2,\ldots,k+1$. If F_X is TSSRI (TSSRD), then we have $T_1 \leq_{hr} (\geq_{hr}) T_1^\star$.*

The next result presents the conditions under which the order $\leq_{rhr}$ is fulfilled by the time-to-failure random variables in the dynamic multiplicative degradation model with increasing mean degradation path $\eta(t)$.

Theorem 5. *Let $P = (p_0,\ldots,p_k)$ and $P^\star = (p_0^\star,\ldots,p_k^\star)$ be two probability vectors such that*

(i) $\frac{p_i^\star}{p_i}$ *is nondecreasing in $i = 0,1,\ldots,k$. If $\bar{F}_X$ is OSSRD (OSSRI), then $T \leq_{rhr} (\geq_{rhr}) T^\star$.*

(ii) $\frac{1-s_i^\star}{1-s_i}$ *is nondecreasing in $i = 1,2,\ldots,k+1$. If F_X is TSSRD (TSSRI), then $T \leq_{rhr} (\geq_{rhr}) T^\star$.*

Proof. The assertion (i) is established if one shows that

$$\frac{F_{T^\star}(t)}{F_T(t)} = \frac{\sum_{i=0}^{k} p_i^\star \bar{F}_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(i)}}{\eta(t)}\right)}{\sum_{i=0}^{k} p_i \bar{F}_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(i)}}{\eta(t)}\right)}$$

is nondecreasing (nonincreasing) in $t > 0$. Let $g(j,i) = p_i$, for $j = 1$ and $g(j,i) = p_i^\star$, for $j = 2$, and also $w(i,t) = \bar{F}_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(i)}}{\eta(t)}\right)$. As a result, $T \leq_{rhr} (\geq_{rhr}) T^\star$, if and only if $w^\star(j,t) := \sum_{i=0}^{k} g(j,i)w(i,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0,+\infty)$. By assumption, $\frac{p_i^\star}{p_i}$ is nondecreasing in i; hence, $g(j,i)$ is TP_2 in (j,i), and further, since $\bar{F}_X$ is OSSRD (OSSRI), and $\eta(t)$ is nondecreasing in $t \geq 0$, thus, for every $i_1 < i_2$,

$$\frac{w(i_2,t)}{w(i_1,t)} = \frac{\bar{F}_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(i_2)}}{\eta(t)}\right)}{\bar{F}_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(i_1)}}{\eta(t)}\right)}$$

is nondecreasing (nonincreasing) in $t \geq 0$, which means $w(i,t)$ is TP_2 (RR_2) in (i,t). Using Lemma 3(i), $w^\star(j,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0,+\infty)$, and this provides the proof of (i). For assertion (ii), we need to demonstrate that

$$\frac{F_{T^\star}(t)}{F_T(t)} = \frac{\sum_{i=1}^{k+1}(1-s_i^\star)\left(F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(i)}}{\eta(t)}\right) - F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(i-1)}}{\eta(t)}\right)\right)}{\sum_{i=1}^{k+1}(1-s_i)\left(F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(i)}}{\eta(t)}\right) - F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(i-1)}}{\eta(t)}\right)\right)}$$

is nondecreasing (nonincreasing) in $t > 0$. Let us define $g(j,i) = 1 - s_i$, for $j = 1$ and $g(j,i) = 1 - s_i^\star$, for $j = 2$, and let us also define $w(i,t) = F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(i)}}{\eta(t)}\right) - F_X\left(\frac{\mathfrak{D}_{\mathfrak{f}(i-1)}}{\eta(t)}\right)$. Now, $T \leq_{rhr} (\geq_{rhr}) T^\star$, if and only if $w^\star(j,t) := \sum_{i=1}^{k} g(j,i)w(i,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0,+\infty)$. By assumption, $\frac{1-s_i^\star}{1-s_i}$ is nondecreasing in i; hence, $g(j,i)$ is TP_2 in (j,i), and in addition, since $\bar{F}_X$ is TSSRD (TSSRI), and $\eta(t)$ is nondecreasing in $t \geq 0$, thus, $w(i,t)$ is TP_2 (RR_2) in (i,t). By Lemma 3(i), $w^\star(j,t)$ is TP_2 (RR_2) in $(j,t) \in \{1,2\} \times [0,+\infty)$, and this proves assertion (ii). □

