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Boundary Controlling Synchronization and Passivity Analysis for Multi-Variable Discrete Stochastic Inertial Neural Networks

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Abstract: The current paper considers discrete stochastic inertial neural networks (SINNs) with reaction diffusions. Firstly, we give the difference form of SINNs with reaction diffusions. Secondly, stochastic synchronization and passivity-based control frames of discrete time and space SINNs are newly formulated. Thirdly, by designing a boundary controller and constructing a Lyapunov-Krasovskii functional, we address decision theorems for stochastic synchronization and passivity-based control for the aforementioned discrete SINNs. Finally, to illustrate our main results, a numerical illustration is provided.

Keywords: coupled networks; passivity-based control; stochastic synchronization; discrete spatial diffusion

MSC: 34D06, 68T07

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1. Introduction

Neural networks (NNs) can be considered as complicated nonlinear models coupled with numerous internal nodes, and they are capable of offering an effective approach to solving many difficult tasks in the fields of engineering. Due to their huge potential in real-world applications, they have become a significant research topic over the last few decades and have garnered increasing interest in many areas of technology (please refer to refs. [1–7]). On the other hand, it is necessary to address practical problems by studying the dynamic properties of non-linear neural networks not only in the over-damped case but also under weakly damped conditions [8]. Hence, inertial neural networks (INNs), which can act as second-order differential systems, have been extensively studied. Additionally, numerous publications have addressed synchronization problems, including finite-time synchronization [9], nonfragile H_∞ synchronization [10], event-triggered impulsive synchronization [11], fuzzy synchronization [12], Mittag-Leffler synchronization [13], and others.

Passivity, as a specific form of dissipativity, constitutes a fundamental characteristic of physical problems. A system is considered passive when dissipative elements are present in the modeled system, and the accumulated energies remain lower than the external input over a certain time span. Consequently, passivity ensures internal stability of the systems. Due to its widespread applicability in mechanical and electrical systems, the concept of passivity has garnered increasing attention, leading to extensive studies on the passivity of nonlinear systems. In the literature [14], Zhou et al. discussed passivity-based boundary control for stochastic delay reaction-diffusion systems with boundary input-output. Padmaja and Balasubramaniam [15] analyzed passivity-based stability in fractional-order delayed gene regulatory networks. By leveraging Lyapunov-Krasovskii functionals, novel linear matrix inequality conditions were developed to guarantee certain levels of passivity performance in the networks. For further details on this topic, please consult the references [16–18].

Widely, NNs were implemented through IC in engineering applications; spatial diffusions invariably occur when electronic motion takes place in an inhomogeneous electromagnetic domain. Therefore, it is important to consider NNs that incorporate the impact of spatial diffusions. In recent years, greater attention has been devoted to NNs with spatial diffusions; please refer to papers [19–24]. Stochastic neural networks have received substantial attention in our everyday reality. Typically, actions of random networks are heavily time- and space-dependent. As a result, reaction diffusion must be taken into account. Relevant research topics are discussed in references [14,19,20,22,25,26], etc. While there have been reports on space-time discrete models [27–29] to date, the problems of synchronization and passivity-based control for discrete-time SINNs involving diffusions have not been explored.

It is well known that discrete systems,(DSs) can be utilized to simulate a wide range of phenomena, including biological dynamics and artificial NNs, among others. In many scenarios, it has been demonstrated that DSs outperform continuous systems. As a result, the theory of DSs holds significant importance; please refer to references [30–38]. Reports [35–38] have explored various types of discrete INNs. However, they have not focused on the effects of other variables, such as spatial variables. Addressing this gap, the present paper investigates the issues of stochastic synchronization and passivity-based control for time and space discrete SINNs by designing a novel boundary controller.

Our main contributions include the following:

(1) Establishment of a discrete space and time SINNs model, which complements the continuous cases in literature [22–24] and the discrete-time cases in literature [35–38].
(2) Unlike prior works in the literature [22–24], a controller is formulated at the boundary to achieve synchronization and passivity-based control of discrete space and time SINNs.

In what follows, Section 2 establishes the discrete space and time SINNs based on prior works in the literature [27,29]. Section 3 discusses synchronization and passivity-based control of the discrete SINNs. In Section 4, in order to illustrate our main results, a numerical illustration is provided. Finally, the conclusions and perspectives are described in Section 5.

