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η-Ricci–Yamabe Solitons along Riemannian Submersions

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Abstract: In this paper, we investigate the geometrical axioms of Riemannian submersions in the context of the η-Ricci–Yamabe soliton (η-RY soliton) with a potential field. We give the categorization of each fiber of Riemannian submersion as an η-RY soliton, an η-Ricci soliton, and an η-Yamabe soliton. Additionally, we consider the many circumstances under which a target manifold of Riemannian submersion is an η-RY soliton, an η-Ricci soliton, an η-Yamabe soliton, or a quasi-Yamabe soliton. We deduce a Poisson equation on a Riemannian submersion in a specific scenario if the potential vector field ω of the soliton is of gradient type $=:\text{grad}(\gamma)$ and provide some examples of an η-RY soliton, which illustrates our finding. Finally, we explore a number theoretic approach to Riemannian submersion with totally geodesic fibers.

Keywords: η-Ricci–Yamabe soliton; Riemannian submersion; Riemannian manifold; homotopy groups

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1. Introduction

Since Riemannian geometry's inception, the idea of Riemannian immersion has been the subject of extensive study. In fact, the Riemannian manifolds that were initially intended to be examined were surfaces embedded in $\mathbb{R}^3$ [1].

Initially, Gray and O'Neill were the first to discuss the "dual" concept of Riemannian submersion and investigated it further. Because of their applications in supergravity, the theory of relativity, and other physical theories, Riemannian submersions have received considerable attention in both mathematics and theoretical physics (see [2–7]). Studies on Riemannian submersion are reported in [8–12].

A soliton, which is related to the geometrical flow of Riemannian (semi-Riemannian) geometry, is a significant symmetry.

However, the theory of geometric flows has emerged as one of the most important geometrical theories for illuminating Riemannian geometric structures. The study of singularities of the flows involves a certain section of solutions when the metric evolves via dilations and diffeomorphisms because they appear as potential singularity models. They are frequently referred to as solitons.

In 1988, Hamilton [13] presented the ideas of Ricci flow and Yamabe flow for the first time. The limit of the solutions for the Ricci flow and the Yamabe flow, respectively, is shown to be the soliton of Ricci and the soliton of Yamabe. Geometric flow theory, including the Ricci flow and Yamabe flow, has drawn the attention of many mathematicians over the past two decades.

Under the term Ricci–Yamabe map, geometers [14] initiated research concerning a novel geometric flow that is a generalization of the Ricci and Yamabe flows. Ricci–Yamabe flow of the type (σ,ρ) is another name for this. The metrics on the Riemannian manifold defined by Guler and Crasmareanu evolve into the Ricci–Yamabe flow [14].

$$\frac{1}{2}\frac{\partial}{\partial t}g(t) = -\sigma S(t) - \frac{\rho}{2}R(t)g(t), \quad g_0 = g(0). \tag{1}$$

An interpolation of solitons between the Ricci and Yamabe soliton is considered in the Ricci–Bourguignon soliton corresponding to Ricci–Bourguignon flow but it depends on a single scalar. Ricci–Yamabe flow can either be a Riemannian flow, a semi-Riemannian flow, or a singular Riemannian flow, depending on the sign of the associated scalars σ and ρ. Such a range of options may be beneficial in various geometrical or physical models, such as the general theory of relativity.

Consequently, the Ricci–Yamabe soliton inevitably appears as the limit of the soliton of the Ricci–Yamabe flow. Ricci–Yamabe solitons are solitons to the Ricci–Yamabe flow that move only by one parameter group of diffeomorphism and scaling. Specifically, a Ricci–Yamabe soliton on the Riemannian manifold, (M,g), is a data $(g,\omega,\tau,\sigma,\rho)$ satisfying

$$\frac{1}{2}\mathcal{L}_\omega g + \sigma S + \left(\tau - \frac{\rho}{2}R\right)g = 0, \tag{2}$$

where the Ricci tensor is S, the scalar curvature is R, and the Lie-derivative along the vector field ω is $\mathcal{L}_\omega$. The manifold (M,g,ω,τ,ν) is referred to as a Ricci–Yamabe shrinker, expander, or stable soliton depending on the constant τ, whether $\tau < 0$, $\tau > 0$ or $\tau = 0$.

As an extension of Ricci and Yamabe solitons, Equation (2) is referred to as a Ricci–Yamabe soliton of kind (σ,ρ). We see that the Ricci–Yamabe solitons of kind $(\sigma,0)$ and $(0,\rho)$ are, respectively, the σ-Ricci solitons and the ρ-Yamabe solitons.

The idea of an η-Ricci soliton described in [15], is an evolutionary abstraction of the Ricci soliton. As a result, we can define the new concept similarly by amending the expression (2) that explains the type of soliton by a multiple of a specific $(0,2)$-tensor field $\eta \otimes \eta$. These findings result in a significantly more comprehensive concept, termed an η-Ricci–Yamabe soliton (briefly an η-RY soliton) of kind (σ,ρ) defined as:

$$\frac{1}{2}\mathcal{L}_\omega g + \sigma S + \left(\tau - \frac{\rho}{2}R\right)g + \nu\eta\otimes\eta = 0, \tag{3}$$

where ν is a constant. Let us reiterate that η-RY solitons of kinds $(\sigma,0)$ or $(1,0)$, $(0,\rho)$, or $(0,1)$-type are an η-Ricci soliton and an η-Yamabe soliton, respectively. For more information about these specific cases, see [16–22].

According to [23], if τ in (3) is replaced with the soliton function, then we may claim that the manifold (M,g) is an almost η-RY soliton [24]. It is important to note that they originate from the Ricci–Bourguignon flow and conformal Ricci flow, which Cantino, Mazzieri and Siddiqi recently examined [25–28]. We refer to (3) as the core equation of an approximately η-RY soliton in this more extended context.

In [22], the authors proved that the total manifold of a Riemannian submersion admits a Ricci soliton. In fact, the η-Ricci–Yamabe soliton is a generalization of the η-Ricci soliton from the proceedings of the η-Yamabe soliton, Yamabe soliton, and Einstein soliton. Therefore, motivated by the previous studies, in this paper, we discuss Riemannian submersions in terms of an η-Ricci–Yamabe soliton.

Example 1. *Let us look at the instance of an Einstein soliton, which produces solutions to Einstein flow that are self-similar (for more details see [26]), so that*

$$\frac{\partial}{\partial t}g(t) = -2\left(S - \frac{R}{2}g\right).$$

As a result, an Einstein soliton appears as the limit of the Einstein flow solution, such that

$$\mathcal{L}_\omega g + 2S + (\tau - \frac{R}{2})g = 0. \tag{4}$$

When comparing Equations (3) and (4) in this situation, we find that $\sigma = 1$ and $\rho = 1$, or its type $(1,1)$, are RY solitons.

Moreover, we note a useful definition:

Definition 1. *A smooth vector field ζ on a Riemannian manifold $(\mathcal{N}, g)$ is said to be a* conformal vector field *if there exists a smooth function φ on $\mathcal{N}$ that satisfies [29]*

$$\mathcal{L}_\zeta g = 2\varphi g, \tag{5}$$

where $\mathcal{L}_\zeta g$ is the Lie derivative of ζ with respect to g. If $\varphi = 0$, then ζ is called a Killing vector field.

2. Riemannian Submersions

We present the additional context for Riemannian submersions (briefly *RS*) in this part.

Let $(\mathcal{N}^n, g)$ and $(\mathcal{B}^m, g_{\mathcal{B}})$ be two Riemannian manifolds (briefly *RS*), endowed with metrics g and $g_{\mathcal{B}}$, wherein $\dim(\mathcal{N}) > \dim(\mathcal{B})$.

A surjective mapping $\pi : (\mathcal{N}, g) \to (\mathcal{B}, g_{\mathcal{B}})$ is called a *Riemannian submersion* [30] if:

(A1)

$$\dim(\mathcal{B}) = \mathrm{Rank}(\pi).$$

In this instance, $\pi^{-1}(s) = \pi_s^{-1}$ is a submanifold $\mathcal{N}$ ($\dim(\mathcal{N}) = t$) and is referred to as a *fiber* for all $s \in \mathcal{B}$, wherein

$$\dim(\mathcal{N}) - t = \dim(\mathcal{B}).$$

If a vector field on $\mathcal{N}$ is always tangent (resp. orthogonal) to fibers, it is said to be *vertical* (resp. *horizontal*). If a vector field P on $\mathcal{N}$ is horizontal and π-related to a vector field P_* on $\mathcal{B}$, then $\pi_*(\mathrm{P}_p) = \mathrm{E}_{*\pi(p)}$ is the basis for all $s \in \mathcal{N}$ and $\mathrm{E} \in \mathcal{B}$, wherein π_* is the differential map of π.

