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Positive Solutions for a System of Hadamard Fractional Boundary Value Problems on an Infinite Interval

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Abstract: Our investigation is devoted to examining the existence, uniqueness, and multiplicity of positive solutions for a system of Hadamard fractional differential equations. This system is defined on an infinite interval and is subject to coupled nonlocal boundary conditions. These boundary conditions encompass both Hadamard fractional derivatives and Riemann–Stieltjes integrals, and the nonlinearities within the system are non-negative functions that may not be bounded. To establish the main results, we rely on the utilization of mathematical theorems such as the Schauder fixed-point theorem, the Banach contraction mapping principle, and the Avery–Peterson fixed-point theorem.

Keywords: Hadamard fractional differential equations; nonlocal coupled boundary conditions; positive solutions; existence; uniqueness; multiplicity

MSC: 34A08; 34B10; 34B15; 34B18

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1. Introduction

Fixed-point theory finds its applications across a spectrum of domains in our lives. Notably, it plays a crucial role in tackling ordinary differential equations, partial differential equations, and, more recently, fractional differential equations. Within these arenas, researchers delve into investigating the existence, uniqueness, and multiplicity of various types of solutions, whether they be positive or otherwise. To achieve this, they harness a multitude of fixed-point theorems. These theorems include the influential Banach contraction mapping principle, the Guo–Krasnosel'skii fixed-point theorem involving cone expansion and compression of norm type, the Schauder fixed-point theorem, the Leray–Schauder alternative, the nonlinear alternative of Leray–Schauder type, the Leggett–Williams theorem, the Avery–Peterson fixed-point theorem, the nonlinear alternative of Leray–Schauder type specifically designed for Kakutani maps, and the Covitz–Nadler fixed-point theorem, among others. In the pursuit of understanding and applying these theorems, noteworthy references encompass well-established books such as [1–8], along with key papers such as [9,10].

In this paper, we analyze the following system comprising nonlinear Hadamard fractional differential equations

$$\begin{cases} {}^{H}D^{\alpha}_{1+}\mathfrak{u}(t) + \mathfrak{a}(t)\mathfrak{f}(t,\mathfrak{u}(t),\mathfrak{v}(t)) = 0, & t \in (1,+\infty), \\ {}^{H}D^{\beta}_{1+}\mathfrak{v}(t) + \mathfrak{b}(t)\mathfrak{g}(t,\mathfrak{u}(t),\mathfrak{v}(t)) = 0, & t \in (1,+\infty), \end{cases} \tag{1}$$

subject to the nonlocal coupled boundary conditions

$$\begin{cases} \mathfrak{u}^{(i)}(1)=0, \;\; i=0,1,\ldots,n-2; \;\; \mathfrak{v}^{(j)}(1)=0, \;\; j=0,1,\ldots,m-2; \\ {}^{H}D_{1+}^{\alpha-1}\mathfrak{u}(+\infty)=\int_1^{+\infty}\mathfrak{u}(s)\,d\mathcal{H}_1(s)+\int_1^{+\infty}\mathfrak{v}(s)\,d\mathcal{H}_2(s); \\ {}^{H}D_{1+}^{\beta-1}\mathfrak{v}(+\infty)=\int_1^{+\infty}\mathfrak{u}(s)\,d\mathcal{K}_1(s)+\int_1^{+\infty}\mathfrak{v}(s)\,d\mathcal{K}_2(s), \end{cases} \tag{2}$$

where $\alpha \in (n-1,n]$, $\beta \in (m-1,m]$, $n,m \in \mathbb{N}$, $n,m \geq 2$, ${}^{H}D_{1+}^{p}$ denotes the Hadamard fractional derivative of order p (for $p=\alpha,\beta,\alpha-1,\beta-1$), the non-negative functions $\mathfrak{a},\mathfrak{b}$ are defined on $[1,+\infty)$, the functions $\mathfrak{f},\mathfrak{g}:[1,+\infty)\times\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ may be unbounded and verify some assumptions, ($\mathbb{R}_+=[0,+\infty)$), and the integrals from the conditions (2) are Riemann–Stieltjes integrals with $\mathcal{H}_1,\mathcal{H}_2,\mathcal{K}_1,\mathcal{K}_2:[1,+\infty)\to\mathbb{R}$ functions of bounded variation. The term "nonlocal" within the context of the boundary conditions signifies that the unknown functions $\mathfrak{u}$ and $\mathfrak{v}$ at the ends of the interval are influenced by their own values within that interval. In the conditions (2), the functions $\mathfrak{u}$ and $\mathfrak{v}$ at infinity exhibit this nonlocal characteristic, as their dependence (mediated by Riemann–Stieltjes integrals) extends over the entire interval $(1,+\infty)$.

Our focus lies in investigating the existence, uniqueness, and multiplicity of positive solutions for the problem described by system (1) and the conditions (2). We consider different assumptions regarding the functions $\mathfrak{a},\mathfrak{b},\mathfrak{f}$, and $\mathfrak{g}$. To accomplish this, we employ mathematical tools such as the Schauder fixed-point theorem, the Banach contraction mapping principle, and the Avery–Peterson fixed-point theorem (see [1,10]). A positive solution of (1), (2) is a pair of functions $(\mathfrak{u}(t),\mathfrak{v}(t))$, $t\in[1,+\infty)$ which satisfies (1) and (2), with $\mathfrak{u}(t)\geq 0$, $\mathfrak{v}(t)\geq 0$, and $\mathfrak{u}(t)>0$ for all $t\in(1,+\infty)$ or $\mathfrak{v}(t)>0$ for all $t\in(1,+\infty)$. The problems in (1), (2), where the functions $\mathfrak{f}$ and $\mathfrak{g}$ are independent of t, continuous, and bounded, was recently investigated in [11], under different assumptions than those used the present paper. In [11], for the proof of the main results, the authors applied the Guo–Krasnosel'skii fixed-point theorem and the Leggett–Williams theorem (see [2,9]). In the paper in [12], the authors studied the positive solutions of system (1) with $\mathfrak{f}(t,u,v)=f(u,v)$, $\mathfrak{g}(t,u,v)=g(u,v)$, $\alpha,\ \beta\in(1,2]$ ($n=m=2$) and bounded nonlinearities f and g, supplemented with the boundary conditions

$$\begin{cases} \mathfrak{u}(1)=0, \;\; {}^{H}D_{1+}^{\alpha-1}\mathfrak{u}(+\infty)=\sum_{i=1}^{m}\lambda_i\,{}^{H}I_{1+}^{\alpha_i}\mathfrak{v}(\eta), \\ \mathfrak{v}(1)=0, \;\; {}^{H}D_{1+}^{\beta-1}\mathfrak{v}(+\infty)=\sum_{j=1}^{n}\sigma_j\,{}^{H}I_{1+}^{\beta_j}\mathfrak{u}(\xi), \end{cases} \tag{3}$$

where λ_i, $\sigma_j>0$ for $i=1,\ldots,m$, $j=1,\ldots,n$, $\eta>1$, $\xi>1$, and ${}^{H}I_{1+}^{k}$ is the Hadamard fractional integral of order k with lower limit 1. As we mentioned in [11], the last conditions for $+\infty$ from (3) are particular cases of the boundary conditions (2). Indeed we can write these conditions as $\sum_{i=1}^{m}\lambda_i\,{}^{H}I_{1+}^{\alpha_i}\mathfrak{v}(\eta)=\int_1^{+\infty}\mathfrak{v}(s)\,d\mathcal{H}_2(s)$, and $\sum_{j=1}^{n}\sigma_j\,{}^{H}I_{1+}^{\beta_j}\mathfrak{u}(\xi)=\int_1^{+\infty}\mathfrak{u}(s)\,d\mathcal{K}_1(s)$, with some bounded variation functions $\mathcal{H}_2$ and $\mathcal{K}_1$, ($\mathcal{H}_1\equiv 0$ and $\mathcal{K}_2\equiv 0$). We also mention the paper in [13], in which the authors investigated the nonlinear Hadamard fractional differential equation with nonlocal boundary conditions

$$\begin{cases} {}^{H}D_{1+}^{\alpha}\mathfrak{u}(t)+\mathfrak{a}(t)\mathfrak{f}(t,\mathfrak{u}(t))=0, \;\; t\in(1,+\infty), \\ \mathfrak{u}(1)=\mathfrak{u}'(1)=0, \;\; {}^{H}D_{1+}^{\alpha-1}\mathfrak{u}(+\infty)=\sum_{i=1}^{m}\alpha_i\,{}^{H}I_{1+}^{\beta_i}\mathfrak{u}(\eta)+b\sum_{j=1}^{n}\sigma_j\mathfrak{u}(\xi_j), \end{cases} \tag{4}$$

where $\mathfrak{f}$ is an unbounded function, $\alpha\in(2,3)$, $b\geq 0$, $\beta_i>0$ and $\alpha_i\geq 0$ for all $i=1,\ldots,m$, $\sigma_j\geq 0$ for all $j=1,\ldots,n$, $1<\eta<\xi_1<\ldots<\xi_n<+\infty$. They studied the existence, uniqueness, and multiplicity of positive solutions for problem (4) using diverse fixed-point theorems. The last boundary condition in (4) can be written as ${}^{H}D_{1+}^{\alpha-1}\mathfrak{u}(+\infty) = \int_1^{+\infty} \mathfrak{u}(s)\, dH(s)$, where $H(s) = \sum_{i=1}^{m} \alpha_i H_i(s) + bH_0(s)$, with

$$H_i(s) = \begin{cases} \dfrac{1}{\Gamma(\beta_i+1)}\left((\ln \eta)^{\beta_i} - \left(\ln \dfrac{\eta}{s}\right)^{\beta_i}\right), & \text{if } 1 \le s \le \eta, \\ \dfrac{1}{\Gamma(\beta_i+1)}(\ln \eta)^{\beta_i}, & \text{if } s \ge \eta, \end{cases}$$

$$H_0(s) = \begin{cases} 0, & s \in [1, \xi_1), \\ \sigma_1, & s \in [\xi_1, \xi_2), \\ \sigma_1 + \sigma_2, & s \in [\xi_2, \xi_3), \\ \vdots \\ \sigma_1 + \sigma_2 + \cdots + \sigma_n, & s \in [\xi_n, +\infty), \end{cases}$$

for $i = 1, \ldots, m$. So, in contrast to the paper in [11], our problems (1), (2) introduce several novel aspects. Firstly, the functions $\mathfrak{f}$ and $\mathfrak{g}$ are allowed to be unbounded. Additionally, the conditions imposed on these functions in the main results differ from those in the aforementioned paper. Lastly, our proof strategy involves applying fixed-point theorems that differ from those used in [11]. In scenarios where the variable t is confined to the finite interval $(0, 1)$, it is worth noting the contribution presented in the paper in [14]. In that work, the authors delve into the exploration of positive solutions within a system of Riemann–Liouville fractional differential equations. Notably, these equations are subject to uncoupled nonlocal boundary conditions, encompassing fractional derivatives and Riemann–Stieltjes integrals. The system's nonlinearities exhibit characteristics such as non-negativity and the potential for singularity with respect to the time variable. In substantiating their central theorems, the authors employ the Guo–Krasnosel'skii fixed-point theorem. This theorem serves as a pivotal tool in establishing the existence of positive solutions in the context of the explored system. For a thorough grasp of Riemann–Liouville, Caputo, Hadamard, Hilfer, and other variants of fractional differential equations and systems, coupled with a range of boundary conditions and their multifaceted applications across diverse fields, we recommend that readers explore the monographs in [15–26], and the papers in [27–31]. These references serve as invaluable resources for gaining deeper insights into this intricate subject matter.

The structure of the paper is outlined as follows. Section 2 provides an overview of key preliminary results from the paper in [11], which will be employed in the subsequent section. In Section 3, we present the existence theorems for problems (1), (2). Section 4 showcases a selection of examples that serve to illustrate our results, and Section 5 contains the conclusions of this paper.

2. Auxiliary Results

This section aims to introduce several essential preliminary results from the paper in [11], which will be utilized in the subsequent section.

We study the system of Hadamard fractional differential equations

$$\begin{cases} {}^{H}D_{1+}^{\alpha}\mathfrak{u}(t) + \mathfrak{h}(t) = 0, & t \in (1, +\infty), \\ {}^{H}D_{1+}^{\beta}\mathfrak{v}(t) + \mathfrak{k}(t) = 0, & t \in (1, +\infty), \end{cases} \tag{5}$$

where $\mathfrak{h}, \mathfrak{k} \in C([1, +\infty), \mathbb{R}_+)$, subject to the boundary conditions (2). We denote by

$$\begin{aligned} &a = \Gamma(\alpha) - \int_1^{+\infty} (\ln \zeta)^{\alpha-1}\, d\mathcal{H}_1(\zeta), \quad b = \int_1^{+\infty} (\ln \zeta)^{\beta-1}\, d\mathcal{H}_2(\zeta), \\ &c = \int_1^{+\infty} (\ln \zeta)^{\alpha-1}\, d\mathcal{K}_1(\zeta), \quad d = \Gamma(\beta) - \int_1^{+\infty} (\ln \zeta)^{\beta-1}\, d\mathcal{K}_2(\zeta), \\ &\Delta = ad - bc. \end{aligned} \tag{6}$$

We also use the notation $I = [1, +\infty)$.