2. Problem Formulation

2.1. SINNs in Discrete Form

Now, our primary focus is dedicated to the time and space discrete SINNs, as noted below

$$\begin{aligned}\Delta^2\mathbf{z}^{[\iota]}_{i,k+1} = {}& (e^{-D_\circ h}+e^{-Ih}-2I)\Delta\mathbf{z}^{[\iota]}_{i,k}+\frac{(I-e^{-D_\circ h})(I-e^{-Ih})}{D_\circ}\Bigg[M\Delta^2_{\hbar}\mathbf{z}^{[\iota-1]}_{i,k}-C\mathbf{z}^{[\iota]}_{i,k}\\ &+Af(\mathbf{z}^{[\iota]}_{i,k})+\alpha\sum_{j=1}^{N}b_{ij}\Gamma\left(\frac{\mathbf{z}^{[\iota]}_{j,k+1}-e^{-Ih}\mathbf{z}^{[\iota]}_{j,k}}{I-e^{-Ih}}\right)+\Xi g(\mathbf{z}^{[\iota]}_{i,k})w_{i,k}+\Lambda\gamma^{[\iota]}_{i,k}+J\Bigg], \quad (1)\end{aligned}$$

where $(\iota, k) \in (0,\ell)_{\mathbf{Z}} \times \mathbf{Z}_0$ and $\ell \in \mathbf{Z}_+$ (here, $\mathbf{Z}$ is the set of integral numbers, $\mathbf{Z}_0 := \{0,1,2,\ldots\}$ and $\mathbf{Z}_+ := \mathbf{Z}_0 \setminus \{0\}$), $\mathbf{z}_i = (\mathbf{z}_{i1},\ldots,\mathbf{z}_{in})^T \in \mathbb{R}^n$ is the state of node i; $i = 1,2,\ldots,N$; $\Delta^2 \mathbf{z}_{i,k+1}^{[\cdot]} = \mathbf{z}_{i,k+2}^{[\cdot]} - 2\mathbf{z}_{i,k+1}^{[\cdot]} + \mathbf{z}_{i,k}^{[\cdot]}$, $\Delta \mathbf{z}_{i,k}^{[\cdot]} = \mathbf{z}_{i,k+1}^{[\cdot]} - \mathbf{z}_{i,k}^{[\cdot]}$ for $k \in \mathbf{Z}_0$;

$$\Delta_{\hbar}^2 \mathbf{z}_{i,\cdot}^{[\cdot]} := \frac{\mathbf{z}_{i,\cdot}^{[\cdot+2]} - 2\mathbf{z}_{i,\cdot}^{[\cdot+1]} + \mathbf{z}_{i,\cdot}^{[\cdot]}}{\hbar^2},$$

$\hbar$ and h of less than 1 denote the space and time steps' length in order; $C = \mathrm{diag}\{c_1, c_2, \ldots, c_n\}$ and $D = \mathrm{diag}\{d_1, d_2, \ldots, d_n\}$ are constant positive definite matrices, $D_\circ = D - I$, I denotes n-order identity matrix; $M \in \mathbb{R}^{n\times n}$ with $|M| \neq 0$, A, Ξ and Λ are the connection weight n-order matrices; $\alpha > 0$ is the coupling strength, $\Gamma \in \mathbb{R}^{n\times n}$ is the inner coupling matrix, and $B = (b_{ij})_{N\times N}$ is the outer coupling configuration matrix satisfying $b_{ij} > 0\,(i \neq j)$ and $b_{ii} = -\sum_{j=1,j\neq i}^{N} b_{ij}$; $f(\cdot)$ and $g(\cdot)$ are n dimensional activation functions; $\gamma_i = (\gamma_{i1},\ldots,\gamma_{in})^T \in \mathbb{R}^n$ is the external input of the node i, $J \in \mathbb{R}^n$ is the external input; $w_{1,k},\ldots,w_{n,k}$, which are scalar mutually independent random variables on complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, are $\mathscr{F}_k := \sigma\{(w_{1,q},\ldots,w_{N,q}) : q = 0,1,\ldots,k\}$-adaptive, independent of $\mathscr{F}_{k-1}$ and satisfy

$$\mathbb{E}w_{j,k} = 0, \quad \mathbb{E}w_{j,k}^2 = 1, \quad \mathbb{E}(w_{i,k}w_{j,k}) = 0\,(i \neq j), \quad \mathbb{E}(w_{j,k}w_{j,k'}) = 0\,(k \neq k')$$

for $k, k' \in \mathbf{Z}_0$, $i, j = 1,2,\ldots,N$. Hereby, $\mathbb{E}$ represents the expectation operator with respect to probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The INNs Equation (1) possesses the following controlled boundary conditions

$$\Delta_{\hbar}\mathbf{z}_{i,k}^{[\iota]}\Big|_{\iota=0} = 0, \quad \Delta_{\hbar}\mathbf{z}_{i,k}^{[\iota]}\Big|_{\iota=\ell-1} = \rho_{i,k}, \tag{2}$$

