Article

Extropy Based on Concomitants of Order Statistics in Farlie-Gumbel-Morgenstern Family for Random Variables Representing Past Life

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Abstract: In this paper, we refined the concept of past extropy measure for concomitants of order statistics from Farlie-Gumbel-Morgenstern family. In addition, cumulative past extropy measure and dynamic cumulative past extropy measure for concomitant of rth order statistic are also conferred and their properties are studied. The problem of estimating the cumulative past extropy is investigated using empirical technique. The validity of the proposed estimator has been emphasized using simulation study.

Keywords: concomitants of order statistics; Farlie-Gumbel-Morgenstern family; past extropy; cumulative past extropy; dynamic cumulative past extropy

MSC: 62B10; 62G30

1. Introduction

Let (X_i, Y_i), $i = 1, 2, \ldots, n$ be independent and identically distributed (iid) random variables (rvs) from a continuous bivariate distribution with cumulative distribution function (cdf) $F(x, y)$ and joint probability density function (pdf) $f(x, y)$. Let $f_X(x)$ and $f_Y(y)$ be the marginal pdf of X and Y respectively, $F_X(x)$ and $F_Y(y)$ be the marginal cdf, respectively. We define $F_Y^{-1}(u) = inf\{y; F_Y(y) \geq u\}$, $u \in [0, 1]$ as the quantile function of $F_Y(y)$. If we arrange Xs in the order $X_1 \leq X_2 \leq \ldots\ldots \leq X_n$, then the Ys associated with these order statistics (OS) are called concomitants of order statistics (COS). We denote $X_{r:n}$ and $Y_{[r:n]}$ for the rth order statistic and concomitant of rth order statistic.

If the members of a random sample are sorted according to corresponding values of another random sample, then the COS arise. The concept of COS has been introduced by [1]. It has important applications in various areas such as in selection problems, inference problems etc. One may refer [2] for more details and applications of COS.

The common approach in the problems of modeling is that to select a family of distributions and then choose the one from the family that is more appropriate to describe the observation. The desirable feature of selecting a family is that it should be flexible. That is, it should accommodate several models that could represent any data situations. The most well-known parametric family from all the available parametric families is the Farlie-Gumbel-Morgenstern (FGM) family, that is mannered by [3]. The FGM family of distributions is constructed in such a way that it includes a joint distribution function along with its marginals, which makes it easier for the analyst to make several assumptions

about marginals. Ref. [4] studied about the new generalized FGM distributions and COS. A multi-parameter Generalized FGM bivariate copula family via Bernstein polynomial has been introduced by [5].

One of the most widely used measure of uncertainty is the entropy measure introduced by [6], which is used to quantify the amount of uncertainty involved in a rv and has several applications in various areas. The Shannon entropy is given by

$$H(X) = -\int_0^\infty f_X(x) \log f_X(x) dx.$$

Interested readers can refer the work by [7] for the characterizations of OS and records of entropy. Ref. [8] introduced new measure of sample entropy of continuous rv. Ref. [9] suggested some estimators of entropy using random sampling. The idea of varentropy is proposed by [10] as a great replacement for the kurtosis measure.

The differential extropy proposed by [11] is considered as the complementary dual of Shannon entropy. It is appeared from a critical analysis to score the forecasting distribution. For a non-negative rv X with pdf $f_X(x)$, differential extropy is given as follows

$$J(X) = -\frac{1}{2}\int_0^\infty f_X^2(x) dx. \tag{1}$$

In the application perspective, one can refer to [12] for the application of extropy in thermodynamics and [13] for the application of extropy in automatic speech recognition. Ref. [14] analysed the characterization results of extropy using OS and record values. Ref. [15] developed some results related to extropy properties of mixed systems. Ref. [16] provided general weighted extropy of ranked set sampling (RSS). They studied some characterization results, stochastic comparison and monotone properties of general weighted extropy.

In order to deal with the situations where uncertainty may be confined to the past life of an event, ref. [17] introduced past entropy for a rv $_tX = (t - x|X < t)$. The past extropy is offered by [18,19] to measure the uncertainty in a past lifetime distribution based on extropy. It is given as

$$J_t(X) = -\frac{1}{2}\int_0^t \left(\frac{f_X(x)}{F_X(t)}\right)^2 dx. \tag{2}$$

From Equation (2), it is clear that $J_t(X) < 0$ and $J_t(X) = J(X)$, for $t = \infty$. It has several applications in information theory, reliability theory, survival analysis etc.

Ref. [20] introduced the concept of varextropy which is a dispersion measure of extropy for residual and past lifetimes of rvs. Ref. [21] explored RSS properties of past extropy and negative cumulative extropy. Ref. [22] made a comparison study between past entropy and past extropy and showed that there are situations where uncertainty related to past life is approximately lower for past extropy measure than past entropy. Thus, it is evident that in some situations of uncertainty, information content is more while using past extropy measure than past entropy for rvs representing past life time.

If the properties of the residual and past lifetime depend on time, they are considered as dynamic. This means that these characteristics vary over time and are not constant. Ref. [23] considered dynamic weighted extropy. Ref. [24] introduced the concept of dynamic survival extropy in the literature. Ref. [25] introduced another uncertainty measure called Dynamic survival past extropy (DSPE) for past lifetime of a system. It is defined for a non-negative rv X with absolutely continuous cdf $F_X(x)$ as

$$J_{DSP_t}(X) = -\frac{1}{2}\int_0^t \left(\frac{F_X(x)}{F_X(t)}\right)^2 dx. \tag{3}$$

It is always less than zero and for $t = \infty$, it became the cumulative past extropy (CPE) which is given by [25], as

$$\mathcal{J}_t = -\frac{1}{2}\int_0^\infty F_X^2(x)dx. \tag{4}$$

Some properties of Fisher information in OS and its concomitants are inspected by [26]. Ref. [27] explored information properties for COS in FGM family. [28] explored properties of extropy for concomitants of generalized OS in FGM family. Ref. [29] explained the concept of ordered variables and their concomitants under extropy via COVID-19 data application. Ref. [30] developed the approach of the extropy of concomitants of generalized OS from Huang–Kotz–Morgenstern bivariate distribution.

The concomitants are mostly used in selection problems where the individuals are chosen according to the X values and associated characteristic is shown using the corresponding the Y values. The flexible nature of the FGM family is built in such a way that once the prior information in the form of marginals is available the bivariate distribution can be constructed with the help of those marginals and correlation between the variables directly. Its flexibility encouraged us to cope up the work in COS from FGM family. Ref. [31] studied residual and past entropy for concomitants of ordered rvs from FGM family. But not any work is available on the literature based on the COS from FGM family in the case of past life time rvs. This motivated us to present the information content and properties of COS for the rvs representing past lifetime.

The structure of the paper is organized as: In Section 2, we obtain the expression for $Y_{[r:n]}$ based on past extropy measure as well as some results related to past extropy of concomitant of rth order statistic are derived. In Sections 3 and 4, the properties of COS in CPE and DSPE measures are studied. Section 5 deals with the estimation of CPE for concomitant of rth order statistic using empirical estimators. And in Section 6, simulation study is exhibited to support the potency of the estimator. Finally, Section 7 provides a conclusion of main findings.

2. Past Extropy for Concomitants of Order Statistics in FGM Family

In this section, we derive measures of past extropy for COS in FGM family.
The cdf and pdf of the FGM family are given by [32] as

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)[1 + \alpha(1 - F_X(x))(1 - F_Y(y))], \tag{5}$$

and

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)[1 + \alpha(2F_X(x) - 1)(2F_Y(y)) - 1], \tag{6}$$

where $-1 \leq \alpha \leq 1$, $f_X(x)$ and $f_Y(y)$ are the marginal pdf of X and Y, respectively. $F_X(x)$ and $F_Y(y)$ are the marginal cdf of X and Y, respectively.

The pdf and cdf of concomitant of rth order statistic $Y_{[r:n]}$ given by [33], respectively are

$$g_{Y_{[r:n]}} = f_Y(y)[1 + \gamma_r(1 - 2F_Y(y))], \tag{7}$$

and

$$G_{Y_{[r:n]}}(y) = F_Y(y)[1 + \gamma_r(1 - F_Y(y))], \tag{8}$$

where $\gamma_r = \alpha\frac{n-2r+1}{n+1}$.

