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# Positive Solutions for Periodic Boundary Value Problems of Fractional Differential Equations with Sign-Changing Nonlinearity and Green's Function

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**Abstract:** In this paper, a class of nonlinear fractional differential equations with periodic boundary condition is investigated. Although the nonlinearity of the equation and the Green's function are sign-changing, the results of the existence and nonexistence of positive solutions are obtained by using the Schaefer's fixed-point theorem. Finally, two examples are given to illustrate the main results.

**Keywords:** fractional differential equation; sign-changing; periodic boundary condition; fixed-point theorem

**MSC:** 34A08; 34B10

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## 1. Introduction

Fractional differential equations (FDEs) have attracted great interests in the past several decades as FDEs are widely used in many fields, see [1–5]. In recent years, many papers have investigated the existence, multiplicity and non-existence of solutions for initial value problems (IVPs) or boundary value problems (BVPs) of various classes of FDEs (conformable FDEs [6], impulsive FDEs [7], coupled system of FDEs [8–10], hybrid FDEs [11–13], fractional relaxation DEs [14], variable-order FDEs [15]); also see the references therein.

BVPs with positive solutions have played a very important role in the study of mathematical physics problems; see [16–19]. There are some very recent interesting results on this topic; see [16,20–25], and the references therein. Bai and Lü [26] studied the existence of positive solutions of the BVP

$$D^{\alpha}_{0+}u(t) + f(t, u(t)) = 0, \quad t \in (0,1), \tag{1}$$

$$u(0) = u(1) = 0, \tag{2}$$

where $1 < \alpha \leq 2$, $D^{\alpha}_{0+}$ is the Riemann–Liouville fractional differentiation, $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function, and $u : [0,1] \to [0,+\infty)$ is the positive solution of (1) and (2). By using the techniques of fixed-point theorems, they obtained some existence results under the conditions that the nonlinearity $f$ and the corresponding Green's function are non-negative. Li et al. [27] considered a class of FDEs with four point boundary condition. By means of the Avery-Peterson theorem, they derived the existence result of positive solutions based on the assumption that the nonlinearity is non-negative.

To the best of our knowledge, in most of the existing studies found in the literature, the non-negative conditions of the nonlinearity or the Green's function are fundamental to obtaining the positive solutions [28]. Hence, a natural question is what would happen if the nonlinearity or the Green's function is sign-changing. Several papers have considered the positive solutions for BVPs with sign-changing nonlinearity and sign-changing Green's function [28–34]. Ma [29] studied the BVP with sign-changing Green's function:

$$u''(t) + a(t)u(t) = \lambda b(t)f(u(t)), \quad t \in (0,T), \tag{3}$$

$$u(0) = u(T), \;\; u'(0) = u'(T), \tag{4}$$

$f$, $a$ and $b$ are given functions, and $\lambda$ is a parameter. Some suitable assumptions of $f$, $a$ and $b$ are imposed, wherein they obtained the existence and nonexistence of positive solutions for the above problem.

Motivated by the above works, this paper considers the periodic BVP with sign-changing nonlinearity and Green's function:

$$({}^C D^{\alpha}_{0+}u)(t) - Mu(t) - \lambda g(t)f(u(t)) = 0, \quad t \in (0,1), \tag{5}$$

$$u(0) = u(1), \;\; u'(0) = u'(1), \tag{6}$$

where $1 < \alpha < 2$, ${}^C D^{\alpha}_{0+}$ is the Caputo fractional derivative (FD), $M > 0$ is a constant, $\lambda$ is a parameter and $g : [0,1] \to [0,\infty)$ is a continuous function, $f : [0,\infty) \to \mathbb{R}$ is a continuous function and $f(0) > 0$. In [3] (Equation (9.37)), Podlubny pointed out, with $\alpha = 1.0315$, the FDE of (5) and (6) is good at depicting the model of a re-heating furnace. The most remarkable feature of the paper is its capability to obtain the results of the existence and nonexistence of positive solutions under the conditions that the nonlinearity $f$ and the Green's function are sign-changing.

The paper is organized as follows. In Section 2, some notations and definitions of fractional calculus are introduced, and a lemma is proven. In Section 3, some useful criteria of existence and nonexistence for the BVPs of (5) and (6) are established. In Section 4, two examples are presented to illustrate the main results. Finally, a conclusion of the paper is presented.

## 2. Preliminaries

**Definition 1** ([2] (p. 69, Equation (2.1.1)). *Let $[a,b]$ be a finite interval on the real axis $\mathbb{R}$. The Riemann–Liouville fractional integral $I^{\alpha}_{a+}f$ of order $\alpha$ is defined by*

$$(I^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \;\; x > a; \; \alpha > 0. \tag{7}$$

**Definition 2** ([2] (p. 70, Equation (2.1.5))). *The Riemann–Liouville fractional derivative $D^{\alpha}_{a+}y$ of order $\alpha$ is defined by*

$$(D^{\alpha}_{a+}y)(x) = \left(\frac{d}{dx}\right)^n I^{n-\alpha}_{a+}y(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1}y(t)dt, \;\; n = [\alpha]+1; x > a, \tag{8}$$

*where $[\alpha]$ means the integral part of $\alpha$.*

**Definition 3** ([2] (pp. 90–91, Equation (2.4.1))). *The Caputo fractional derivative ${}^C D^{\alpha}_{a+}y(x)$ of order $\alpha$ on $[a,b]$ is defined via the above Riemann–Liouville fractional derivatives by*

$$({}^C D^{\alpha}_{a+}y)(x) = \left(D^{\alpha}_{a+}\left[y(t) - \sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(t-a)^k\right]\right)(x), \tag{9}$$