where $\Delta_{\hbar}\mathbf{z}_{i,k}^{[\cdot]} := \frac{1}{\hbar}(\mathbf{z}_{i,k}^{[\cdot+1]} - \mathbf{z}_{i,k}^{[\cdot]})$ and $\rho_{i,k}$ denotes the control input, $k \in \mathbf{Z}_0$, $i = 1,2,\ldots,N$. Further, the initial condition of the INNs Equation (1) is given by

$$\mathbf{z}_{i,0}^{\langle\iota\rangle} = \varphi_{i,0}^{\langle\iota\rangle}, \quad \Delta\mathbf{z}_{i,0}^{\langle\iota\rangle} = \tilde{\varphi}_{i,0}^{\langle\iota\rangle}, \quad \forall \iota \in [0,\ell]_{\mathbf{Z}}, \tag{3}$$

where $\varphi_{i,0}^{\langle\cdot\rangle}$ and $\tilde{\varphi}_{i,0}^{\langle\cdot\rangle}$ are $\mathscr{F}_0$-adaptive and $\mathscr{F}_1$-adaptive, respectively, $i = 1,2,\ldots,N$.

Let $\mathbf{z}_{i,k}^{[\iota]} = \mathbf{z}_i(\iota\hbar, kh)$ for $(\iota, k) \in [0,\ell]_{\mathbf{Z}} \times \mathbf{Z}_0$. So discrete space and time INNs Equation (1) provides a full discretization scheme for the following stochastic INNs with reaction diffusions

$$\begin{aligned}\frac{\partial^2 \mathbf{z}_i(x,t)}{\partial t^2} &= -D\frac{\partial \mathbf{z}_i(x,t)}{\partial t} + M\frac{\partial^2 \mathbf{z}_i(x,t)}{\partial x^2} - C\mathbf{z}_i(x,t) + Af(\mathbf{z}_i(x,t)) \\ &\quad + \alpha\sum_{j=1}^{N} b_{ij}\Gamma\left(\frac{\partial \mathbf{z}_j(x,t)}{\partial t} + \mathbf{z}_j(x,t)\right) + \Xi g(\mathbf{z}_i(x,t))\frac{\mathrm{d}\mathbf{B}_i(t)}{\mathrm{d}t} + \Lambda\gamma_i(x,t) + J,\end{aligned} \tag{4}$$

where $(x,t) \in (0,L) \times [0,+\infty)$ with $L = \ell\hbar$, $\mathbf{B}_i$ is a one-dimensional Brownian motion on some complete probability space, $i = 1,2,\ldots,N$.

Recently, continuous-time INNs Equation (4) with reaction diffusions has been studied by a few authors (see refs. [21–24]) and the corresponding discrete networks have been discussed in reports [27,29]. The different approach of INNs Equation (1) is similar to those in refs. [27,29].

Hereon, INNs Equation (1) can be regarded as slaver networks and the isolated node $\mathbf{w} \in \mathbb{R}^n$ satisfies the master networks below

$$\begin{cases} \Delta^2 \mathbf{w}_{k+1}^{[\iota]} = (e^{-D_\circ h} + e^{-Ih} - 2I)\Delta \mathbf{w}_k^{[\iota]} + \dfrac{(I - e^{-D_\circ h})(I - e^{-Ih})}{D_\circ} \\ \qquad \times \Big[M\Delta_\hbar^2 \mathbf{w}_k^{[\iota-1]} - C\mathbf{w}_k^{[\iota]} + Af(\mathbf{w}_k^{[\iota]}) + \Xi g(\mathbf{w}_k^{[\iota]}) w_{i,k} + J \Big], \\ \Delta_\hbar \mathbf{w}_k^{[\iota]} \Big|_{\iota=0} = \Delta_\hbar \mathbf{w}_k^{[\iota]} \Big|_{\iota=\ell} = 0, \quad \forall(\iota,k) \in (0,\ell)_{\mathbf{Z}} \times \mathbf{Z}_0. \end{cases} \tag{5}$$

The initial condition of INNs Equation (5) is described as

$$\mathbf{w}_0^{\langle\iota\rangle} = \phi_0^{\langle\iota\rangle}, \quad \Delta \mathbf{w}_0^{\langle\iota\rangle} = \tilde{\phi}_0^{\langle\iota\rangle}, \quad \forall \iota \in [0,\ell]_{\mathbf{Z}}, \tag{6}$$

where $\phi_0^{\langle\cdot\rangle}$ and $\tilde{\phi}_0^{\langle\cdot\rangle}$ are $\mathscr{F}_0$-adaptive and $\mathscr{F}_1$-adaptive, respectively.