The projections on the vertical distribution $\mathcal{K}er\pi_*$ and the horizontal distribution $\mathcal{K}er\pi_*^{\perp}$ will be indicated by the symbols V (briefly vdV) and H (briefly hdH), respectively.

The manifold $(\mathcal{N}, g)$ is regarded as the *total manifold*, and the manifold $(\mathcal{B}, g_{\mathcal{B}})$ is regarded as the *base manifold*, as is customary.

(A2) The size of the horizontal vectors are preserved by π_*.

These requirements are similar to claiming that the differential map of π_*, restricted to $\mathcal{K}er\pi_*^{\perp}$, is a linear isometry. We obtain the following information if P and Q are the fundamental vector fields, connected to $\mathrm{P}_{\mathcal{B}}$ and $\mathrm{Q}_{\mathcal{B}}$ by π:

1. $g(\mathrm{P}, \mathrm{Q}) = g_{\mathcal{B}}(\mathrm{P}_{\mathcal{B}}, \mathrm{Q}_{\mathcal{B}}) \circ \pi$,
2. $h[\mathrm{P}, \mathrm{Q}]$ is the basic vector field π-connected to $[\mathrm{P}_{\mathcal{B}}, \mathrm{Q}_{\mathcal{B}}]$,
3. $h(\nabla_{\mathrm{P}}\mathrm{Q})$ is the basic vector field π-connected to $\nabla^{\mathcal{B}}{}_{\mathrm{P}_{\mathcal{B}}}\mathrm{Q}_{\mathcal{B}}$.

In the case of each vertical vector field $\{V, [I, \mathrm{J}]\}$ is vertical.

O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$, which are described below:

$$\mathcal{T}_{\mathrm{I}}\mathrm{J} = V\nabla_{V\mathrm{I}}H\mathrm{J} + H\nabla_{V\mathrm{I}}V\mathrm{J}, \tag{6}$$

$$\mathcal{A}_{\mathrm{I}}\mathrm{J} = V\nabla_{H}\mathrm{I}H\mathrm{J} + H\nabla_{H\mathrm{I}}V\mathrm{J} \tag{7}$$

if any vector fields I and J exist on $\mathcal{N}$, where ∇ denotes the Levi–Civita connection of g. The skew-symmetric operators on the tangent bundle of $\mathcal{N}$ that project the vdV and the hdH are evidently $\mathcal{T}_{\mathrm{I}}$ and A_J.

If G, K are vertical vector fields on $\mathcal{N}$ and P, Q are horizontal vector fields, then we obtain

$$\mathcal{T}_{\mathrm{G}}\mathrm{K} = \mathcal{T}_{\mathrm{K}}\mathrm{G}, \tag{8}$$

$$\mathcal{A}_{\mathrm{P}}\mathrm{Q} = -\mathcal{A}_{\mathrm{Q}}\mathrm{P} = \frac{1}{2}V[\mathrm{P}, \mathrm{Q}]. \tag{9}$$

Alternatively, we discover from (6) and (7)

$$\nabla_{\mathrm{G}}\mathrm{K} = \mathcal{T}_{\mathrm{G}}\mathrm{K} + \hat{\nabla}_{\mathrm{G}}\mathrm{K}, \tag{10}$$

$$\nabla_{\mathrm{G}}\mathrm{P} = \mathcal{T}_{\mathrm{G}}\mathrm{P} + H\nabla_{\mathrm{G}}\mathrm{P}, \tag{11}$$

$$\nabla_{\mathrm{P}}\mathrm{G} = \mathcal{A}_{\mathrm{P}}\mathrm{G} + V\nabla_{\mathrm{P}}\mathrm{G}, \tag{12}$$

$$\nabla_{\mathrm{P}}\mathrm{Q} = H\nabla_{\mathrm{P}}\mathrm{Q} + \mathcal{A}_{\mathrm{P}}\mathrm{Q}, \tag{13}$$

wherein $\hat{\nabla}_{\mathrm{G}}\mathrm{K} = V\nabla_{\mathrm{G}}\mathrm{K}$. Additionally, we have

$$H\nabla_{\mathrm{G}}\mathrm{P} = \mathcal{A}_{\mathrm{P}}\mathrm{G}$$

where P is basic. It is not hard to see that $\mathcal{A}$ acts on the hdH and estimates of the resistance to the integrability of this distribution while $\mathcal{T}$ operates on the fibers as the second basic form. We refer to the book [8] as well as the paper by O'Neill [30] for more information about the RS.

3. Characteristics of Curvatures on Riemannian Submersions

The following useful Riemannian submersion (RS) curvature properties are covered in this section:

Proposition 1. *For an RS π, the Riemannian curvatures of the total manifold, the base manifold, and each fiber of π denoted by R^T, $R^{\mathcal{B}}$ and $\hat{R}$, respectively, then we have*

$$R^T(\mathrm{I},\mathrm{J},\mathrm{G},\mathrm{H}) = \hat{R}(\mathrm{I},\mathrm{J},\mathrm{G},\mathrm{H}) + g(\mathcal{T}_{\mathrm{J}}\mathrm{H},\mathcal{T}_{\mathrm{I}}\mathrm{G}) - g(\mathcal{T}_{\mathrm{I}}\mathrm{H},\mathcal{T}_{\mathrm{J}}\mathrm{G}), \tag{14}$$

$$R^T(\mathrm{P},\mathrm{Q},\mathrm{R},\mathrm{L}) = R^{\mathcal{B}}(\mathrm{P}_{\mathcal{B}},\mathrm{Q}_{\mathcal{B}},\mathrm{R}_{\mathcal{B}},\mathrm{L}_{\mathcal{B}})\circ\pi + 2g(\mathcal{A}_{\mathrm{P}}\mathrm{Q},\mathcal{A}_{\mathrm{R}}\mathrm{L}) \tag{15}$$

$$-g(\mathcal{A}_{\mathrm{Q}}\mathrm{R},\mathcal{A}_{\mathrm{P}}\mathrm{L}) + g(\mathcal{A}_{\mathrm{P}}\mathrm{R},\mathcal{A}_{\mathrm{R}}\mathrm{L}).$$

for any $\mathrm{I},\mathrm{J},\mathrm{G},\mathrm{H} \in \Gamma V(\mathcal{N})$ *and* $\mathrm{P},\mathrm{Q},\mathrm{R},\mathrm{L} \in \Gamma H(\mathcal{N})$.

Proposition 2. *For an RS π, Ricci curvatures of $(\mathcal{N},g)$, $(\mathcal{B},g_{\mathcal{B}})$ and any fiber of π are denoted by S, S^N and $\hat{S}$, respectively. Then, we have*

$$S(\mathrm{I},\mathrm{J}) = \hat{S}(\mathrm{I},\mathrm{J}) + g(N,\mathcal{T}_{\mathrm{I}}\mathrm{J}) - \sum_{i=1}^{n} g\big((\nabla_{\mathrm{P}_i}\mathcal{T})(\mathrm{I},\mathrm{J}),\mathrm{P}_i\big) - g\big(\mathcal{A}_{\mathrm{P}_1}\mathrm{I},\mathcal{A}_{\mathrm{P}_1}\mathrm{J}\big) \tag{16}$$

$$S(\mathrm{P},\mathrm{Q}) = S^{\mathcal{B}}(\mathrm{P}^{\mathcal{B}},\mathrm{Q}^{\mathcal{B}})\circ\pi - \frac{1}{2}\{g(\nabla_{\mathrm{P}}N,\mathrm{Q}) + g(\nabla_{\mathrm{Q}}N,\mathrm{P})\}, \tag{17}$$

$$+2\sum_{i=1}^{n} g(\mathcal{A}_{\mathrm{P}}\mathrm{P}_i,\mathcal{A}_{\mathrm{Q}}\mathrm{P}_i) + \sum_{j=1}^{r} g(\mathcal{T}_{\mathrm{P}_i}\mathrm{P},\mathcal{T}_{\mathrm{P}_i}\mathrm{Q}),$$

$$S(\mathrm{I},\mathrm{P}) = -g(\nabla_{\mathrm{I}}N,\mathrm{P}) + \sum_{j} g((\nabla_{\mathrm{I}_i}\mathcal{T})(\mathrm{I}_j,\mathrm{E}),\mathrm{P}) \tag{18}$$

$$-\sum_{i=1}^{n}\{g((\nabla_{\mathrm{P}_i}\mathcal{A})(\mathrm{P}_i,\mathrm{P}),\mathrm{I}) + 2g(\mathcal{A}_{\mathrm{P}_i}\mathrm{P},\mathcal{T}_{\mathrm{I}}\mathrm{P}_i)\}$$

where $\{\mathrm{I}_i\}$ *and* $\{\mathrm{P}_i\}$ *are the orthonormal basis of vdV and hdH, respectively, and* $\mathrm{I},\mathrm{E} \in \Gamma V(\mathcal{N})$, $\mathrm{P},\mathrm{Q} \in \Gamma H(\mathcal{N})$.