*where $n = [\alpha]+1$ for $\alpha \notin \mathbb{N}_0$; $n = \alpha$ for $\alpha \in \mathbb{N}_0$, $\mathbb{N}_0 = \{0,1,\cdots\}$*

**Lemma 1** ([2] (p. 230)). *The Cauchy problem*

$$({}^C D^{\alpha}_{a+}y)(x) - My(x) = f(x) \;\; (a < x < b; \; n-1 < \alpha < n; \; n \in \mathbb{N}; \; M \in \mathbb{R}; \;\; f(x) \in C[a,b]), \tag{10}$$

$$y^{(k)}(a) = b_k \;\; (b_k \in \mathbb{R}; \; k = 0,1,\cdots,n-1), \tag{11}$$

*has a unique solution*

$$y(x) = \sum_{j=0}^{n-1} b_j (x-a)^j E_{\alpha,j+1}(M(x-a)^\alpha) + \int_a^x (x-t)^{\alpha-1} E_{\alpha,\alpha}(M(x-t)^\alpha) f(t) dt, \quad (12)$$

*where* $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}$ *is the Mittag-Leffler (ML) function.*

Next, we shall prove a lemma which is very useful in proving our main results.

**Lemma 2.** *Assume that* $M > 0$ *satisfies*

$$(1 - E_{\alpha,1}(M))^2 \neq \frac{1}{\alpha} E_{\alpha,\alpha}(M) E_{\alpha,2}(M) \quad (13)$$

*Then, the BVP*

$$({}^C D_{0+}^\alpha u)(t) - Mu(t) = f(t), \;\; t \in (0,1), \;\; 1 < \alpha < 2, \;\; f(t) \in C[0,1], \quad (14)$$

$$u(0) = u(1), \;\; u'(0) = u'(1), \quad (15)$$

*has a unique solution*

$$u(t) = \int_0^1 G(t,s) f(s) ds, t \in [0,1], \quad (16)$$

*where*

$$G(t,s) = \begin{cases} \frac{(1-E_{\alpha,1}(M))E_{\alpha,1}(Mt^\alpha) + \frac{t}{\alpha}E_{\alpha,\alpha}(M)E_{\alpha,2}(Mt^\alpha)}{F(M)}(1-s)^{\alpha-1}E_{\alpha,\alpha}(M(1-s)^\alpha) \\ + \frac{E_{\alpha,2}(M)E_{\alpha,1}(Mt^\alpha) + t(1-E_{\alpha,1}(M))E_{\alpha,2}(Mt^\alpha)}{F(M)}(1-s)^{\alpha-2}E_{\alpha,\alpha-1}(M(1-s)^\alpha) \\ + (t-s)^{\alpha-1}E_{\alpha,\alpha}(M(t-s)^\alpha), & s \le t, \\ \frac{(1-E_{\alpha,1}(M))E_{\alpha,1}(Mt^\alpha) + \frac{t}{\alpha}E_{\alpha,\alpha}(M)E_{\alpha,2}(Mt^\alpha)}{F(M)}(1-s)^{\alpha-1}E_{\alpha,\alpha}(M(1-s)^\alpha) \\ + \frac{E_{\alpha,2}(M)E_{\alpha,1}(Mt^\alpha) + t(1-E_{\alpha,1}(M))E_{\alpha,2}(Mt^\alpha)}{F(M)}(1-s)^{\alpha-2}E_{\alpha,\alpha-1}(M(1-s)^\alpha), & t < s, \end{cases} \quad (17)$$

*and*

$$F(M) = (1 - E_{\alpha,1}(M))^2 - \frac{1}{\alpha} E_{\alpha,\alpha}(M) E_{\alpha,2}(M). \quad (18)$$

**Proof.** By Lemma 1, we can obtain the solution for the problem of (14), subject to the following initial conditions:

$$u(0) = b_0, \;\; u'(0) = b_1 \quad (19)$$

is

$$u(t) = b_0 E_{\alpha,1}(Mt^\alpha) + b_1 t E_{\alpha,2}(Mt^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(M(t-s)^\alpha) f(s) ds. \quad (20)$$

Using the properties of the ML function (see p. 42 of [2]):

$$\frac{d}{dt}(t^{\beta-1} E_{\alpha,\beta}(Mt^\alpha)) = t^{\beta-2} E_{\alpha,\beta-1}(Mt^\alpha), \;\; \beta = 2, \alpha, \quad (21)$$

$$\frac{d}{dt}(E_{\alpha,1}(Mt^\alpha)) = E^2_{\alpha,1+\alpha}(Mt^\alpha) = \frac{1}{\alpha} E_{\alpha,\alpha}(Mt^\alpha), \quad (22)$$

we have

$$u'(t) = b_0 \frac{1}{\alpha} E_{\alpha,\alpha}(Mt^{\alpha}) + b_1 E_{\alpha,1}(Mt^{\alpha}) + \int_0^t (t-s)^{\alpha-2} E_{\alpha,\alpha-1}(M(t-s)^{\alpha}) f(s) ds. \tag{23}$$