Let $\mathbf{u}_i = \mathbf{z}_i - \mathbf{w}$, then the error networks of INNs Equations (1) and (5) are described by

$$\begin{cases} \Delta^2 \mathbf{u}_{i,k+1}^{[\iota]} = (e^{-D_\circ h} + e^{-Ih} - 2I)\Delta \mathbf{u}_{i,k}^{[\iota]} + \dfrac{(I - e^{-D_\circ h})(I - e^{-Ih})}{D_\circ} \Big[M\Delta_\hbar^2 \mathbf{u}_{i,k}^{[\iota-1]} - C\mathbf{u}_{i,k}^{[\iota]} \\ \qquad + A\tilde{f}(\mathbf{u}_{i,k}^{[\iota]}) + \alpha \sum_{j=1}^{N} b_{ij} \Gamma \Big(\dfrac{\mathbf{u}_{j,k+1}^{[\iota]} - e^{-Ih}\mathbf{u}_{j,k}^{[\iota]}}{I - e^{-Ih}} \Big) + \Xi \tilde{g}(\mathbf{u}_{i,k}^{[\iota]}) w_{i,k} + \Lambda \gamma_{i,k}^{[\iota]} \Big], \\ \Delta_\hbar \mathbf{u}_{i,k}^{[\iota]} \Big|_{\iota=0} = 0, \quad \Delta_\hbar \mathbf{u}_{i,k}^{[\iota]} \Big|_{\iota=\ell-1} = \rho_{i,k}, \quad \forall(\iota,k) \in (0,\ell)_{\mathbf{Z}} \times \mathbf{Z}_0, \end{cases} \tag{7}$$

where $\tilde{f}(\mathbf{u}_i) := f(\mathbf{z}_i) - f(\mathbf{w})$ and $\tilde{g}(\mathbf{u}_i) := g(\mathbf{z}_i) - g(\mathbf{w})$, $i = 1,2,\ldots,N$. With the help of Equations (3) and (6), the initial condition for INNs in Equation (7) can be derived, as depicted by

$$\mathbf{u}_{i,0}^{\langle\iota\rangle} = \varphi_{i,0}^{\langle\iota\rangle} - \phi_0^{\langle\iota\rangle}, \quad \Delta \mathbf{u}_{i,0}^{\langle\iota\rangle} = \tilde{\varphi}_{i,0}^{\langle\iota\rangle} - \tilde{\phi}_0^{\langle\iota\rangle}, \quad \forall \iota \in [0,\ell]_{\mathbf{Z}}, i = 1,2,\ldots,N. \tag{8}$$

To study INNs Equation (1) effectively, let

$$\mathbf{u}_{i,k+1}^{[\iota]} = e^{-Ih}\mathbf{u}_{i,k}^{[\iota]} + \varepsilon(I - e^{-Ih})\mathbf{v}_{i,k}^{[\iota]}, \quad \forall(\iota,k) \in (0,\ell)_{\mathbf{Z}} \times \mathbf{Z}, \tag{9}$$

where $\varepsilon > 0$ is a controlling parameter, which can be adjusted freely, $i = 1,2,\ldots,N$. Then, the first equation in INNs Equation (7) is changed into

$$\begin{aligned} \mathbf{v}_{i,k+1}^{[\iota]} = {} & e^{-D_\circ h}\mathbf{v}_{i,k}^{[\iota]} + \frac{I - e^{-D_\circ h}}{D_\circ} \Big[M_\varepsilon \Delta_\hbar^2 \mathbf{u}_{i,k}^{[\iota-1]} + C_\varepsilon \mathbf{u}_{i,k}^{[\iota]} \\ & + A_\varepsilon \tilde{f}(\mathbf{u}_{i,k}^{[\iota]}) + \alpha \sum_{j=1}^{N} b_{ij} \Gamma \mathbf{v}_{j,k}^{[\iota]} + \Xi_\varepsilon \tilde{g}(\mathbf{u}_{i,k}^{[\iota]}) w_{i,k} + \Lambda_\varepsilon \gamma_{i,k}^{[\iota]} \Big], \end{aligned} \tag{10}$$

$\forall(\iota,k) \in (0,\ell)_{\mathbf{Z}} \times \mathbf{Z}$, $C_\varepsilon = \varepsilon^{-1}(D - C - I)$, $M_\varepsilon = \varepsilon^{-1}M$, $A_\varepsilon = \varepsilon^{-1}A$, $\Xi_\varepsilon = \varepsilon^{-1}\Xi$, $\Lambda_\varepsilon = \varepsilon^{-1}\Lambda$, $i = 1,2,\ldots,N$.