Lemma 1 ([11])**.** *Assume that a, b, c, $d \in \mathbb{R}$, and the functions $\mathfrak{h}, \mathfrak{k} \in C([1,+\infty), \mathbb{R}_+)$ satisfy the conditions $\int_1^{+\infty} \mathfrak{h}(\zeta)\,\frac{d\zeta}{\zeta} < +\infty$ and $\int_1^{+\infty} \mathfrak{k}(\zeta)\,\frac{d\zeta}{\zeta} < +\infty$. If $\Delta \neq 0$, then the solution of problems (5), (2) is given by*

$$\begin{cases} \mathfrak{u}(t) = \int_1^{+\infty} \mathcal{G}_1(t,\zeta)\mathfrak{h}(\zeta)\,\frac{d\zeta}{\zeta} + \int_1^{+\infty} \mathcal{G}_2(t,\zeta)\mathfrak{k}(\zeta)\,\frac{d\zeta}{\zeta}, & t \in I, \\ \mathfrak{v}(t) = \int_1^{+\infty} \mathcal{G}_3(t,\zeta)\mathfrak{h}(\zeta)\,\frac{d\zeta}{\zeta} + \int_1^{+\infty} \mathcal{G}_4(t,\zeta)\mathfrak{k}(\zeta)\,\frac{d\zeta}{\zeta}, & t \in I, \end{cases} \tag{7}$$

where the Green functions $\mathcal{G}_i,\ i = 1,\ldots,4$ are given by

$$\begin{aligned} \mathcal{G}_1(t,\zeta) &= g_\alpha(t,\zeta) + \frac{(\ln t)^{\alpha-1}}{\Delta}\left(d\int_1^{+\infty} g_\alpha(\tau,\zeta)\,d\mathcal{H}_1(\tau) + b\int_1^{+\infty} g_\alpha(\tau,\zeta)\,d\mathcal{K}_1(\tau)\right), \\ \mathcal{G}_2(t,\zeta) &= \frac{(\ln t)^{\alpha-1}}{\Delta}\left(d\int_1^{+\infty} g_\beta(\tau,\zeta)\,d\mathcal{H}_2(\tau) + b\int_1^{+\infty} g_\beta(\tau,\zeta)\,d\mathcal{K}_2(\tau)\right), \\ \mathcal{G}_3(t,\zeta) &= \frac{(\ln t)^{\beta-1}}{\Delta}\left(c\int_1^{+\infty} g_\alpha(\tau,\zeta)\,d\mathcal{H}_1(\tau) + a\int_1^{+\infty} g_\alpha(\tau,\zeta)\,d\mathcal{K}_1(\tau)\right), \\ \mathcal{G}_4(t,\zeta) &= g_\beta(t,\zeta) + \frac{(\ln t)^{\beta-1}}{\Delta}\left(c\int_1^{+\infty} g_\beta(\tau,\zeta)\,d\mathcal{H}_2(\tau) + a\int_1^{+\infty} g_\beta(\tau,\zeta)\,d\mathcal{K}_2(\tau)\right), \end{aligned} \tag{8}$$

with

$$\begin{aligned} g_\alpha(t,\zeta) &= \frac{1}{\Gamma(\alpha)}\begin{cases} (\ln t)^{\alpha-1} - \left(\ln\frac{t}{\zeta}\right)^{\alpha-1}, & 1 \le \zeta \le t, \\ (\ln t)^{\alpha-1}, & 1 \le t \le \zeta, \end{cases} \\ g_\beta(t,\zeta) &= \frac{1}{\Gamma(\beta)}\begin{cases} (\ln t)^{\beta-1} - \left(\ln\frac{t}{\zeta}\right)^{\beta-1}, & 1 \le \zeta \le t, \\ (\ln t)^{\beta-1}, & 1 \le t \le \zeta. \end{cases} \end{aligned} \tag{9}$$

Additionally, we employ the notations

$$\Lambda_1 = \frac{d}{\Delta},\ \ \Lambda_2 = \frac{b}{\Delta},\ \ \Lambda_3 = \frac{c}{\Delta},\ \ \Lambda_4 = \frac{a}{\Delta}. \tag{10}$$

Lemma 2 ([11])**.** *Suppose that the functions $\mathcal{H}_i$, $\mathcal{K}_i$, $i = 1,2$ are nondecreasing functions, a, $d \in \mathbb{R}_+$, b, $c \in \mathbb{R}$, and $\Delta > 0$, and let $\theta > 1$. Then the functions g_α, g_β, and $\mathcal{G}_i$, $i = 1,\ldots,4$ (given by (9) and (8)) are continuous on $I \times I$, and satisfy the following inequalities for all $t, \zeta \in I$:*

(a) $0 \le g_\alpha(t,\zeta) \le \frac{1}{\Gamma(\alpha)}(\ln t)^{\alpha-1};\ \ 0 \le g_\beta(t,\zeta) \le \frac{1}{\Gamma(\beta)}(\ln t)^{\beta-1}$;

(b) $0 \le \frac{g_\alpha(t,\zeta)}{1+(\ln t)^{\alpha-1}} \le \frac{1}{\Gamma(\alpha)};\ \ 0 \le \frac{g_\beta(t,\zeta)}{1+(\ln t)^{\beta-1}} \le \frac{1}{\Gamma(\beta)}$;

(c) $\mathcal{G}_i(t,\zeta) \ge 0,\ \ i = 1,\ldots,4$;

(d) $\frac{\mathcal{G}_i(t,\zeta)}{1+(\ln t)^{\alpha-1}} \le \Lambda_i,\ \ i = 1,2;\ \ \frac{\mathcal{G}_j(t,\zeta)}{1+(\ln t)^{\beta-1}} \le \Lambda_j,\ \ j = 3,4$;

(e) $\min_{t\in[\theta,+\infty)} \frac{\mathcal{G}_1(t,\zeta)}{1+(\ln t)^{\alpha-1}} \ge \frac{(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})}$
$\times\left(d\int_1^{+\infty} g_\alpha(\tau,\zeta)\,d\mathcal{H}_1(\tau) + b\int_1^{+\infty} g_\alpha(\tau,\zeta)\,d\mathcal{K}_1(\tau)\right)$;

(f) $\min_{t\in[\theta,+\infty)} \frac{\mathcal{G}_2(t,\zeta)}{1+(\ln t)^{\alpha-1}} = \frac{(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})}$
$\times\left(d\int_1^{+\infty} g_\beta(\tau,\zeta)\,d\mathcal{H}_2(\tau) + b\int_1^{+\infty} g_\beta(\tau,\zeta)\,d\mathcal{K}_2(\tau)\right)$;

(g) $\min_{t\in[\theta,+\infty)} \frac{\mathcal{G}_3(t,\zeta)}{1+(\ln t)^{\beta-1}} = \frac{(\ln\theta)^{\beta-1}}{\Delta(1+(\ln\theta)^{\beta-1})}$
$\times\left(c\int_1^{+\infty} g_\alpha(\tau,\zeta)\,d\mathcal{H}_1(\tau) + a\int_1^{+\infty} g_\alpha(\tau,\zeta)\,d\mathcal{K}_1(\tau)\right)$;

(h) $\min_{t\in[\theta,+\infty)} \frac{\mathcal{G}_4(t,\zeta)}{1+(\ln t)^{\beta-1}} \ge \frac{(\ln\theta)^{\beta-1}}{\Delta(1+(\ln\theta)^{\beta-1})}$
$\times\left(c\int_1^{+\infty} g_\beta(\tau,\zeta)\,d\mathcal{H}_2(\tau) + a\int_1^{+\infty} g_\beta(\tau,\zeta)\,d\mathcal{K}_2(\tau)\right)$.

Under the assumptions of Lemma 2, we obtain that $a,\ d > 0$ and $b,\ c \geq 0$, so $\Lambda_i,\ i = 1,\ldots,4$ given by (10) satisfies the inequalities $\Lambda_1,\ \Lambda_4 > 0$, and $\Lambda_2,\ \Lambda_3 \geq 0$.

We present now the main assumptions that we will use in our results.

$(A1)$ $\alpha \in (n-1,n]$, $\beta \in (m-1,m]$, $n,m \in \mathbb{N}$, $n,m \geq 2$, $\mathcal{H}_i, \mathcal{K}_i : I \to \mathbb{R}$, $i = 1,2$ are nondecreasing functions, $a,d \in \mathbb{R}_+$, $b,c \in \mathbb{R}$, and $\Delta > 0$ (given by (6)).

$(A2)$ The functions $\mathfrak{a},\ \mathfrak{b} : I \to \mathbb{R}_+$ are not identical zero on any subinterval of I, and $0 < \int_1^{+\infty} \mathfrak{a}(\tau)\,\frac{d\tau}{\tau} < +\infty, 0 < \int_1^{+\infty} \mathfrak{b}(\tau)\,\frac{d\tau}{\tau} < +\infty$.

$(A3)$ The functions $\mathfrak{f},\ \mathfrak{g} : I \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the conditions

(i) $\mathfrak{f}(t,0,0) \not\equiv 0$, $\mathfrak{g}(t,0,0) \not\equiv 0$ on any subinterval of I;

(ii) $\mathfrak{f}(\cdot,u,v),\ \mathfrak{g}(\cdot,u,v)$ are measurable for every $(u,v) \in \mathbb{R}_+ \times \mathbb{R}_+$;

(iii) $\mathfrak{f}(t,\cdot,\cdot)$, $\mathfrak{g}(t,\cdot,\cdot)$ are continuous on $\mathbb{R}_+ \times \mathbb{R}_+$, for a.e. $t \in I$;

(iv) For any $r > 0$, there exist the functions $\varphi_r,\ \psi_r : I \to \mathbb{R}_+$ with $\int_1^{+\infty} \varphi_r(\tau)\mathfrak{a}(\tau)\,\frac{d\tau}{\tau} < +\infty$ and $\int_1^{+\infty} \psi_r(\tau)\mathfrak{b}(\tau)\,\frac{d\tau}{\tau} < +\infty$, such that

$$\begin{aligned} &\mathfrak{f}(t,(1+(\ln t)^{\alpha-1})u,(1+(\ln t)^{\beta-1})v \leq \varphi_r(t), \\ &\mathfrak{g}(t,(1+(\ln t)^{\alpha-1})u,(1+(\ln t)^{\beta-1}v) \leq \psi_r(t), \end{aligned}$$

for all $u,v \in [0,r]$ and a.e. $t \in I$.

We introduce the space

$$\mathcal{X}_1 = \left\{ \mathfrak{u} \in C(I,\mathbb{R}),\ \sup_{t\in I} \frac{|\mathfrak{u}(t)|}{1+(\ln t)^{\alpha-1}} < +\infty \right\},$$

with the norm $\|\mathfrak{u}\|_1 = \sup_{t\in I} \frac{|\mathfrak{u}(t)|}{1+(\ln t)^{\alpha-1}}$, the space

$$\mathcal{X}_2 = \left\{ \mathfrak{v} \in C(I,\mathbb{R}),\ \sup_{t\in I} \frac{|\mathfrak{v}(t)|}{1+(\ln t)^{\beta-1}} < +\infty \right\},$$

with the norm $\|\mathfrak{v}\|_2 = \sup_{t\in I} \frac{|\mathfrak{v}(t)|}{1+(\ln t)^{\beta-1}}$, and the space $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ with the norm $\|(\mathfrak{u},\mathfrak{v})\| = \|\mathfrak{u}\|_1 + \|\mathfrak{v}\|_2$. The spaces $(\mathcal{X}_1, \|\cdot\|_1)$, $(\mathcal{X}_2, \|\cdot\|_2)$ and $(\mathcal{X}, \|(\cdot,\cdot)\|)$ are Banach spaces (see [32], Lemma 2.7).

We define now the positive cone $\mathcal{P} \subset \mathcal{X}$ by

$$\mathcal{P} = \{(\mathfrak{u},\mathfrak{v}) \in \mathcal{X},\ \mathfrak{u}(t) \geq 0,\ \mathfrak{v}(t) \geq 0,\ \forall\, t \in I\},$$

and the operator $\mathcal{F} : \mathcal{P} \to \mathcal{X}$ by $\mathcal{F}(\mathfrak{u},\mathfrak{v}) = (\mathcal{F}_1(\mathfrak{u},\mathfrak{v}), \mathcal{F}_2(\mathfrak{u},\mathfrak{v}))$, $(\mathfrak{u},\mathfrak{v}) \in \mathcal{P}$, where the operators $\mathcal{F}_1 : \mathcal{P} \to \mathcal{X}_1$ and $\mathcal{F}_2 : \mathcal{P} \to \mathcal{X}_2$ are defined by

$$\begin{aligned} \mathcal{F}_1(\mathfrak{u},\mathfrak{v})(t) &= \int_1^{+\infty} \mathcal{G}_1(t,\zeta)\mathfrak{a}(\zeta)\mathfrak{f}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta} + \int_1^{+\infty} \mathcal{G}_2(t,\zeta)\mathfrak{b}(\zeta)\mathfrak{g}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta}, \\ \mathcal{F}_2(\mathfrak{u},\mathfrak{v})(t) &= \int_1^{+\infty} \mathcal{G}_3(t,\zeta)\mathfrak{a}(\zeta)\mathfrak{f}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta} + \int_1^{+\infty} \mathcal{G}_4(t,\zeta)\mathfrak{b}(\zeta)\mathfrak{g}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta}, \end{aligned}$$

for all $t \in I$ and $(\mathfrak{u},\mathfrak{v}) \in \mathcal{P}$.

Based on Lemma 2 and relations (7), we easily deduce that $(\mathfrak{u},\mathfrak{v})$ is a solution of problems (1), (2) if and only if $(\mathfrak{u},\mathfrak{v})$ is a fixed point of operator $\mathcal{F}$.

Using Lemma 2.8 from [32], and similar arguments to those used in the proof of Lemma 7 from [11], and Lemma 3.4 from [13], we obtain the following result.

Lemma 3. *If $(A1)$–$(A3)$ hold, then the operator $\mathcal{F} : \mathcal{P} \to \mathcal{P}$ is completely continuous, that is, continuous and it maps bounded sets into relatively compact sets.*

3. Existence of Positive Solutions

In this section, we present our main theorems that pertain to the existence of positive solutions for the problems (1), (2).

Our first existence theorem is the next one based on the Schauder fixed-point theorem.

Theorem 1. *Assume that $(A1)$–$(A3)$ hold. In addition, we suppose*

$(A4)$ *there exist non-negative functions* $\mathfrak{c}$, $\mathfrak{d}$, $\mathfrak{l}$, $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{m}$ *with*

$$\mathfrak{c}^* = \int_1^{+\infty} \mathfrak{c}(t)\mathfrak{a}(t)\,\frac{dt}{t} < +\infty,\ \mathfrak{d}^* = \int_1^{+\infty} \mathfrak{d}(t)\mathfrak{a}(t)\,\frac{dt}{t} < +\infty,$$
$$\mathfrak{l}^* = \int_1^{+\infty} \mathfrak{l}(t)\mathfrak{a}(t)\,\frac{dt}{t} < +\infty,\ \mathfrak{p}^* = \int_1^{+\infty} \mathfrak{p}(t)\mathfrak{b}(t)\,\frac{dt}{t} < +\infty,$$
$$\mathfrak{q}^* = \int_1^{+\infty} \mathfrak{q}(t)\mathfrak{b}(t)\,\frac{dt}{t} < +\infty,\ \mathfrak{m}^* = \int_1^{+\infty} \mathfrak{m}(t)\mathfrak{b}(t)\,\frac{dt}{t} < +\infty,$$

such that

$$\mathfrak{f}(t,u,v) \le \frac{\mathfrak{c}(t)u}{1+(\ln t)^{\alpha-1}} + \frac{\mathfrak{d}(t)v}{1+(\ln t)^{\beta-1}} + \mathfrak{l}(t),$$
$$\mathfrak{g}(t,u,v) \le \frac{\mathfrak{p}(t)u}{1+(\ln t)^{\alpha-1}} + \frac{\mathfrak{q}(t)v}{1+(\ln t)^{\beta-1}} + \mathfrak{m}(t),$$

for all $(t,u,v) \in I \times \mathbb{R}_+ \times \mathbb{R}_+$.

If

$$(\Lambda_1+\Lambda_3)(\mathfrak{c}^*+\mathfrak{d}^*) + (\Lambda_2+\Lambda_4)(\mathfrak{p}^*+\mathfrak{q}^*) < 1, \tag{11}$$

then the boundary value problems (1), (2) has at least one positive solution $(\mathfrak{u}(t),\mathfrak{v}(t)),\ t \in I$.

Proof. We take a positive number R_0 satisfying the condition

$$R_0 \ge \frac{(\Lambda_1+\Lambda_3)\mathfrak{l}^* + (\Lambda_2+\Lambda_4)\mathfrak{m}^*}{1-(\Lambda_1+\Lambda_3)(\mathfrak{c}^*+\mathfrak{d}^*) - (\Lambda_2+\Lambda_4)(\mathfrak{p}^*+\mathfrak{q}^*)},$$

and we define the set $\Omega_1 = \{(\mathfrak{u},\mathfrak{v}) \in \mathcal{P},\ \|(\mathfrak{u},\mathfrak{v})\| \le R_0\}$.