From (15), (20) and (23), it implies that:

$$u(1) = b_0 = b_0 E_{\alpha,1}(M) + b_1 E_{\alpha,2}(M) + \int_0^1 (1-s)^{\alpha-1} E_{\alpha,\alpha}(M(1-s)^{\alpha}) f(s) ds, \tag{24}$$

$$u'(1) = b_1 = b_0 \frac{1}{\alpha} E_{\alpha,\alpha}(M) + b_1 E_{\alpha,1}(M) + \int_0^1 (1-s)^{\alpha-2} E_{\alpha,\alpha-1}(M(1-s)^{\alpha}) f(s) ds. \tag{25}$$

Since (13) holds, it implies $F(M) \neq 0$. Thus:

$$\begin{aligned} u(t) &= \frac{(1-E_{\alpha,1}(M))E_{\alpha,1}(Mt^{\alpha}) + \frac{t}{\alpha}E_{\alpha,\alpha}(M)E_{\alpha,2}(Mt^{\alpha})}{F(M)} \int_0^1 (1-s)^{\alpha-1} E_{\alpha,\alpha}(M(1-s)^{\alpha}) f(s) ds \\ &+ \frac{E_{\alpha,2}(M)E_{\alpha,1}(Mt^{\alpha}) + t(1-E_{\alpha,1}(M))E_{\alpha,2}(Mt^{\alpha})}{F(M)} \int_0^1 (1-s)^{\alpha-2} E_{\alpha,\alpha-1}(M(1-s)^{\alpha}) f(s) ds \\ &+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(M(t-s)^{\alpha}) f(s) ds \\ &= \int_0^1 G(t,s) f(s) ds. \end{aligned} \tag{26}$$

□

**Remark 1.** *If $f(\cdot) \in \mathbb{C}[0,1]$, then the improper integral in Lemma 2 is:*

$$\int_0^1 (1-s)^{\alpha-2} E_{\alpha,\alpha-1}(M(1-s)^{\alpha}) f(s) ds < \infty. \tag{27}$$

## 3. Main Results

**Lemma 3.** *Let*

$$E_{\alpha,1}(M) > E_{\alpha,2}(M) + 1, E_{\alpha,1}(M) > \frac{1}{\alpha} E_{\alpha,\alpha}(M) + 1 \tag{28}$$

*holds. Suppose that*

*(i) $h : \mathbb{R} \to \mathbb{R}$ is a continuous function and $|h(\cdot)| \leq N$ for some constant $N > 0$.*
*(ii) $g : [0,1] \to [0,\infty)$ is a continuous function.*

*Then, for every $\lambda \in \mathbb{R}$, the BVP*

$$({}^C D_{0+}^{\alpha} u)(t) - Mu(t) - \lambda g(t) h(u(t)) = 0, \quad t \in (0,1), \tag{29}$$

$$u(0) = u(1), \quad u'(0) = u'(1), \tag{30}$$

*has a solution $u_\lambda \in \mathbb{X}$, where $\mathbb{X}$ is the Banach space $\mathbb{C}[0,1]$ with the norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$.*

**Proof.** Consider the operator $\Lambda_\lambda : \mathbb{X} \to \mathbb{X}$ defined by:

$$\Lambda_\lambda u(t) = \lambda \int_0^1 G(t,s) g(s) h(u(s)) ds, \quad t \in [0,1]. \tag{31}$$

From Lemma 2, we can obtain and determine that the solutions of the BVPs (29) and (30) are fixed points of $\Lambda_\lambda$. Next, we will prove that all the fixed points of $\Lambda_\lambda$ are solutions of the BVPs (29) and (30). In fact, Let $u(t) = \Lambda_\lambda u(t)$. Then

$$u(t) = b_0 E_{\alpha,1}(Mt^\alpha) + b_1 t E_{\alpha,2}(Mt^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(M(t-s)^\alpha)\lambda g(s) h(u(s)) ds, \quad (32)$$

where

$$b_0 = \frac{(1-E_{\alpha,1}(M)) \int_0^1 (1-s)^{\alpha-1} E_{\alpha,\alpha}(M(1-s)^\alpha)\lambda g(s)h(u(s))ds + E_{\alpha,2}(M)\int_0^1 (1-s)^{\alpha-2} E_{\alpha,\alpha-1}(M(1-s)^\alpha)\lambda g(s)h(u(s))ds}{F(M)}, \quad (33)$$

$$b_1 = \frac{\frac{1}{\alpha} E_{\alpha,\alpha}(M) \int_0^1 (1-s)^{\alpha-1} E_{\alpha,\alpha}(M(1-s)^\alpha)\lambda g(s)h(u(s))ds + (1-E_{\alpha,1}(M))\int_0^1 (1-s)^{\alpha-2} E_{\alpha,\alpha-1}(M(1-s)^\alpha)\lambda g(s)h(u(s))ds}{F(M)}. \quad (34)$$

Hence, from Lemma 1, we know that $u(t)$ satisfies the problem of (29), subject to the following conditions:

$$u(0) = b_0, \quad u'(0) = b_1. \quad (35)$$

Moreover, through (32)–(34), together with the properties of the ML function (21) and (22), we can obtain $u(t)$, which satisfies (30). Thus, $u(t)$ is a solution of the BVPs (29) and (30).