The vector forms of INNs Equations (9) and (10) are written as

$$\begin{cases} \mathbf{e}_{u,k+1}^{[\iota]} = (e^{-Ih})_{\otimes}\mathbf{e}_{u,k}^{[\iota]} + \varepsilon(I - e^{-Ih})_{\otimes}\mathbf{e}_{v,k}^{[\iota]}, \\ \mathbf{e}_{v,k+1}^{[\iota]} = (e^{-D_\circ h})_{\otimes}\mathbf{e}_{v,k}^{[\iota]} + \left[\dfrac{(I - e^{-D_\circ h})M_\varepsilon}{D_\circ}\right]_{\otimes} \Delta_{\hbar}^2 \mathbf{e}_{u,k}^{[\iota-1]} \\ \qquad + \left[\dfrac{(I - e^{-D_\circ h})C_\varepsilon}{D_\circ}\right]_{\otimes} \mathbf{e}_{u,k}^{[\iota]} + \left[\dfrac{(I - e^{-D_\circ h})A_\varepsilon}{D_\circ}\right]_{\otimes} \mathbf{F}(\mathbf{e}_{u,k}^{[\iota]}) \\ \qquad + \alpha\left[\dfrac{(I - e^{-D_\circ h})\Gamma}{D_\circ}\right]_{\otimes B} \mathbf{e}_{v,k}^{[\iota]} + \left[\dfrac{(I - e^{-D_\circ h})\Lambda_\varepsilon}{D_\circ}\right]_{\otimes} \gamma_k^{[\iota]} \\ \qquad + \left[\dfrac{(I - e^{-D_\circ h})\Xi_\varepsilon}{D_\circ}\right]_{\otimes w_k} \mathbf{G}(\mathbf{e}_{u,k}^{[\iota]}), \\ \Delta_{\hbar}\mathbf{e}_{u,k}^{[\iota]}\Big|_{\iota=0} = 0, \quad \Delta_{\hbar}\mathbf{e}_{u,k}^{[\iota]}\Big|_{\iota=\ell-1} = \rho_k, \end{cases} \tag{11}$$

where

$$\mathbf{e}_u = (\mathbf{u}_1, \ldots, \mathbf{u}_N)^T, \quad \mathbf{e}_v = (\mathbf{v}_1, \ldots, \mathbf{v}_N)^T,$$
$$\mathbf{F}(\mathbf{e}_u) := (\tilde{f}(\mathbf{u}_1), \ldots, \tilde{f}(\mathbf{u}_N))^T, \quad \mathbf{G}(\mathbf{e}_u) := (\tilde{g}(\mathbf{u}_1), \ldots, \tilde{g}(\mathbf{u}_N))^T,$$
$$w = \mathrm{diag}(w_1, \ldots, w_N)^T, \quad \gamma = (\gamma_1, \ldots, \gamma_N)^T, \quad \rho = (\rho_1, \ldots, \rho_N)^T,$$

I_N denotes the N-order identity matrix. Hereby, $(A)_{\otimes} := I_N \otimes A$ and $(A)_{\otimes B} := B \otimes A$. In accordance with Equations (8) and (9), the initial condition of INNs Equation (11) is expressed by

$$\mathbf{e}_{u,0}^{\langle\iota\rangle} = \psi_0^{\langle\iota\rangle}, \quad \mathbf{e}_{v,0}^{\langle\iota\rangle} = \varepsilon^{-1}\left[(I - e^{-Ih})^{-1}\right]_{\otimes}\tilde{\psi}_0^{\langle\iota\rangle} + \varepsilon^{-1}I_N \otimes \psi_0^{\langle\iota\rangle}, \tag{12}$$

where $\iota \in [0, \ell]_{\mathbf{Z}}, i = 1, 2, \ldots, N$, $\psi_0^{\langle\cdot\rangle} = (\varphi_{1,0}^{\langle\cdot\rangle} - \phi_0^{\langle\cdot\rangle}, \ldots, \varphi_{N,0}^{\langle\cdot\rangle} - \phi_0^{\langle\cdot\rangle})^T$ and $\tilde{\psi}_0^{\langle\cdot\rangle} = (\tilde{\varphi}_{1,0}^{\langle\cdot\rangle} - \tilde{\phi}_0^{\langle\cdot\rangle}, \ldots, \tilde{\varphi}_{N,0}^{\langle\cdot\rangle} - \tilde{\phi}_0^{\langle\cdot\rangle})^T$. Throughout this article, supposing that

$$\sum_{\iota=1}^{\ell-1} \mathbb{E}\left\|\varphi_{i,0}^{\langle\iota\rangle}\right\|^2 < \infty, \quad \sum_{\iota=1}^{\ell-1} \mathbb{E}\left\|\phi_0^{\langle\iota\rangle}\right\|^2 < \infty, \quad \sum_{\iota=1}^{\ell-1} \mathbb{E}\left\|\tilde{\varphi}_{i,0}^{\langle\iota\rangle}\right\|^2 < \infty, \quad \sum_{\iota=1}^{\ell-1} \mathbb{E}\left\|\tilde{\phi}_0^{\langle\iota\rangle}\right\|^2 < \infty$$

for $i = 1, 2, \ldots, N$. Based on Equation (12), we have

$$\sum_{\iota=1}^{\ell-1} \mathbb{E}\left\|\mathbf{e}_{u,0}^{\langle\iota\rangle}\right\|^2 < \infty, \quad \sum_{\iota=1}^{\ell-1} \mathbb{E}\left\|\mathbf{e}_{v,0}^{\langle\iota\rangle}\right\|^2 < \infty. \tag{13}$$