Using (16) and (17), we derive the following:

Proposition 3. *In an RS π , the vertical scalar curvature R_V and the horizontal scalar curvature R_H are provided as*

$$R_V = \sum_{k=1}^{s} S(\mathrm{I}_k,\mathrm{I}_k) = \sum_{k=1}^{s} \hat{S}(\mathrm{I}_k,\mathrm{I}_k) + g(N,\mathcal{T}_{\mathrm{I}_k}\mathrm{I}_k)$$

$$-\sum_{i=1}^{n} d((\nabla_{\mathrm{P}_i}\mathcal{T})(\mathrm{I}_k,\mathrm{I}_k),\mathrm{P}_i) - g(\mathcal{A}_{\mathrm{P}_i}\mathrm{I}_k,\mathcal{A}_{\mathrm{P}_i}I_k), \tag{19}$$

$$R_H = \sum_{i=1}^{r} S(\mathrm{P}_i,\mathrm{P}_i) = \sum_{i=1}^{n}\Big\{ S^{\mathcal{B}}(\mathrm{P}_i^{\mathcal{B}},\mathrm{P}_i^{\mathcal{B}})\circ\pi - \frac{1}{2}\{g(\nabla_{\mathrm{P}_i}N,\mathrm{P}_i) + g(\nabla_{\mathrm{P}_i}N,\mathrm{P}_i)\},$$

$$+2\sum_{i=1}^{n} g(\mathcal{A}_{\mathrm{P}}\mathrm{P}_i,\mathcal{A}_{\mathrm{Q}}\mathrm{P}_i) + \sum_{j=1}^{r} g(\mathcal{T}_{\mathrm{P}_i}\mathrm{P},\mathcal{T}_{\mathrm{P}_i}\mathrm{Q}). \tag{20}$$

Now, Equations (19) and (20) entail that

$$R_V = \hat{R} + \|N\|^2 - \mathrm{div}(\mathrm{N}) - \|\mathcal{A}\|^2, \tag{21}$$

$$R_H = (R^{\mathcal{B}}\circ\pi) + \|\mathcal{T}\|^2 + 2\|\mathcal{A}\|^2 - \mathrm{div}(N), \tag{22}$$

Adopting (21) and (22), we turn up the scalar curvature R of the base manifold $(\mathcal{B},g_{\mathcal{B}})$

$$R = \hat{R} + (R^{\mathcal{B}}\circ\pi) + \|N\|^2 + \|\mathcal{A}\|^2 + \|\mathcal{T}\|^2 - 2\mathrm{div}(N). \tag{23}$$

In addition, the mean curvature vector field $\mathbf{H}$ for every fiber of RS is given by $r\mathbf{H} = N$, where N is a horizontal vector field, such that

$$N = \sum_{j=1}^{r} \mathcal{T}_{\mathrm{I}_j}\mathrm{I}_j. \tag{24}$$

Additionally, any fiber π dimension is indicated by the prefix r, and the orthonormal basis for vdV is $\{\mathrm{E}_1,\mathrm{E}_2,\cdots,\mathrm{E}_r\}$. We emphasize that all fibers of RS must be minimal, if, and only if, the horizontal vector field N vanishes. From (24), we obtain

$$g(\nabla_Z N,\mathrm{P}) = \sum_{j=1}^{r} g((\nabla_Z\mathcal{T})(\mathrm{I}_j,\mathrm{I}_j),\mathrm{P}) \tag{25}$$

for any $Z\in\Gamma(T\mathcal{N})$ and $\mathrm{P}\in\Gamma H(\mathcal{N})$.

Any horizontal vector field P divergence on $\Gamma H(\mathcal{N})$, and denoted by $\mathrm{div}(\mathcal{P})$, is determined by

$$\mathrm{div}(\mathrm{P}) = \sum_{i=1}^{n} g(\nabla_{\mathrm{P}_i}\mathrm{P},\mathrm{P}_i), \tag{26}$$

where the orthonormal basis of the horizontal space $\Gamma H(\mathcal{N})$ is $\{\mathrm{P}_1,\mathrm{P}_2,\cdots,\mathrm{P}_n\}$. Thus, taking into account (26), we have

$$\mathrm{div}(N) = \sum_{i=1}^{n}\sum_{j=1}^{r} g(\nabla_{\mathrm{P}_i}\mathcal{T})(\mathrm{I}_j,\mathrm{I}_j),\mathrm{P}_i). \tag{27}$$

4. η-Ricci–Yamabe Solitons in Riemannian Submersions

This section discusses the η-RY soliton of kind- (σ,ρ) on RS $\pi:(\mathcal{N},g)\longrightarrow(\mathcal{B},g_{\mathcal{B}})$ from Riemannian manifolds and the characteristics of fiber of such RS with target manifold $(\mathcal{B},g_{\mathcal{B}})$. Throughout the study, RS stands for a Riemannian submersion between Riemannian manifolds. We discover the following conclusions as a result of Equations (10) to (13) in the case of an RS:

Theorem 1. *If $\pi : (\mathcal{N}, g) \longrightarrow (\mathcal{B}, g_{\mathcal{B}})$ is an RS. Then, the*
1. vdV is parallel with respect to the connection ∇, if the horizontal components $\mathcal{T}_{\mathrm{I}}\mathrm{J}$ and $\mathcal{A}_{\mathrm{P}}\mathrm{I}$ are eliminated, identically.
2. hdH is parallel with respect to the connection ∇, if the vertical components $\mathcal{T}_{\mathrm{I}}\mathrm{P}$ and $\mathcal{A}_{\mathrm{P}}\mathrm{Q}$ are eliminated, identically,
for any $\mathrm{P}, \mathrm{Q} \in \Gamma H(\mathcal{N})$ and $\mathrm{I}, \mathrm{J} \in \Gamma V(\mathcal{N})$.

Since $(\mathcal{N}, g)$ is an η-RY soliton, then, by (3), we find

$$2\sigma S(\mathrm{I}, \mathrm{J}) + (2\tau - \rho R) g(\mathrm{I}, \mathrm{J}) + 2\nu\eta(\mathrm{I})\eta(\mathrm{J}) + (\mathcal{L}_{\omega} g)(\mathrm{I}, \mathrm{J}) = 0 \tag{28}$$

for each $\mathrm{I}, \mathrm{I} \in \Gamma V(\mathcal{N})$. Adopting (16), we have

$$2\sigma \hat{S}(\mathrm{I}, \mathrm{J}) + g(N, \mathcal{T}_{\mathrm{I}}\mathrm{J}) + \{g(\nabla_{\mathrm{I}}\omega, \mathrm{J}) + g(\nabla_{\mathrm{J}}\omega, \mathrm{I})\} \tag{29}$$

$$- \sum_{i=1}^{n} g((\nabla_{\mathrm{P}_t}\mathcal{T})(\mathrm{I}, \mathrm{F}), \mathrm{P}_i) - g(\mathcal{A}_{\mathrm{P}_t}\mathrm{I}, \mathcal{A}_{\mathrm{P}_t}\mathrm{J}) + (2\tau - \rho R) g(I, \mathrm{J}) + 2\nu\eta(\mathrm{I})\eta(\mathrm{J}) = 0$$

wherein ∇ is a Levi–Civita connection on $\mathcal{N}$ and $\{\mathrm{P}_i\}$ denotes an orthonormal basis of the hdH. The following equation is then obtained by using Theorem 1, the Equations (7) and (10),

$$2\sigma \hat{S}(\mathrm{I}, \mathrm{J}) + [\hat{d}(\hat{\nabla}_{\mathrm{I}}\omega, \mathrm{J}) + \hat{g}(\hat{\nabla}_{\mathrm{I}}\omega, \mathrm{I})] \tag{30}$$

$$+ (2\tau - \rho \hat{R\rceil}_V) \hat{g}(\mathrm{I}, \mathrm{J}) + 2\nu\eta(\mathrm{I})\eta(\mathrm{J}) = 0,$$

for every $\mathrm{I}, \mathrm{J} \in \Gamma V(\mathcal{N})$. Using (21), we find

$$2\sigma \hat{S}(\mathrm{I}, \mathrm{J}) + [\hat{g}(\hat{\nabla}_{\mathrm{I}}\omega, \mathrm{J}) + \hat{g}(\hat{\nabla}_{\mathrm{J}}\omega, \mathrm{I})] \tag{31}$$

$$+ (2\tau - \rho \hat{R} + \|N\|^2 - \|\mathcal{A}\|^2 - \mathrm{div}(N)) \hat{g}(\mathrm{I}, \mathrm{J}) + 2\nu\eta(\mathrm{I})\eta(\mathrm{J}) = 0.$$