Next, we use the Schaefer's fixed-point theorem to consider the fixed points of $\Lambda_\lambda$. Here, (a) we will prove that $\Lambda_\lambda$ is a continuous operator. Denote $\{u_n\}$ to be a sequence, which satisfy $u_n \to u$,

$$\begin{aligned} |\Lambda_\lambda u_n(t) - \Lambda_\lambda u(t)| &\le |\lambda| \int_0^1 |G(t,s)| g(s) |h(u_n(s)) - h(u(s))| ds \\ &\le |\lambda| \int_0^1 \frac{2E_{\alpha,1}(M)-1}{(\alpha-1)F(M)} \Big( (E_{\alpha,1}(M)-1)(1-s) + E_{\alpha,2}(M)(\alpha-1) \Big) \\ &\quad \cdot (1-s)^{\alpha-2} E_{\alpha,\alpha-1}(M(1-s)^\alpha) g(s) |h(u_n(s)) - h(u(s))| ds \\ &\le |\lambda| \cdot \frac{2E_{\alpha,1}(M)-1}{(\alpha-1)F(M)} (E_{\alpha,1}(M) - 1 + E_{\alpha,2}(M)) \\ &\quad \cdot \int_0^1 (1-s)^{\alpha-2} E_{\alpha,\alpha-1}(M(1-s)^\alpha) g(s) |h(u_n(s)) - h(u(s))| ds. \end{aligned} \quad (36)$$

From the definition of the ML function, it achieves

$$\int_0^1 (1-s)^{\alpha-2} E_{\alpha,\alpha-1}(M(1-s)^\alpha) ds = \int_0^1 (1-s)^{\alpha-2} \sum_{k=0}^{\infty} \frac{M^k (1-s)^{\alpha k}}{\Gamma(\alpha k + \alpha - 1)} ds = E_{\alpha,\alpha}(M) \quad (37)$$

is bounded. Note that $h$ and $g$ are both continuous, and so we obtain

$$\|\Lambda_\lambda u_n - \Lambda_\lambda u\| \to 0, \qquad n \to \infty. \quad (38)$$

Thus, $\Lambda_\lambda$ is a continuous operator.

(b) We shall show that $\Lambda_\lambda$ is uniformly bounded in $\mathbb{X}$. For each $u \in \mathbb{X}$,

$$\begin{aligned} |\Lambda_\lambda u| &\le |\lambda| \int_0^1 |G(t,s)| g(s) |h(u(s))| ds \\ &\le |\lambda| \cdot \frac{(2E_{\alpha,1}(M)-1)(E_{\alpha,1}(M)-1+E_{\alpha,2}(M))}{(\alpha-1)F(M)} \|g\| E_{\alpha,\alpha}(M) N \end{aligned} \quad (39)$$

This implies that $\Lambda_\lambda$ is uniformly bounded.

(c) We will verify that $\Lambda_\lambda$ is equicontinuous in $\mathbb{X}$. For each $t_1, t_2 \in [0,1]$, $t_1 < t_2$:

$$\begin{aligned}
&|\Lambda_\lambda u(t_2) - \Lambda_\lambda u(t_1)| \\
&\leq |\lambda \int_0^1 G(t_2,s)g(s)h(u(s))ds - \lambda \int_0^1 G(t_1,s)g(s)h(u(s))ds| \\
&\leq |\lambda| \tfrac{(E_{\alpha,1}(M)-1)(E_{\alpha,1}(Mt_2^\alpha)-E_{\alpha,1}(Mt_1^\alpha))+\frac{1}{\alpha}E_{\alpha,\alpha}(M)(t_2E_{\alpha,2}(Mt_2^\alpha)-t_1E_{\alpha,2}(Mt_1^\alpha))}{F(M)} \\
&\quad \cdot \int_0^1 (1-s)^{\alpha-1}E_{\alpha,\alpha}(M(1-s)^\alpha)g(s)h(u(s))ds \\
&\quad +|\lambda| \tfrac{E_{\alpha,2}(M)(E_{\alpha,1}(Mt_2^\alpha)-E_{\alpha,1}(Mt_1^\alpha))+(E_{\alpha,1}(M)-1)(t_2E_{\alpha,2}(Mt_2^\alpha)-t_1E_{\alpha,2}(Mt_1^\alpha))}{F(M)} \\
&\quad \cdot \int_0^1 (1-s)^{\alpha-2}E_{\alpha,\alpha-1}(M(1-s)^\alpha)g(s)h(u(s))ds \\
&\quad +|\lambda| \int_0^{t_2} (t_2-s)^{\alpha-1}E_{\alpha,\alpha}(M(t_2-s)^\alpha)g(s)h(u(s))ds \\
&\quad -|\lambda| \int_0^{t_1} (t_1-s)^{\alpha-1}E_{\alpha,\alpha}(M(t_1-s)^\alpha)g(s)h(u(s))ds.
\end{aligned} \tag{40}$$

Note that

$$\begin{aligned}
&\int_0^{t_2} (t_2-s)^{\alpha-1}E_{\alpha,\alpha}(M(t_2-s)^\alpha)ds - \int_0^{t_1} (t_1-s)^{\alpha-1}E_{\alpha,\alpha}(M(t_1-s)^\alpha)ds \\
&= \int_0^{t_2} \sum_{k=0}^{\infty} \tfrac{M^k(t_2-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)}ds - \int_0^{t_1} \sum_{k=0}^{\infty} \tfrac{M^k(t_1-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)}ds \\
&= \sum_{k=0}^{\infty} \tfrac{M^k}{\Gamma(\alpha k+\alpha)} \int_0^{t_2} (t_2-s)^{\alpha k+\alpha-1}ds - \sum_{k=0}^{\infty} \tfrac{M^k}{\Gamma(\alpha k+\alpha)} \int_0^{t_1} (t_1-s)^{\alpha k+\alpha-1}ds \\
&= \sum_{k=0}^{\infty} \tfrac{M^k}{\Gamma(\alpha k+\alpha+1)} t_2^{\alpha k+\alpha} - \sum_{k=0}^{\infty} \tfrac{M^k}{\Gamma(\alpha k+\alpha+1)} t_1^{\alpha k+\alpha} \\
&= t_2^\alpha E_{\alpha,\alpha+1}(Mt_2^\alpha) - t_1^\alpha E_{\alpha,\alpha+1}(Mt_1^\alpha).
\end{aligned} \tag{41}$$

Therefore, the right hand side of (40) $\to 0$ as $t_1 \to t_2$. Then, $\Lambda_\lambda$ is equicontinuous in $\mathbb{X}$. Due to (a), (b), (c) and the Arzela–Ascoli theorem, we can determine that $\Lambda_\lambda$ is completely continuous.