We show firstly that $\mathcal{F}(\Omega_1) \subset \Omega_1$. For this, let $(\mathfrak{u},\mathfrak{v}) \in \Omega_1$, that is $\|(\mathfrak{u},\mathfrak{v})\| \le R_0$, which implies $\|\mathfrak{u}\|_1 \le R_0$ and $\|\mathfrak{v}\|_2 \le R_0$, or $0 \le \frac{\mathfrak{u}(t)}{1+(\ln t)^{\alpha-1}} \le R_0$ and $0 \le \frac{\mathfrak{v}(t)}{1+(\ln t)^{\beta-1}} \le R_0$ for all $t \in I$. Then, by using Lemma 2, we obtain for all $t \in I$

$$\begin{aligned}
&\frac{\mathcal{F}_1(\mathfrak{u},\mathfrak{v})(t)}{1+(\ln t)^{\alpha-1}} = \int_1^{+\infty} \frac{\mathcal{G}_1(t,\zeta)}{1+(\ln t)^{\alpha-1}}\mathfrak{a}(\zeta)\mathfrak{f}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta}\\
&\quad + \int_1^{+\infty} \frac{\mathcal{G}_2(t,\zeta)}{1+(\ln t)^{\alpha-1}}\mathfrak{b}(\zeta)\mathfrak{g}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta}\\
&\le \Lambda_1 \int_1^{+\infty} \mathfrak{a}(\zeta)\mathfrak{f}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta} + \Lambda_2 \int_1^{+\infty} \mathfrak{b}(\zeta)\mathfrak{g}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta}\\
&\le \Lambda_1 \int_1^{+\infty} \mathfrak{a}(\zeta)\left[\frac{\mathfrak{c}(\zeta)\mathfrak{u}(\zeta)}{1+(\ln t)^{\alpha-1}} + \frac{\mathfrak{d}(\zeta)\mathfrak{v}(\zeta)}{1+(\ln t)^{\beta-1}} + \mathfrak{l}(\zeta)\right]\frac{d\zeta}{\zeta}\\
&\quad + \Lambda_2 \int_1^{+\infty} \mathfrak{b}(\zeta)\left[\frac{\mathfrak{p}(\zeta)\mathfrak{u}(\zeta)}{1+(\ln t)^{\alpha-1}} + \frac{\mathfrak{q}(\zeta)\mathfrak{v}(\zeta)}{1+(\ln t)^{\beta-1}} + \mathfrak{m}(\zeta)\right]\frac{d\zeta}{\zeta}\\
&\le \Lambda_1 R_0 \int_1^{+\infty} \mathfrak{a}(\zeta)\mathfrak{c}(\zeta)\,\frac{d\zeta}{\zeta} + \Lambda_1 R_0 \int_1^{+\infty} \mathfrak{a}(\zeta)\mathfrak{d}(\zeta)\,\frac{d\zeta}{\zeta} + \Lambda_1 \int_1^{+\infty} \mathfrak{a}(\zeta)\mathfrak{l}(\zeta)\,\frac{d\zeta}{\zeta}\\
&\quad + \Lambda_2 R_0 \int_1^{+\infty} \mathfrak{b}(\zeta)\mathfrak{p}(\zeta)\,\frac{d\zeta}{\zeta} + \Lambda_2 R_0 \int_1^{+\infty} \mathfrak{b}(\zeta)\mathfrak{q}(\zeta)\,\frac{d\zeta}{\zeta} + \Lambda_2 \int_1^{+\infty} \mathfrak{b}(\zeta)\mathfrak{m}(\zeta)\,\frac{d\zeta}{\zeta}\\
&= \Lambda_1(\mathfrak{c}^* R_0 + \mathfrak{d}^* R_0 + \mathfrak{l}^*) + \Lambda_2(\mathfrak{p}^* R_0 + \mathfrak{q}^* R_0 + \mathfrak{m}^*).
\end{aligned}$$

So we find

$$\|\mathcal{F}_1(\mathfrak{u},\mathfrak{v})\|_1 = \sup_{t\in I} \frac{\mathcal{F}_1(\mathfrak{u},\mathfrak{v})(t)}{1+(\ln t)^{\alpha-1}} \le R_0(\Lambda_1\mathfrak{c}^* + \Lambda_1\mathfrak{d}^* + \Lambda_2\mathfrak{p}^* + \Lambda_2\mathfrak{q}^*) + \Lambda_1\mathfrak{l}^* + \Lambda_2\mathfrak{m}^*.$$

In a similar manner, for all $t \in I$, we have

$$\begin{aligned}
\frac{\mathcal{F}_2(\mathfrak{u},\mathfrak{v})(t)}{1+(\ln t)^{\beta-1}} &= \int_1^{+\infty} \frac{\mathcal{G}_3(t,\zeta)}{1+(\ln t)^{\beta-1}} \mathfrak{a}(\zeta)\mathfrak{f}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta)) \frac{d\zeta}{\zeta} \\
&+ \int_1^{+\infty} \frac{\mathcal{G}_4(t,\zeta)}{1+(\ln t)^{\beta-1}} \mathfrak{b}(\zeta)\mathfrak{g}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta)) \frac{d\zeta}{\zeta} \\
&\leq \Lambda_3 \int_1^{+\infty} \mathfrak{a}(\zeta)\mathfrak{f}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta)) \frac{d\zeta}{\zeta} + \Lambda_4 \int_1^{+\infty} \mathfrak{b}(\zeta)\mathfrak{g}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta)) \frac{d\zeta}{\zeta} \\
&\leq \Lambda_3 \int_1^{+\infty} \mathfrak{a}(\zeta)\left[\frac{\mathfrak{c}(\zeta)\mathfrak{u}(\zeta)}{1+(\ln t)^{\alpha-1}} + \frac{\mathfrak{d}(\zeta)\mathfrak{v}(\zeta)}{1+(\ln t)^{\beta-1}} + \mathfrak{l}(\zeta)\right] \frac{d\zeta}{\zeta} \\
&+ \Lambda_4 \int_1^{+\infty} \mathfrak{b}(\zeta)\left[\frac{\mathfrak{p}(\zeta)\mathfrak{u}(\zeta)}{1+(\ln t)^{\alpha-1}} + \frac{\mathfrak{q}(\zeta)\mathfrak{v}(\zeta)}{1+(\ln t)^{\beta-1}} + \mathfrak{m}(\zeta)\right] \frac{d\zeta}{\zeta} \\
&\leq \Lambda_3 R_0 \int_1^{+\infty} \mathfrak{a}(\zeta)\mathfrak{c}(\zeta) \frac{d\zeta}{\zeta} + \Lambda_3 R_0 \int_1^{+\infty} \mathfrak{a}(\zeta)\mathfrak{d}(\zeta) \frac{d\zeta}{\zeta} + \Lambda_3 \int_1^{+\infty} \mathfrak{a}(\zeta)\mathfrak{l}(\zeta) \frac{d\zeta}{\zeta} \\
&+ \Lambda_4 R_0 \int_1^{+\infty} \mathfrak{b}(\zeta)\mathfrak{p}(\zeta) \frac{d\zeta}{\zeta} + \Lambda_4 R_0 \int_1^{+\infty} \mathfrak{b}(\zeta)\mathfrak{q}(\zeta) \frac{d\zeta}{\zeta} + \Lambda_4 \int_1^{+\infty} \mathfrak{b}(\zeta)\mathfrak{m}(\zeta) \frac{d\zeta}{\zeta} \\
&= \Lambda_3(\mathfrak{c}^* R_0 + \mathfrak{d}^* R_0 + \mathfrak{l}^*) + \Lambda_4(\mathfrak{p}^* R_0 + \mathfrak{q}^* R_0 + \mathfrak{m}^*).
\end{aligned}$$

Then we deduce

$$\|\mathcal{F}_2(\mathfrak{u},\mathfrak{v})\|_2 = \sup_{t\in I} \frac{\mathcal{F}_2(\mathfrak{u},\mathfrak{v})(t)}{1+(\ln t)^{\beta-1}} \leq R_0(\Lambda_3\mathfrak{c}^* + \Lambda_3\mathfrak{d}^* + \Lambda_4\mathfrak{p}^* + \Lambda_4\mathfrak{q}^*) + \Lambda_3\mathfrak{l}^* + \Lambda_4\mathfrak{m}^*.$$

Therefore, by using condition (11), we obtain

$$\begin{aligned}
&\|\mathcal{F}(\mathfrak{u},\mathfrak{v})\| = \|\mathcal{F}_1(\mathfrak{u},\mathfrak{v})\|_1 + \|\mathcal{F}_2(\mathfrak{u},\mathfrak{v})\|_2 \\
&\leq R_0((\Lambda_1+\Lambda_3)(\mathfrak{c}^*+\mathfrak{d}^*) + (\Lambda_2+\Lambda_4)(\mathfrak{p}^*+\mathfrak{q}^*)) + (\Lambda_1+\Lambda_3)\mathfrak{l}^* + (\Lambda_2+\Lambda_4)\mathfrak{m}^* \leq R_0.
\end{aligned}$$

So we conclude that $\mathcal{F}(\Omega_1) \subset \Omega_1$.

By Lemma 3, we know that operator $\mathcal{F} : \Omega_1 \to \Omega_1$ is completely continuous. By using the Schauder fixed-point theorem, we deduce that $\mathcal{F}$ has a fixed point $(\mathfrak{u},\mathfrak{v}) \in \mathcal{P}$ with $\|(\mathfrak{u},\mathfrak{v})\| \leq R_0$, which, by assumptions $(A2)$, $(A3)$, is a positive solution of problems (1), (2). □

The second existence theorem is the following one, which is based on the Banach contraction mapping principle.

Theorem 2. *Assume that $(A1)$–$(A3)$ hold. In addition, we suppose*

$(A5)$ there exist non-negative functions χ_i, $i = 1,\ldots,4$ with

$$\begin{aligned}
\chi_1^* &= \int_1^{+\infty} \mathfrak{a}(t)\chi_1(t)\frac{dt}{t} < +\infty, \quad \chi_2^* = \int_1^{+\infty} \mathfrak{a}(t)\chi_2(t)\frac{dt}{t} < +\infty, \\
\chi_3^* &= \int_1^{+\infty} \mathfrak{b}(t)\chi_3(t)\frac{dt}{t} < +\infty, \quad \chi_4^* = \int_1^{+\infty} \mathfrak{b}(t)\chi_4(t)\frac{dt}{t} < +\infty,
\end{aligned}$$

such that

$$\begin{aligned}
|\mathfrak{f}(t,u_1,v_1) - \mathfrak{f}(t,u_2,v_2)| &\leq \frac{\chi_1(t)}{1+(\ln t)^{\alpha-1}}|u_1-u_2| + \frac{\chi_2(t)}{1+(\ln t)^{\beta-1}}|v_1-v_2|, \\
|\mathfrak{g}(t,u_1,v_1) - \mathfrak{g}(t,u_2,v_2)| &\leq \frac{\chi_3(t)}{1+(\ln t)^{\alpha-1}}|u_1-u_2| + \frac{\chi_4(t)}{1+(\ln t)^{\beta-1}}|v_1-v_2|,
\end{aligned}$$

for all $t \in I$, $u_i, v_i \in \mathbb{R}_+$, $i = 1,2$.

If

$$\widetilde{\Upsilon}_0 := \widetilde{\Upsilon}_1 + \widetilde{\Upsilon}_2 < 1, \tag{12}$$

where $\widetilde{\Upsilon}_1 = \max\{\Lambda_1\chi_1^* + \Lambda_2\chi_3^*, \Lambda_1\chi_2^* + \Lambda_2\chi_4^*\}$, $\widetilde{\Upsilon}_2 = \max\{\Lambda_3\chi_1^* + \Lambda_4\chi_3^*, \Lambda_3\chi_2^* + \Lambda_4\chi_4^*\}$, then the boundary value problems (1), (2) has a unique positive solution $(\mathfrak{u}^*,\mathfrak{v}^*) \in \mathcal{P}$. Moreover, *for any* $(\mathfrak{u}_0, \mathfrak{v}_0) \in \mathcal{P}$*, the sequence* $((\mathfrak{u}_n, \mathfrak{v}_n))_{n\geq 0}$ *defined by* $(\mathfrak{u}_n, \mathfrak{v}_n) = \mathcal{F}(\mathfrak{u}_{n-1}, \mathfrak{v}_{n-1}),\ n \geq 1$ *converges to* $(\mathfrak{u}^*, \mathfrak{v}^*)$*, as* $n \to +\infty$*. In addition, we have the following error estimate:*

$$\|(\mathfrak{u}_n, \mathfrak{v}_n) - (\mathfrak{u}^*, \mathfrak{v}^*)\| \leq \frac{\widetilde{\Upsilon}_0^n}{1 - \widetilde{\Upsilon}_0} \|(\mathfrak{u}_1, \mathfrak{v}_1) - (\mathfrak{u}_0, \mathfrak{v}_0)\|. \tag{13}$$

Proof. By using Lemma 2 and assumption $(A5)$, for any $(\mathfrak{u}_1, \mathfrak{v}_1)$, $(\mathfrak{u}_2, \mathfrak{v}_2) \in \mathcal{P}$, we obtain