Defining $R = \hat{R} + \|N\|^2 - \|\mathcal{A}\|^2 - \mathrm{div}(N)$, then, the Equation (31) follows;

$$2\sigma \hat{S}(\mathrm{I}, \mathrm{J}) + (2\tau - \rho R) \hat{g}(\mathrm{I}, \mathrm{J}) + [\hat{g}(\hat{\nabla}_{\mathrm{I}}\omega, \mathrm{J}) + \hat{g}(\hat{\nabla}_{\mathrm{J}}\omega, \mathrm{I})] + 2\nu\eta(\mathrm{I})\eta(\mathrm{J}) = 0. \tag{32}$$

Let us mention here the "vertical potential vector field" (in brief $VPVF$) and the "horizontal potential vector field" ($HPVF$). Hence, we generate the following results:

Theorem 2. *Let $(\mathcal{N}, g, \omega, \tau, \nu, \sigma, \rho)$ be an η-RY soliton of kind-(σ, ρ) with a $VPVF$ ω and π be an RS from the Riemannian manifolds. If the vdV is parallel, then every fiber in an RS is an η-RY soliton.*

Remark 1. *Now, for $\sigma = 1, \rho = 0$ and $\nu \neq 0$, then, from (30), we find*

$$2\hat{S}(\mathrm{I}, \mathrm{J}) + [\hat{g}(\hat{\nabla}_{\mathrm{I}}\omega, \mathrm{J}) + \hat{g}(\hat{\nabla}_{\mathrm{I}}\omega, \mathrm{J})] + 2\tau\hat{g}(\mathrm{I}, \mathrm{J}) + 2\nu\eta(\mathrm{I})\eta(\mathrm{J}) = 0. \tag{33}$$

Therefore, one can obtain the following

Theorem 3. *Let $(\mathcal{N}, g, \omega, \tau, \nu, \sigma)$ be an η-Ricci soliton of kind-$(1, 0)$ with $VPVF$ ω and π be a RS. If the vdV is parallel, then every fiber in an RS is an η-Ricci soliton.*

Remark 2. *Next, setting $\sigma = 0, \rho = 1$ and $\nu \neq 0$, so (30) entails that*

$$[\hat{g}(\hat{\nabla}_{\mathrm{I}}\omega, \mathrm{J}) + \hat{g}(\hat{\nabla}_{\mathrm{I}}\omega, \mathrm{J})] + (2\tau - R)\hat{g}(\mathrm{I}, \mathrm{J}) + 2\nu\eta(\mathrm{I})\eta(\mathrm{J}) = 0, \tag{34}$$

Therefore, one can obtain the following outcome:

Theorem 4. *Let $(\mathcal{N}, g, \omega, \tau, \nu, \rho)$ be an η-RY soliton of kind-$(0,1)$ with a $VPVF$ ω and π be a RS. If the vdV is parallel, then every fiber in an RS is a η-Yamabe soliton.*

So, if the total space $(\mathcal{N}, g)$ of RS $\pi : (\mathcal{N}, g) \longrightarrow (\mathcal{B}, g_{\mathcal{B}})$ admits, an η-RY soliton of kind-(σ, ρ), now, in view of (3) and (16), we obtain

$$\{g(\nabla_{\mathrm{I}}\omega, \mathrm{J}) + g(\nabla_{\mathrm{J}}\omega, \mathrm{I})\} + 2\sigma\hat{S}(\mathrm{I}, \mathrm{J}) + \sum_{j=1}^{r} g(\mathcal{T}_{\mathrm{I}_j}\mathrm{I}_j, \mathcal{T}_{\mathrm{I}}\mathrm{J}) \tag{35}$$
$$- \sum_{i=1}^{n} d((\nabla_{\mathrm{P}_i}\mathcal{T})(\mathrm{I}, \mathrm{J}), \mathrm{P}_i) - g(\mathcal{A}_{\mathrm{P}_i}\mathrm{I}, \mathcal{A}_{\mathrm{P}_i}\mathrm{J}) + (2\tau - \rho\hat{R})\hat{d}(\mathrm{I}, \mathrm{J}) + 2\nu\eta(\mathrm{I})\eta(\mathrm{J}) = 0$$

where $\mathrm{I}, \mathrm{J} \in \Gamma V(\mathcal{N})$. In addition, an η-RY soliton $(\mathcal{N}, g, \omega, \tau, \nu)$ of kind- (σ, ρ) admits totally umbilical fibers and adopting (10) in (35), we obtain

$$\{g(\hat{\nabla}_{\mathrm{I}}\omega, \mathrm{G}) + g(\hat{\nabla}_{\mathrm{G}}\omega, \mathrm{I})\} + 2\sigma\hat{S}(\mathrm{I}, \mathrm{G}) + \sum_{j=1}^{r} g(\mathcal{T}_{\mathrm{I}_j}\mathrm{I}_j, \mathcal{T}_{\mathrm{I}}\mathrm{G}) \tag{36}$$
$$- \sum_{i=1}^{n}\{(\nabla_{\mathrm{P}_i}g)(\mathrm{I}, \mathrm{G})g(\mathrm{K}, \mathrm{P}_i) - g(\nabla_{\mathrm{P}_i}\mathrm{K}, \mathrm{P}_i)\hat{g}(\mathrm{I}, \mathrm{G})\}$$
$$- \sum_{i=1}^{n} g(\mathcal{A}_{\mathrm{P}_i}\mathrm{I}, \mathcal{A}_{\mathrm{P}_i}\mathrm{G}) + (2\tau - \rho R|_{\mathrm{H}})\hat{g}(\mathrm{I}, \mathrm{G}) + 2\nu\eta(\mathrm{I})\eta(\mathrm{G}) = 0.$$

Since with integrable hdH, we derive,

$$(\mathcal{L}_{\omega}\hat{g})(\mathrm{I}, \mathrm{G}) + 2\sigma\hat{S}(\mathrm{I}, \mathrm{G}) - \sum_{i=1}^{n} g(\nabla_{\mathrm{P}_i}\mathrm{K}, \mathrm{P}_i)\hat{g}(\mathrm{I}, \mathrm{G}) \tag{37}$$
$$+ r\|W\|^2\hat{g}(\mathrm{I}, \mathrm{G}) + (2\tau - \rho\hat{R})\hat{g}(\mathrm{I}, \mathrm{G}) + 2\nu\eta(\mathrm{I})\eta(\mathrm{G}) = 0$$

wherein K is the mean curvature vector of any fiber of π. By (26), we derive

$$(\mathcal{L}_{\omega}\hat{g})(\mathrm{I}, \mathrm{G}) + 2\sigma\hat{S}(\mathrm{I}, \mathrm{G}) + [2\tau - \rho(\hat{R} - \mathrm{div}(N) + r\|N\|^2]\hat{g}(\mathrm{I}, \mathrm{G}) + 2\nu\eta(\mathrm{I})\eta(\mathrm{G}) = 0. \tag{38}$$

We observe, that every fiber for π is an almost η-RY soliton. As a result, one can state the following outcome:

Theorem 5. *If $(\mathcal{N}, g, \omega, \tau, \nu, \sigma, \rho)$ be an η-RY soliton of kind-(σ, ρ) with a VPF ω and π be an RS with totally umbilical fibers and the hdH is integrable, then every fiber in an RS is an almost η-RY soliton.*

Furthermore, the following results are obtained:

Theorem 6. *If $(\mathcal{N}, g, \omega, \tau, \nu, \sigma, \rho)$ is an η-RY soliton of kind-$(1,0)$ with a VPF, ω and π are an RS with totally umbilical fibers, and the hdH is integrable, then every fiber in a RS is an almost η-Ricci soliton.*

Proof. Fix $\sigma = 1, \rho = 0, \nu \neq 0$ and from (38) we derive the required outcomes. □

Theorem 7. *If $(\mathcal{N}, g, \omega, \tau, \nu, \sigma, \rho)$ is an η-RY soliton of kind-$(0,1)$ with a VPF ω and π is an RS with totally umbilical fibers and the hdH is integrable, then every fiber in an RS is an almost η-quasi Yamabe soliton.*

Proof. Putting $\sigma = 0, \rho = 1, \nu \neq 0$ and using (38), we gain the following: □

Assuming once more the Theorem 5, we arrive at the following corollaries:

Corollary 1. *If $(\mathcal{N}, g, \omega, \tau, \nu, \sigma, \rho)$ is an η-RY soliton of kind-(σ, ρ) and π is an RS, and the hdH is integrable, and if every fiber of π is totally umbilical and admits constant mean curvature, then any fiber in RS is an almost η-RY soliton,*

Corollary 2. *If $(\mathcal{N}, g, \omega, \tau, \nu, \sigma, \rho)$ is an η-RY soliton of kind-(σ, ρ) and π is an RS, such that the hdH is integrable, and if every fiber of π is totally geodesic, then any fiber of an RS is an almost η-RY soliton,*