(d) It remains to show that the set $\Omega = \{u \in \mathbb{X} | u = \mu\Lambda_\lambda u,\ 0 < \mu < 1\}$ is bounded. Let $u \in \Omega$. Then, $u = \mu\Lambda_\lambda u,\ 0 < \mu < 1$. For each $t \in [0,1]$, we have

$$|u(t)| = |\mu\Lambda_\lambda u(t)| \leq |\lambda| \frac{(2E_{\alpha,1}(M)-1)(E_{\alpha,1}(M)-1+E_{\alpha,2}(M))}{(\alpha-1)F(M)} \|g\| E_{\alpha,\alpha}(M)N. \tag{42}$$

Hence, $\Omega$ is bounded. Through the Schaefer's fixed-point theorem, we can discern that $\Lambda_\lambda$ has a fixed point. □

**Remark 2.** *The function $G(\cdot,\cdot)$ defined by (17) may change sign on $(0,1)\times(0,1)$.*

In fact, for $s \leq t$:

$$\begin{aligned}
G(t,s) &= \tfrac{(1-E_{\alpha,1}(M))E_{\alpha,1}(Mt^\alpha)+\frac{t}{\alpha}E_{\alpha,\alpha}(M)E_{\alpha,2}(Mt^\alpha)}{F(M)}(1-s)^{\alpha-1}E_{\alpha,\alpha}(M(1-s)^\alpha) \\
&\quad + \tfrac{E_{\alpha,2}(M)E_{\alpha,1}(Mt^\alpha)+t(1-E_{\alpha,1}(M))E_{\alpha,2}(Mt^\alpha)}{F(M)}(1-s)^{\alpha-2}E_{\alpha,\alpha-1}(M(1-s)^\alpha) \\
&\quad +(t-s)^{\alpha-1}E_{\alpha,\alpha}(M(t-s)^\alpha).
\end{aligned}$$

Note that

$$E_{\alpha,\alpha-1}(M(1-s)^{\alpha}) = \sum_{k=0}^{\infty} \frac{M^k(1-s)^{\alpha k}}{\Gamma(\alpha k+\alpha-1)} \geq (\alpha-1)\sum_{k=0}^{\infty} \frac{M^k(1-s)^{\alpha k}}{\Gamma(\alpha k+\alpha)} = (\alpha-1)E_{\alpha,\alpha}(M(1-s)^{\alpha}), \tag{43}$$

we have

$$G(0,0) \leq \frac{(1-E_{\alpha,1}(M)) + (\alpha-1)E_{\alpha,2}(M)}{(\alpha-1)F(M)} E_{\alpha,\alpha-1}(M) \leq 0, \tag{44}$$

$$\begin{aligned} G(1,s) &= \tfrac{(1-E_{\alpha,1}(M))E_{\alpha,1}(M)+\frac{1}{\alpha}E_{\alpha,\alpha}(M)E_{\alpha,2}(M)}{F(M)}(1-s)^{\alpha-1}E_{\alpha,\alpha}(M(1-s)^{\alpha}) \\ &\quad + \tfrac{E_{\alpha,2}(M)E_{\alpha,1}(M)+(1-E_{\alpha,1}(M))E_{\alpha,2}(M)}{F(M)}(1-s)^{\alpha-2}E_{\alpha,\alpha-1}(M(1-s)^{\alpha}) \\ &\quad + (1-s)^{\alpha-1}E_{\alpha,\alpha}(M(1-s)^{\alpha}) \\ &\geq \tfrac{(1-E_{\alpha,1}(M))(1-s)+(\alpha-1)E_{\alpha,2}(M)}{F(M)}(1-s)^{\alpha-2}E_{\alpha,\alpha}(M(1-s)^{\alpha}). \end{aligned} \tag{45}$$

Therefore, $G(1,s) \geq 0$ for $s \geq 1 - \frac{(\alpha-1)E_{\alpha,2}(M)}{E_{\alpha,1}(M)-1}$. Thus, we can determine that $G(t,s)$ change sign on $(0,1)\times(0,1)$.

In the following, we denote $G^+(t,s) = \max\{G(t,s),0\}, t,s \in [0,1]$ as the positive parts of $G$, and denote $G^-(t,s) = \max\{-G(t,s),0\}, t,s \in [0,1]$ as the negative parts of $G$, where $G$ is Green's function of the BVPs (5) and (6).