$$\begin{aligned}
&\left|\frac{\mathcal{F}_1(\mathfrak{u}_1, \mathfrak{v}_1)(t)}{1+(\ln t)^{\alpha-1}} - \frac{\mathcal{F}_1(\mathfrak{u}_2, \mathfrak{v}_2)(t)}{1+(\ln t)^{\alpha-1}}\right| \\
&= \left|\int_1^{+\infty} \frac{\mathcal{G}_1(t,\zeta)}{1+(\ln t)^{\alpha-1}} \mathfrak{a}(\zeta)\mathfrak{f}(\zeta, \mathfrak{u}_1(\zeta), \mathfrak{v}_1(\zeta)) \frac{d\zeta}{\zeta}\right. \\
&\quad + \int_1^{+\infty} \frac{\mathcal{G}_2(t,\zeta)}{1+(\ln t)^{\alpha-1}} \mathfrak{b}(\zeta)\mathfrak{g}(\zeta, \mathfrak{u}_1(\zeta), \mathfrak{v}_1(\zeta)) \frac{d\zeta}{\zeta} \\
&\quad - \int_1^{+\infty} \frac{\mathcal{G}_1(t,\zeta)}{1+(\ln t)^{\alpha-1}} \mathfrak{a}(\zeta)\mathfrak{f}(\zeta, \mathfrak{u}_2(\zeta), \mathfrak{v}_2(\zeta)) \frac{d\zeta}{\zeta} \\
&\quad \left. - \int_1^{+\infty} \frac{\mathcal{G}_2(t,\zeta)}{1+(\ln t)^{\alpha-1}} \mathfrak{b}(\zeta)\mathfrak{g}(\zeta, \mathfrak{u}_2(\zeta), \mathfrak{v}_2(\zeta)) \frac{d\zeta}{\zeta}\right| \\
&\leq \int_1^{+\infty} \frac{\mathcal{G}_1(t,\zeta)}{1+(\ln t)^{\alpha-1}} \mathfrak{a}(\zeta)|\mathfrak{f}(\zeta, \mathfrak{u}_1(\zeta), \mathfrak{v}_1(\zeta)) - \mathfrak{f}(\zeta, \mathfrak{u}_2(\zeta), \mathfrak{v}_2(\zeta))| \frac{d\zeta}{\zeta} \\
&\quad + \int_1^{+\infty} \frac{\mathcal{G}_2(t,\zeta)}{1+(\ln t)^{\alpha-1}} \mathfrak{b}(\zeta)|\mathfrak{g}(\zeta, \mathfrak{u}_1(\zeta), \mathfrak{v}_1(\zeta)) - \mathfrak{g}(\zeta, \mathfrak{u}_2(\zeta), \mathfrak{v}_2(\zeta))| \frac{d\zeta}{\zeta} \\
&\leq \Lambda_1 \int_1^{+\infty} \mathfrak{a}(\zeta)\left(\frac{\chi_1(\zeta)}{1+(\ln\zeta)^{\alpha-1}}|\mathfrak{u}_1(\zeta) - \mathfrak{u}_2(\zeta)| + \frac{\chi_2(\zeta)}{1+(\ln\zeta)^{\beta-1}}|\mathfrak{v}_1(\zeta) - \mathfrak{v}_2(\zeta)|\right)\frac{d\zeta}{\zeta} \\
&\quad + \Lambda_2 \int_1^{+\infty} \mathfrak{b}(\zeta)\left(\frac{\chi_3(\zeta)}{1+(\ln\zeta)^{\alpha-1}}|\mathfrak{u}_1(\zeta) - \mathfrak{u}_2(\zeta)| + \frac{\chi_4(\zeta)}{1+(\ln\zeta)^{\beta-1}}|\mathfrak{v}_1(\zeta) - \mathfrak{v}_2(\zeta)|\right)\frac{d\zeta}{\zeta} \\
&\leq \Lambda_1(\chi_1^*\|\mathfrak{u}_1 - \mathfrak{u}_2\|_1 + \chi_2^*\|\mathfrak{v}_1 - \mathfrak{v}_2\|_2) + \Lambda_2(\chi_3^*\|\mathfrak{u}_1 - \mathfrak{u}_2\|_1 + \chi_4^*\|\mathfrak{v}_1 - \mathfrak{v}_2\|_2) \\
&= (\Lambda_1\chi_1^* + \Lambda_2\chi_3^*)\|\mathfrak{u}_1 - \mathfrak{u}_2\|_1 + (\Lambda_1\chi_2^* + \Lambda_2\chi_4^*)\|\mathfrak{v}_1 - \mathfrak{v}_2\|_2 \\
&\leq \max\{\Lambda_1\chi_1^* + \Lambda_2\chi_3^*, \Lambda_1\chi_2^* + \Lambda_2\chi_4^*\}(\|\mathfrak{u}_1 - \mathfrak{u}_2\|_1 + \|\mathfrak{v}_1 - \mathfrak{v}_2\|_2) \\
&= \widetilde{\Upsilon}_1\|(\mathfrak{u}_1, \mathfrak{v}_1) - (\mathfrak{u}_2, \mathfrak{v}_2)\|, \ \ \forall\, t \in I.
\end{aligned}$$

In a similar manner, we find

$$\begin{aligned}
&\left|\frac{\mathcal{F}_2(\mathfrak{u}_1, \mathfrak{v}_1)(t)}{1+(\ln t)^{\beta-1}} - \frac{\mathcal{F}_2(\mathfrak{u}_2, \mathfrak{v}_2)(t)}{1+(\ln t)^{\beta-1}}\right| \\
&= \left|\int_1^{+\infty} \frac{\mathcal{G}_3(t,\zeta)}{1+(\ln t)^{\beta-1}} \mathfrak{a}(\zeta)\mathfrak{f}(\zeta, \mathfrak{u}_1(\zeta), \mathfrak{v}_1(\zeta)) \frac{d\zeta}{\zeta}\right. \\
&\quad + \int_1^{+\infty} \frac{\mathcal{G}_4(t,\zeta)}{1+(\ln t)^{\beta-1}} \mathfrak{b}(\zeta)\mathfrak{g}(\zeta, \mathfrak{u}_1(\zeta), \mathfrak{v}_1(\zeta)) \frac{d\zeta}{\zeta} \\
&\quad - \int_1^{+\infty} \frac{\mathcal{G}_3(t,\zeta)}{1+(\ln t)^{\beta-1}} \mathfrak{a}(\zeta)\mathfrak{f}(\zeta, \mathfrak{u}_2(\zeta), \mathfrak{v}_2(\zeta)) \frac{d\zeta}{\zeta} \\
&\quad \left. - \int_1^{+\infty} \frac{\mathcal{G}_4(t,\zeta)}{1+(\ln t)^{\beta-1}} \mathfrak{b}(\zeta)\mathfrak{g}(\zeta, \mathfrak{u}_2(\zeta), \mathfrak{v}_2(\zeta)) \frac{d\zeta}{\zeta}\right| \\
&\leq \int_1^{+\infty} \frac{\mathcal{G}_3(t,\zeta)}{1+(\ln t)^{\beta-1}} \mathfrak{a}(\zeta)|\mathfrak{f}(\zeta, \mathfrak{u}_1(\zeta), \mathfrak{v}_1(\zeta)) - \mathfrak{f}(\zeta, \mathfrak{u}_2(\zeta), \mathfrak{v}_2(\zeta))| \frac{d\zeta}{\zeta} \\
&\quad + \int_1^{+\infty} \frac{\mathcal{G}_4(t,\zeta)}{1+(\ln t)^{\beta-1}} \mathfrak{b}(\zeta)|\mathfrak{g}(\zeta, \mathfrak{u}_1(\zeta), \mathfrak{v}_1(\zeta)) - \mathfrak{g}(\zeta, \mathfrak{u}_2(\zeta), \mathfrak{v}_2(\zeta))| \frac{d\zeta}{\zeta} \\
&\leq \Lambda_3 \int_1^{+\infty} \mathfrak{a}(\zeta)\left(\frac{\chi_1(\zeta)}{1+(\ln\zeta)^{\alpha-1}}|\mathfrak{u}_1(\zeta) - \mathfrak{u}_2(\zeta)| + \frac{\chi_2(\zeta)}{1+(\ln\zeta)^{\beta-1}}|\mathfrak{v}_1(\zeta) - \mathfrak{v}_2(\zeta)|\right)\frac{d\zeta}{\zeta} \\
&\quad + \Lambda_4 \int_1^{+\infty} \mathfrak{b}(\zeta)\left(\frac{\chi_3(\zeta)}{1+(\ln\zeta)^{\alpha-1}}|\mathfrak{u}_1(\zeta) - \mathfrak{u}_2(\zeta)| + \frac{\chi_4(\zeta)}{1+(\ln\zeta)^{\beta-1}}|\mathfrak{v}_1(\zeta) - \mathfrak{v}_2(\zeta)|\right)\frac{d\zeta}{\zeta} \\
&\leq \Lambda_3(\chi_1^*\|\mathfrak{u}_1 - \mathfrak{u}_2\|_1 + \chi_2^*\|\mathfrak{v}_1 - \mathfrak{v}_2\|_2) + \Lambda_4(\chi_3^*\|\mathfrak{u}_1 - \mathfrak{u}_2\|_1 + \chi_4^*\|\mathfrak{v}_1 - \mathfrak{v}_2\|_2) \\
&= (\Lambda_3\chi_1^* + \Lambda_4\chi_3^*)\|\mathfrak{u}_1 - \mathfrak{u}_2\|_1 + (\Lambda_3\chi_2^* + \Lambda_4\chi_4^*)\|\mathfrak{v}_1 - \mathfrak{v}_2\|_2 \\
&\leq \max\{\Lambda_3\chi_1^* + \Lambda_4\chi_3^*, \Lambda_3\chi_2^* + \Lambda_4\chi_4^*\}(\|\mathfrak{u}_1 - \mathfrak{u}_2\|_1 + \|\mathfrak{v}_1 - \mathfrak{v}_2\|_2) \\
&= \widetilde{\Upsilon}_2\|(\mathfrak{u}_1, \mathfrak{v}_1) - (\mathfrak{u}_2, \mathfrak{v}_2)\|, \ \ \forall\, t \in I.
\end{aligned}$$

From the above inequalities, we deduce

$$\begin{aligned}
&\|\mathcal{F}(\mathfrak{u}_1,\mathfrak{v}_1)-\mathcal{F}(\mathfrak{u}_2,\mathfrak{v}_2)\| \\
&=\|\mathcal{F}_1(\mathfrak{u}_1,\mathfrak{v}_1)-\mathcal{F}_1(\mathfrak{u}_2,\mathfrak{v}_2)\|_1+\|\mathcal{F}_2(\mathfrak{u}_1,\mathfrak{v}_1)-\mathcal{F}_2(\mathfrak{u}_2,\mathfrak{v}_2)\|_2 \\
&=\sup_{t\in I}\left|\frac{\mathcal{F}_1(\mathfrak{u}_1,\mathfrak{v}_1)(t)-\mathcal{F}_1(\mathfrak{u}_2,\mathfrak{v}_2)(t)}{1+(\ln t)^{\alpha-1}}\right|+\sup_{t\in I}\left|\frac{\mathcal{F}_2(\mathfrak{u}_1,\mathfrak{v}_1)(t)-\mathcal{F}_2(\mathfrak{u}_2,\mathfrak{v}_2)(t)}{1+(\ln t)^{\beta-1}}\right| \\
&\le(\widetilde{\Upsilon}_1+\widetilde{\Upsilon}_2)\|(\mathfrak{u}_1,\mathfrak{v}_1)-(\mathfrak{u}_2,\mathfrak{v}_2)\|=\widetilde{\Upsilon}_0\|(\mathfrak{u}_1,\mathfrak{v}_1)-(\mathfrak{u}_2,\mathfrak{v}_2)\|.
\end{aligned}$$

By condition (12), we infer that $\mathcal{F}$ is a contraction operator. By using the Banach contraction mapping principle, we conclude that $\mathcal{F}$ has a unique fixed point $(\mathfrak{u}^*,\mathfrak{v}^*)\in\mathcal{P}$, which is the unique positive solution of problems (1), (2). In addition, for $(\mathfrak{u}_0,\mathfrak{v}_0)\in\mathcal{P}$, the sequence $((\mathfrak{u}_n,\mathfrak{v}_n))_{n\ge 0}$ defined by $(\mathfrak{u}_n,\mathfrak{v}_n)=\mathcal{F}(\mathfrak{u}_{n-1},\mathfrak{v}_{n-1}),\ n\ge 1$ converges to $(\mathfrak{u}^*,\mathfrak{v}^*)$, as $n\to+\infty$. From the proof of the Banach theorem, we obtain the error estimate (13). □

Our third existence result is based on the fixed-point theorem of Avery and Peterson (see Theorem 10 from [10], or Theorem 2.1 from [13]). We will use the notations of the functionals from Theorem 2.1 from [13].

For $\theta>1$, we introduce firstly the following constants

$$\begin{aligned}
&a_*=\int_\theta^{+\infty}\mathfrak{a}(\omega)\,\frac{d\omega}{\omega},\ \ b_*=\int_\theta^{+\infty}\mathfrak{b}(\omega)\,\frac{d\omega}{\omega},\\
&\widetilde{V}_1=\frac{d}{\Gamma(\alpha)}\int_1^\theta(\ln\omega)^{\alpha-1}\,d\mathcal{H}_1(\omega)+\frac{b}{\Gamma(\alpha)}\int_1^\theta(\ln\omega)^{\alpha-1}\,d\mathcal{K}_1(\omega),\\
&\widetilde{V}_2=\frac{d}{\Gamma(\beta)}\int_1^\theta(\ln\omega)^{\beta-1}\,d\mathcal{H}_2(\omega)+\frac{b}{\Gamma(\beta)}\int_1^\theta(\ln\omega)^{\beta-1}\,d\mathcal{K}_2(\omega),\\
&\widetilde{V}_3=\frac{c}{\Gamma(\alpha)}\int_1^\theta(\ln\omega)^{\alpha-1}\,d\mathcal{H}_1(\omega)+\frac{a}{\Gamma(\alpha)}\int_1^\theta(\ln\omega)^{\alpha-1}\,d\mathcal{K}_1(\omega),\\
&\widetilde{V}_4=\frac{c}{\Gamma(\beta)}\int_1^\theta(\ln\omega)^{\beta-1}\,d\mathcal{H}_2(\omega)+\frac{a}{\Gamma(\beta)}\int_1^\theta(\ln\omega)^{\beta-1}\,d\mathcal{K}_2(\omega),\\
&V_1=\frac{\widetilde{V}_1(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})},\ \ V_2=\frac{\widetilde{V}_2(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})},\\
&V_3=\frac{\widetilde{V}_3(\ln\theta)^{\beta-1}}{\Delta(1+(\ln\theta)^{\beta-1})},\ \ V_4=\frac{\widetilde{V}_4(\ln\theta)^{\beta-1}}{\Delta(1+(\ln\theta)^{\beta-1})},\\
&\Upsilon_1=a_*(V_1+V_3),\ \ \Upsilon_2=b_*(V_2+V_4).
\end{aligned}\tag{14}$$

Theorem 3. *Assume that $(A1)$–$(A3)$ hold, and there exists $\theta>1$ such that $a_*,\ b_*>0$, and $\widetilde{V}_i>0$, $i=1,\dots,4$. In addition, we suppose that there exist non-negative functions h, k, and positive constants $\varrho_i,\ i=1,\dots,3$ with $0<\varrho_1<\varrho_2<\varrho_3$, and $0<h^*=\int_1^{+\infty}h(\zeta)\mathfrak{a}(\zeta)\,\frac{d\zeta}{\zeta}<+\infty$, $0<k^*=\int_1^{+\infty}k(\zeta)\mathfrak{b}(\zeta)\,\frac{d\zeta}{\zeta}<+\infty$, such that*

$(A6)$ $\mathfrak{f}(t,(1+(\ln t)^{\alpha-1})u,(1+(\ln t)^{\beta-1})v)<\frac{\varrho_1}{2\Upsilon_3}h(t)$, *and*
$\mathfrak{g}(t,(1+(\ln t)^{\alpha-1})u,(1+(\ln t)^{\beta-1})v)<\frac{\varrho_1}{2\Upsilon_4}k(t)$,
for all $t\in I,\ \ u,v\ge 0,\ \ u+v\le\varrho_1$;

$(A7)$ $\mathfrak{f}(t,(1+(\ln t)^{\alpha-1})u,(1+(\ln t)^{\beta-1})v)>\frac{\varrho_2}{2\Upsilon_1}$, *and*
$\mathfrak{g}(t,(1+(\ln t)^{\alpha-1})u,(1+(\ln t)^{\beta-1})v)>\frac{\varrho_2}{2\Upsilon_2}$,
for all $t\in[\theta,+\infty),\ \ u,v\ge 0,\ \varrho_2\le u+v\le\varrho_3$,

$(A8)$ $\mathfrak{f}(t,(1+(\ln t)^{\alpha-1})u,(1+(\ln t)^{\beta-1})v)\le\frac{\varrho_3}{2\Upsilon_3}h(t)$, *and*
$\mathfrak{g}(t,(1+(\ln t)^{\alpha-1})u,(1+(\ln t)^{\beta-1})v)\le\frac{\varrho_3}{2\Upsilon_4}k(t)$,
for all $t\in I,\ \ u,v\ge 0,\ \ u+v\le\varrho_3$,

where $\Upsilon_3=h^*(\Lambda_1+\Lambda_3),\ \ \Upsilon_4=k^*(\Lambda_2+\Lambda_4)$, and Υ_1, Υ_2 are given by (14). Then the boundary value problems (1), (2) have at least three positive solutions $(\mathfrak{u}_i(t),\mathfrak{v}_i(t)),\ \ t\in I,\ \ i=1,\dots,3$ with $\|(\mathfrak{u}_i,\mathfrak{v}_i)\|\le\varrho_3$, $i=1,\dots,3$.