Remark 3. *In light of Corollaries 1 and 2, we can derive identical results for an almost η-Ricci soliton and an almost η-quasi Yamabe soliton.*

Next, we obtain the following:

Theorem 8. *If $(\mathcal{N}, g, Z, \tau, \nu, \sigma, \rho)$ is an η-RY soliton of kind-(σ, ρ) with a VPF $Z \in \Gamma(TM)$ and π is an RS and the hdH is parallel, then the following holds:*

1. *$(\mathcal{B}, g_{\mathcal{B}})$ is an η-Einstein if Z is a VVF,*
2. *$(\mathcal{B}, g_{\mathcal{B}})$ is an η-RY soliton with VPF $Z_{\mathcal{B}}$ if U is HVF, such that $\pi_* Z = Z_{\mathcal{B}}$.*

Proof. As far as $(\mathcal{N}, g)$, the total space of RS π admits an η-RY soliton of kind- (σ, ρ) with a VPF $Z \in \Gamma(T\mathcal{N})$; then, utilizing (3) and (17), we gain

$$[g(\nabla_{\rm P} U, {\rm Q}) + g(\nabla_{\rm Q} U, {\rm P})] + 2\sigma S_{\mathcal{B}}({\rm P}_{\rm B}, \mathcal{Q}_{\mathcal{B}}) \circ \pi - (d(\nabla_{\mathcal{P}} N, \mathcal{Q}) + d(\nabla_{\mathcal{Q}} N, \mathcal{P})) \tag{39}$$
$$+2\sum_{i=1}^{n} d(\mathcal{A}_{\mathcal{P}} {\rm P}_i, \mathcal{A}_{\rm Q} {\rm P}_i) + \sum_{j=1}^{r} g(\mathcal{T}_{{\rm I}_j} {\rm P}, \mathcal{T}_{{\rm I}_j} {\rm Q}) + (2\tau - \rho R) g({\rm P}, {\rm Q}) + 2\nu \eta({\rm P}) \eta({\rm Q}) = 0$$

wherein ${\rm P}_{\mathcal{B}}$ and ${\rm Q}_{\mathcal{B}}$ are π-connected to P and Q, respectively, for any ${\rm P}, {\rm Q} \in \Gamma H(\mathcal{N})$.

Utilizing Theorems (1) to (39), we derive

$$[g(\nabla_{\rm P} Z, {\rm Q}) + g(\nabla_{\rm Q} Z, {\rm P})] + 2\sigma S_{\mathcal{B}}({\rm P}_{\mathcal{B}}, {\rm Q}_{\mathcal{B}}) \circ \pi \tag{40}$$
$$+(2\tau - \rho R) g({\rm P}, {\rm Q}) + 2\nu \eta({\rm P}) \eta({\rm Q}) = 0.$$

1. If Z is a VVF, from (12), it follows

$$[g(\mathcal{A}_{\rm P} Z, {\rm Q}) + g(\mathcal{A}_{\rm Q} Z, {\rm P})] + 2\sigma S_{\mathcal{B}}({\rm P}_{\mathcal{B}}, {\rm Q}_{\mathcal{B}}) \circ \pi \tag{41}$$
$$+(2\tau - \rho R|_{\mathcal{V}}) g({\rm P}, {\rm Q}) + 2\nu \eta({\rm P}) \eta({\rm Q}) = 0.$$

Since H is parallel, we obtain

$$S_{\mathcal{B}}({\rm P}_{\mathcal{B}}, {\rm Q}_{\mathcal{B}}) \circ \pi = a g({\rm P}, {\rm Q}) + b \eta({\rm P}) \eta({\rm Q}) = 0. \tag{42}$$

This proves that $(\mathcal{B}, g_{\mathcal{B}})$ is an η-Einstein, wherein $a = -(\tau - \frac{R|_{\mathcal{V}}}{2})$ and $b = -\nu$.

2. If Z is a horizontal vector field, from (40), we obtain

$$(\mathcal{L}_Z g)({\rm P}, {\rm Q}) + 2\sigma S_{\mathcal{B}}({\rm P}_{\mathcal{B}}, {\rm Q}_{\mathcal{B}}) \circ \pi + (2\tau - \rho R_{\rm H}) g({\rm P}, {\rm Q}) + 2\nu \eta({\rm P}) \eta({\rm Q}) = 0. \tag{43}$$

It is observed that the total space $(\mathcal{B}, g_{\mathcal{B}})$ is an η-RY soliton with the PVF ${\rm E}_{\mathcal{B}}$ lying horizontally. □

Now, from (43) and assuming that the vector field Z is horizontal, we can state the following:

Theorem 9. *Let $(\mathcal{N}, g, Z, \tau, \nu, \rho)$ be an η-RY soliton of kind-$(0, \rho)$, which admits the PVF $Z \in \Gamma(T\mathcal{N})$, and π be an RS. If the hdH is parallel and the vector field Z is horizontal, then $(\mathcal{B}, g_{\mathcal{B}})$ is an η-quasi-Yamabe soliton with HPVF $P_{\mathcal{B}}$, such that*

$$(\mathcal{L}_Z g)(P,Q) + (2\tau - \rho\Big\{(R^N \circ \pi) + \|\mathcal{T}\|^2 + 2\|\mathcal{A}\|^2 - \mathrm{div}(N)\Big\})g(P,Q) + 2\nu\eta(P)\eta(Q) = 0. \quad (44)$$

Once more combining Theorem (1) and (17), we arrive at the following result:

Lemma 1. *If $(\mathcal{N}, g, \zeta, \tau, \nu, \sigma, \rho)$ is an η-RY soliton on RS π that admits HPVF ζ, such that H is parallel, then the vector field N on hdH is Killing.*

Since $(\mathcal{N}, g, \zeta, \tau, \nu)$ is an η-RY soliton of kind-(σ, ρ), and again using (17) in (3), we find that

$$(\mathcal{L}_\zeta g)(P,Q) + 2\sigma S^{\mathcal{B}}(P_{\mathcal{B}}, Q_{\mathcal{B}}) \circ \pi - \{g(\nabla_P N, Q) + g(\nabla_Q N, \mathcal{P})\} \quad (45)$$

$$+2\sum_i g(\mathcal{A}_P P_i, \mathcal{A}_Q P_i) + \sum_j g(\mathcal{T}_{Z_j} P, \mathcal{T}_{Z_j} Q) + (2\tau - \rho R|_H)g(P,Q) + 2\nu\eta(P)\eta(Q) = 0.$$

For any $P, Q \in \Gamma H(\mathcal{N})$, where $\{\mathcal{P}_i\}$ denotes an orthonormal basis of H. Equation (45) is derived from Theorem 1 as follows:

$$(\mathcal{L}_\zeta d)(\mathcal{P}, \mathcal{Q}) + 2\sigma S^{\mathcal{B}}(\mathcal{P}_{\mathcal{B}}, \mathcal{Q}_{\mathcal{B}}) \circ \pi + (2\tau - \rho R|_H)d(\mathcal{P}, \mathcal{Q}) + 2\nu\eta(\mathcal{P})\eta(\mathcal{Q}) = 0. \quad (46)$$

We may determine that ζ is a conformal Killing vector field ($CKVF$) because the Riemannian manifold $(\mathcal{B}, d_{\mathcal{N}})$ is an η-Einstein. As a result, we can state the following outcome:

Theorem 10. *Let $(\mathcal{B}, d, \zeta, \tau, \nu, \sigma, \rho)$ be an η-RY soliton of kind (σ, ρ) on RS to an η-Einstein which admits HPVF ζ, such that hdH is parallel. Then, the vector field ζ on hdH is CKVF.*

5. Examples

Example 2. *Let $\mathcal{N}^6 = \{(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) | \theta_6 \neq 0\}$ be a 6-dimensional differentiable manifold where (θ_i) signifies the standard coordinates of a point in $\mathbb{R}^6$, and $i = 1, 2, 3, 4, 5, 6$.*

Let

$$\delta_1 = \partial\theta_1, \quad \delta_2 = \partial\theta_2, \quad \delta_3 = \partial\theta_3,$$

$$\delta_4 = \partial\theta_4, \quad \delta_5 = \partial\theta_5, \quad \delta_6 = \partial\theta_6$$

be the basis for the tangent space $T(\mathcal{N}^6)$ since it consists of a set of linearly independent vector fields at each point of the manifold $\mathcal{N}^6$. A definite positive metric d on $\mathcal{N}^6$ is defined as follows: with $i, j = 1, 2, 3, 4, 5, 6$, and it is defined as

$$d = \sum_{i.j=1}^{6} dx\theta_i \otimes d\theta_j.$$

Let γ be a 1-form such that $\gamma(U) = d(U, P)$ where $\delta_6^\sharp = P$. Thus, $(\mathcal{N}^6, d)$ is a Riemannian manifold. In addition, $\bar{\nabla}$ is the Levi–Civita connection with respect to d. Then, we have

$$[\delta_1, \delta_2] = 0, \quad [\delta_1, \delta_6] = \delta_1, \quad [\delta_2, \delta_6] = \delta_2, \quad [\delta_3, \delta_6] = \delta_3,$$

$$[\delta_4, \delta_6] = \delta_4, \quad [\delta_5, \delta_6] = \delta_6, \quad [\delta_i, \delta_j] = 0,$$

where $1 \leq i \neq j \leq 5$.
The induced connection $\hat{\nabla}$ for the metric $\hat{g}$ is described as

$$2g(\hat{\nabla}_{UV,W}) = Ug(V,W) + Vg(W,U) - Wg(U,V) \\ - g(U,[V,W]) - g(V,[U,W]) + g(W,[U,V]),$$

where the metric g corresponds to the Levi–Civita connection denoted by the symbol ∇.