**Theorem 1.** *Let* (28) *hold. Assume that $g$ satisfies*

*(A1)* $\min\{\int_0^1 G^-(t,s)g(s)ds \mid t \in (0,1)\} > 0$;

*(A2) There exists $\varepsilon > 0$, such that*

$$\int_0^1 (G^+(t,s) - (1+\varepsilon)G^-(t,s))g(s)ds > 0, \quad t \in [0,1].$$

*Hence, there exists a constant $\lambda_0 > 0$, for $\lambda \in (0,\lambda_0)$, and the BVPs* (5)–(6) *have a positive solution.*

**Proof.** Let $K > 0$ and define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(u) = \begin{cases} f(0), & u \leq 0, \\ f(u), & 0 < u \leq K, \\ f(K), & K < u. \end{cases} \tag{46}$$

Then, $|h(u)| \leq N = \max\limits_{0\leq u\leq K} f(u)$ is bounded. Through Lemma 3, the problem (29) and (30) has a solution $u_\lambda \in \mathbb{X}$.

Let $\varkappa > 0$. Then, by the continuity of $h$, we can deduce that there exists a $\sigma \in (0,K)$, and

$$h(0) - h(0)\varkappa < h(u) < h(0) + h(0)\varkappa, \quad |u| < \sigma. \tag{47}$$

From (39),

$$\begin{aligned} |u_\lambda(t)| &\leq |\lambda| \textstyle\int_0^1 |G(t,s)| g(s) |h(u_\lambda(s))| ds \\ &\leq |\lambda| \cdot \tfrac{(2E_{\alpha,1}(M)-1)(E_{\alpha,1}(M)-1+E_{\alpha,2}(M))}{(\alpha-1)F(M)} \|g\| E_{\alpha,\alpha}(M) N, \end{aligned} \tag{48}$$

it follows that there exists

$$\lambda_0 = \frac{(\alpha - 1)F(M)\sigma}{(2E_{\alpha,1}(M) - 1)(E_{\alpha,1}(M) - 1 + E_{\alpha,2}(M))\|g\|E_{\alpha,\alpha}(M)N} > 0 \tag{49}$$

such that for $\lambda \in (0, \lambda_0)$, we have $\|u_\lambda\| \le \sigma$, and

$$\begin{aligned} u_\lambda(t) &= \lambda \int_0^1 G(t,s)g(s)h(u_\lambda(s))ds \\ &= \lambda \int_0^1 (G^+(t,s) - G^-(t,s))g(s)h(u_\lambda(s))ds \\ &> \lambda \int_0^1 G^+(t,s)g(s)(h(0) - h(0)\varkappa)ds - \lambda \int_0^1 G^-(t,s)g(s)(h(0) + h(0)\varkappa)ds \\ &= \lambda h(0)(1-\varkappa) \int_0^1 (G^+(t,s)g(s) - \tfrac{1+\varkappa}{1-\varkappa} G^-(t,s)g(s))ds \\ &= \lambda h(0)(1-\varkappa) \int_0^1 (G^+(t,s)g(s) - (1+\varepsilon)G^-(t,s)g(s))ds \\ &\quad + \lambda h(0)(1-\varkappa) \int_0^1 ((1+\varepsilon)G^-(t,s)g(s) - \tfrac{1+\varkappa}{1-\varkappa} G^-(t,s)g(s))ds \\ &> \lambda h(0)(1-\varkappa) \int_0^1 G^-(t,s)g(s)ds \Big( (1+\varepsilon) - \tfrac{1+\varkappa}{1-\varkappa} \Big) > 0. \end{aligned} \tag{50}$$

Consequently, $0 < u_\lambda \le K$, for $t \in [0,1]$. Therefore, the BVPs (5) and (6) have a positive solution. □

Denote

$$\beta(t) = \int_0^1 G(t,s)g(s)ds, \quad \beta_1(t) = \int_0^1 G(t,s)g(s)\beta(s)ds, t \in [0,1]. \tag{51}$$

**Theorem 2.** *Let* (28) *and* $(A1)$ *hold. Furthermore, assume* $f$ *is bounded and* $f$ *is* $C^2$ *in some neighborhood of* 0*, and:*

*(A3) There exits* $t_0 \in [0,1]$ *such that* $\beta(t_0) = 0$*.*

*(A4)* $\beta_1(t_0)f'(0) < 0$*.*

*Then, the BVPs* (5) *and* (6) *have no positive solutions for* $\lambda \to 0^+$*.*

**Proof.** As $f$ is bounded, the BVPs (5) and (6) have a solution $u_\lambda(t)$ via Lemma 3. Let $u_\lambda(t) = \lambda\varrho(t)$. Then, $\varrho(t)$ satisfies

$$({}^C D_{0+}^\alpha \varrho)(t) - M\varrho(t) - g(t)f(\lambda\varrho(t)) = 0, \quad t \in (0,1), \tag{52}$$

$$\varrho(0) = \varrho(1), \quad \varrho'(0) = \varrho'(1), \tag{53}$$

and $\varrho(t) = \int_0^1 G(t,s)g(s)f(\lambda\varrho(s))ds$. Through the Lebesgue dominated convergence theorem, it implies that

$$\varrho(t) \to f(0)\beta(t), \quad \lambda \to 0^+. \tag{54}$$

First, we consider that there exists a constant $t^* \in [0,1]$, and $\beta(t^*) < 0$. Thus, $u_\lambda(t^*) = \lambda\varrho(t^*) < 0$, $\lambda \to 0^+$.