In the setting of Theorem 5, if $\frac{p_i^\star}{p_i}$ is nondecreasing in $i = 0,1,\ldots,k$, then $\frac{1-s_i^\star}{1-s_i}$ is nondecreasing in $i = 1,2,\ldots,k+1$. Lemma 3(i) can be used to prove it. Let us set $g(j,i) = p_i^\star$, for $j = 2$ and $g(j,i) = p_i$ for $j = 1$, when $i = 0,1,\ldots,k$. Set $w(i,t) = I[i \leq t-1]$, where $t = 1,2,\ldots,k+1$ and $i = 0,1,\ldots,k$. Since $\frac{p_i^\star}{p_i}$ is nondecreasing in $i = 0,1,\ldots,k$, thus $g(j,i)$ is TP_2 in (j,i), and also $w(i,t) = I[i \leq t]$ is TP_2 in (i,t). Thus, $w^\star(j,i) := \sum_{i=0}^{k} g(j,i)w(i,t)$ is TP_2 in (j,t), i.e., $\frac{1-s_i^\star}{1-s_i}$ is nondecreasing in $i = 1,2,\ldots,k+1$. Therefore, the condition on probabilities in Theorem 5(ii) is weaker than the condition on probabilities in Theorem 5(i). Moreover, if F_X is TSSRD (TSSRI), then $\bar{F}_X$ is OSSRD (OSSRI). This means that the random effects distribution condition in Theorem 5(ii) is stronger than the random effect distribution condition in Theorem 5(i).

The following theorem imposes conditions on the order $\leq_{rhr}$ between the random variables for the time to failure in the dynamic multiplicative degradation model with decreasing mean degradation path $\eta(t)$. The proof, which is similar to the proof of Theorem 5, has been omitted.

Theorem 6. *Let $\Pi = (\pi_0,\ldots,\pi_k)$ and $\Pi^\star = (\pi_0^\star,\ldots,\pi_k^\star)$ be two probability vectors such that*

(i) $\frac{\pi_i^\star}{\pi_i}$ *is nondecreasing in $i = 0,1,\ldots,k$. If F_X is OSSRI (OSSRD), then we have $T_1 \leq_{rhr} (\geq_{rhr}) T_1^\star$.*

(ii) $\frac{1-s_i^\star}{1-s_i}$ *is nondecreasing in $i = 1,2,\ldots,k+1$. If F_X is TSSRI (TSSRD), then we have $T_1 \leq_{rhr} (\geq_{rhr}) T_1^\star$.*

4. Examples

In this section, we investigate and test the random effects distribution conditions to satisfy the ordering properties in Section 3 with some typical random effects distribution functions listed in Bae et al. [7]. These functions are appropriate functions that arise in most practical situations as Bae et al. [7] confirm. We prove that the applicable standard distributions for the random variation X are within the framework of the theorems in Section 3.

Before giving the examples, we state the following lemma.

Lemma 5. *Let f_X, F_X, and $\bar{F}_X$ be the PDF, CDF, and SF of random variation X around $\eta(t)$. Then,*

(i) If F_X is TSSRD (TSSRI), then F_X is OSSRD (OSSRI).
(ii) F_X is TSSRD (TSSRI), if and only if $\bar{F}_X$ is TSSRD (TSSRI).
(iii) If f_X is OSSRD (OSSRI), then F_X and $\bar{F}_X$ are TSSRD (TSSRI).

Proof. The proof of (i) is obvious (see the lines after Definition 2). To prove (ii), it is enough to observe that for all $t_i \geq s_i \geq 0, i = 1,2$ and $t_2 \geq t_1$ and $s_2 \geq s_1$, it holds that:

$$\frac{F_X(t_2x) - F_X(t_1x)}{F_X(s_2x) - F_X(s_1x)} = \frac{\bar{F}_X(t_2x) - F_X(t_1x)}{\bar{F}_X(s_2x) - \bar{F}_X(s_1x)}.$$