The following equations are obtained by combining Koszul's formula with (10).

$$\hat{\nabla}_{\delta_1}\delta_1 = \delta_6,\ \hat{\nabla}_{\delta_2}\delta_2 = \delta_6,\ \hat{\nabla}_{\delta_3}\delta_3 = \delta_6,\ \hat{\nabla}_{\delta_4}\delta_4 = \delta_6, \hat{\nabla}_{\delta_5}\delta_5 = \delta_6 \tag{47}$$

$$\hat{\nabla}_{\delta_6}\delta_6 = 0,\ \hat{\delta}_{\delta_6}\delta_i = 0,\ \hat{\nabla}_{\delta_i}\delta_6 = \delta_i,\ 1 \leq i \leq 5$$

wherein $1 \leq i, j \leq 5$, we have $\hat{\nabla}_{\delta_i}\delta_i = 0$.

The non-vanishing components of $\hat{R}$, $\hat{S}$, and $\hat{R}$ of the fiber may now be computed from Equations (14) and (47).

$$\hat{R}(\delta_1,\delta_2)\delta_1 = \delta_2, \quad \hat{R}(\delta_1,\delta_2)\delta_2 = -\delta_1, \quad \hat{R}(\delta_1,\delta_3)\delta_1 = -\delta_3, \hat{R}(\delta_1,\delta_3)\delta_3 = \delta_1 \tag{48}$$

$$\hat{R}(\delta_1,\delta_4)\delta_1 = -\delta_4, \quad \hat{R}(\delta_1,\delta_4)\delta_4 = \delta_1, \quad \hat{R}(\delta_1,\delta\delta_5)\delta_1 = -\delta_5, \hat{R}(\delta_1,\delta_5)\delta_5 = \delta_1$$

$$\hat{R}(\delta_1,\delta_6)\delta_1 = -\delta_6, \quad \hat{R}(\delta_1,\delta_6)\delta_6 = -\delta_1, \quad \hat{R}(\delta_2,\delta_3)\delta_2 = -\delta_3, \hat{R}(\delta_2,\delta_3)\delta_3 = \delta_2$$

$$\hat{R}(\delta_2,\delta_4)\delta_2 = \delta_4, \quad \hat{R}(\delta_2,\delta_4)\delta_4 = -\delta_2, \quad \hat{R}(\delta_2,\delta_5)\delta_2 = \delta_5, \hat{R}(\delta_2,\delta_5)\delta_5 = -\delta\delta_2$$

$$\hat{R}(\delta_2,\delta_6)\delta_2 = \delta_6, \quad \hat{R}(\delta_2,\delta_6)\delta_6 = -\delta_2, \quad \hat{R}(\delta_3,\delta_4)\delta_3 = \delta_4, \hat{R}(\delta_3,\delta_4)\delta_4 = \delta_5$$

$$\hat{R}(\delta_3,\delta_5)\delta_5 = -\delta_3, \quad \hat{R}(\delta_3,\delta_6)\delta_3 = -\delta_6, \quad \hat{R}(\delta_3,\delta_6)\delta_3 = -\delta_6, \hat{R}(\delta_3,\delta_6)\delta_6 = -\delta_3$$

$$\hat{R}(\delta_4,\delta_5)\delta_4 = \delta_5, \quad \hat{R}(\delta_4,\delta_5)\delta_5 = -\delta_4, \quad \hat{R}(\delta_4,\delta_6)\delta_4 = -\delta_6,$$

$$\hat{R}(\delta_4,\delta_6)\delta_6 = -\delta_4, \quad \hat{R}(\delta_5,\delta_6)\delta_5 = -\delta_6, \quad \hat{R}(\delta_5,\delta_6)\delta_6 = -\delta_5.$$

$$\hat{S}(\delta_i,\delta_j) = \begin{bmatrix} -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix}.$$

$$\hat{R} = Trace(\hat{S}) = -20. \tag{49}$$

From Equation (16), we have

$$\frac{1}{2}[\hat{g}(\hat{\nabla}_{\delta_i}\delta_6,\delta_i) + \hat{g}(\hat{\nabla}_{\delta_i}\delta_6,\delta_i)] + \sigma\hat{S}(\delta_i,\delta_i) + (\tau - \frac{1}{2}\rho\hat{R})\hat{g}(\delta_i,\delta_i) + 2\nu\Omega^i_j = 0 \tag{50}$$

wherein, for all $i \in \{1,2,3,4,5,6\}$. Thus, $\tau = 10\rho - 3\sigma - 1$ and $\nu = 23\sigma - 30\rho - 1$, and the data $(\hat{g}, \delta_6, \tau, \nu, \sigma, \rho)$ is an η-RY soliton, verified by Equation (16). Therefore, the data $(\omega, \hat{d}, \tau, \nu, \sigma, \rho)$ admits increasing, decreasing and stable η-RY solitons referring to $(3\sigma + 1) > 10\rho$, $(3\alpha + 1) < 10\rho$ or $(3\sigma + 1) = 10\rho$, respectively

The two basic instances for a specific value of σ and ρ are as follows:

Case 1. For an η-Ricci–Yamabe soliton of type (σ, ρ), if $\sigma = 1, \rho = 0$, we gain $\tau = -4$ and $\nu = 22$. Then, we say $(\hat{d}, \delta_6, \tau, \nu, 1, 0)$ is an η-Ricci soliton which is shrinking. This case illustrates Theorem 3.
Case 2. For an η-RY soliton of kind (σ, ρ) if $\sigma = 0, \rho = 0$, we derive $\tau = 9$ and $\nu = -31$; then, we have the data $(\hat{g}, \delta_6, \tau, \nu, 0, 1)$ is an η-Yamabe soliton is expanding. This illustrates Theorem 4.

Example 3. *Let* $\pi : \mathbb{R}^6 \rightarrow \mathbb{R}^3$ *be a submersion defined by*

$$\pi(x_1, x_2, ...x_6) = (y_1, y_2, y_3),$$

where

$$y_1 = \frac{x_1 + x_2}{\sqrt{2}},\ y_2 = \frac{x_3 + x_4}{\sqrt{2}} \text{ and } y_3 = \frac{x_5 + x_6}{\sqrt{2}}.$$

The Jacobi matrix of π has rank 3 at that point. This indicates that π is a submersion. Simple calculations produce

$$(\mathcal{K}er\pi_*) = Span\{V_1 = \frac{1}{\sqrt{2}}(-\partial x_1 + \partial x_2), V_2 = \frac{1}{\sqrt{2}}(-\partial x_3 + \partial x_4),$$
$$V_3 = \frac{1}{\sqrt{2}}(-\partial x_5 + \partial x_6)\},$$

and

$$(\mathcal{K}er\pi_*)^{\perp} = Span\{H_1 = \frac{1}{\sqrt{2}}(\partial x_1 + \partial x_2), H_2 = \frac{1}{\sqrt{2}}(\partial x_3 + \partial x_4),$$
$$H_3 = \frac{1}{\sqrt{2}}(\partial x_5 + \partial x_6)\},$$

Also, direct computation yields

$$\pi_*(H_1) = \partial y_1, \pi_*(H_2) = \partial y_2 \text{ and } \pi_*(H_3) = \partial y_3.$$

It is easy to observe that

$$g_{\mathbb{R}^6}(H_i, H_i) = g_{\mathbb{R}^3}(\pi_*(H_i), \pi_*(H_i)),\ i = 1, 2, 3$$

Hence, ψ is a RS.