Proof. For θ given in the assumptions of this theorem, we introduce the concave functional $\Upsilon \in C(\mathcal{P}, \mathbb{R}_+)$, the convex functionals Λ, $\Theta \in C(\mathcal{P}, \mathbb{R}_+)$, and the functional $\Xi \in C(\mathcal{P}, \mathbb{R}_+)$ in the following way

$$\begin{aligned}
&\Upsilon(\mathfrak{u},\mathfrak{v}) = \inf_{t\in[\theta,+\infty)} \left(\frac{\mathfrak{u}(t)}{1+(\ln t)^{\alpha-1}} + \frac{\mathfrak{v}(t)}{1+(\ln t)^{\beta-1}} \right), \ \forall (\mathfrak{u},\mathfrak{v}) \in \mathcal{P}, \\
&\Lambda(\mathfrak{u},\mathfrak{v}) = \Theta(\mathfrak{u},\mathfrak{v}) = \Xi(\mathfrak{u},\mathfrak{v}) = \|(\mathfrak{u},\mathfrak{v})\|, \ \forall (\mathfrak{u},\mathfrak{v}) \in \mathcal{P}.
\end{aligned}$$

We have $\Xi(\lambda(\mathfrak{u},\mathfrak{v})) = \lambda\|(\mathfrak{u},\mathfrak{v})\|$, for all $(\mathfrak{u},\mathfrak{v}) \in \mathcal{P}$ and $\lambda \in [0,1]$. In addition, we find $\Upsilon(\mathfrak{u},\mathfrak{v}) \le \Xi(\mathfrak{u},\mathfrak{v})$ and $\|(\mathfrak{u},\mathfrak{v})\| \le \Lambda(\mathfrak{u},\mathfrak{v})$ for all $(\mathfrak{u},\mathfrak{v}) \in \overline{P(\Lambda,\varrho_3)}$, where $\overline{P(\Lambda,\varrho_3)}$ is the closure of the set $P(\Lambda,\varrho_3) \overset{def}{=} \{(\mathfrak{u},\mathfrak{v}) \in \mathcal{P}, \ \Lambda(\mathfrak{u},\mathfrak{v}) < \varrho_3\}$, that is, $\overline{P(\Lambda,\varrho_3)} = \{(\mathfrak{u},\mathfrak{v}) \in \mathcal{P}, \ \Lambda(\mathfrak{u},\mathfrak{v}) \le \varrho_3\}$.

We will prove that $\mathcal{F} : \overline{P(\Lambda,\varrho_3)} \to \overline{P(\Lambda,\varrho_3)}$. For this, let $(\mathfrak{u},\mathfrak{v}) \in \overline{P(\Lambda,\varrho_3)}$. So $\Lambda(\mathfrak{u},\mathfrak{v}) = \|(\mathfrak{u},\mathfrak{v})\| \le \varrho_3$. With $(A8)$, we obtain

$$\begin{aligned}
\|\mathcal{F}_1(\mathfrak{u},\mathfrak{v})\|_1 &= \sup_{t\in I} \frac{\mathcal{F}_1(\mathfrak{u},\mathfrak{v})(t)}{1+(\ln t)^{\alpha-1}} \\
&= \sup_{t\in I} \left(\int_1^{+\infty} \frac{\mathcal{G}_1(t,\zeta)}{1+(\ln t)^{\alpha-1}} \mathfrak{a}(\zeta)\mathfrak{f}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta)) \frac{d\zeta}{\zeta} \right. \\
&\quad \left. + \int_1^{+\infty} \frac{\mathcal{G}_2(t,\zeta)}{1+(\ln t)^{\alpha-1}} \mathfrak{b}(\zeta)\mathfrak{g}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta)) \frac{d\zeta}{\zeta} \right) \\
&\le \Lambda_1 \int_1^{+\infty} \mathfrak{a}(\zeta)\mathfrak{f}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta)) \frac{d\zeta}{\zeta} + \Lambda_2 \int_1^{+\infty} \mathfrak{b}(\zeta)\mathfrak{g}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta)) \frac{d\zeta}{\zeta} \\
&\le \Lambda_1 \frac{\varrho_3}{2\Upsilon_3} \int_1^{+\infty} \mathfrak{a}(\zeta)h(\zeta) \frac{d\zeta}{\zeta} + \Lambda_2 \frac{\varrho_3}{2\Upsilon_4} \int_1^{+\infty} \mathfrak{b}(\zeta)k(\zeta) \frac{d\zeta}{\zeta} \\
&= \Lambda_1 \frac{\varrho_3}{2\Upsilon_3} h^* + \Lambda_2 \frac{\varrho_3}{2\Upsilon_4} k^*, \\
\|\mathcal{F}_2(\mathfrak{u},\mathfrak{v})\|_2 &= \sup_{t\in I} \frac{\mathcal{F}_2(\mathfrak{u},\mathfrak{v})(t)}{1+(\ln t)^{\beta-1}} \\
&= \sup_{t\in I} \left(\int_1^{+\infty} \frac{\mathcal{G}_3(t,\zeta)}{1+(\ln t)^{\beta-1}} \mathfrak{a}(\zeta)\mathfrak{f}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta)) \frac{d\zeta}{\zeta} \right. \\
&\quad \left. + \int_1^{+\infty} \frac{\mathcal{G}_4(t,\zeta)}{1+(\ln t)^{\beta-1}} \mathfrak{b}(\zeta)\mathfrak{g}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta)) \frac{d\zeta}{\zeta} \right) \\
&\le \Lambda_3 \int_1^{+\infty} \mathfrak{a}(\zeta)\mathfrak{f}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta)) \frac{d\zeta}{\zeta} + \Lambda_4 \int_1^{+\infty} \mathfrak{b}(\zeta)\mathfrak{g}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta)) \frac{d\zeta}{\zeta} \\
&\le \Lambda_3 \frac{\varrho_3}{2\Upsilon_3} \int_1^{+\infty} \mathfrak{a}(\zeta)h(\zeta) \frac{d\zeta}{\zeta} + \Lambda_4 \frac{\varrho_3}{2\Upsilon_4} \int_1^{+\infty} \mathfrak{b}(\zeta)k(\zeta) \frac{d\zeta}{\zeta} \\
&= \Lambda_3 \frac{\varrho_3}{2\Upsilon_3} h^* + \Lambda_4 \frac{\varrho_3}{2\Upsilon_4} k^*.
\end{aligned}$$

Then we deduce

$$\begin{aligned}
\Lambda(\mathcal{F}(\mathfrak{u},\mathfrak{v})) &= \|\mathcal{F}(\mathfrak{u},\mathfrak{v})\| = \|\mathcal{F}_1(\mathfrak{u},\mathfrak{v})\|_1 + \|\mathcal{F}_2(\mathfrak{u},\mathfrak{v})\|_2 \\
&\le \frac{(\Lambda_1+\Lambda_3)\varrho_3 h^*}{2\Upsilon_3} + \frac{(\Lambda_2+\Lambda_4)\varrho_3 k^*}{2\Upsilon_4} = \varrho_3.
\end{aligned}$$

Therefore, $\mathcal{F}(\overline{P(\Lambda,\varrho_3)}) \subset \overline{P(\Lambda,\varrho_3)}$.

Next, by Lemma 3, we infer that $\mathcal{F} : \overline{P(\Lambda,\varrho_3)} \to \overline{P(\Lambda,\varrho_3)}$ is completely continuous. We choose $\widetilde{\varrho}_3 \in (\varrho_2,\varrho_3)$, and in what follows we will verify the conditions (i)–(iii) of Theorem 2.1 from [13].

For (i), we show firstly that the set $\{(\mathfrak{u},\mathfrak{v}) \in P(\Lambda,\Theta,\Upsilon,\varrho_2,\widetilde{\varrho}_3,\varrho_3), \ \Upsilon(\mathfrak{u},\mathfrak{v}) > \varrho_2\}$ is a nonempty set, where $P(\Lambda,\Theta,\Upsilon,\varrho_2,\widetilde{\varrho}_3,\varrho_3) \overset{def}{=} \{(\mathfrak{u},\mathfrak{v}) \in \mathcal{P}, \ \Upsilon(\mathfrak{u},\mathfrak{v}) \ge \varrho_2, \ \Theta(\mathfrak{u},\mathfrak{v}) \le \widetilde{\varrho}_3, \ \Lambda(\mathfrak{u},\mathfrak{v}) \le \varrho_3\}$. We consider the element

$$(\mathfrak{u}_0(t),\mathfrak{v}_0(t)) = \left(\frac{\varrho_2+\widetilde{\varrho}_3}{4}(1+(\ln t)^{\alpha-1}), \frac{\varrho_2+\widetilde{\varrho}_3}{4}(1+(\ln t)^{\beta-1}) \right), \ \ t \in I.$$

We obtain that $\|(\mathfrak{u}_0,\mathfrak{v}_0)\| = \|\mathfrak{u}_0\|_1 + \|\mathfrak{v}_0\|_2 = \dfrac{\varrho_2+\widetilde{\varrho}_3}{2} < \widetilde{\varrho}_3$, and $\Upsilon(\mathfrak{u}_0,\mathfrak{v}_0) = \dfrac{\varrho_2+\widetilde{\varrho}_3}{2} > \varrho_2$. Then, $(\mathfrak{u}_0,\mathfrak{v}_0) \in P(\Lambda,\Theta,\Upsilon,\varrho_2,\widetilde{\varrho}_3,\varrho_3)$, with $\Upsilon(\mathfrak{u}_0,\mathfrak{v}_0) > \varrho_2$. In addition, for any $(\mathfrak{u},\mathfrak{v}) \in P(\Lambda,\Theta,\Upsilon,\varrho_2,\widetilde{\varrho}_3,\varrho_3)$, by using $(A7)$, we have

$$\begin{aligned}
\Upsilon(\mathcal{F}(\mathfrak{u},\mathfrak{v})) &= \inf_{t\in[\theta,+\infty)}\left(\frac{\mathcal{F}_1(\mathfrak{u}(t),\mathfrak{v}(t))}{1+(\ln t)^{\alpha-1}} + \frac{\mathcal{F}_2(\mathfrak{u}(t),\mathfrak{v}(t))}{1+(\ln t)^{\beta-1}}\right)\\
&\geq \inf_{t\in[\theta,+\infty)}\frac{\mathcal{F}_1(\mathfrak{u}(t),\mathfrak{v}(t))}{1+(\ln t)^{\alpha-1}} + \inf_{t\in[\theta,+\infty)}\frac{\mathcal{F}_2(\mathfrak{u}(t),\mathfrak{v}(t))}{1+(\ln t)^{\beta-1}}\\
&= \inf_{t\in[\theta,+\infty)}\left(\int_1^{+\infty}\frac{\mathcal{G}_1(t,\zeta)}{1+(\ln t)^{\alpha-1}}\mathfrak{a}(\zeta)\mathfrak{f}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta}\right.\\
&\quad\left.+\int_1^{+\infty}\frac{\mathcal{G}_2(t,\zeta)}{1+(\ln t)^{\alpha-1}}\mathfrak{b}(\zeta)\mathfrak{g}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta}\right)\\
&\quad+\inf_{t\in[\theta,+\infty)}\left(\int_1^{+\infty}\frac{\mathcal{G}_3(t,\zeta)}{1+(\ln t)^{\beta-1}}\mathfrak{a}(\zeta)\mathfrak{f}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta}\right.\\
&\quad\left.+\int_1^{+\infty}\frac{\mathcal{G}_4(t,\zeta)}{1+(\ln t)^{\beta-1}}\mathfrak{b}(\zeta)\mathfrak{g}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta}\right)\\
&\geq \inf_{t\in[\theta,+\infty)}\int_1^{+\infty}\frac{\mathcal{G}_1(t,\zeta)}{1+(\ln t)^{\alpha-1}}\mathfrak{a}(\zeta)\mathfrak{f}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta}\\
&\quad+\inf_{t\in[\theta,+\infty)}\int_1^{+\infty}\frac{\mathcal{G}_2(t,\zeta)}{1+(\ln t)^{\alpha-1}}\mathfrak{b}(\zeta)\mathfrak{g}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta}\\
&\quad+\inf_{t\in[\theta,+\infty)}\int_1^{+\infty}\frac{\mathcal{G}_3(t,\zeta)}{1+(\ln t)^{\beta-1}}\mathfrak{a}(\zeta)\mathfrak{f}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta}\\
&\quad+\inf_{t\in[\theta,+\infty)}\int_1^{+\infty}\frac{\mathcal{G}_4(t,\zeta)}{1+(\ln t)^{\beta-1}}\mathfrak{b}(\zeta)\mathfrak{g}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta}\\
&\geq \inf_{t\in[\theta,+\infty)}\int_\theta^{+\infty}\frac{\mathcal{G}_1(t,\zeta)}{1+(\ln t)^{\alpha-1}}\mathfrak{a}(\zeta)\mathfrak{f}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta}\\
&\quad+\inf_{t\in[\theta,+\infty)}\int_\theta^{+\infty}\frac{\mathcal{G}_2(t,\zeta)}{1+(\ln t)^{\alpha-1}}\mathfrak{b}(\zeta)\mathfrak{g}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta}\\
&\quad+\inf_{t\in[\theta,+\infty)}\int_\theta^{+\infty}\frac{\mathcal{G}_3(t,\zeta)}{1+(\ln t)^{\beta-1}}\mathfrak{a}(\zeta)\mathfrak{f}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta}\\
&\quad+\inf_{t\in[\theta,+\infty)}\int_\theta^{+\infty}\frac{\mathcal{G}_4(t,\zeta)}{1+(\ln t)^{\beta-1}}\mathfrak{b}(\zeta)\mathfrak{g}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta}\\
&\geq \int_\theta^{+\infty}\inf_{t\in[\theta,+\infty)}\frac{\mathcal{G}_1(t,\zeta)}{1+(\ln t)^{\alpha-1}}\mathfrak{a}(\zeta)\mathfrak{f}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta}\\
&\quad+\int_\theta^{+\infty}\inf_{t\in[\theta,+\infty)}\frac{\mathcal{G}_2(t,\zeta)}{1+(\ln t)^{\alpha-1}}\mathfrak{b}(\zeta)\mathfrak{g}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta}\\
&\quad+\int_\theta^{+\infty}\inf_{t\in[\theta,+\infty)}\frac{\mathcal{G}_3(t,\zeta)}{1+(\ln t)^{\beta-1}}\mathfrak{a}(\zeta)\mathfrak{f}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta}\\
&\quad+\int_\theta^{+\infty}\inf_{t\in[\theta,+\infty)}\frac{\mathcal{G}_4(t,\zeta)}{1+(\ln t)^{\beta-1}}\mathfrak{b}(\zeta)\mathfrak{g}(\zeta,\mathfrak{u}(\zeta),\mathfrak{v}(\zeta))\,\frac{d\zeta}{\zeta}\\
&> V_1\int_\theta^{+\infty}\mathfrak{a}(\zeta)\frac{\varrho_2}{2\Upsilon_1}\,\frac{d\zeta}{\zeta} + V_2\int_\theta^{+\infty}\mathfrak{b}(\zeta)\frac{\varrho_2}{2\Upsilon_2}\,\frac{d\zeta}{\zeta}\\
&\quad+V_3\int_\theta^{+\infty}\mathfrak{a}(\zeta)\frac{\varrho_2}{2\Upsilon_1}\,\frac{d\zeta}{\zeta} + V_4\int_\theta^{+\infty}\mathfrak{b}(\zeta)\frac{\varrho_2}{2\Upsilon_2}\,\frac{d\zeta}{\zeta}\\
&= \frac{V_1\varrho_2+V_3\varrho_2}{2\Upsilon_1}\int_\theta^{+\infty}\mathfrak{a}(\zeta)\,d\zeta + \frac{V_2\varrho_2+V_4\varrho_2}{2\Upsilon_2}\int_\theta^{+\infty}\mathfrak{b}(\zeta)\,d\zeta\\
&= \frac{(V_1+V_3)\varrho_2}{2\Upsilon_1}a_* + \frac{(V_2+V_4)\varrho_2}{2\Upsilon_2}b_* = \varrho_2.
\end{aligned}$$

So $\Upsilon(\mathcal{F}(\mathfrak{u},\mathfrak{v})) > \varrho_2$, that is, assumption (i) is satisfied.