Proposition 1. *Let (X_i, Y_i), $i = 1, 2, \ldots, n$ be a bivariate random sample from FGM family and $Y_{[r:n]}$ denotes the concomitant of the rth order statistic arising from FGM family, the past extropy of $Y_{[r:n]}$ denoted as $J_t(Y_{[r:n]})$ is given by*

$$J_t(Y_{[r:n]}) = \left(\frac{1}{1 + \gamma_r(1 - F_Y(t))}\right)^2 \left[(1 + \gamma_r)^2 J_t(Y) + \frac{2\gamma_r(1 + \gamma_r)}{F_Y(t)}E\left(U_t f_Y(F_Y^{-1}(U_t))\right)\right.$$
$$\left. - \frac{2\gamma_r^2}{F_Y(t)}E\left(U_t^2 f_Y(F_Y^{-1}(U_t))\right)\right], \tag{9}$$

where $J_t(Y)$ is the past extropy of rv Y, $\gamma_r = \alpha \frac{n-2r+1}{n+1}$ and U_t is a uniformly distributed rv on $(0, F_Y(t))$ with pdf $\frac{1}{F_Y(t)}$.

Proof. For $Y_{[r:n]}$, from Equation (2) we have

$$J_t(Y_{[r:n]}) = -\frac{1}{2} \int_0^t \frac{g^2_{Y_{[r:n]}}(y)}{G^2_{Y_{[r:n]}}(t)} dy, \tag{10}$$

where $g_{Y_{[r:n]}}$ and $G_{Y_{[r:n]}}$ are defined in Equations (7) and (8), respectively.

Then,

$$J_t(Y_{[r:n]}) = -\frac{1}{2} \int_0^t \frac{f_Y^2(y)[1 + \gamma_r(1 - 2F_Y(y))]^2}{G^2_{Y_{[r:n]}}(t)} dy$$

$$= -\frac{1}{2} \int_0^t \frac{f_Y^2(y)\left[1 + 2\gamma_r(1 - 2F_Y(y)) + \gamma_r^2(1 - 2F_Y(y))^2\right]}{G^2_{Y_{[r:n]}}(t)} dy$$

$$= -\frac{1}{2} \int_0^t \frac{f_Y^2(y)[1 + 2\gamma_r(1 - 2F_Y(y)) + \gamma_r^2(1 - 4F_Y(y) + 4F_Y^2(y))]}{G^2_{Y_{[r:n]}}(t)} dy.$$

Expanding and rearranging, we get

$$J_t(Y_{[r:n]}) = -\frac{1}{2G^2_{Y_{[r:n]}}(t)} \left[(1+\gamma_r)^2 \int_0^t f_Y^2(y)dy - 4\gamma_r(1+\gamma_r) \int_0^t F_Y(y)f_Y^2(y)dy \right.$$

$$\left. + 4\gamma_r^2 \int_0^t F_Y^2(y)f_Y^2(y)dy \right].$$

Using Equation (8) it will be

$$J_t(Y_{[r:n]}) = -\frac{1}{2} \left(\frac{1}{1 + \gamma_r(1 - F_Y(t))} \right)^2 \left[\frac{(1+\gamma_r)^2}{F_Y^2(t)} \int_0^t f_Y^2(y)dy \right.$$

$$\left. - \frac{4\gamma_r(1+\gamma_r)}{F_Y^2(t)} \int_0^t F_Y(y)f_Y^2(y)dy + \frac{4\gamma_r^2}{F_Y^2(t)} \int_0^t F_Y^2(y)f_Y^2(y)dy \right]$$

$$= \left(\frac{1}{1 + \gamma_r(1 - F_Y(t))} \right)^2 \left[-\frac{(1+\gamma_r)^2}{2F_Y^2(t)} \int_0^{F_Y(t)} f_Y(F_Y^{-1}(u))du \right.$$

$$\left. + \frac{2\gamma_r(1+\gamma_r)}{F_Y^2(t)} \int_0^{F_Y(t)} u f_Y(F_Y^{-1}(u))du - \frac{2\gamma_r^2}{F_Y^2(t)} \int_0^{F_Y(t)} u^2 f_Y(F_Y^{-1}(u))du \right]$$

$$= \left(\frac{1}{1 + \gamma_r(1 - F_Y(t))} \right)^2 \left[(1+\gamma_r)^2 J_t(Y) + \frac{2\gamma_r(1+\gamma_r)}{F_Y^2(t)} \int_0^{F_Y(t)} u f_Y(F_Y^{-1}(u))du \right.$$

$$\left. - \frac{2\gamma_r^2}{F_Y^2(t)} \int_0^{F_Y(t)} u^2 f_Y(F_Y^{-1}(u))du \right]$$

$$= \left(\frac{1}{1 + \gamma_r(1 - F_Y(t))} \right)^2 \left[(1+\gamma_r)^2 J_t(Y) + \frac{2\gamma_r(1+\gamma_r)}{F_Y(t)} E\left(U_t f_Y(F_Y^{-1}(U_t)) \right) \right.$$

$$\left. - \frac{2\gamma_r^2}{F_Y(t)} E\left(U_t^2 f_Y(F_Y^{-1}(U_t)) \right) \right].$$

Hence, the theorem is proved. □

Remark 1. *When $\alpha = 0$ in Equation (9), then $\gamma_r = 0$, which implies $J_t(Y_{[r:n]}) = J_t(Y)$.*

Remark 2. *If $r = 1$ and $r = n$, we get the concomitants of the 1st and nth OS of a random sample of size n. Then, the past extropy measure for COS $Y_{[1:n]}$ and $Y_{[n:n]}$, respectively are given by*

$$
J_t(Y_{[1:n]}) = \left(\frac{1}{1 + C_{n,\alpha}(1 - F_Y(t))} \right)^2 \left[(1 + C_{n,\alpha})^2 J_t(Y) \right.
$$
$$
\left. + \frac{2C_{n,\alpha}(1 - C_{n,\alpha})}{F_Y(t)} E\left(U_t f_Y(F_Y^{-1}(U_t)) \right) - \frac{2C_{n,\alpha}^2}{F_Y(t)} E\left(U_t^2 f_Y(F_Y^{-1}(U_t)) \right) \right],
$$

(11)

and

$$
J_t(Y_{[n:n]}) = \left(\frac{1}{1 - C_{n,\alpha}(1 - F_Y(t))} \right)^2 \left[(1 - C_{n,\alpha})^2 J_t(Y) \right.
$$
$$
\left. - \frac{2C_{n,\alpha}(1 - C_{n,\alpha})}{F_Y(t)} E\left(U_t f_Y(F_Y^{-1}(U_t)) \right) - \frac{2C_{n,\alpha}^2}{F_Y(t)} E\left(U_t^2 f_Y(F_Y^{-1}(U_t)) \right) \right],
$$

(12)

where U_t is a rv distributed uniformly on $(0, F_Y(t))$ and $C_{n,\alpha} = \alpha \frac{n-1}{n+1}$.

Example 1. *Let (X_i, Y_i), $i = 1, 2, \ldots, n$ be a bivariate random sample arising from FGM bivariate exponential distribution with cdf*

$$
F(x, y) = \left(1 - e^{-\frac{x}{\theta_1}} \right) \left(1 - e^{-\frac{y}{\theta_2}} \right) \left(1 + \alpha e^{-\frac{x}{\theta_1} - \frac{y}{\theta_2}} \right),
$$

(13)

where $-1 \leq \alpha \leq 1$, $x, y > 0$, $\theta_1, \theta_2 > 0$. Then,

$$
J_t(Y_{[r:n]}) = -\frac{-2e^{-\frac{t}{\theta_2}}(-4 + \gamma_r)\gamma_r + 6\gamma_r^2 + e^{\frac{2t}{\theta_2}}(3 + \gamma_r(2 + \gamma_r)) + e^{\frac{3t}{\theta_2}}(3 + \gamma_r(2 + \gamma_r))}{12(-1 + e^{\frac{t}{\theta_2}})\theta_2(e^{\frac{t}{\theta_2}} + \gamma_r)^2}.
$$

(14)

Figure 1 shows some plots of $J_t\left(Y_{[r:n]}\right)$ for a sample of size $n = 50$ and the first order statistic $r = 1$ with various selections of θ_2 for FGM bivariate exponential distribution.