To prove assertion (iii), it suffices to establish that if f_X is OSSRD (OSSRI), then F_X is TSSRD (TSSRI) because this is equivalent to $\bar{F}_X$ being TSSRD (TSSRI) from assertion (ii). We have

$$\frac{F_X(t_2x) - F_X(t_1x)}{F_X(s_2x) - F_X(s_1x)} = \frac{\int_{t_1x}^{t_2x} f_X(u)du}{\int_{s_1x}^{s_2x} f_X(u)du} = \frac{\int_{t_1}^{t_2} f_X(xy)dy}{\int_{s_1}^{s_2} f_X(xy)dy}.$$

The ratio $\frac{F_X(t_2x)-F_X(t_1x)}{F_X(s_2x)-F_X(s_1x)}$ is nonincreasing (nondecreasing) in $x \geq 0$ for all $t_i \geq s_i \geq 0$, $i = 1,2$ and $t_2 \geq t_1$ and $s_2 \geq s_1$, if and only if $w^\star(j,x) := \int_0^{+\infty} g(j,y)w(y,x)\, dy$ is RR_2 (TP_2) in $(j,x) \in \{1,2\} \times [0,+\infty)$, where $g(j,y) = I[s_1 < y \leq s_2]$ for $j = 1$, and $g(j,y) = I[t_1 < y \leq t_2]$ for $j = 2$ and $w(y,x) = f_X(xy)$. It is not hard to prove that $g(j,y)$ is TP_2 in (j,y), and also since f_X is OSSRD (OSSRI), thus $w(y,x)$ is RR_2 (TP_2) in (y,x). Hence, by Lemma 3(ii) the required result follows. □

The following examples show that the results of Theorems 2–5 and Theorem 6 can be applied to several typical standard distributions for the random variation X.

Example 1. *(X is Weibull-distributed). Suppose that X has SF $\bar{F}_X(x) = \exp(-(\lambda x)^\alpha)$, where $\lambda > 0$ and $\alpha > 0$. The PDF of X is $f_X(x) = \alpha\lambda^\alpha x^{\alpha-1}\exp(-(\lambda x)^\alpha)$. Thus,*

$$\frac{f_X(tx)}{f_X(x)} = t^{\alpha-1}\exp((\lambda x)^\alpha(1 - t^\alpha))$$

which is decreasing in $x \geq 0$, for all $t > 1$; thus, f_X is OSSRD, and as a result of Lemma 5(iii), F_X is TSSRD, and $\bar{F}_X$ is TSSRD.

Example 2. *(X is gamma-distributed). Assume that X has PDF $f_X(x) = \frac{\lambda^\gamma x^{\gamma-1}\exp(-\lambda x)}{\Gamma(\gamma)}$, where $\gamma > 0$ and $\lambda > 0$. We obtain*

$$\frac{f_X(tx)}{f_X(x)} = t^{\gamma-1}\exp((\lambda x)(1 - t)),$$

which is decreasing in $x \geq 0$, for every $t > 1$, i.e., f_X is OSSRD and by Lemma 5(iii), F_X is TSSRD, and $\bar{F}_X$ is also TSSRD.

Example 3. *(X is log-logistically distributed). Let us take X as a random variable with PDF $f_X(x) = \frac{\beta e^\alpha x^{\beta-1}}{(1+e^\alpha x^\beta)^2}$, for $\beta > 0$. We can derive*

$$\frac{f_X(tx)}{f_X(x)} = t^{\beta-1}\left(\frac{1 + e^\alpha x^\beta}{1 + e^\alpha (tx)^\beta}\right)^2,$$

which is decreasing in $x \geq 0$, for every $t > 1$, and this means f_X is OSSRD, which by Lemma 5(iii) implies that F_X is TSSRD, and $\bar{F}_X$ is also TSSRD.

The following example gives an application of Theorem 1.

Example 4. *Suppose $W(t)$ is a degradation process with an increasing mean degradation path. Assume that T denotes the time to failure of a device and that $T^\star$ denotes the time to failure after applying a burn-in strategy. This strategy omits devices that fail before their degradation reaches $\mathfrak{D}_{\mathfrak{f}(\mathfrak{1})}$. If $T_1 := \inf\{t \geq 0 \mid W(t) > \mathfrak{D}_{\mathfrak{f}(\mathfrak{1})}\}$, then*

$$p_0^\star = P(0 \leq T^\star < T_1) = 0, p_1^\star = P(T_1 \leq T^\star < +\infty) = 1,$$

and also we assume that

$$p_0 = P(0 \leq T < T_1) > 0, p_1 = P(T_1 \leq T < +\infty) < 1.$$

Since, $P \preceq P^\star$, with $P = (p_0, p_1)$ and $P^\star = (0, 1)$, thus, according to Theorem 1(i), $T \leq_{st} T^\star$. Note that $\frac{p_0^\star}{p_0} < \frac{p_1^\star}{p_1}$; therefore, if X is OSSRD, then by Theorem 2(i), $T \leq_{lr} T^\star$.