Next, we consider $\beta(t) \geq 0,\ t \in [0,1]$. Since (A3), (A4) and $f$ are continuous in 0, we have

$$\begin{aligned}
\varrho(t_0) &= \int_0^1 G(t_0,s)g(s)f(\lambda\varrho(s))ds \\
&= \int_0^1 G(t_0,s)g(s)\Big(f(0)+\lambda f'(0)\varrho(s)+\tfrac{\lambda^2 f''(\xi)}{2}\varrho^2(s)\Big)ds \\
&= f(0)\beta(t_0)+\lambda f'(0)\int_0^1 G(t_0,s)g(s)\varrho(s)ds+\tfrac{\lambda^2 f''(\xi)}{2}\int_0^1 G(t_0,s)g(s)\varrho^2(s)ds \\
&= \lambda f'(0)\int_0^1 G(t_0,s)g(s)\varrho(s)ds+\tfrac{\lambda^2 f''(\xi)}{2}\int_0^1 G(t_0,s)g(s)\varrho^2(s)ds, \quad \xi>0,
\end{aligned} \tag{55}$$

and it implies that

$$\frac{\varrho(t_0)}{\lambda} \to f'(0)\int_0^1 G(t_0,s)g(s)f(0)\beta(s)ds = f(0)f'(0)\beta_1(t_0) < 0, \ \text{for} \ \lambda \to 0^+. \tag{56}$$

Thus, $u_\lambda(t_0) = \lambda\varrho(t_0) < 0$, $\lambda \to 0^+$.

Therefore, the BVPs (5) and (6) have no positive solutions for $\lambda \to 0^+$. □

## 4. Examples

**Example 1.** *Consider*

$$({}^C D_{0+}^{1.5}u)(t) - 2u(t) - \lambda(\sin u(t) + 1) = 0, \quad t \in (0,1), \tag{57}$$

$$u(0) = u(1), \ \ u'(0) = u'(1), \tag{58}$$

*with $\lambda$ as a parameter, $M = 2$, $\alpha = 1.5$, $g(t) = 1$ and $f(u(t)) = \sin u(t) + 1$. Then, $g$ and $f$ are continuous functions and $g(t) > 0, t \in [0,1]$, $f(0) = 1 > 0$.*

Through computing, we have

$$E_{1.5,1}(2) = 3.3487, \ \ E_{1.5,2}(2) = 1.7997, \ \ E_{1.5,1.5}(2) = 2.5483, \tag{59}$$

$$F(2) = (1 - E_{1.5,1}(2))^2 - \frac{2}{3}E_{1.5,1.5}(2)E_{1.5,2}(2) = 2.4589 > 0, \tag{60}$$

and

$$E_{1.5,1}(2) > E_{1.5,2}(2) + 1, \ \ E_{1.5,1}(2) > \frac{2}{3}E_{1.5,1.5}(2) + 1. \tag{61}$$

Then, (28) and (A1) are satisfied, and

$$G(t,s) = \begin{cases}
\frac{(-2.3487)E_{1.5,1}(2t^{1.5})+\frac{2t}{3}\times 2.5483E_{1.5,2}(2t^{1.5})}{2.4589}(1-s)^{0.5}E_{1.5,1.5}(2(1-s)^{1.5}) \\
+\frac{1.7997E_{1.5,1}(2t^{1.5})+t(-2.3487)E_{1.5,2}(2t^{1.5})}{2.4589}(1-s)^{-0.5}E_{1.5,0.5}(2(1-s)^{1.5}) \\
+(t-s)^{0.5}E_{1.5,1.5}(2(t-s)^{1.5}), & s \leq t, \\
\frac{(-2.3487)E_{1.5,1}(2t^{1.5})+\frac{2t}{3}\times 2.5483E_{1.5,2}(2t^{1.5})}{2.4589}(1-s)^{0.5}E_{1.5,1.5}(2(1-s)^{1.5}) \\
+\frac{1.7997E_{1.5,1}(2t^{1.5})+t(-2.3487)E_{1.5,2}(2t^{1.5})}{2.4589}(1-s)^{-0.5}E_{1.5,0.5}(2(1-s)^{1.5}), & t < s.
\end{cases} \tag{62}$$

Let

$$\beta(t) = \int_0^1 G(t,s)ds = 0.743E_{1.5,1}(2t^{1.5}) - 1.624tE_{1.5,2}(2t^{1.5}) + t^{1.5}E_{1.5,2.5}(2t^{1.5}), \quad t \in [0,1]. \tag{63}$$

From Figure 1, we can obtain $\beta(t) > 0$. It implies that there exists $\varepsilon > 0$, and (A2) holds. Thus, all conditions of Theorem 1 are satisfied.

Let $K = \frac{\pi}{2} > 0$. From Theorem 1, we have

$$h(u) = \begin{cases} 1, & u \leq 0, \\ \sin u + 1, & 0 < u \leq \frac{\pi}{2}, \\ 2, & \frac{\pi}{2} < u. \end{cases}$$

Then, $|h(u)| \leq N = 2$. Let $\varkappa = 0.01$. Through (47), we can choose $\sigma = 0.005$. Thus, there exists a constant $\lambda_0 = 5.1 \times 10^{-5}$ defined by (49), and the BVPs (57) and (58) have a positive solution for $\lambda \in (0, \lambda_0)$.