For (ii), we will prove that $\Upsilon(\mathcal{F}(\mathfrak{u},\mathfrak{v})) > \varrho_2$ for any $(\mathfrak{u},\mathfrak{v}) \in P(\Lambda,\Upsilon,\varrho_2,\varrho_3) \overset{def}{=} \{(\mathfrak{u},\mathfrak{v}) \in \mathcal{P},\ \Upsilon(\mathfrak{u},\mathfrak{v}) \geq \varrho_2,\ \Lambda(\mathfrak{u},\mathfrak{v}) \leq \varrho_3\}$ with $\Theta(\mathcal{F}(\mathfrak{u},\mathfrak{v})) > \widetilde{\varrho}_3$. So, let $(\mathfrak{u},\mathfrak{v}) \in P(\Lambda,\Upsilon,\varrho_2,\varrho_3)$ with $\|\mathcal{F}(\mathfrak{u},\mathfrak{v})\| > \widetilde{\varrho}_3$. In a similar manner as that used in the proof of (i), we obtain that $\Upsilon(\mathcal{F}(\mathfrak{u},\mathfrak{v})) > \varrho_2$.

For (iii), we will show that $(0,0) \notin R(\Lambda, \Xi, \varrho_1, \varrho_3)$, and $\Xi(\mathcal{F}(\mathfrak{u}, \mathfrak{v})) < \varrho_1$ for any $(\mathfrak{u}, \mathfrak{v}) \in R(\Lambda, \Xi, \varrho_1, \varrho_3)$ with $\Xi(\mathfrak{u}, \mathfrak{v}) = \varrho_1$. Here, $R(\Lambda, \Xi, \varrho_1, \varrho_3) \stackrel{def}{=} \{(\mathfrak{u}, \mathfrak{v}) \in \mathcal{P},\ \Xi(\mathfrak{u}, \mathfrak{v}) \geq \varrho_1,\ \Lambda(\mathfrak{u}, \mathfrak{v}) \leq \varrho_3\}$. Because $\Xi(0,0) = 0 < \varrho_1$, we find that $(0,0) \notin R(\Lambda, \Xi, \varrho_1, \varrho_3)$. Moreover, for $(\mathfrak{u}, \mathfrak{v}) \in R(\Lambda, \Xi, \varrho_1, \varrho_3)$ with $\Xi(\mathfrak{u}, \mathfrak{v}) = \varrho_1$, we infer as in the first part of the proof, by using $(A6)$, that $\|\mathcal{F}(\mathfrak{u}, \mathfrak{v})\| < \varrho_1$.

So, by applying Theorem 2.1 from [13] (with $\tau_1 = \varrho_1$, $\tau_2 = \varrho_2$, $\tau_3 = \widetilde{\varrho}_3$, $\tau_4 = \varrho_3$, $M = 1$, $T = \mathcal{F}$, $P = \mathcal{P}$), we conclude that operator $\mathcal{F}$ has at least three fixed points $(\mathfrak{u}_i, \mathfrak{v}_i) \in \overline{P(\Lambda, \varrho_3)}$, $i = 1, \ldots, 3$, which are positive solutions of our problems (1), (2). In addition, by Theorem 2.1 from [13], we obtain $\|(\mathfrak{u}_i, \mathfrak{v}_i)\| \leq \varrho_3$, $i = 1, \ldots, 3$, $\Upsilon(\mathfrak{u}_1, \mathfrak{v}_1) > \varrho_2$, $\|(\mathfrak{u}_2, \mathfrak{v}_2)\| > \varrho_1$, $\Upsilon(\mathfrak{u}_2, \mathfrak{v}_2) < \varrho_2$ and $\|(\mathfrak{u}_3, \mathfrak{v}_3)\| < \varrho_1$. $\square$

4. Examples

Let $\alpha = \frac{7}{3}$, $(n = 3)$, $\beta = \frac{16}{5}$, $(m = 4)$, $\mathcal{H}_1(\zeta) = \{2/3,\ \zeta \in [1,4);\ 7/6,\ \zeta \in [4, +\infty)\}$, $\mathcal{H}_2(\zeta) = \{1/(20),\ \zeta \in [1,2);\ 1/(20)(\zeta-1)^{25/6},\ \zeta \in [2,3);\ 2^{13/6}/5,\ \zeta \in [3, +\infty)\}$, $\mathcal{K}_1(\zeta) = \{7^{9/8}/(72 \cdot 2^{1/8}),\ \zeta \in [1,4);\ 1/(36)(\zeta - 1/2)^{9/8},\ \zeta \in [4,9);\ 17^{9/8}/(72 \cdot 2^{1/8}),\ \zeta \in [9, +\infty)\}$, $\mathcal{K}_2(\zeta) = \{1,\ \zeta \in [1,3);\ 25/(23),\ \zeta \in [3,7);\ 2/(183)(\zeta^{3/2} - 7^{3/2}) + 25/(23),\ \zeta \in [7,11);\ 2/(183)(11^{3/2} - 7^{3/2}) + 25/(23),\ \zeta \in [11, +\infty)\}$.

We consider the system of fractional differential equations

$$\begin{cases} {}^{H}D^{7/3}_{1+}\mathfrak{u}(t) + \mathfrak{a}(t)\mathfrak{f}(t, \mathfrak{u}(t), \mathfrak{v}(t)) = 0, \ \ t \in (1, +\infty), \\ {}^{H}D^{16/5}_{1+}\mathfrak{v}(t) + \mathfrak{b}(t)\mathfrak{g}(t, \mathfrak{u}(t), \mathfrak{v}(t)) = 0, \ \ t \in (1, +\infty), \end{cases} \tag{15}$$

subject to the nonlocal coupled boundary conditions

$$\begin{cases} \mathfrak{u}(1) = \mathfrak{u}'(1) = 0, \ \ \mathfrak{v}(1) = \mathfrak{v}'(1) = \mathfrak{v}''(1) = 0, \\ {}^{H}D^{4/3}_{1+}\mathfrak{u}(+\infty) = \dfrac{1}{2}\mathfrak{u}(4) + \dfrac{5}{24}\displaystyle\int_2^3 (\zeta-1)^{19/6}\mathfrak{v}(\zeta)\, d\zeta, \\ {}^{H}D^{11/5}_{1+}\mathfrak{v}(+\infty) = \dfrac{1}{32}\displaystyle\int_4^9 \left(\zeta - \frac{1}{2}\right)^{1/8}\mathfrak{u}(\zeta)\, d\zeta + \dfrac{2}{23}\mathfrak{v}(3) + \dfrac{1}{61}\displaystyle\int_7^{11} \zeta^{1/2}\mathfrak{v}(\zeta)\, d\zeta. \end{cases} \tag{16}$$

By using the Mathematica program, we obtain $a \approx 0.41776194$, $b \approx 0.81063207$, $c \approx 0.44499194$, $d \approx 1.20259595$, and $\Delta \approx 0.14167408 > 0$. Therefore, assumption $(A1)$ is satisfied. In addition, we find $\Lambda_1 = d/\Delta \approx 8.48846859$, $\Lambda_2 = b/\Delta \approx 5.72180947$, $\Lambda_3 = c/\Delta \approx 3.14095529$, and $\Lambda_4 = a/\Delta \approx 2.94875357$.

Example 1. *We consider the functions*

$$\begin{aligned} \mathfrak{a}(t) &= \frac{1}{(t-1/3)^{1/2}}, \ \ \mathfrak{b}(t) = \frac{t+1}{(t-1/4)^{4/3}}, \\ \mathfrak{f}(t,u,v) &= \frac{(t+1)^2 e^{-2t+1} u}{17(1+(\ln t)^{4/3} + u)} + \frac{(t-1)^4 e^{-3t} v}{8(1+(\ln t)^{11/5})} + \frac{t^{10/3}}{t^3+2}, \\ \mathfrak{g}(t,u,v) &= \frac{\sqrt{\pi} t^3 e^{-2t} u}{26(1+(\ln t)^{4/3})} + \frac{(t+2)e^{-4t+3} v}{19(1+(\ln t)^{11/5} + 2v)} + \frac{t^{5/4}}{t+1}, \end{aligned} \tag{17}$$

for all $t \in I$, $u, v \in \mathbb{R}_+$. We derive $\int_1^{+\infty} \mathfrak{a}(\zeta)\, \frac{d\zeta}{\zeta} \approx 2.13208425$, and $\int_1^{+\infty} \mathfrak{b}(\zeta)\, \frac{d\zeta}{\zeta} \approx 4.2319539$. So assumption $(A2)$ is also satisfied. For the functions $\mathfrak{f}$ and $\mathfrak{g}$, the assumptions $(A3)$ (i), (ii), and (iii) are easily verified.

These functions satisfied the inequalities

$$\begin{aligned} \mathfrak{f}(t,u,v) &\leq \frac{\mathfrak{c}(t)u}{1+(\ln t)^{4/3}} + \frac{\mathfrak{d}(t)v}{1+(\ln t)^{11/5}} + \mathfrak{l}(t), \\ \mathfrak{g}(t,u,v) &\leq \frac{\mathfrak{p}(t)u}{1+(\ln t)^{4/3}} + \frac{\mathfrak{q}(t)v}{1+(\ln t)^{11/5}} + \mathfrak{m}(t), \end{aligned}$$

for all $t \in I$, $u, v \in \mathbb{R}_+$, *where*

$$\begin{aligned}&\mathfrak{c}(t) = \tfrac{1}{17}(t+1)^2 e^{-2t+1},\ \ \mathfrak{d}(t) = \tfrac{1}{8}(t-1)^4 e^{-3t},\ \ \mathfrak{l}(t) = \tfrac{t^{10/3}}{t^3+2},\\ &\mathfrak{p}(t) = \tfrac{1}{26}\sqrt{\pi} t^3 e^{-2t},\ \ \mathfrak{q}(t) = \tfrac{1}{19}(t+2)e^{-4t+3},\ \ \mathfrak{m}(t) = \tfrac{t^{5/4}}{t+1},\ \ \forall\, t \in I.\end{aligned}$$

We obtain here $\mathfrak{c}^* \approx 0.0440089$, $\mathfrak{d}^* \approx 0.00017564$, $\mathfrak{p}^* \approx 0.01982375$, $\mathfrak{q}^* \approx 0.03071373$, $\mathfrak{l}^* \approx 5.77600681$, *and* $\mathfrak{m}^* \approx 12.36540194$.

In addition, for any $r > 0$ *and for all* $t \in I$, $u, v \in [0, r]$, *we find*

$$\begin{aligned}&\mathfrak{f}(t, (1+(\ln t)^{4/3})u, (1+(\ln t)^{11/5})v) \le \tfrac{r}{1+r}\mathfrak{c}(t) + r\mathfrak{d}(t) + \mathfrak{l}(t) =: \varphi_r(t),\\ &\mathfrak{g}(t, (1+(\ln t)^{4/3})u, (1+(\ln t)^{11/5})v) \le r\mathfrak{p}(t) + \tfrac{r}{1+2r}\mathfrak{q}(t) + \mathfrak{m}(t) =: \psi_r(t).\end{aligned}$$

Then

$$\begin{aligned}&\int_1^{+\infty} \varphi_r(t)\mathfrak{a}(t)\,\frac{dt}{t} = \frac{r\mathfrak{c}^*}{1+r} + r\mathfrak{d}^* + \mathfrak{l}^* < +\infty,\\ &\int_1^{+\infty} \psi_r(t)\mathfrak{b}(t)\,\frac{dt}{t} = r\mathfrak{p}^* + \frac{r\mathfrak{q}^*}{1+2r} + \mathfrak{m}^* < +\infty,\end{aligned}$$

that is, assumption $(A3)$ *iv) is satisfied.*

Because $(\Lambda_1 + \Lambda_3)(\mathfrak{c}^* + \mathfrak{d}^*) + (\Lambda_2 + \Lambda_4)(\mathfrak{p}^* + \mathfrak{q}^*) \approx 0.952 < 1$, *by Theorem 1 we deduce that problems (15), (16) with the nonlinearities (17) have at least one positive solution* $(\mathfrak{u}(t), \mathfrak{v}(t))$, $t \in [1, +\infty)$.