(a) $\alpha = -0.7$　　　　　　　　　　**(b)** $\alpha = 0.9$
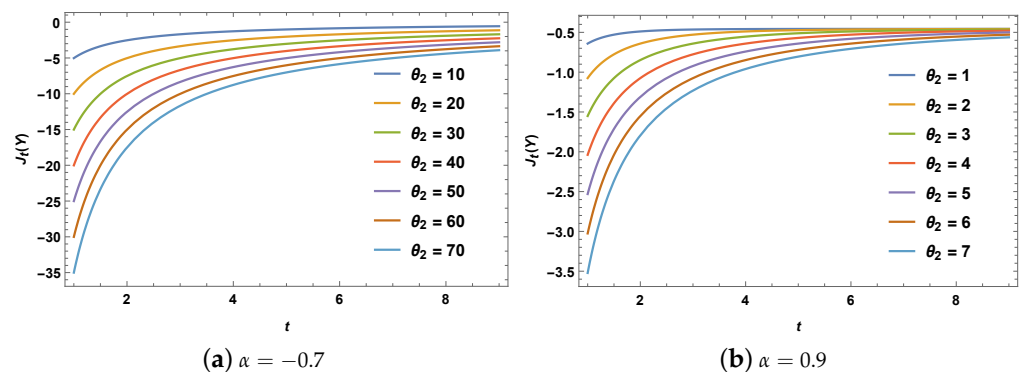
Figure 1. The plots of $J_t\left(Y_{[r:n]}\right)$ for $n = 50, r = 1$ with various values of θ_2 for FGM bivariate exponential distribution.

It is clear that $J_t\left(Y_{[r:n]}\right)$ values are negative for all cases considered in this figure. Using Equation (14), the difference between concomitants of nth and 1st OS for past extropy measure denoted as D_t is given by

$$D_t = J_t(Y_{[n:n]}) - J_t(Y_{[1:n]})$$

$$= \frac{e^{\frac{t}{\theta_2}}(-1+e^{\frac{t}{\theta_2}})(-1+n^2)\alpha\left[e^{\frac{t}{\theta_2}}(1+n)^2 - 2(-1+n)^2\alpha^2 - e^{\frac{t}{\theta_2}}(-1+n)^2\alpha^2\right]}{3\left[e^{\frac{2t}{\theta_2}}(1+n)^2 - (-1+n)^2\alpha^2\right]^2\theta_2}, \tag{15}$$

which is positive when $0 < \alpha \le 1$, is negative when $-1 \le \alpha < 0$, and is zero for $\alpha = 0$.

Example 2. *For FGM bivariate logistic distribution with cdf*

$$F(x,y) = (1+e^{-x})^{-1}(1+e^{-y})^{-1}\left(1+\alpha\frac{e^{-x-y}}{(1+e^{-x})(1+e^{-y})}\right), \tag{16}$$

where $-1 \le \alpha \le 1$, $-\infty < x < \infty$ and $-\infty < y < \infty$, the past extropy is given by

$$J_t(Y_{[r:n]}) = -\frac{e^{-2t}(-1+e^t)}{480(1+e^t)(1+e^t+\gamma_r^2)}\left(20 + 120e^t + 200e^{2t} + 120e^{3t} + 20e^{4t} + 15\gamma_r\right.$$

$$\left. +90e^t\gamma_r - 90e^{3t}\gamma_r - 15e^{4t}\gamma_r + 4\gamma_r^2 + 24e^t\gamma_r^2 - 56e^{2t}\gamma_r^2 + 24e^{3t}\gamma_r^2 + 4e^{4t}\gamma_r^2\right). \tag{17}$$

Using Equation (17), D_t is obtained as

$$D_t = e^{-2t}\alpha\left(25(1+n)(-1+n)^3 - 15e^{6t}(1+n)(-1+n)^3 - 7\alpha^2(1+n)(-1+n)^3\right.$$

$$-5e^{2t}(-1+n^2)\left(-35(1+n)^2 + 2(-1+n)^2\alpha^2\right) - 10e^t(-1+n^2)$$

$$\left(-11(1+n)^2 + 2(-1+n)^2\alpha^2\right) - 2e^{5t}(-1+n)^2\left(65(1+n)^2 + 4(-1+n)^2\alpha^2\right)$$

$$+20e^{3t}(-1+n^2)\left((1+n)^2 + 5(-1+n)^2\alpha^2\right)$$

$$\left.-5e^{4t}(-1+n)^2\left(37(1+n)^2 + 11(-1+n)^2\alpha^2\right)\right) \Bigg/ \left(240((1+n)(1+e^t) + (-1+n)\alpha)^2\right.$$

$$\left.((1+n)(1+e^t) + \alpha(1-n))^2\right). \tag{18}$$

From Equation (18), it is transparent that D_t is positive for $0 < \alpha \le 1$ and $n > 1$, is negative for $-1 \le \alpha < 0$ and $n > 1$. It is zero when $\alpha = 0$ and $n = 1$.

Figure 2 presents plots of the $J_t\left(Y_{[r:n]}\right)$ for a sample of size $n = 50$ to the first order statistic $r = 1$ with FGM bivariate logistic distribution.
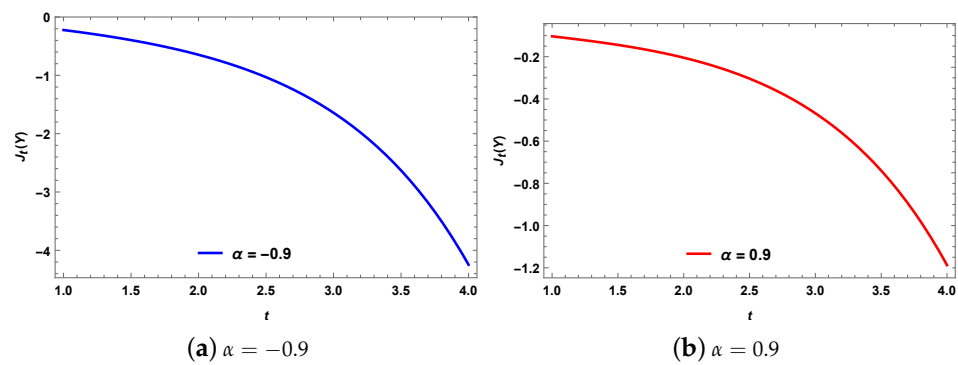
Figure 2. Plots of $J_t\left(Y_{[r:n]}\right)$ for $n = 50, r = 1$ for the FGM bivariate logistic distribution.

Figure 2 indicates that the values of $J_t\left(Y_{[r:n]}\right)$ are negative and its value depends on the parameter α.

Corollary 1. *Let* $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ *be a bivariate sample of size n coming from FGM family. Then, the past extropy for concomitant of rth order statistic for $\alpha > 0$ is same as past extropy for concomitant of $(n - r + 1)$th order statistic with $\alpha < 0$.*

Proof. Let us denote γ_r as $\gamma_{r,n,\alpha}$ and $J_t(Y_{[r:n:\alpha]})$ be the past extropy of concomitant of rth order statistic for any α. From the definition of γ_r it is clear that

$$\gamma_{r,n,\alpha} = \gamma_{n-2r+1,n,-\alpha}.$$

Thus, $J_t(Y_{[r:n:\alpha]}) = J_t(Y_{[n-2r+1:n:-\alpha]})$. Hence the corollary is attained. □

Corollary 2. *Let* $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ *be a bivariate sample of size n coming from FGM family. If n is odd, then the concomitant of median of X observations in the case of past extropy measure is same as past extropy of Y.*

Proof. In the case, if n is odd the concomitant of median of X observations is $Y_{[\frac{n+1}{2}:n]}$. Again, we can say that $\gamma_r = 0$ for $r = \frac{n+1}{2}$ and n is odd. Thus, $J_t(Y_{[\frac{n+1}{2}:n]}) = J_t(Y)$. The corollary is thus obtained. □

Theorem 1. *Let $Y_{[r:n]}$ be the concomitant of rth order statistic arising from FGM family, then the upper bound for past extropy of $Y_{[r:n]}$ is given by*

$$J_t(Y_{[r:n]}) \leq \frac{2\gamma_r(1 + \gamma_r)}{F_Y(t)} E[U_t f_Y(F_Y^{-1}(U_t))],$$

where $U_t \in U(0, F_Y(t))$.