The current discussion will establish a boundary controller to synchronize and passivity-based control the master INNs Equations (5) and slave INNs (1), which will be demonstrated in Section 3.

Hereon, we need the following assumption for activation functions.

(F) L_f and L_g are n-order matrices ensuring

$$[f(x) - f(y)]^T[f(x) - f(y)] \leq (x - y)^T L_f (x - y),$$
$$[g(x) - g(y)]^T[g(x) - g(y)] \leq (x - y)^T L_g (x - y), \quad \forall x, y \in \mathbb{R}^n.$$

2.2. Some Important Inequalities

Lemma 1 ([39])**.** *Let $X, Y \in \mathbb{R}^m$. Then $X^TY + Y^TX \leq \alpha X^TX + \frac{1}{\alpha}Y^TY$ for any $\alpha > 0$.*

Lemma 2 ([40]). *If $X : [0, \ell]_{\mathbf{Z}} \to \mathbb{R}^m$, $P \in \mathbb{R}^{m\times m}$, one has*

$$\sum_{\iota=1}^{\ell-1} X_\iota^T P\Delta^2 X_{\iota-1} = X_\iota^T P\Delta X_{\iota-1}\Big|_1^\ell - \sum_{\iota=1}^{\ell-1}\Delta X_\iota^T P\Delta X_\iota.$$

Lemma 3 ([41,42]). *If $X : [0, \ell]_{\mathbf{Z}} \to \mathbb{R}^m$, $P \in \mathbb{R}^{m\times m}$, $P \geq 0$, and $X_0 = 0$, one has*

$$\nu_\ell \sum_{\iota=0}^{\ell} X_\iota^T P X_\iota \leq \sum_{\iota=0}^{\ell-1}\Delta X_\iota^T P\Delta X_\iota \leq \mu_\ell \sum_{\iota=0}^{\ell} X_\iota^T P X_\iota,$$

where $\mu_\ell = 4\cos^2\frac{\pi}{2\ell+1}$ and $\nu_\ell = 4\sin^2\frac{\pi}{2(2\ell+1)}$.

Lemma 4 ([41,43]). *If $X : [1, \ell]_{\mathbf{Z}} \to \mathbb{R}^m$, $P \in \mathbb{R}^{m\times m}$, $P \geq 0$, one has*

$$\kappa_\ell \sum_{\iota=1}^{\ell} X_\iota^T P X_\iota \leq \sum_{\iota=1}^{\ell-1}\Delta X_\iota^T P\Delta X_\iota + (X_1 + X_\ell)^T P(X_1 + X_\ell),$$

$$\sum_{\iota=1}^{\ell-1}\Delta X_\iota^T P\Delta X_\iota + \left[X_1 + (-1)^\ell X_\ell\right]^T P\left[X_1 + (-1)^\ell X_\ell\right] \leq (4-\kappa_\ell)\sum_{\iota=1}^{\ell} X_\iota^T P X_\iota.$$

Using Lemma 3, we get

$$\sum_{\iota=0}^{\ell-2}\Delta_\hbar^2 \mathbf{e}_{u,k}^{[\iota]T} P\Delta_\hbar^2 \mathbf{e}_{u,k}^{[\iota]} \leq \frac{\mu_{\ell-1}}{\hbar^2}\sum_{\iota=0}^{\ell-1}\Delta_\hbar \mathbf{e}_{u,k}^{[\iota]T} P\Delta_\hbar \mathbf{e}_{u,k}^{[\iota]}, \quad \forall k \in \mathbf{Z}_0, \tag{14}$$

where P is defined as in Lemma 3.