Example 2. *We consider the functions*

$$\begin{aligned}&\mathfrak{a}(t) = \frac{t+1}{(t-1/5)^{5/4}},\ \ \mathfrak{b}(t) = \frac{t}{(t-1/2)^{8/7}},\\ &\mathfrak{f}(t,u,v) = \frac{3(t+2)^3 e^{-7t+4}}{19(1+(\ln t)^{4/3})}\sqrt{u^2+5} + \frac{(t-1)^2 e^{-3t+1}}{12(1+(\ln t)^{11/5})}\sin^2(v+2) + \frac{t^{17/8}}{t^2+1},\\ &\mathfrak{g}(t,u,v) = \frac{t^2 e^{-4t}}{5(1+(\ln t)^{4/3})}\arctan u + \frac{(t+3)^4 e^{-6t+1}}{8(1+(\ln t)^{11/5})}\sqrt{v^2+1} + \frac{t^{10/9}}{t+4},\end{aligned} \tag{18}$$

for all $t \in I$, $u, v \in \mathbb{R}_+$. *We obtain* $\int_1^{+\infty} \mathfrak{a}(\zeta)\,\frac{d\zeta}{\zeta} \approx 5.16137094$ *and* $\int_1^{+\infty} \mathfrak{b}(\zeta)\,\frac{d\zeta}{\zeta} \approx 7.72862659$. *So assumption* $(A2)$ *is satisfied. The assumptions* $(A3)$ *(i), (ii), (iii) for the functions* $\mathfrak{f}$ *and* $\mathfrak{g}$ *are also verified.*

For $r > 0$, *and for all* $t \in I$, $u, v \in [0, r]$, *we derive the inequalities*

$$\begin{aligned}&\mathfrak{f}(t, (1+(\ln t)^{4/3})u, (1+(\ln t)^{11/5})v) \le \varphi_r(t),\\ &\mathfrak{g}(t, (1+(\ln t)^{4/3})u, (1+(\ln t)^{11/5})v) \le \psi_r(t),\end{aligned}$$

where

$$\begin{aligned}\varphi_r(t) &= \frac{(t-1)^2 e^{-3t+1}(1+(\ln t)^{11/5})}{12} r^2 + r\left[\frac{3(t+2)^3 e^{-7t+4}}{19} + \frac{(t-1)^2 e^{-3t+1}}{3}\right]\\ &\quad + \frac{3\sqrt{5}(t+2)^3 e^{-7t+4}}{19(1+(\ln t)^{4/3})} + \frac{(t-1)^2 e^{-3t+1}}{3(1+(\ln t)^{11/5})} + \frac{t^{17/8}}{t^2+1},\\ \psi_r(t) &= r\left(\frac{t^2 e^{-4t}}{5} + \frac{(t+3)^4 e^{-6t+1}}{8}\right) + \frac{(t+3)^4 e^{-6t+1}}{8(1+(\ln t)^{11/5})} + \frac{t^{10/9}}{t+4}.\end{aligned}$$

We deduce

$$\begin{aligned}&\textstyle\int_1^{+\infty} \varphi_r(t)\mathfrak{a}(t)\,\frac{dt}{t} \approx 0.00095981 r^2 + 0.07455534 r + 8.80152353 < +\infty,\\ &\textstyle\int_1^{+\infty} \psi_r(t)\mathfrak{b}(t)\,\frac{dt}{t} \approx 0.07137629 r + 30.31740004 < +\infty,\end{aligned}$$

so assumption $(A3)$ *(iv) is satisfied.*

The functions $\mathfrak{f}$ *and* $\mathfrak{g}$ *satisfy the inequalities*

$$|\mathfrak{f}(t,u_1,v_1)-\mathfrak{f}(t,u_2,v_2)| \le \frac{\chi_1(t)}{1+(\ln t)^{4/3}}|u_1-u_2| + \frac{\chi_2(t)}{1+(\ln t)^{11/5}}|v_1-v_2|,$$
$$|\mathfrak{g}(t,u_1,v_1)-\mathfrak{g}(t,u_2,v_2)| \le \frac{\chi_3(t)}{1+(\ln t)^{4/3}}|u_1-u_2| + \frac{\chi_4(t)}{1+(\ln t)^{11/5}}|v_1-v_2|,$$

for all $t\in I,\ u_i,v_i\in\mathbb{R}_+,\ i=1,2$, *where*

$$\chi_1(t)=\frac{3(t+2)^3e^{-7t+4}}{19},\ \chi_2(t)=\frac{(t-1)^2e^{-3t+1}}{6},$$
$$\chi_3(t)=\frac{t^2e^{-4t}}{5},\ \chi_4(t)=\frac{(t+3)^4e^{-6t+1}}{8},\ \forall t\in I.$$

We find $\chi_1^*\approx 0.07166196$, $\chi_2^*\approx 0.00144669$, $\chi_3^*\approx 0.00202314$, *and* $\chi_4^*\approx 0.06935316$. *We also obtain* $\widetilde{\Upsilon}_1\approx 0.61987628$, $\widetilde{\Upsilon}_2\approx 0.23105274$, *and* $\widetilde{\Upsilon}_0\approx 0.851<1$. *Hence, by Theorem 2, we conclude that problems (15), (16) with the nonlinearities (18) have a unique positive solution* $(\mathfrak{u}^*(t),\mathfrak{v}^*(t))$, $t\in[1,+\infty)$. *For a given element* $(\mathfrak{u}_0,\mathfrak{v}_0)$, *this solution can be approximated using the sequence* $(\mathfrak{u}_n(t),\mathfrak{v}_n(t))_{n\ge 0}$ *defined by* $\mathfrak{u}_{n+1}(t)=\mathcal{F}_1(\mathfrak{u}_n(t),\mathfrak{v}_n(t))$, $\mathfrak{v}_{n+1}(t)=\mathcal{F}_2(\mathfrak{u}_n(t),\mathfrak{v}_n(t))$, $n\ge 0$, *or equivalently*

$$\mathfrak{u}_{n+1}(t)=\int_1^{+\infty}\mathcal{G}_1(t,\zeta)\mathfrak{a}(\zeta)\mathfrak{f}(\zeta,\mathfrak{u}_n(\zeta),\mathfrak{v}_n(\zeta))\,\frac{d\zeta}{\zeta}+\int_1^{+\infty}\mathcal{G}_2(t,\zeta)\mathfrak{b}(\zeta)\mathfrak{g}(\zeta,\mathfrak{u}_n(\zeta),\mathfrak{v}_n(\zeta))\,\frac{d\zeta}{\zeta},$$
$$\mathfrak{v}_{n+1}(t)=\int_1^{+\infty}\mathcal{G}_3(t,\zeta)\mathfrak{a}(\zeta)\mathfrak{f}(\zeta,\mathfrak{u}_n(\zeta),\mathfrak{v}_n(\zeta))\,\frac{d\zeta}{\zeta}+\int_1^{+\infty}\mathcal{G}_4(t,\zeta)\mathfrak{b}(\zeta)\mathfrak{g}(\zeta,\mathfrak{u}_n(\zeta),\mathfrak{v}_n(\zeta))\,\frac{d\zeta}{\zeta}.$$

If we consider $(\mathfrak{u}_0,\mathfrak{v}_0)=(0,0)$, *then, by Lemma 3 from [11], we find for* $(\mathfrak{u}_1,\mathfrak{v}_1)$ *the following formulas:*

$$\begin{aligned}
\mathfrak{u}_1(t) &= -\tfrac{1}{\Gamma(7/3)}\int_1^t\left(\ln\tfrac{t}{\zeta}\right)^{4/3}\mathfrak{a}(\zeta)\mathfrak{f}(\zeta,0,0)\,\tfrac{d\zeta}{\zeta}\\
&\quad+\tfrac{(\ln t)^{4/3}}{\Delta}\Bigg[d\int_1^{+\infty}\mathfrak{a}(\zeta)\mathfrak{f}(\zeta,0,0)\,\tfrac{d\zeta}{\zeta}-\tfrac{d}{\Gamma(7/3)}\int_1^{+\infty}\left(\int_1^{\zeta}\left(\ln\tfrac{\zeta}{\tau}\right)^{4/3}\mathfrak{a}(\tau)\mathfrak{f}(\tau,0,0)\,\tfrac{d\tau}{\tau}\right)d\mathcal{H}_1(\zeta)\\
&\quad-\tfrac{d}{\Gamma(16/5)}\int_1^{+\infty}\left(\int_1^{\zeta}\left(\ln\tfrac{\zeta}{\tau}\right)^{11/5}\mathfrak{b}(\tau)\mathfrak{g}(\tau,0,0)\,\tfrac{d\tau}{\tau}\right)d\mathcal{H}_2(\zeta)\\
&\quad+b\int_1^{+\infty}\mathfrak{b}(\zeta)\mathfrak{g}(\zeta,0,0)\,\tfrac{d\zeta}{\zeta}-\tfrac{b}{\Gamma(7/3)}\int_1^{+\infty}\left(\int_1^{\zeta}\left(\ln\tfrac{\zeta}{\tau}\right)^{4/3}\mathfrak{a}(\tau)\mathfrak{f}(\tau,0,0)\,\tfrac{d\tau}{\tau}\right)d\mathcal{K}_1(\zeta)\\
&\quad-\tfrac{b}{\Gamma(16/5)}\int_1^{+\infty}\left(\int_1^{\zeta}\left(\ln\tfrac{\zeta}{\tau}\right)^{11/5}\mathfrak{b}(\tau)\mathfrak{g}(\tau,0,0)\,\tfrac{d\tau}{\tau}\right)d\mathcal{K}_2(\zeta)\Bigg],\ t\in I,\\
\mathfrak{v}_1(t) &= -\tfrac{1}{\Gamma(16/5)}\int_1^t\left(\ln\tfrac{t}{\zeta}\right)^{11/5}\mathfrak{b}(\zeta)\mathfrak{g}(\zeta,0,0)\,\tfrac{d\zeta}{\zeta}\\
&\quad+\tfrac{(\ln t)^{11/5}}{\Delta}\Bigg[a\int_1^{+\infty}\mathfrak{b}(\zeta)\mathfrak{g}(\zeta,0,0)\,\tfrac{d\zeta}{\zeta}-\tfrac{a}{\Gamma(7/3)}\int_1^{+\infty}\left(\int_1^{\zeta}\left(\ln\tfrac{\zeta}{\tau}\right)^{4/3}\mathfrak{a}(\tau)\mathfrak{f}(\tau,0,0)\,\tfrac{d\tau}{\tau}\right)d\mathcal{K}_1(\zeta)\\
&\quad-\tfrac{a}{\Gamma(16/5)}\int_1^{+\infty}\left(\int_1^{\zeta}\left(\ln\tfrac{\zeta}{\tau}\right)^{11/5}\mathfrak{b}(\tau)\mathfrak{g}(\tau,0,0)\,\tfrac{d\tau}{\tau}\right)d\mathcal{K}_2(\zeta)\\
&\quad+c\int_1^{+\infty}\mathfrak{a}(\zeta)\mathfrak{f}(\zeta,0,0)\,\tfrac{d\zeta}{\zeta}-\tfrac{c}{\Gamma(7/3)}\int_1^{+\infty}\left(\int_1^{\zeta}\left(\ln\tfrac{\zeta}{\tau}\right)^{4/3}\mathfrak{a}(\tau)\mathfrak{f}(\tau,0,0)\,\tfrac{d\tau}{\tau}\right)d\mathcal{H}_1(\zeta)\\
&\quad-\tfrac{c}{\Gamma(16/5)}\int_1^{+\infty}\left(\int_1^{\zeta}\left(\ln\tfrac{\zeta}{\tau}\right)^{11/5}\mathfrak{b}(\tau)\mathfrak{g}(\tau,0,0)\,\tfrac{d\tau}{\tau}\right)d\mathcal{H}_2(\zeta)\Bigg],\ t\in I.
\end{aligned}$$

In addition, the error estimate is

$$\|(\mathfrak{u}_n,\mathfrak{v}_n)-(\mathfrak{u}^*,\mathfrak{v}^*)\|\le\frac{\widetilde{\Upsilon}_0^n}{1-\widetilde{\Upsilon}_0}\|(\mathfrak{u}_1,\mathfrak{v}_1)\|.$$

Example 3. *We consider the functions*

$$\mathfrak{a}(t) = \frac{t+2}{(t-1/6)^{11/8}},\ \ \mathfrak{b}(t) = \frac{t+4}{(t-3/10)^{15/11}},\ \ t \in I,$$

$$\mathfrak{f}(t,u,v) = \begin{cases} \frac{1}{123}t^{1/4}\Big[\frac{1}{3} + \frac{u}{6(1+(\ln t)^{4/3}+u)} + \frac{v}{7(1+(\ln t)^{11/5}+v)}\Big], \\ \quad t \in I,\ u,v \geq 0,\ 0 \leq \frac{u}{1+(\ln t)^{4/3}} + \frac{v}{1+(\ln t)^{11/5}} < 1, \\ \frac{1}{123}t^{1/4}\Big[\frac{1}{3} + \frac{1}{6}\Big(1+315\Big(\arctan\Big(\frac{u}{1+(\ln t)^{4/3}} + \frac{v}{1+(\ln t)^{11/5}}\Big) - \frac{\pi}{4}\Big)\Big)\frac{u}{1+(\ln t)^{4/3}+u} \\ \quad + \frac{1}{7}\Big(1+237\cos^2\frac{\pi}{2}\Big(\frac{u}{1+(\ln t)^{4/3}} + \frac{v}{1+(\ln t)^{11/5}}\Big)\Big)\frac{v}{1+(\ln t)^{11/5}+v}\Big], \\ \quad t \in I,\ u,v \geq 0,\ 1 < \frac{u}{1+(\ln t)^{4/3}} + \frac{v}{1+(\ln t)^{11/5}} \leq 4, \\ \frac{1}{123}t^{1/4}\Big[\frac{1}{3} + \frac{1}{6}\big(1+315\big(\arctan 4 - \frac{\pi}{4}\big)\big)\frac{u}{1+(\ln t)^{4/3}+u} \\ \quad + \frac{238}{7}\frac{v}{1+(\ln t)^{11/5}+v}\Big],\ t \in I,\ u,v \geq 0,\ \frac{u}{1+(\ln t)^{4/3}} + \frac{v}{1+(\ln t)^{11/5}} > 4, \end{cases} \tag{19}$$

$$\mathfrak{g}(t,u,v) = \begin{cases} \frac{1}{147}t^{2/11}\Big[\frac{1}{2} + \frac{u}{5(1+(\ln t)^{4/3}+u)} + \frac{v}{3(1+(\ln t)^{11/5}+v)}\Big], \\ \quad t \in I,\ u,v \geq 0,\ 0 \leq \frac{u}{1+(\ln t)^{4/3}} + \frac{v}{1+(\ln t)^{11/5}} < 1, \\ \frac{1}{147}t^{2/11}\Big[\frac{1}{2} + \frac{1}{5}\Big(1+283\Big(1-\sin\frac{\pi}{2}\Big(\frac{u}{1+(\ln t)^{4/3}} + \frac{v}{1+(\ln t)^{11/5}}\Big)\Big)\Big)\frac{u}{1+(\ln t)^{4/3}+u} \\ \quad + \frac{1}{3}\Big(1+376\Big(1-\frac{1}{2\sqrt{u/(1+(\ln t)^{4/3})+v/(1+(\ln t)^{11/5})-1}}\Big)\Big)\frac{v}{1+(\ln t)^{11/5}+v}\Big], \\ \quad t \in I,\ u,v \geq 0,\ 1 < \frac{u}{1+(\ln t)^{4/3}} + \frac{v}{1+(\ln t)^{11/5}} \leq 4, \\ \frac{1}{147}t^{2/11}\Big[\frac{1}{2} + \frac{284}{5}\frac{u}{1+(\ln t)^{4/3}+u} + \frac{755}{9}\frac{v}{1+(\ln t)^{11/5}+v}\Big], \\ \quad t \in I,\ u,v \geq 0,\ \frac{u}{1+(\ln t)^{4/3}} + \frac{v}{1+(\ln t)^{11/5}} > 4. \end{cases}$$