Proof. Since $J_t(Y) \leq 0$, the inequality can be obtained directly from Proposition 1. □

Theorem 2. *Let $\kappa_f = E\left(f_Y^2(Y)|Y < t\right)$. If $Y_{[r:n]}$ is the concomitant of rth order statistic obtained from FGM family, the lower bound for $J_t(Y_{[r:n]})$ is*

$$J_t(Y_{[r:n]}) \geq -\frac{1}{2G_{Y_{[r:n]}}^2(t)} \kappa_f^{\frac{1}{2}} \left[(1 + \gamma_r)^5 - \frac{(1 + \gamma_r - 2F_Y(t)\gamma_r)}{10\gamma_r} \right]^{1/2}.$$

Proof. From Equations (7) and (10) we have

$$J_t(Y_{[r:n]}) = -\frac{1}{2G^2_{Y_{[r:n]}}(t)} \int_0^t f_Y^2(y)[1 + \gamma_r(1 - 2F_Y(y))]^2 \, dy$$

$$= -\frac{1}{2G^2_{Y_{[r:n]}}(t)} \int_0^{F_Y(t)} f_Y(F_Y^{-1}(u))(1 + \gamma_r(1 - 2u))^2 du.$$

Use Cauchy-Schwartz inequality to get

$$J_t(Y_{[r:n]}) \geq -\frac{1}{2G^2_{Y_{[r:n]}}(t)} \left[\int_0^{F_Y(t)} f_Y^2(F_Y^{-1}(u))du\right]^{\frac{1}{2}} \left[\int_0^{F_Y(t)} (1 + \gamma_r(1 - 2u))^4 du\right]^{\frac{1}{2}}$$

$$= -\frac{1}{2G^2_{Y_{[r:n]}}(t)} \kappa_f^{\frac{1}{2}} \left[(1 + \gamma_r)^5 - \frac{(1 + \gamma_r - 2F_Y(t)\gamma_r)^5}{10\gamma_r}\right]^{1/2},$$

which completes the proof. $\quad \square$

3. Cumulative Past Extropy of Concomitants of Order Statistics in FGM Family

In this section, we introduce the concept of CPE for concomitant of rth order statistic. The CPE can be defined only for the rvs that have bounded range of possible values since it gives negative infinity for any rv with unbounded support. So, we limit the definition of CPE to bounded rvs. For a non-negative rv X with bounded range of values $(0, b)$, the CPE is given by

$$\bar{J}_t(X) = -\frac{1}{2} \int_0^b F_X^2(x)dx.$$

Proposition 2. *Let (X_i, Y_i), $i = 1, 2, \ldots, n$ be a bivariate random sample from FGM family. Then, the concomitant of $Y_{[r:n]}$ in the case of CPE is given by*

$$\bar{J}_t(Y_{[r:n]}) = (1 + \gamma_r)^2 \bar{J}_t(Y) + \gamma_r(1 + \gamma_r)E\left(\frac{U^3}{f_Y(F_Y^{-1}(U))}\right) - \frac{\gamma_r^2}{2}E\left(\frac{U^4}{f_Y(F_Y^{-1}(U))}\right), \quad (19)$$

where U is a rv uniformly distributed on $(0, 1)$, $\bar{J}_t(Y)$ is the CPE of rv Y, and $\gamma_r = \alpha\frac{n-2r+1}{n+1}$.

Proof. For a rv Y bounded on $(0, b)$ and from the definition of $\bar{J}_t$ one may have

$$\bar{J}_t(Y_{[r:n]}) = -\frac{1}{2} \int_0^b G^2_{Y_{[r:n]}}(y)dy$$

$$= -\frac{1}{2} \int_0^b F_Y^2(y)\left[1 + \gamma_r(1 - F_Y(y))\right]^2 dy$$

$$= -\frac{1}{2} \int_0^b F_Y^2(y)\left[1 + 2\gamma_r(1 - F_Y(y)) + \gamma_r^2(1 - 2F_Y(y) + F_Y^2(y))\right]dy$$

$$= -\frac{1}{2}\left[(1 + \gamma_r)^2 \int_0^b F_Y^2(y)dy - 2\gamma_r(1 + \gamma_r)\int_0^b F_Y^3(y) + \gamma_r^2\int_0^b F_Y^4(y)dy\right]$$

$$= -\frac{(1 + \gamma_r)^2}{2} \int_0^1 \frac{u^2}{f_Y(F_Y^{-1}(u))}du + \gamma_r(1 + \gamma_r)\int_0^1 \frac{u^3}{f_Y(F_Y^{-1}(u))}du$$

$$- \frac{\gamma_r^2}{2} \int_0^1 \frac{u^4}{f_Y(F_Y^{-1}(u))}du$$

$$= (1 + \gamma_r)^2 \bar{J}_t(Y) + \gamma_r(1 + \gamma_r)E\left(\frac{U^3}{f_Y(F_Y^{-1}(U))}\right) - \frac{\gamma_r^2}{2}E\left(\frac{U^4}{f_Y(F_Y^{-1}(U))}\right).$$

Hence, attained the result. □

Remark 3. *In Equation (19), when $\alpha = 0$ then $\gamma_r = 0$, which implies that $\mathcal{J}_t(Y_{[r:n]}) = \mathcal{J}_t(Y)$.*

Remark 4. *The CPE of concomitants of the 1st and nth OS, respectively, are given by*

$$\mathcal{J}_t(Y_{[1:n]}) = \left[(1 + C_{n,\alpha})^2 \mathcal{J}_t(Y) + C_{n,\alpha}(1 + C_{n,\alpha})E\left(\frac{U^3}{f_Y(F_Y^{-1}(U))} \right) - \frac{C_{n,\alpha}^2}{2}E\left(\frac{U^4}{f_Y(F_Y^{-1}(U))} \right) \right],$$

and

$$\mathcal{J}_t(Y_{[n:n]}) = \left[(1 - C_{n,\alpha})^2 \mathcal{J}_t(Y) - C_{n,\alpha}(1 - C_{n,\alpha})E\left(\frac{U^3}{f_Y(F_Y^{-1}(U))} \right) - \frac{C_{n,\alpha}^2}{2}E\left(\frac{U^4}{f_Y(F_Y^{-1}(U))} \right) \right],$$

where $U \in U(0,1)$ and $C_{n,\alpha} = \alpha \frac{n-1}{n+1}$.
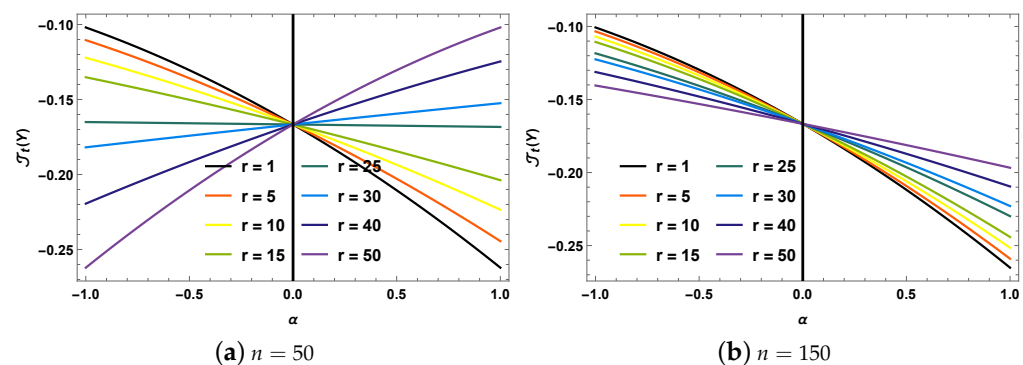
Example 3. *For FGM bivariate uniform distribution with cdf given by*

$$F(x,y) = \frac{x}{\theta_1} \frac{y}{\theta_2} \left[1 + \alpha \left(1 - \frac{x}{\theta_1} \right) \left(1 - \frac{y}{\theta_2} \right) \right],$$

where $0 < \theta_1 < 1$, $0 < \theta_2 < 1$ and $-1 \le \alpha \le 1$, the expression for $\mathcal{J}_t(Y_{[r:n]})$ with parameters $\theta_1 = \theta_2 = 1$ is

$$\mathcal{J}_t(Y_{[r:n]}) = -\frac{1}{2} \left[\frac{\gamma_r^2 + 5\gamma_r + 10}{30} \right].$$

Figure 3 presents plots of $\mathcal{J}_t\left(Y_{[r:n]}\right)$ for sample sizes $n = 50, 150$ selected from FGM bivariate uniform distribution to different OS $r = 1, 5, 10, 15, 25, 30, 40, 50$.

(a) $n = 50$ **(b)** $n = 150$

Figure 3. Plots of $\mathcal{J}_t\left(Y_{[r:n]}\right)$ for various selections of r based on the FGM bivariate uniform distribution.

Again, from Figure 3 it is obvious that the values of $\mathcal{J}_t\left(Y_{[r:n]}\right)$ are negative and the shape of the $\mathcal{J}_t\left(Y_{[r:n]}\right)$ depends on the sample size and the rth order statistic. Let β_t be the difference between concomitant of nth order statistic and concomitant of 1st order statistic. Then,

$$\beta_t = \frac{\alpha}{6} \left(\frac{1-n}{1+n} \right),$$

which is positive, negative or zero whenever $0 < \alpha \le 1$ and $n > 1$, $-1 \le \alpha < 0$ and $n > 1$ or $\alpha = 0$ and $n = 1$, respectively.