3. Stochastic Synchronization and Passivity-Based Control

The slave INNs Equation (1) is said to be stochastically synchronized with the master INNs Equation (5) if the error vector networks Equation (11) achieves globally asymptotically stability in mean square, i.e.,

$$\lim_{k\to\infty}\sum_{\iota=1}^{\ell-1}\mathbb{E}\left\|\mathbf{e}_{u,k}^{[\iota]}\right\|^2 = 0 = \lim_{k\to\infty}\sum_{\iota=1}^{\ell-1}\mathbb{E}\left\|\mathbf{e}_{v,k}^{[\iota]}\right\|^2.$$

3.1. Stochastic Synchronization

Define

$$\rho_k = -\sum_{\iota=1}^{\ell-1}\Theta_\otimes \mathbf{e}_{u,k}^{[\iota]}, \quad \forall k \in \mathbf{Z}_0, \tag{15}$$

where $\Theta \in \mathbb{R}^{n\times n}$. Set $\mathcal{D} := \frac{I-e^{-D_\circ h}}{D_\circ}$.

Theorem 1. *Assuming that* **(F)** *is valid, and $\varepsilon > 0$ is given in advance, $\mathcal{D}$ and M_ε are nonsingular. The slaver INNs Equation (1) stochastically synchronizes with the master INNs Equation (5); in other words, model Equation (11) is globally mean-squared asymptotically stable if it has positive constants λ_f, λ_g and n-order matrices $P > 0$, $Q > 0$, $H > 0$, $K > 0$ such that*

$$\mathbb{O} := \begin{bmatrix} \mathbb{O}_{11} & \mathbb{O}_{12} & \mathbb{O}_{13} & \mathbb{O}_{14} & \mathbb{O}_{15} & \mathbb{O}_{16} \\ * & \mathbb{O}_{22} & \mathbb{O}_{23} & \mathbb{O}_{24} & \mathbb{O}_{25} & \mathbb{O}_{26} \\ * & * & \mathbb{O}_{33} & \mathbb{O}_{34} & \mathbb{O}_{35} & \mathbb{O}_{36} \\ * & * & * & \mathbb{O}_{44} & \mathbb{O}_{45} & \mathbb{O}_{46} \\ * & * & * & * & \mathbb{O}_{55} & \mathbb{O}_{56} \\ * & * & * & * & * & \mathbb{O}_{66} \end{bmatrix} < 0,$$

where

$$\mathbb{O}_{11} = -\frac{1}{\hbar} sym(C_\varepsilon K)_\otimes + \Big[e^{-Ih} P e^{-Ih} - P\Big]_\otimes + \Big[C_\varepsilon \mathcal{D} Q \mathcal{D} C_\varepsilon\Big]_\otimes + \lambda_f (L_f)_\otimes + \lambda_g (L_g)_\otimes,$$

$$\mathbb{O}_{12} = \varepsilon\Big[e^{-Ih} P(I - e^{-Ih})\Big]_\otimes + \Big[e^{-D_\circ h} Q \mathcal{D} C_\varepsilon\Big]_\otimes^T + \alpha\Big[C_\varepsilon \mathcal{D} Q \mathcal{D} \Gamma\Big]_{\otimes B}, \quad \mathbb{O}_{13} = -\frac{1}{\hbar}(C_\varepsilon K)_\otimes,$$

$$\mathbb{O}_{15} = \Big[C_\varepsilon \mathcal{D} Q \mathcal{D} A_\varepsilon\Big]_\otimes, \quad \mathbb{O}_{25} = \Big[e^{-D_\circ h} Q \mathcal{D} A_\varepsilon\Big]_\otimes + \alpha\Big[A_\varepsilon^T \mathcal{D} Q \mathcal{D} \Gamma\Big]_{\otimes B}^T,$$

$$\mathbb{O}_{22} = -Q_\otimes + \varepsilon^2\Big[(I - e^{-Ih}) P (I - e^{-Ih})\Big]_\otimes + \alpha sym\Big[e^{-D_\circ h} Q \mathcal{D} \Gamma\Big]_{\otimes B}$$
$$+2\Big[e^{-D_\circ h} Q e^{-D_\circ h}\Big]_\otimes + 2\alpha^2\Big[\Gamma^T \mathcal{D} Q \mathcal{D} \Gamma\Big]_{\otimes_{B^T B}},$$

$$\mathbb{O}_{33} = -H_\otimes, \quad \mathbb{O}_{44} = -sym\Big[C_\varepsilon \mathcal{D} Q \mathcal{D} M_\varepsilon\Big]_\otimes + \frac{4\mu_{\ell-1}}{\hbar^2}\Big[M_\varepsilon^T \mathcal{D} Q \mathcal{D} M_\varepsilon\Big]_\otimes + \frac{\hbar^2 \ell}{\kappa_\ell} H_\otimes,$$

$$\mathbb{O}_{55} = -\lambda_f I_\otimes + 2\Big[A_\varepsilon^T \mathcal{D} Q \mathcal{D} A_\varepsilon\Big]_\otimes, \quad \mathbb{O}_{66} = -\lambda_g I_\otimes + \Big[\Xi_\varepsilon^T \mathcal{D} Q \mathcal{D} \Xi_\varepsilon\Big]_\otimes,$$