Then we obtain

$$\begin{aligned} &\mathfrak{f}(t,(1+(\ln t)^{4/3})u,(1+(\ln t)^{11/5})v) \\ &= \begin{cases} \frac{1}{123}t^{1/4}\Big[\frac{1}{3} + \frac{u}{6(1+u)} + \frac{v}{7(1+v)}\Big],\ t \in I,\ u,v \geq 0,\ 0 \leq u+v < 1, \\ \frac{1}{123}t^{1/4}\Big[\frac{1}{3} + \frac{1}{6}\big(1+315\big(\arctan(u+v) - \frac{\pi}{4}\big)\big)\frac{u}{1+u} \\ \quad + \frac{1}{7}\big(1+237\cos^2\frac{\pi}{2}(u+v)\big)\frac{v}{1+v}\Big],\ t \in I,\ u,v \geq 0,\ 1 < u+v \leq 4, \\ \frac{1}{123}t^{1/4}\Big[\frac{1}{3} + \frac{1}{6}\big(1+315\big(\arctan 4 - \frac{\pi}{4}\big)\big)\frac{u}{1+u} + \frac{238}{7}\frac{v}{1+v}\Big] \\ \quad t \in I,\ u,v \geq 0,\ u+v > 4, \end{cases} \\ &\mathfrak{g}(t,(1+(\ln t)^{4/3})u,(1+(\ln t)^{11/5})v) \\ &= \begin{cases} \frac{1}{147}t^{2/11}\Big[\frac{1}{2} + \frac{u}{5(1+u)} + \frac{v}{3(1+v)}\Big],\ t \in I,\ u,v \geq 0,\ 0 \leq u+v < 1, \\ \frac{1}{147}t^{2/11}\Big[\frac{1}{2} + \frac{1}{5}\big(1+283\big(1-\sin\frac{\pi}{2}(u+v)\big)\big)\frac{u}{1+u} \\ \quad + \frac{1}{3}\Big(1+376\Big(1-\frac{1}{2\sqrt{u+v-1}}\Big)\Big)\frac{v}{1+v}\Big],\ t \in I,\ u,v \geq 0,\ 1 < u+v \leq 4, \\ \frac{1}{147}t^{2/11}\Big[\frac{1}{2} + \frac{284}{5}\frac{u}{1+u} + \frac{755}{9}\frac{v}{1+v}\Big],\ t \in I,\ u,v \geq 0,\ u+v > 4. \end{cases} \end{aligned}$$

We also find $\int_1^{+\infty} \mathfrak{a}(\zeta)\,\frac{d\zeta}{\zeta} \approx 4.53433833$ *and* $\int_1^{+\infty} \mathfrak{b}(\zeta)\,\frac{d\zeta}{\zeta} \approx 6.99005054$*. So assumption* $(A2)$ *is satisfied. The assumptions* $(A3)$ *(i), (ii), and (iii) for the functions* $\mathfrak{f}$ *and* $\mathfrak{g}$ *are easily verified.*

For $r > 0$*, and for all* $t \in I,\ u,v \in [0,r]$*, we deduce the inequalities*

$$\begin{aligned} \mathfrak{f}(t,(1+(\ln t)^{4/3})u,(1+(\ln t)^{11/5}v)) &\leq \frac{1}{123}t^{1/4}\Big[\frac{1}{3} + \frac{1}{6}\Big(1+\frac{315\pi}{4}\Big)\frac{r}{1+r} + \frac{238}{7}\frac{r}{1+r}\Big] =: \varphi_r(t), \\ \mathfrak{g}(t,(1+(\ln t)^{4/3})u,(1+(\ln t)^{11/5}v)) &\leq \frac{1}{147}t^{2/11}\Big[\frac{1}{2} + \frac{567}{5}\frac{r}{1+r} + \frac{377}{3}\frac{r}{1+r}\Big] =: \psi_r(t). \end{aligned}$$

Because we have $\int_1^{+\infty} \varphi_r(t)\mathfrak{a}(t)\frac{dt}{t} < +\infty$, $\int_1^{+\infty} \psi_r(t)\mathfrak{b}(t)\frac{dt}{t} < +\infty$*, then the assumption* $(A3)$ *(iv) is satisfied.*

We choose $\theta = 5$, $\varrho_1 = 1$, $\varrho_2 = 4$, $\varrho_3 = 1000$*, and we consider the functions* $h(t) = \frac{1}{3}t^{1/4}$, $k(t) = \frac{1}{2}t^{2/11}$, $t \in I$*. We obtain* $h^* \approx 3.41851186$, $k^* \approx 5.15683344$, $a_* \approx 1.64041821$, $b_* \approx 1.90949909$*. We also find* $\widetilde{V}_1 \approx 0.82418924$, $\widetilde{V}_2 \approx 0.43794179$, $\widetilde{V}_3 \approx 0.31130063$, $\widetilde{V}_4 \approx 0.16724753$, $V_1 \approx 3.80180825$, $V_2 \approx 2.02013159$, $V_3 \approx 1.62641651$, $V_4 \approx 0.87379887$, $\Upsilon_1 \approx 8.90455874$, $\Upsilon_2 \approx 5.52595759$, $\Upsilon_3 \approx 39.75532345$*, and* $\Upsilon_4 \approx 44.71264939$.

Moreover, we deduce

$$\begin{aligned}
&\mathfrak{f}(t,(1+(\ln t)^{4/3})u,(1+(\ln t)^{11/5}v)) < \tfrac{41}{10332}t^{1/4} \approx 0.00396825t^{1/4} \\
&\quad < \tfrac{1}{6\Upsilon_3}t^{1/4} \approx 0.00419231t^{1/4}, \ \forall\, t\in I, \ u,v\geq 0, \ u+v\leq 1, \\
&\mathfrak{g}(t,(1+(\ln t)^{4/3})u,(1+(\ln t)^{11/5}v)) < \tfrac{23}{4410}t^{2/11} \approx 0.00521542t^{2/11} \\
&\quad < \tfrac{1}{4\Upsilon_4}t^{2/11} \approx 0.00559126t^{2/11}, \ \forall\, t\in I, \ u,v\geq 0, \ u+v\leq 1, \\
&\mathfrak{f}(t,(1+(\ln t)^{4/3})u,(1+(\ln t)^{11/5}v)) \\
&\quad \geq \tfrac{1}{123}5^{1/4}\left[\tfrac{1}{3}+\tfrac{4}{5}\min\left\{\tfrac{1}{6}\left(1+315\left(\arctan 4-\tfrac{\pi}{4}\right)\right),\tfrac{238}{7}\right\}\right] \approx 0.28161534 \\
&\quad > \tfrac{2}{\Upsilon_1} \approx 0.22460405, \ \forall\, t\geq 5, \ u,v\geq 0, \ 4\leq u+v\leq 1000, \\
&\mathfrak{g}(t,(1+(\ln t)^{4/3})u,(1+(\ln t)^{11/5}v)) \\
&\quad \geq \tfrac{1}{147}5^{2/11}\left[\tfrac{1}{2}+\tfrac{4}{5}\min\left\{\tfrac{284}{5},\tfrac{755}{9}\right\}\right] \approx 0.41875414 \\
&\quad > \tfrac{2}{\Upsilon_2} \approx 0.36192822, \quad \forall\, t\geq 5, \ u,v\geq 0, \ 4\leq u+v\leq 1000, \\
&\mathfrak{f}(t,(1+(\ln t)^{4/3})u,(1+(\ln t)^{11/5}v)) \\
&\quad \leq \tfrac{1}{123}t^{1/4}\left[\tfrac{1}{3}+\tfrac{1}{6}\left(1+315\left(\arctan 4-\tfrac{\pi}{4}\right)\right)\tfrac{1000}{1001}+\tfrac{238}{7}\tfrac{1000}{1001}\right] \approx 0.51064673t^{1/4} \\
&\quad \leq \tfrac{1000}{6\Upsilon_3}t^{1/4} \approx 4.19231067t^{1/4}, \ \forall\, t\in I, \ u,v\geq 0, \ u+v\leq 1000, \\
&\mathfrak{g}(t,(1+(\ln t)^{4/3})u,(1+(\ln t)^{11/5}v)) \\
&\quad \leq \tfrac{1}{147}t^{2/11}\left(\tfrac{1}{2}+\tfrac{567}{5}\tfrac{1000}{1001}+\tfrac{755}{9}\tfrac{1000}{1001}\right) \approx 1.34416188t^{2/11} \\
&\quad \leq \tfrac{1000}{4\Upsilon_4}t^{2/11} \approx 5.59125893t^{2/11}, \ \forall\, t\in I, \ u,v\geq 0, \ u+v\leq 1000,
\end{aligned}$$

that is, assumptions $(A6)$, $(A7)$, *and* $(A8)$ *are satisfied.*

By Theorem 3, we conclude that problems (15), (16) with the nonlinearities (19) have at least three positive solutions $(\mathfrak{u}_i(t),\mathfrak{v}_i(t)), \ t\in I, \ i=1,\ldots,3,$ *with*

$$\sup_{t\in I}\frac{\mathfrak{u}_i(t)}{1+(\ln t)^{4/3}}+\sup_{t\in I}\frac{\mathfrak{v}_i(t)}{1+(\ln t)^{11/5}}\leq 1000, \ i=1,\ldots,3.$$

5. Conclusions

In this research paper, we focused on investigating the presence, uniqueness, and multiplicity of positive solutions for a system of Hadamard fractional differential equations (1) on an infinite interval. The system was supplemented with nonlocal coupled boundary conditions (2), which incorporate fractional derivatives and Riemann–Stieltjes integrals. It is worth noting that unlike the previous work presented in the paper [11], the nonlinearities in our system (1) are allowed to be unbounded. Furthermore, we employed different function conditions compared to those in [11], and we utilized various fixed-point theorems, including the Schauder fixed-point theorem (for the existence of positive solutions, in Theorem 1), the Banach contraction mapping principle (for the existence and uniqueness of positive solution, in Theorem 2), and a fixed-point theorem introduced by Avery and Peterson (refer to [10] for the existence of at least three positive solutions in Theorem 3). In the second-to-last section of our paper, we presented three illustrative examples that effectively showcase the main three outcomes of our research. Moving forward, we intend to explore the investigation of other systems of fractional equations, involving fractional derivatives of different types, subject to diverse boundary conditions.

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References

1. Granas, A.; Dugundji, J. *Fixed Point Theory*; Springer: New York, NY, USA, 2003.
2. Guo, D.; Lakshmikantham, V. *Nonlinear Problems in Abstract Cones*; Academic Press: New York, NY, USA, 1988.
3. Khamsi, M.A.; Kirk, W.A. *An Introduction to Metric Spaces and Fixed Point Theory*; John Wiley and Sons, Inc.: New York, NY, USA, 2001.
4. Krasnosel'skii, M.A. Some problems of nonlinear analysis. *Amer. Math. Soc. Transl.* **1958**, *10*, 345–409.
5. Krasnosel'skii, M.A. *Positive Solution of Operator Equations*; Noordhoff: Groningen, The Netherlands, 1964.
6. Krasnosel'skii, M.A.; Zabreiko, P.P. *Geometrical Methods of Nonlinear Analysis*; Springer: New York, NY, USA, 1984.
7. Subrahmanyam, P.V. *Elementary Fixed Point Theorems*; Springer: Singapore, 2019.
8. Debnath, P.; Konwar, N.; Radenovic, S. (Eds.) *Metric Fixed Point Theory, Applications in Science, Engineering and Behavioral Sciences*; Springer: Singapore, 2021.
9. Leggett, R.; Williams, L. Multiple positive fixed points of nonlinear operators on ordered Banach spaces. *Indiana Univ. Math. J.* **1979**, *28*, 673–688. [CrossRef]
10. Avery, R.I.; Peterson, A.C. Three positive fixed points of nonlinear operators on ordered Banach spaces. *Comput. Math. Appl.* **2001**, *42*, 313–322. [CrossRef]
11. Luca, R.; Tudorache, A. On a system of Hadamard fractional differential equations with nonlocal boundary conditions on an infinite interval. *Fractal Fract.* **2023**, *7*, 458. [CrossRef]
12. Tariboon, J.; Ntouyas, S.K.; Asawasamrit, S.; Promsakon, C. Positive solutions for Hadamard differential systems with fractional integral conditions on an unbounded domain. *Open Math.* **2017**, *15*, 645–666.
13. Zhang, W.; Liu, W. Existence, uniqueness, and multiplicity results on positive solutions for a class of Hadamard-type fractional boundary value problem on an infinite interval. *Math. Methods Appl. Sci.* **2020**, *43*, 2251–2275. [CrossRef]
14. Tudorache, A.; Luca, R. Positive solutions for a singular fractional boundary value problem. *Math. Methods Appl. Sci.* **2020**, *43*, 10190–10203. [CrossRef]
15. Ahmad, B.; Alsaedi, A.; Ntouyas, S.K.; Tariboon, J. *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*; Springer: Cham, Switzerland, 2017.
16. Ahmad, B.; Henderson, J.; Luca, R. *Boundary Value Problems for Fractional Differential Equations and Systems*; Trends in Abstract and Applied Analysis 9; World Scientific: Hackensack, NJ, USA, 2021.
17. Baleanu, D.; Diethelm, K.; Scalas, E.; Trujillo, J.J. *Fractional Calculus Models and Numerical Methods*; Series on Complexity, Nonlinearity and Chaos; World Scientific: Boston, MA, USA, 2012.
18. Baleanu, D.; Tenreiro Machado, J.A.; Luo, A.C.J. (Eds.) *Fractional Dynamics and Control*; Springer: New York, NY, USA, 2012.
19. Das, S. *Functional Fractional Calculus for System Identification and Controls*; Springer: New York, NY, USA, 2008.
20. Henderson, J.; Luca, R. *Boundary Value Problems for Systems of Differential, Difference and Fractional Equations. Positive Solutions*; Elsevier: Amsterdam, The Netherlands, 2016.
21. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies, 204; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006.
22. Klafter, J.; Lim, S.C.; Metzler, R. (Eds.) *Fractional Dynamics in Physics*; World Scientific: Singapore, 2011.
23. Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999.
24. Sabatier, J.; Agrawal, O.P.; Machado, J.A.T. (Eds.) *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*; Springer: Dordrecht, The Netherlands, 2007.
25. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives. Theory and Applications*; Gordon and Breach: Yverdon, Switzerland, 1993.
26. Zhou, Y.; Wang, J.R.; Zhang, L. *Basic Theory of Fractional Differential Equations*, 2nd ed.; World Scientific: Singapore, 2016.
27. Butkovskii, A.G.; Postnov, S.S.; Postnova, E.A. Fractional integro-differential calculus and its control-theoretical applications. I. Mathematical fundamentals and the problem of interpretation. *Autom. Remote Control* **2013**, *74*, 543–574. [CrossRef]
28. Butkovskii, A.G.; Postnov, S.S.; Postnova, E.A. Fractional integro-differential calculus and its control-theoretical applications. II. Fractional dynamic systems: Modeling and hardware implementation. *Autom. Remote Control* **2013**, *74*, 725–749. [CrossRef]
29. Makhlouf, A.B.; Mchiri, L. Some results on the study of Caputo-Hadamard fractional stochastic differential equations. *Chaos Solitons Fractals* **2022**, *155*, 111757. [CrossRef]
30. Liu, J.; Wei, W.; Wang, J.; Xu, W. Limit behavior of the solution of Caputo-Hadamard fractional stochastic differential equations. *Appl. Math. Lett.* **2023**, *140*, 108586. [CrossRef]

31. Rhaima, M. Ulam type stability for Caputo-Hadamard fractional functional stochastic differential equations with delay. *Math. Methods Appl. Sci.* **2023**, *46*, 10995–11006. [CrossRef]
32. Thiramanus, P.; Ntouyas, S.K.; Tariboon, J. Positive solutions for Hadamard fractional differential equations on infinite domain. *Adv. Differ. Equ.* **2016**, *2016*, 1–18. [CrossRef]