Corollary 3. *For a bivariate sample from FGM family, if n is odd then the CPE for concomitant of median of X observations is same as CPE of rv Y.*

Corollary 4. *For a bivariate sample obtained from FGM family, the CPE for concomitant of rth order statistic for $\alpha > 0$ is same as CPE of $(n - r + 1)$th order statistic for $\alpha < 0$.*

Theorem 3. *For a bivariate sample (X_i, Y_i), $i = 1, 2, \ldots, n$, lower bound for $\mathcal{J}_t(Y_{[r:n]})$ is given by*

$$\mathcal{J}_t(Y_{[r:n]}) \geq -\frac{1}{2}\left[1 + 2\gamma_r^2 + \frac{\gamma_r^4}{5}\right]^{\frac{1}{2}}\left[E\left(\frac{U^4}{f_Y^2(F_Y^{-1}(U))}\right)\right]^{\frac{1}{2}},$$

where $U \in U(0, 1)$.

Proof.

$$\mathcal{J}_t(Y_{[r:n]}) = -\frac{1}{2}\int_0^b G_{Y_{[r:n]}}^2(y)dy$$

$$= -\frac{1}{2}\int_0^b F_Y^2(y)\left[1 + \gamma_r(1 - F_Y(y))\right]^2 dy$$

$$= -\frac{1}{2}\int_0^1 \frac{u^2}{f_Y(F_Y^{-1}(u))}\left[1 + \gamma_r(1 - F_Y(y))\right]^2 du$$

By using Cauchy-Schwartz inequality, we get

$$\mathcal{J}_t(Y_{[r:n]}) \geq -\frac{1}{2}\left[\int_0^1 (1 + \gamma_r(1 - u))^4\right]^{\frac{1}{2}}\left[\int_0^1 \frac{u^4}{f_Y^2(F_Y^{-1}(u))}\right]^{\frac{1}{2}}$$

$$= -\frac{1}{2}\left[1 + 2\gamma_r^2 + \frac{\gamma_r^4}{5}\right]^{\frac{1}{2}}\left[E\left(\frac{U^4}{f_Y^2(F_Y^{-1}(U))}\right)\right]^{\frac{1}{2}}.$$

Hence the proof is attained. □

4. Dynamic Survival Past Extropy for Concomitants of Order Statistics in FGM Family

Here, we derive the expression for measures of DSPE for concomitant of rth order statistic.

Proposition 3. *Let $(X_1, Y_1), (X_2, Y_2), \ldots \ldots, (X_n, Y_n)$ be a bivariate sample of size n from FGM family. The concomitant of rth order statistic for DSPE measure, $J_{DSP_t}(Y_{[r:n]})$ is*

$$J_{DSP_t}(Y_{[r:n]}) = \left(\frac{1}{1 + \gamma_r(1 - F_Y(t))}\right)^2\left[(1 + \gamma_r)^2 J_{DSP_t}(Y)\right.$$

$$\left. + \frac{\gamma_r(1 + \gamma_r)}{F_Y(t)}E\left(\frac{U_t^3}{f_Y(F_Y^{-1}(U_t))}\right) - \frac{\gamma_r^2}{2F_Y(t)}E\left(\frac{U_t^4}{f_Y(F_Y^{-1}(U_t))}\right)\right], \tag{20}$$

where $J_{DSP_t}(Y)$ is DSPE of rv Y, $U_t \in U(0, F_Y(t))$ and $\gamma_r = \alpha\frac{n - 2r + 1}{n + 1}$.

Proof. To prove the proposition, from Equation (3) we have

$$J_{DSP_t}(Y_{[r:n]}) = -\frac{1}{2}\int_0^t \frac{G_{Y_{[r:n]}}^2(y)}{G_{Y_{[r:n]}}^2(t)}dy,$$

where is $G^2_{Y_{[r:n]}}(y)$ defined in Equation (8). Then,

$$J_{DSP_t}(Y_{[r:n]}) = -\frac{1}{2}\int_0^t \frac{F_Y^2(y)[1+\gamma_r(1-F_Y(y))]^2}{G^2_{Y_{[r:n]}}(t)}dy$$

$$= -\frac{1}{2}\int_0^t \frac{F_Y^2(y)[1+2\gamma_r(1-F_Y(y))+\gamma_r^2(1-2F_Y(y)+F_Y^2(y))]}{G^2_{Y_{[r:n]}}(t)}dy.$$

Rearranging, we get

$$J_{DSP_t}(Y_{[r:n]}) = -\frac{1}{2G^2_{Y_{[r:n]}}(t)}\left[(1+\gamma_r)^2\int_0^t F_Y^2(y)dy - 2\gamma_r(1+\gamma_r)\int_0^t F_Y^3(y) + \gamma_r^2\int_0^t F_Y^4(y)dy\right].$$

Using Equation (8), this equation will be

$$J_{DSP_t}(Y_{[r:n]}) = -\frac{1}{2}\left(\frac{1}{1+\gamma_r(1-F_Y(t))}\right)^2\left[\frac{(1+\gamma_r)^2}{F_Y^2(t)}\int_0^t F_Y^2(y)dy\right.$$

$$\left. -\frac{2\gamma_r(1+\gamma_r)}{F_Y^2(t)}\int_0^t F_Y^3(y)dy + \frac{\gamma_r^2}{F_Y^2(t)}\int_0^t F_Y^4(y)dy\right]$$

$$= \left(\frac{1}{1+\gamma_r(1-F_Y(t))}\right)^2\left[-\frac{(1+\gamma_r)^2}{2F_Y^2(t)}\int_0^{F_Y(t)}\frac{u^2}{f_Y(F_Y^{-1}(u))}du\right.$$

$$\left. +\frac{\gamma_r(1+\gamma_r)}{F_Y^2(t)}\int_0^{F_Y(t)}\frac{u^3}{f_Y(F_Y^{-1}(u))}du - \frac{\gamma_r^2}{2F_Y^2(t)}\int_0^{F_Y(t)}\frac{u^4}{f_Y(F_Y^{-1}(u))}du\right]$$

$$= \left(\frac{1}{1+\gamma_r(1-F_Y(t))}\right)^2\left[(1+\gamma_r)^2 J_{DSP_t}(Y)\right.$$

$$\left. +\frac{\gamma_r(1+\gamma_r)}{F_Y^2(t)}\int_0^{F_Y(t)}\frac{u^3}{f_Y(F_Y^{-1}(u))}du - \frac{\gamma_r^2}{2F_Y^2(t)}\int_0^{F_Y(t)}\frac{u^4}{f_Y(F_Y^{-1}(u))}du\right]$$

$$= \left(\frac{1}{1+\gamma_r(1-F_Y(t))}\right)^2\left[(1+\gamma_r)^2 J_{DSP_t}(Y)\right.$$

$$\left. +\frac{\gamma_r(1+\gamma_r)}{F_Y(t)}E\left(\frac{U_t^3}{f_Y(F_Y^{-1}(U_t))}\right) - \frac{\gamma_r^2}{2F_Y(t)}E\left(\frac{U_t^4}{f_Y(F_Y^{-1}(U_t))}\right)\right].$$

Hence, the result is attained. $\square$

Remark 5. *When $\alpha = 0$ in Equation (20), then $\gamma_r = 0$, which implies $J_{DSP_t}(Y_{[r:n]}) = J_{DSP_t}(Y)$.*

Remark 6. *The concomitants of 1st and nth OS of a random sample of size n are obtained using $r = 1, r = n$. Then, the DSPE measure for concomitants of the corresponding OS, respectively are*

$$J_{DSP_t}(Y_{[1:n]}) = \left(\frac{1}{1+C_{n,\alpha}(1-F_Y(t))}\right)^2\left[(1+C_{n,\alpha})^2 J_{DSP_t}(Y)\right.$$

$$\left. +\frac{C_{n,\alpha}(1+C_{n,\alpha})}{F_Y(t)}E\left(\frac{U_t^3}{f_Y(F_Y^{-1}(U_t))}\right) - \frac{C_{n,\alpha}^2}{2F_Y(t)}E\left(\frac{U_t^4}{f_Y(F_Y^{-1}(U_t))}\right)\right],$$

(21)

and

$$
\begin{aligned}
J_{DSP_t}(Y_{[n:n]}) = \left(\frac{1}{1 - C_{n,\alpha}(1 - F_Y(t))} \right)^2 & \left[(1 - C_{n,\alpha})^2 J_{DSP_t}(Y) \right. \\
& \left. - \frac{C_{n,\alpha}(1 - C_{n,\alpha})}{F_Y(t)} E\left(\frac{U_t^3}{f_Y(F_Y^{-1}(U_t))} \right) - \frac{C_{n,\alpha}^2}{2F_Y(t)} E\left(\frac{U_t^4}{f_Y(F_Y^{-1}(U_t))} \right) \right],
\end{aligned}
\tag{22}
$$

where U_t is a rv distributed uniformly on $(0, F_Y(t))$ and $C_{n,\alpha} = \alpha \frac{n-1}{n+1}$.