Next, we estimate the components of $\hat{R}$, $\hat{S}$ and $\hat{R}$ for $\mathcal{K}er\pi_*$ and $\mathcal{K}er\pi_*^{\perp}$, respectively. For the vertical space, we gain

$$\hat{R}(V_1, V_2)V_1 = -2V_2, \quad \hat{R}(V_1, V_2)V_2 = 2V_1, \quad \hat{R}(V_1, V_3)V_1 = -2V_3 \tag{51}$$

$$\hat{R}(V_1, V_2)V_3 = V_1, \quad \hat{R}(V_2, V_3)V_3 = V_2, \quad \hat{R}(V_2, V_3)V_2 = V_2.$$

$$\hat{S}(V_i^{\sharp}, V_j^{\sharp}) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\hat{R} = Trace(\hat{S}) = 5. \tag{52}$$

Using (3), we find $\tau = \frac{5\rho}{2} - \sigma$ and $\nu = \alpha$. Therefore, $(\mathcal{K}er\psi_*, g)$ admits the increasing, decreasing and stable η-RY solitons referring to $\frac{5\rho}{2} < \sigma$, $\frac{5\rho}{2} > \sigma$ or $\frac{5\rho}{2} = \sigma$, respectively.

Moreover, we also have the following cases for particular values of α and β, such as:
Case 1 . In an η-RY soliton of type (σ, ρ) for $\sigma = 1, \sigma = 0$, we find $\tau = -2$ and $\nu = 1$, then $(\mathcal{K}er\psi_*, d)$ admitting a shrinking η-Ricci soliton.
Case 2. In an η-RY soliton of type (σ, τ) for $\sigma = 0, \rho = 1$, we find $\tau = \frac{5}{2}$ and $\nu = 0$; then, we have $(\mathcal{K}er\pi_*, g)$ admitting an expanding Yamabe soliton.

In a similar way, for the horizontal space, we derive

$$R^{\mathcal{B}}(\pi_*(H_1), \pi_*(H_2))\pi_*(H_1) = \frac{1}{2}(\partial x_3 + \partial x_4), \qquad R^{\mathcal{B}}(\pi_*(H_1), \pi_*(H_3))\pi_*(H_3) = \frac{1}{\sqrt{2}}(\partial x_6 - \partial x_5),$$

$$R^{\mathcal{B}}(\pi_*(H_1), \pi_*(H_3))\pi_*(H_1) = \frac{1}{2}\partial x_6, \qquad R^{\mathcal{B}}(\pi_*(H_2), \pi_*(H_3))\pi_*(H_2) = (\frac{1}{\sqrt{2}} - 1)\partial x_6,$$

$$R^{\mathcal{B}}(\pi_*(H_2), \pi_*(H_3))\pi_*(H_3) = -\frac{1}{2}(\partial x_3 + \partial x_4),$$
$$R^{\mathcal{B}}(\pi_*(H_1), \pi_*(H_2))\pi_*(H_2) = \frac{1}{2\sqrt{2}}(\partial x_1 + \partial x_2).$$

and

$$S^{\mathcal{B}}(\pi_* H_i, \pi_* H_j) = \begin{bmatrix} -\frac{3}{2\sqrt{2}} & 0 & 0 \\ 0 & -\frac{3}{2\sqrt{2}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

$$R^{\mathcal{B}} = Trace(S^{\mathcal{B}}) = -2\sqrt{2}. \tag{53}$$

Again using (3), we derive $\tau = \frac{3\sigma}{2\sqrt{2}} - \sqrt{(2)}\rho$ and $\nu = -\frac{\sigma}{2\sqrt{2}}$. Therefore, $((\mathcal{K}er\pi_*^{\perp}), g)$ admits the expanding, shrinking and steady η-RY solitons referring to $\frac{53\sigma}{2\sqrt{2}} > \sqrt{(2)}\rho$, $\frac{3\sigma}{2\sqrt{2}} < \sqrt{(2)}\rho$ or $\frac{3\sigma}{2\sqrt{2}} = \sqrt{(2)}\rho$, respectively.

Also, we have obtained the following cases for particular values of σ and ρ, such as:

Case 1. In an η-RY soliton of type (σ, ρ) for $\sigma = 1, \rho = 0$, we find $\tau = \frac{3}{2\sqrt{2}}$ and $\nu = -\frac{1}{2\sqrt{2}}$; then, $((\mathcal{K}er\pi_*^{\perp}), g)$ is admitting an expanding η-Ricci soliton.

Case 2. In an η-RY soliton of type (σ, ρ) for $\sigma = 0, \rho = 1$, we find $\tau = -\sqrt{2}$ and $\nu = 0$; then, we have $((\mathcal{K}er\pi_*^{\perp}), g)$ is admitting a shrinking Yamabe soliton.

6. η-Ricci–Yamabe Soliton with a Potential Vector Field $\omega = grad(\gamma)$

Let the potential vector field $\omega = grad(\gamma)$ on $\mathcal{N}$; then, $(\mathcal{N}, g, \omega, \tau, \nu, \sigma, \rho)$ is said to be a gradient η-RY soliton, which is indicated by $(\mathcal{N}, g, \omega, \tau, \nu, \sigma, \rho)$.

Now, consider the equation η-Ricci–Yamabe soliton for an r-dimensional fiber in RS.

$$2\sigma\hat{S}(\mathrm{I}, \mathrm{J}) = -[\hat{g}(\hat{\nabla}_{\mathrm{I}}\omega, \mathrm{J}) + \hat{g}(\hat{\nabla}_{\mathrm{J}}\omega, \mathrm{I})] - (2\tau - \rho R)\hat{d}(\mathrm{I}, \mathrm{J}) - 2\nu\eta(\mathrm{I})\eta(\mathrm{J}). \tag{54}$$

Contracting the Equation (54), we obtain

$$\mathrm{div}(\omega) = -r\tau + R\Big(\frac{\rho}{2} - \sigma\Big) - \nu. \tag{55}$$

As a result, the following theorems exist:

Theorem 11. *If $(\mathcal{N}, g, \omega, \tau, \nu, \sigma, \rho)$ is an η-RY soliton of kind (σ, ρ) with gradient PVF $\omega = grad(\gamma)$, and the vd$\mathcal{V}$ is parallel, then every fiber in RS is an η-RY soliton, and the Poisson equation satisfied by γ becomes*

$$\Delta(\gamma) = -r\tau + R\Big(\frac{\rho}{2} - \sigma\Big) - \nu. \tag{56}$$

Theorem 12. *Let $(\mathcal{N}, g, \omega, \tau, \nu, \rho)$ be an η-RY soliton of kind $(0, \rho)$ with gradient PVF $\omega = grad(\gamma)$ and the vd$\mathcal{V}$ is parallel, then every fiber in RS is an η-Yamabe soliton, and the Poisson equation satisfied by γ becomes*

$$\Delta(\gamma) = -r\tau + R\left(\frac{\rho}{2}\right) - \nu. \tag{57}$$

Remark 4. *If $\nu = 0$ in (56) and (57), we can easily obtain similar types of results for the RY soliton and Yamabe soliton from Theorems (11) and (12), respectively.*

7. Physical Applications of Solitons

As far as a physically relevant model having a solitonic solution is concerned, the theory of collapse condensates with the inter-atomic attraction and spin-orbit coupling (SOC) [31], which is a fundamentally important effect in physical models, chiefly, Bose–Einstein condensates (BEC) [32]. The SOC emulation proceeds by mapping the spinor wave function of electrons into a pseudo-spinor mean-field wave function in BEC, whose components represent two atomic states in the condensate. While SOC in bosonic gases is a linear effect, there is interplay with the intrinsic BEC non-linearity, including several types of one dimensional $(1D)$ solitons [33]. An experimental realization of SOC in two-dimensional $(2D)$ geometry has been reported too [34], which suggests, in particular, the possibility of creation of a $2D$ gap soliton [35], supported by a combination of SOC and a spatially periodic field.

A fundamental problem that impedes the creation of $2D$ and $3D$ solitons in BES, nonlinear optics, and other nonlinear settings, is that the ubiquitous cubic self-attraction, which usually rise to solitons, simultaneously derives the critical and supercritical collapse in the $2D$ and $3D$ cases, respectively [36]. Although SOC modifies the conditions of the existence of the solutions and of the blow-up, it does not arrest the collapse completely [33]. The collapse destabilizes formally existing solitons, which results in stabilization of $2D$ and $3D$ solitons [32].

In the presence of SOC, the evolution of the wave function is described by a system-coupled nonlinear PDE in the Schrödinger form [37]

$$i\hbar\frac{\partial\Psi}{\partial} = \left[-\frac{\hbar^2}{2M}\Delta + \hat{H}_{so} + \frac{1}{2}(B.\hat{\sigma}) - g_2|\Psi|^2\right]\Psi, \tag{58}$$

where M is the mass of the particle, $\hat{H}_{so}$ is the SOC Hamiltonian, B is the effective magnetic field, $\hat{\sigma}$ is the spin operator and g_2 is the coupling constant.

The key point in understanding the role of the SOC in the collapse process is the modified velocity

$$v = k + \nabla_k \hat{H}_{so}, \tag{59}$$

where $k = -i\frac{\partial}{\partial r}$, including the velocity and $\nabla_k \hat{H}_{so}$ $(\nabla_k \equiv \frac{\partial}{\partial k})$, are directly related to the particle spin.