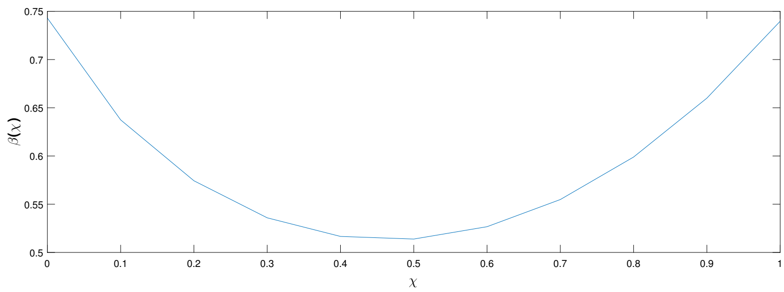


**Figure 1.** Image of $\beta(t)$ in Example 4.1.

**Example 2.** *Consider*

$$({}^{C}D_{0+}^{1.5}u)(t) - 5u(t) - \lambda f(u(t)) = 0, \quad t \in (0,1), \tag{64}$$

$$u(0) = u(1), \;\; u'(0) = u'(1), \tag{65}$$

*with $\lambda$ as a parameter, $M = 5$, $\alpha = 1.5$, $g(t) = 1$, $f : [0,\infty) \to \mathbb{R}$ is a continuous function and $f(0) > 0$.*

By computing, we have

$$E_{1.5,1}(5) = 12.4573, \;\; E_{1.5,2}(5) = 4.1355, \;\; E_{1.5,1.5}(5) = 7.2468, \tag{66}$$

$$\digamma(5) = (1 - E_{1.5,1}(5))^2 - \frac{2}{3}E_{1.5,1.5}(5)E_{1.5,2}(5) = 111.2903 > 0, \tag{67}$$

and

$$E_{1.5,1}(5) > E_{1.5,2}(5) + 1, \;\; E_{1.5,1}(5) > \frac{2}{3}E_{1.5,1.5}(5) + 1. \tag{68}$$

Then, (28) and (A1) are satisfied, and

$$G(t,s)=\begin{cases}\frac{(-11.4573)E_{1.5,1}(5t^{1.5})+\frac{2t}{3}\times 7.2468E_{1.5,2}(5t^{1.5})}{111.2903}(1-s)^{0.5}E_{1.5,1.5}(5(1-s)^{1.5})\\ +\frac{4.1355E_{1.5,1}(5t^{1.5})+t(-11.4573)E_{1.5,2}(5t^{1.5})}{111.2903}(1-s)^{-0.5}E_{1.5,0.5}(5(1-s)^{1.5})\\ +(t-s)^{0.5}E_{1.5,1.5}(5(t-s)^{1.5}), & s\le t,\\ \frac{(-11.4573)E_{1.5,1}(5t^{1.5})+\frac{2t}{3}\times 7.2468E_{1.5,2}(5t^{1.5})}{111.2903}(1-s)^{0.5}E_{1.5,1.5}(5(1-s)^{1.5})\\ +\frac{4.1355E_{1.5,1}(5t^{1.5})+t(-11.4573)E_{1.5,2}(5t^{1.5})}{111.2903}(1-s)^{-0.5}E_{1.5,0.5}(5(1-s)^{1.5}), & t<s.\end{cases} \quad (69)$$

From Theorem 2, it results in

$$\beta(t)=\int_0^1 G(t,s)ds=0.0338E_{1.5,1}(5t^{1.5})-0.6462tE_{1.5,2}(5t^{1.5})+t^{1.5}E_{1.5,2.5}(5t^{1.5}),\quad t\in[0,1]. \quad (70)$$

It is easy to achieve $\beta(0)=0.0338$ and $\beta(0.1)=-0.0052$. As $\beta(t)$ is continuous with respect to $t$, we can conclude that there exists $t_0\in(0,0.1)\subseteq[0,1]$, such that $\beta(t_0)=0$. Via MATLAb, we know that $t_0=0.082333631804161$. Thus, (A3) is satisfied. Figure 2 is the visual representation of $\beta(t)$. In fact, there is another $t_0=0.884554959489226\in[0,1]$, such that $\beta(t_0)=0$.

Since

$$\beta_1(t_0)=\int_0^1 G(t_0,s)\beta(s)ds, \quad (71)$$

we take $f(u)=-\sin u+1$ if $\beta_1(t_0)>0$, and take $f(u)=\sin u+1$ if $\beta_1(t_0)<0$. Then $f$ is bounded and $f$ is $C^2$ in some neighborhood of 0. Hence, (A4) is satisfied.

Thus, all conditions of Theorem 2 are satisfied. Consequently, the BVPs (64) and (65) have no positive solutions for $\lambda\to 0^+$.

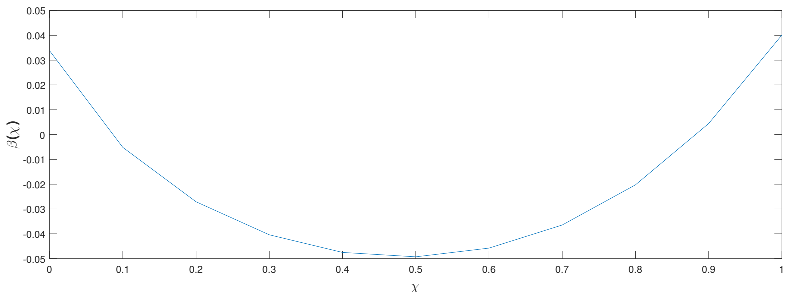


**Figure 2.** Image of $\beta(t)$ in Example 4.2.

## 5. Conclusions

In this paper, the existence and nonexistence of the positive solutions of periodic boundary conditions for FDEs are studied. The most remarkable feature of the paper is that the main results are obtained under the conditions that the nonlinearity $f$ and the Green's function are sign-changing. Some sufficient conditions are established to ensure the existence of positive solutions for small values of $\lambda$. The paper also provides some sufficient conditions to determine ranges of $\lambda$ for which no positive solution exists. At the foundation of this paper, one can consider the positive solutions for FDEs involving a p-Laplacian operator, and can also conduct further research on eigenvalue problems of FDEs.

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