Example 4. *If (X, Y) is a bivariate sample from FGM bivariate exponential distribution with the cdf given in Equation (13), then*

$$
J_{DSP_t}(Y_{[r:n]}) = \frac{6e^{\frac{2t}{\theta_2}}\left(\theta_2 - 4e^{\frac{t}{\theta_2}}\theta_2 + e^{\frac{2t}{\theta_2}}(3\theta_2 - 2t)\right) - \left(-1 + e^{\frac{t}{\theta_2}}\right)^3 \theta_2 \gamma_r\left(3\gamma_r + e^{\frac{t}{\theta_2}}(8 + \gamma_r)\right)}{24\left(-1 + e^{\frac{t}{\theta_2}}\right)^2\left(e^{\frac{t}{\theta_2}} + \gamma_r\right)^2}.
\tag{23}
$$

Figure 4 presents some plots of $J_{DSPt}\left(Y_{[r:n]}\right)$ for a sample of size $n = 50, r = 1$ selected from FGM bivariate exponential distribution with $\alpha = -1, 1$.
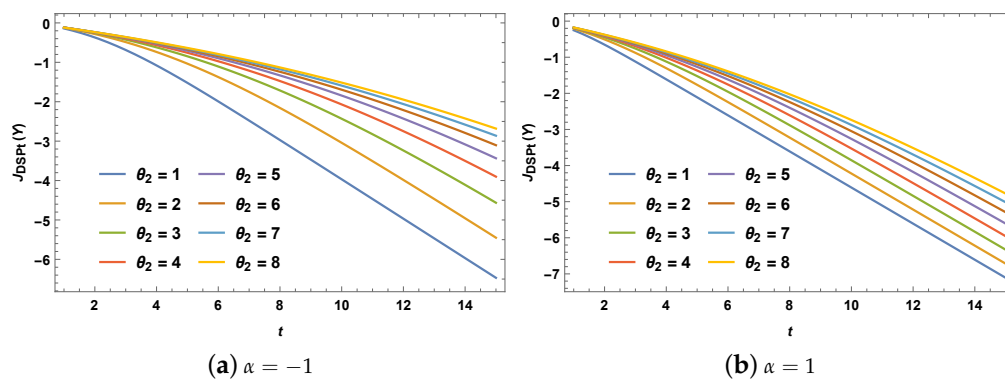
Figure 4. Plots of $J_{DSPt}\left(Y_{[r:n]}\right)$ for $\alpha = -1, 1$ based on the FGM bivariate exponential distribution.

From Figure 4 it is obvious that the values of $J_{DSPt}\left(Y_{[r:n]}\right)$ are negative and its shape depends on the parameter values $\alpha = -1, 1$. Let δ_t be the difference between DSPE of concomitants of nth and 1st OS. It is given by

$$
\begin{aligned}
\delta_t = & \frac{e^{\frac{t}{\theta_2}}(-1 + n^2)\alpha}{6\left(-1 + e^{\frac{t}{\theta_2}}\right)^2\left(e^{\frac{t}{\theta_2}}(1 + n) + (-1 + n)\alpha\right)^2\left(e^{\frac{t}{\theta_2}}(1 + n) + (1 - n)\alpha\right)^2} \left(4e^{\frac{5t}{\theta_2}}(1 + n)^2\theta_2 \right. \\
& - (1 + n)^2\alpha^2\theta_2 + 4e^{\frac{t}{\theta_2}}(-1 + n)^2\alpha^2\theta_2 + 4e^{\frac{3t}{\theta_2}}\left(-3(1 + n)^2 + (-1 + n)^2\alpha^2\right)\theta_2 \\
& - 2e^{\frac{2t}{\theta_2}}\left(-(1 + n)^2 + 3(-1 + n)^2\alpha^2\right)\theta_2 - e^{\frac{4t}{\theta_2}}\left(12(1 + n)^2 t + \left(-6(1 + n)^2 + (-1 + n)^2\alpha^2\right)\theta_2\right) \right).
\end{aligned}
$$

Hence, δ_t is positive when $0 < \alpha \leq 1$ and $n > 1$, is negative when $-1 \leq \alpha < 0$ and $n > 1$, is zero when $\alpha = 0$ and $n = 1$.

Example 5. *Let the bivariate sample is selected from FGM bivariate logistic distribution with cdf given in Equation (16), the DSPE of $Y_{[r:n]}$ is given by*

$$J_{DSP_t}(Y_{[r:n]}) = -\frac{1}{24(1+e^t)^3\left(1+\left(1-\frac{1}{1+e^{-t}}\right)\gamma_r\right)^2}\left(\left(1+e^{-t}\right)^2\left(-6(1+e^{-t})\right)^2\left(-1+log(4)\right.\right.$$

$$\left.\left. + e^t(1+log(4)) - 2(1+e^t)log(1+e^t)\right) + (-1+e^t)\gamma_r\left(3+\gamma_r+4e^t(3+\gamma_r)+e^{2t}(9+\gamma_r)\right)\right).$$

Next,

$$\delta_t = \frac{e^{-2t}(1+e^t)(-1+n^2)\alpha}{\left(12\left(1+n+e^t(1+n)+(-1+n)\alpha\right)^2\left(1+n+e^t(1+n)+(1-n)\alpha\right)^2\right)}\left[\left(-15\right.\right.$$

$$+ 9e^{4t}(1+n)^2 - \alpha^2 + 24log2 + n\left(-30-15n+2\alpha^2-n\alpha^2+24(2+n)log2\right) + e^{3t}\left(24-2\alpha^2\right.$$

$$\left. + 24log2 + 2n\left(24+12n+2\alpha^2-n\alpha^2+12(2+n)log2\right)\right) + 3e^{2t}\left(2+\alpha^2\right.$$

$$\left. + 24log2 + n\left(4+2n-2\alpha^2+n\alpha^2+24(2+n)log2\right)\right) + 24e^t(1+n)^2$$

$$\left.\left.(-1+log8) - 24(1+e^t)^3(1+n)^2log(1+e^t)\right)\right]$$

is positive for $n > 1$ when $0 < \alpha \leq 1$, negative in the range $-1 \leq \alpha < 0$ for $n > 1$ and zero when $n = 1$ with $\alpha = 0$.

Corollary 5. *For a bivariate sample obtained from FGM family, the DSPE for concomitant of rth order statistic for $\alpha > 0$ is same as DSPE of $(n-r+1)$th order statistic for $\alpha < 0$.*

Proof. The proof is similar to the proof of Corollary (1). Hence, $J_{DSP_{t(\alpha)}}(Y_{[r:n]}) = J_{DSP_{t(-\alpha)}}(Y_{[n-r+1:n]})$. □

Corollary 6. *For a bivariate sample from FGM family, if n is odd then the DSPE for concomitant of median of X observations is same as DSPE of rv Y.*

Proof. The proof is similar to the proof of Corollary (2) and hence omitted. □

Theorem 4. *The upper bound for DSPE of $Y_{[r:n]}$ is given by*

$$J_{DSP_t}(Y_{[r:n]}) \leq \frac{\gamma_r(1+\gamma_r)}{F_Y(t)}E\left[U_t(f_Y(F_Y^{-1}(U_t)))\right],$$

where $U_t \in U(0, F_Y(t))$ and $Y_{[r:n]}$ is the concomitant of rth order statistic arising from FGM family.