The novel time-to-failure degradation-based model proposed in this paper can be adopted by experts in statistics. The model includes some parameters, including the parameters in $\eta(t)$, the mean degradation path in the multiplicative degradation model (see, e.g., Bae et al. [7] for some typical shapes), the proportions $p_0, p_1, \ldots, p_k$, the failure probabilities of the device subject to degradation, and the amounts $\mathfrak{D}_{\mathfrak{f}(\mathfrak{1})}, \mathfrak{D}_{\mathfrak{f}(\mathfrak{2})}, \ldots, \mathfrak{D}_{\mathfrak{f}(\mathfrak{k})}$ as the limits of the degradation values. The problem of estimating these parameters using sample data on the degradation process $\{W(t) = X\eta(t) : t \geq 0\}$ and also using time-to-failure observations of devices undergoing degradation is an interesting and challenging study. In previous degradation-based time-to-failure models, there was traditionally a threshold for degradation that was assumed to be a predetermined value determined by empirical experimentation on products with high reliability. However, in the context of the new model in this work, which considers products with arbitrary reliability, it is not straightforward to determine the limits of degradation, i.e., the amounts of $\mathfrak{D}_{\mathfrak{f}(\mathfrak{1})}, \mathfrak{D}_{\mathfrak{f}(\mathfrak{2})}, \ldots, \mathfrak{D}_{\mathfrak{f}(\mathfrak{k})}$. Therefore, in such a situation, whether these parameters can be estimated is a key question. The potentially proposed estimation methods and statistical inference procedures can be investigated through simulation studies and also through the application of real datasets. However, in the context of applied probability theory, which is the basis of the present work, stochastic orderings are commonly used as a tool to make inferences about a population in two typical states without any data about the population in these states. Therefore, the results obtained in this work contribute to the stochastic comparison of the lifetime of two devices under degradation in the context of a new time-to-failure degradation model to evaluate the device with higher reliability. The properties and stochastic ordering results are obtained from the conditions attached to the parameters of the new time-to-failure model, so that after estimating the parameters of the model, one can choose a preferred strategy among two existing strategies that leads to better performance.

5. Conclusions

In this work, we achieved two goals. The first was to develop a novel time-to-failure model to fit the lifetime of devices under a typical degradation process, namely the multiplicative degradation model $W(t) = X\eta(t)$. The basic idea was that the device failure probability is constant at successive intervals as the degradation amount is increased (decreased). It was shown that the time to failure according to the model follows a well-known classical mixed model (Lemma 1). The second objective was to obtain some stochastic ordering properties under variation of probabilities in two different settings and to find conditions under which the device having a stochastically higher lifetime is identified. The degradation intervals were assumed to be fixed in both cases, the mean degradation function $\eta(t)$ was also assumed to be fixed, and the random variation X around $\eta(t)$ was assumed to follow a common distribution function in both cases. The usual stochastic ordering holds when a majorization property between the probability vectors is satisfied. It can be concluded that the reliability of the device under degradation decreases accordingly when the probabilities in one case are more distributed compared to the other cases. For the stronger stochastic orderings such as the likelihood ratio ordering, the hazard rate ordering, and the reverse hazard rate ordering, it has been clarified that in addition to the conditions required to classify the probability vectors from the two settings, additional conditions must be imposed on the distribution function of X. We have shown by some examples that the conditions on the distribution function of X are satisfied for some typical applicable standard distributions.

In future work, we may consider other settings or frameworks to detect devices under degradation that have more reliability. For example, the lower and upper bounds of the degradation intervals can be chosen as (random or nonrandom) variables, and the distribution function of X as well as the mean degradation amount around them can vary. The aging properties of the new time-to-failure model can also be studied, which is useful in model selection for geostrategies.

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