Let the first form Rashaba spin-orbit coupling

$$\hat{H}_{so} \equiv \hat{H}_R = \alpha(k_x\hat{\sigma}_y - k_y\hat{\sigma}_x), \tag{60}$$

with coupling constant α and $k = (k_x, k_y)$. The corresponding spin-dependent term in the velocity operators in Equation (59) becomes (for more details see [33])

$$\frac{\partial\hat{H}_R}{\partial k_x} = \alpha\hat{\sigma}_y, \qquad \frac{\partial\hat{H}_R}{\partial k_y} = -\alpha\hat{\sigma}_x. \tag{61}$$

In particular, in the $2D$ case, the nonlinear Schrödinger equation with cubic self-attraction term gives rise to degenerate families of the fundamental *Townes solitons* [38] with vorticity $S = 0$, which means decaying solutions. Hence, Townes solitons, that play the role of separation between the type of dynamical behavior, are the completable unstable and total norm of the spinor wave function that does not exceed a critical value. Further, it also produces stable dipole and quadrupole bound states of fundamental solitons with opposite signs.

8. Application of Riemannian Submersions to Number Theory

The Hopf fibration [39] is a Riemannian submersion $\pi : (\mathcal{N}^n, g) \to (\mathcal{B}^b, g_{\mathcal{B}})$ with totally geodesic fibers. In addition, a large class of Riemannian submersions are Riemannian submersions between spheres of higher dimensions, such as

$$\pi : \mathbb{S}^{r+m} \longrightarrow \mathbb{S}^m$$

whose fibers have dimension m. The Hopf fibration asserts that the fibration generalizes the idea of a fiber bundle and plays a significant role in algebraic topology, number theory and groups theory [40].

Every fiber in a fibration is closely connected to the homotopy group and satisfies the homotopy property [41]. The homotopy group of spheres $\mathbb{S}^n$ essentially describes how several spheres of different dimensions may twist around one another. For the j-th homotopy group $\Phi_j(\mathbb{S}^r)$, the j-dimensional sphere $\mathbb{S}^j$ can be mapped continuously to the r-dimensional sphere $\mathbb{S}^r$.

Now, we can make the following remark :

Remark 5. *To determine the homotopy groups for positive k using the formula $\pi_{r+k}(\mathbb{S}^r)$. The homotopy groups $\pi_{r+k}(\mathbb{S}r)$ with $r > k+1$ are known as stable homotopy groups of spheres and are denoted by $\pi_k^{\mathbb{S}}$; they are finite abelian groups for $k \neq 0$. In view of Freudenthal's suspension theorem [42], the groups are known as unstable homotopy groups of spheres for $r \leq k+1$.*

Now, in the light of Corollary 2 and using the above facts (5), we gain the following outcomes.

Theorem 13. *If $(\mathcal{N}, g, \omega, \tau, \nu, \sigma, \rho)$ is an η-RY soliton of kind (σ, ρ) and π is an RS , such that the hdH is integrable, if every fiber of π is totally geodesic and any fiber of RS is an almost η-RY soliton, then the homotopy group of RS is $\pi_n(\mathcal{B}^b)$.*

Example 4. *Let us adopt the example (3); we have Riemannian submersion ,*

$$\pi : \mathbb{R}^6 \cong \mathbb{S}^6 \to \mathbb{R}^3 \cong \mathbb{S}^3$$

defined in (3).

Then, according to Hopf-fibration of the fiber bundle, we have homotopy groups

$$\pi_6(\mathbb{S}^3) = \pi_{3+3}\mathbb{S}^3. \tag{62}$$

Therefore, the above remark entails that $r \leq k+1$ i.e., $3 \leq 3+1$. Thus, the homotopy groups $\pi_6(\mathbb{S}^3)$ are unstable homotopy groups.

Remark 6. *For a prime number p, the homotopy p-exponent of a topological space $\mathcal{T}$, denoted by $Exp_p(U)$, is defined to be a largest $e \in \mathbb{N} = \{0, 1, 2, \cdots\}$ such that some homotopy group $\Phi_j(\mathcal{T})$ has an element of order p^e. Cohen et al. [43] proved that the*

$$Exp_p(\mathcal{S}^{2n+1}) = n \quad if \quad p \neq 2.$$

For a prime number p and an integer z, the p-adic order of z is given by $Ord_p(z) = sup\{z \in \mathbb{N} : p^z | z\}$.

Through the above observation, in 2007, Davis and Sun proved an interesting inequality in terms of homotopy groups. For more details see ([44] Theorem 1.1 Page 2). According to these authors, for any prime p and $z = 2, 3, \cdots$ some homotopy group $\pi_i(SU(n))$ contains an element of order $p^{n-1+Ord_p(\lfloor n/p \rfloor !)}$, i.e., then the strong and elegant lower bound for the homotopy p-exponent of a homotopy group is

$$Exp_p(SU(n)) \geq n - 1 + Ord_p\left(\left\lfloor \frac{n}{p} \right\rfloor !\right), \tag{63}$$

where $S(U)(n)$ is a special unitary group of degree n.

Therefore, using Davis and Sun's result (Theorem 1.1 [44]) with Theorem 13, we gain an interesting inequality

Theorem 14. *For any prime number p and $s = 2, 3, \cdots$, some homotopy group $\pi_n(\mathcal{B}^b)$ of Riemannian submersion π with totally geodesic fiber where the fiber is an almost η-RY-soliton of π, contains an element of order $p^{s-1+Ord_p(\lfloor s/p \rfloor !)}$, we derive the inequality*

$$Exp_p(\pi_n(\mathcal{B}^b)) \geq n - 1 + Ord_p\left(\left\lfloor \frac{b}{p} \right\rfloor !\right). \tag{64}$$

Example 5. *Again considering the case of example (4), we have that a homotopy group of Riemannian submersion π with totally geodesic fiber is $\pi_6(\mathbb{R}^3)$. Equation (14) also holds for homotopy group $\pi_6(\mathbb{R}^3)$ of Riemannian submersion π such that*

$$Exp_p(\pi_6(\mathbb{R}^3)) \geq 2 + Ord_p\left(\left\lfloor \frac{3}{p} \right\rfloor !\right). \tag{65}$$

The geometric interpretation of the Hopf fibration can be obtained considering rotations of the 2-sphere in 3-dimensional space. Therefore, the rotation group $SO(3)$, spin group $Spin(3)$, diffeomorphic to the 3-sphere and $Spin(3)$, can be identified with the special unitary group $SU(2)$. Indeed, there are p-local equivalences

$$SO(3) \cong Spin(3) \cong SU(2).$$

Thus, in view (65), we obtain

$$Exp_p(SU(2)) \geq 2 + Ord_p\left(\left\lfloor \frac{2}{p} \right\rfloor !\right). \tag{66}$$

$$Exp_p(Spin(3)) \geq 2 + Ord_p\left(\left\lfloor \frac{3}{p} \right\rfloor !\right). \tag{67}$$

Remark 7. *Each homotopy group is the product of cyclic groups of order p. In [45] Hirsi, a useful classification of homotopy groups of spheres is provided. Again, in light of example (4) $\pi_6(\mathbb{R}^3) = \pi_{3+3}(\mathbb{R}^3) = 12 = 2^2.3 = \mathbb{Z}_{12} = \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ or $\mathbb{Z}_4 \times \mathbb{Z}_3$.*

Remark 8. *In [46], Herstien noted the following facts about any group of order type p^2q:*
1. If G is a group of order p^2q, p, q are primes, then group G has a non-trivial normal subgroup.
2. If G is a group of order p^2q, p, q are primes, then either a p-Sylow subgroup or a q-Sylow subgroup of G must be normal.

Therefore, in light of the above remarks, we can make the following remark:

Remark 9. *The order of a homotopy group $\pi_6(\mathbb{R}^3)$ of Riemanian submersion ψ can be expressed as $2^2.3$. Therefore, The homotopy group $\pi_6(\mathbb{R}^3)$ of Riemanian submersion π has a non-trivial normal* *subgroup. In addition, the homotopy group $\pi_6(\mathbb{R}^3)$ of Riemanian submersion π with a 2-Sylow subgroup or a 3-Sylow subgroup of $\pi_6(\mathbb{R}^3)$ must be normal.*

Remark 10. *In light of Remark 9, we can also find some results for the p-Sylow subgroup of the group of spin of Riemannian submersion and the unitary group of Riemannian submersion. These facts distinguish this manuscript from previously published works based on submersion.*

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Abbreviations

η-RY soliton	η-Ricci–Yamabe soliton
RS	Riemannian submersion
vdV	vertical distribution vector field
hdH	horizontal distribution vector field
PVF	potential vector field
HPVF	horizontal potential vector field
CKVF	conformal Killing vector field

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