Proof. Since the DSPE ≤ 0, by using Proposition (3) we can obtain the inequality straightly. Hence, the theorem is attained. □

Theorem 5. *If $Y_{[r:n]}$ denotes the concomitant of rth order statistic arising from FGM family, then the lower bound of DSPE is*

$$J_{DSP_t}(Y_{[r:n]}) \geq -\frac{1}{2}\left(\frac{1}{1+\gamma_r(1-F_Y(t))}\right)^2\left[\frac{1}{F_Y^2(t)}\frac{(1+\gamma_r)^5-(-1+(-1+F_Y(t))\gamma_r)^5}{5\gamma_r}\right]^{\frac{1}{2}}$$

$$\left[\frac{1}{F_Y(t)}E\left(\frac{U_t^4}{f_Y^2(F_Y^{-1}(U_t))}\right)\right]^{\frac{1}{2}},$$

where $U_t \in U(0, F_Y(t))$.

Proof. Based on Equation (3), we have

$$
J_{DSP_t}(Y_{[r:n]}) = -\frac{1}{2}\int_0^t \frac{G_{Y_{[r:n]}}^2(y)}{G_{Y_{[r:n]}}^2(t)}dy
$$

$$
= -\frac{1}{2}\left(\frac{1}{F_Y^2(t)(1+\gamma_r(1-F_Y(t)))}\right)^2 \int_0^t F_Y^2(y)[1+\gamma_r(1-F_Y(y))]^2 dy
$$

$$
= -\frac{1}{2}\left(\frac{1}{F_Y^2(t)(1+\gamma_r(1-F_Y(t)))}\right)^2 \int_0^{F_Y(t)} \frac{u^2((1+\gamma_r(1-u))^2}{f_Y(F_Y^{-1}(u))}du.
$$

Using Cauchy-Schwartz inequality to get

$$
J_{DSP_t}(Y_{[r:n]}) \geq -\frac{1}{2}\left(\frac{1}{(1+\gamma_r(1-F_Y(t)))}\right)^2 \left[\frac{1}{F_Y^2(t)}\int_0^{F_Y(t)}\frac{u^4}{f_Y^2(F_Y^{-1}(u))}\right]^{\frac{1}{2}}
$$

$$
\left[\frac{1}{F_Y^2(t)}\int_0^{F_Y(t)}(1+\gamma_r(1-u))^4\right]^{\frac{1}{2}}
$$

$$
= -\frac{1}{2}\left(\frac{1}{(1+\gamma_r(1-F_Y(t)))}\right)^2 \left[\frac{1}{F_Y(t)}E\left(\frac{U_t^4}{f_Y^2(F_Y^{-1}(U_t))}\right)\right]^{\frac{1}{2}}
$$

$$
\left[\frac{1}{F_Y^2(t)}\frac{(1+\gamma_r)^5-(-1+(-1+F_Y(t))\gamma_r)^5}{5\gamma_r}\right]^{\frac{1}{2}}.
$$

Hence, the theorem is proved. $\square$

5. Estimation of CPE for Concomitant of *r*th Order Statistic

Here, the problem of estimating the CPE of *r*th order statistic using the empirical CPE is considered. Let (X_i, Y_i) be a sequence obtained from the FGM family. Using Equation (19), the empirical CPE of $Y_{[r:n]}$ is derived as

$$
\hat{\mathcal{J}}(Y_{[r:n]}) = -\frac{1}{2}\left[(1+\gamma_r)^2\int_0^\infty \hat{F}_Y^2(y)dy - 2\gamma_r(1+\gamma_r)\int_0^\infty \hat{F}_Y^3(y)dy + \gamma_r^2\int_0^\infty \hat{F}_Y^4(y)dy\right]
$$

$$
= -\frac{1}{2}\left[(1+\gamma_r)^2\sum_{i=1}^{n-1}\int_{Z_i}^{Z_{i+1}}\hat{F}_Y^2(y)dy - 2\gamma_r(1+\gamma_r)\sum_{i=1}^{n-1}\int_{Z_i}^{Z_{i+1}}\hat{F}_Y^3(y)dy\right.
$$

$$
\left. + \gamma_r^2\sum_{i=1}^{n-1}\int_{Z_i}^{Z_{i+1}}\hat{F}_Y^4(y)dy\right]
$$

$$
= -\frac{1}{2}\left[(1+\gamma_r)^2\sum_{i=1}^{n-1}U_i\left(\frac{i}{n}\right)^2 - 2\gamma_r(1+\gamma_r)\sum_{i=1}^{n-1}U_i\left(\frac{i}{n}\right)^3 + \gamma_r^2\sum_{i=1}^{n-1}U_i\left(\frac{i}{n}\right)^4\right]
$$

$$
= -\frac{1}{2}\sum_{i=1}^{n-1}U_i\left(\frac{i}{n}\right)^2\left[(1+\gamma_r)^2 - 2\gamma_r(1+\gamma_r)\left(\frac{i}{n}\right) + \gamma_r^2\left(\frac{i}{n}\right)^2\right]
$$

$$
= -\frac{1}{2}\sum_{i=1}^{n-1}U_i\left(\frac{i}{n}\right)^2\left[1+\gamma_r\left(1-\frac{i}{n}\right)\right]^2,
$$

(24)

where $U_i = Z_{i+1} - Z_i$, $i = 1, 2, \ldots, n-1$ are the sample spacings based on rv Y.

Example 6. *For FGM bivariate uniform distribution, the spacings U_i are independent beta distributed with parameters 1 and n. Then,*

$$E\left(\hat{\mathfrak{I}}(Y_{[r:n]})\right) = -\frac{1}{2(n+1)} \sum_{i=1}^{n-1} \left(\frac{i}{n}\right)^2 \left(1 + \gamma_r \left(1 - \frac{i}{n}\right)\right)^2, \tag{25}$$

and

$$Var\left(\hat{\mathfrak{I}}(Y_{[r:n]})\right) = \frac{n}{4(n+1)^2(n+2)} \sum_{i=1}^{n-1} \left(\frac{i}{n}\right)^4 \left(1 + \gamma_r \left(1 - \frac{i}{n}\right)\right)^4. \tag{26}$$

We figured the values of $E\left(\hat{\mathfrak{I}}(Y_{[2:n]})\right)$ and $Var\left(\hat{\mathfrak{I}}(Y_{[2:n]})\right)$ for different sample sizes $n = 5, 15, 25, 50$ for $\alpha = -1, -0.5, 0.5, 1$ in the case of FGM bivariate uniform distribution, and the results are given in Tables 1 and 2.

Table 1. Values of $E\left(\hat{\mathfrak{I}}(Y_{[2:n]})\right)$ for FGM bivariate uniform distribution with $\theta_1 = \theta_2 = 1$.

n	$\alpha = -1$	$\alpha = -0.5$	$\alpha = 0.5$	$\alpha = 1$
5	−0.0793	−0.0892	−0.1114	−0.1237
15	−0.0914	−0.1140	−0.1549	−0.2080
25	−0.0945	−0.1197	−0.1723	−0.2299
50	−0.0971	−0.1243	−0.1816	−0.2476

Table 2. Values of $Var\left(\hat{\mathfrak{I}}(Y_{[2:n]})\right)$ for FGM bivariate uniform distribution with $\theta_1 = \theta_2 = 1$.

n	$\alpha = -1$	$\alpha = -0.5$	$\alpha = 0.5$	$\alpha = 1$
5	0.0019	0.0023	0.0033	0.0040
15	0.0012	0.0016	0.0029	0.0039
25	0.0008	0.0011	0.0021	0.0029
50	0.0004	0.0006	0.0012	0.0017

The following properties can be attained from Tables 1 and 2:

- For a fixed value of n, the values of $E\left(\hat{\mathfrak{I}}(Y_{[2:n]})\right)$ are decreasing as the values of α increases, whereas the values of $Var\left(\hat{\mathfrak{I}}(Y_{[2:n]})\right)$ are increasing with α.
- For a fixed α, both the values of $E\left(\hat{\mathfrak{I}}(Y_{[2:n]})\right)$ and $Var\left(\hat{\mathfrak{I}}(Y_{[2:n]})\right)$ are decreasing with the increasing value of n.
- As n tends to infinity, the value of $Var\left(\hat{\mathfrak{I}}(Y_{[2:n]})\right)$ tends to zero.

6. Simulation

A Monte Carlo simulation study is conducted to validate the above proposed empirical estimator of CPE. Data is obtained through random generation following the FGM bivariate uniform distribution with parameter values of θ_1 and θ_2 set to 1. The study involves varying values of α to analyze and explore different scenarios using the generated data. Empirical CPE, theoretical CPE, bias, and mean squared error (MSE) for $Y_{[r:n]}$ are calculated for different values of sample size n and specific α. These computations are performed at various OS, allowing for an analysis between empirical and theoretical values while also assessing bias and accuracy for different scenarios, which are showed in Table 3 and Figure 5.

From Table 3, it is clear that as the sample size n increases, both the bias and MSE decrease, and in the case of the MSE, it approaches zero. This trend indicates that the estimator is performing well in this scenario. The decreasing bias implies that the estimator tends to be more accurate as the sample size becomes larger. The diminishing MSE suggests

that the estimator's predictions are closer to the true values with increasing sample size, signifying improved estimation accuracy and efficiency.

The following conclusions can be obtained from the Figure 5. They are,

- In all the cases discussed in the figure, both $\mathcal{J}_t(Y_{[r:n]})$ and $\hat{\mathcal{J}}(Y_{[r:n]})$ behaves alike. Both the values are almost similar for various values of OS.
- For $\alpha < 0$, both the theoretical and empirical CPE of $Y_{[r:n]}$ show a decreasing trend as the value of r increases.
- On the other hand, for $\alpha > 0$, both the functions $\mathcal{J}_t(Y_{[r:n]})$ and $\hat{\mathcal{J}}(Y_{[r:n]})$ exhibit an increasing pattern as r falls within this range.

Table 3. Theoretical value, Estimated value, bias and MSE of CPE of $Y_{[r:n]}$ for different values of r, α and n.

r	α	n	$\mathcal{J}_t(Y_{[r:n]})$	$\hat{\mathcal{J}}(Y_{[r:n]})$	bias	MSE
	0.2	50	−0.1825	−0.1747	0.0078	0.0005
	0.2	100	−0.1832	−0.1782	0.0050	0.0002
	0.2	200	−0.1836	−0.1815	0.0020	0.0001
	0.5	50	−0.2086	−0.1991	0.0094	0.0005
2	0.5	100	−0.2105	−0.2064	0.0040	0.0002
	0.5	200	−0.2115	−0.2092	0.0022	0.0001
	0.8	50	−0.2371	−0.2280	0.0091	0.0005
	0.8	100	−0.2405	−0.2360	0.0044	0.0003
	0.8	200	−0.2422	−0.2399	0.0023	0.0001
	0.2	50	−0.1811	−0.1718	0.0093	0.0005
	0.2	100	−0.1825	−0.1788	0.0037	0.0002
	0.2	200	−0.1832	−0.1807	0.0025	0.0001
	0.5	50	−0.2047	−0.1958	0.0089	0.0005
4	0.5	100	−0.2085	−0.2046	0.0039	0.0002
	0.5	200	−0.2105	−0.2081	0.0023	0.0001
	0.8	50	−0.2304	−0.2209	0.0095	0.0005
	0.8	100	−0.2370	−0.2321	0.0049	0.0003
	0.8	200	−0.2405	−0.2378	0.0026	0.0001
	0.2	50	−0.1770	−0.1678	0.0092	0.0004
	0.2	100	−0.1804	−0.1763	0.0041	0.0002
	0.2	200	−0.1822	−0.1801	0.0020	0.0001
	0.5	50	−0.1935	−0.1851	0.0083	0.0005
10	0.5	100	−0.2027	−0.1982	0.0045	0.0002
	0.5	200	−0.2075	−0.2051	0.0024	0.0001
	0.8	50	−0.2111	−0.2034	0.0076	0.0005
	0.8	100	−0.2269	−0.2227	0.0042	0.0002
	0.8	200	−0.2353	−0.2334	0.0018	0.0001
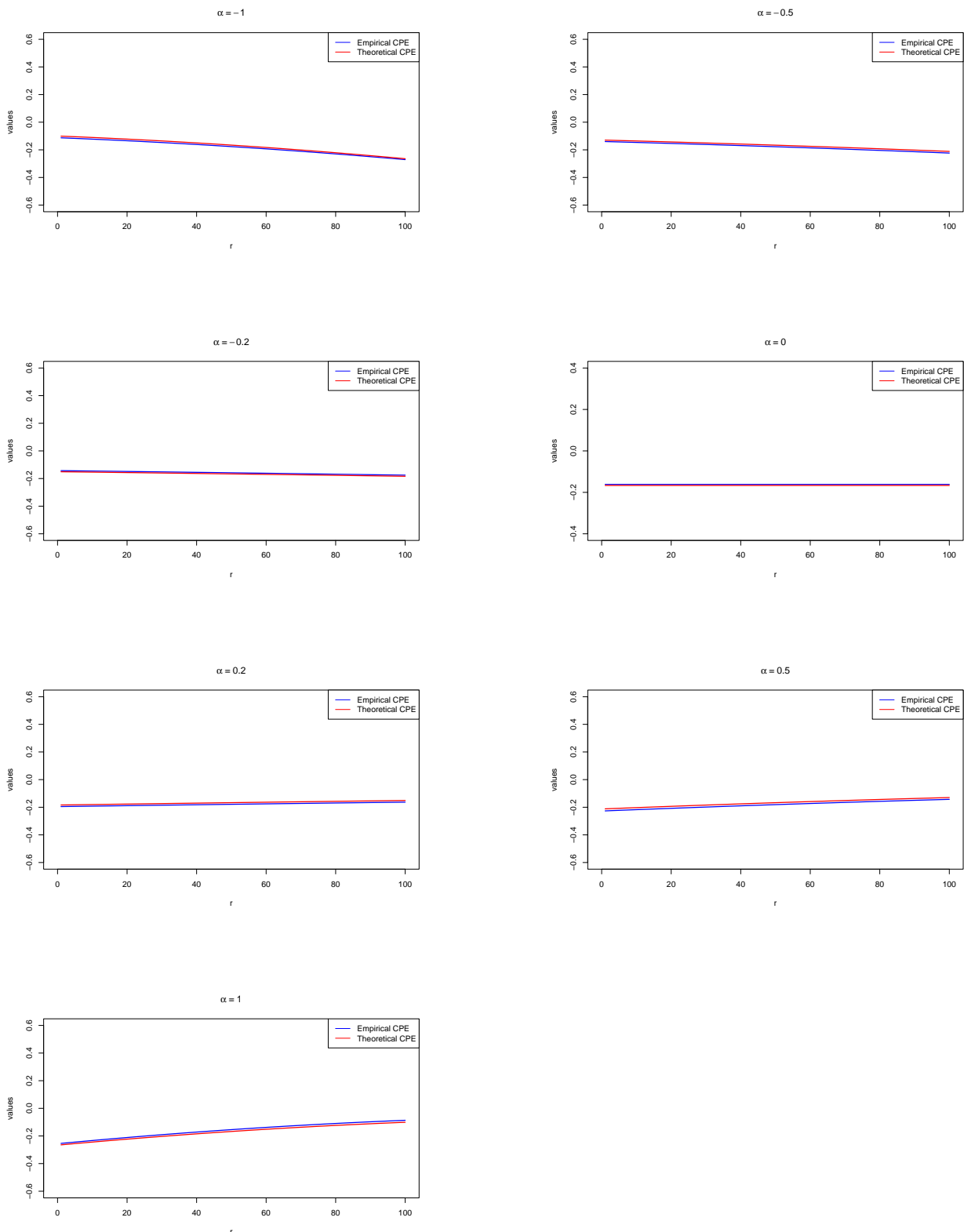
Figure 5. CPE of $Y_{[r:n]}$ (red) and empirical CPE of $Y_{[r:n]}$ (blue) in the case of FGM bivariate uniform distribution for $n = 100$.

7. Conclusions and Future Works

In this work, we have considered the properties of past extropy, CPE, DSPE for concomitant of rth order statistic in FGM family. Several properties and bounds related to

the above three measures for COS are developed. Estimation of CPE has been done using empirical estimation. The validity of the proposed estimators has been backed through numerical calculations on FGM uniform distribution and simulation study. From the simulation study, it can be concluded that the empirical CPE of $Y_{[r:n]}$ exhibits a behavior similar to the theoretical CPE of $Y_{[r:n]}$. When $\alpha < 0$, as r increases, both the theoretical and empirical CPE of $Y_{[r:n]}$ decrease, whereas for $\alpha > 0$, both the theoretical and empirical CPE of $Y_{[r:n]}$ increase with decreasing r in this range. The proposed estimator is considered good based on its MSE performance. This suggests that the estimator is capable of providing reliable and satisfactory estimates, making it a favorable choice for the given scenario.

COS play a crucial role in diverse selection processes. One significant application of COS lies in sampling procedures, including RSS, double sampling, and others. Investigating the information content of different information measures using these sampling designs in terms of COS could be a valuable addition to existing literature. To conduct such a study, COS can be derived from various bivariate families like FGM and Sarmanov.

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