$\mathbb{O}_{14} = \mathbb{O}_{16} = \mathbb{O}_{23} = \mathbb{O}_{24} = \mathbb{O}_{26} = \mathbb{O}_{34} = \mathbb{O}_{35} = \mathbb{O}_{36} = \mathbb{O}_{45} = \mathbb{O}_{46} = \mathbb{O}_{56} = 0$. *Here* $sym(A) = A + A^T$. *The controller gain*

$$\Theta = \Big[\mathcal{D} Q \mathcal{D} M_\varepsilon\Big]^{-1} K.$$

Proof. Let us define a Lyapunov-Krasovskii function, which is described by

$$V_k = V_{1,k} + V_{2,k},$$

where

$$V_{1,k} = \sum_{\iota=1}^{\ell-1} \mathbf{e}_{u,k}^{[\iota]T} (I_N \otimes P) \mathbf{e}_{u,k}^{[\iota]}, \quad V_{2,k} = \sum_{\iota=1}^{\ell-1} \mathbf{e}_{v,k}^{[\iota]T} (I_N \otimes Q) \mathbf{e}_{v,k}^{[\iota]}, \quad \forall k \in \mathbf{Z}_0.$$

In the line with the first segment of the error networks Equation (11), we can derive

$$\begin{aligned} \mathbb{E}[\Delta V_{1,k}] &= \mathbb{E} \sum_{\iota=1}^{\ell-1} \mathbf{e}_{u,k+1}^{[\iota]T} (I_N \otimes P) \mathbf{e}_{u,k+1}^{[\iota]} - V_{1,k} \\ &= \mathbb{E} \sum_{\iota=1}^{\ell-1} \mathbf{e}_{u,k}^{[\iota]T} \Big[e^{-Ih} P e^{-Ih} - P\Big]_\otimes \mathbf{e}_{u,k}^{[\iota]} + \varepsilon \mathbb{E} \sum_{\iota=1}^{\ell-1} sym\Big\{\mathbf{e}_{u,k}^{[\iota]T} \Big[e^{-Ih} P (I - e^{-Ih})\Big]_\otimes \mathbf{e}_{v,k}^{[\iota]}\Big\} \\ &+ \varepsilon^2 \mathbb{E} \sum_{\iota=1}^{\ell-1} \mathbf{e}_{v,k}^{[\iota]T} \Big[(I - e^{-Ih}) P (I - e^{-Ih})\Big]_\otimes \mathbf{e}_{v,k}^{[\iota]}, \quad \forall k \in \mathbf{Z}_0. \end{aligned} \tag{16}$$

According to the second equation of networks Equation (11), we get

$$
\begin{aligned}
\mathbb{E}[V_{2,k+1}] &= \mathbb{E}\sum_{\iota=1}^{\ell-1} \mathbf{e}_{v,k+1}^{[\iota]T}(I_N \otimes Q)\mathbf{e}_{v,k+1}^{[\iota]} \\
&= \underbrace{\mathbb{E}\sum_{\iota=1}^{\ell-1} \mathbf{e}_{v,k}^{[\iota]T}\left[e^{-D\circ h}Qe^{-D\circ h}\right]_{\otimes}\mathbf{e}_{v,k}^{[\iota]}}_{\mathcal{U}_{1,k}} + \underbrace{\mathbb{E}\sum_{\iota=1}^{\ell-1} sym\left\{\mathbf{e}_{v,k}^{[\iota]T}\left[e^{-D\circ h}Q\mathcal{D}M_\varepsilon\right]_{\otimes}\Delta_{\hbar}^{2}\mathbf{e}_{u,k}^{[\iota-1]}\right\}}_{\mathcal{U}_{2,k}} \\
&\quad + \underbrace{\mathbb{E}\sum_{\iota=1}^{\ell-1} sym\left\{\mathbf{e}_{v,k}^{[\iota]T}\left[e^{-D\circ h}Q\mathcal{D}C_\varepsilon\right]_{\otimes}\mathbf{e}_{u,k}^{[\iota]}\right\}}_{\mathcal{U}_{3,k}} \\
&\quad + \underbrace{\mathbb{E}\sum_{\iota=1}^{\ell-1} sym\left\{\mathbf{e}_{v,k}^{[\iota]T}\left[e^{-D\circ h}Q\mathcal{D}A_\varepsilon\right]_{\otimes}\mathbf{F}(\mathbf{e}_{u,k}^{[\iota]})\right\}}_{\mathcal{U}_{4,k}} \\
&\quad + \underbrace{\alpha\mathbb{E}\sum_{\iota=1}^{\ell-1} sym\left\{\mathbf{e}_{v,k}^{[\iota]T}\left[e^{-D\circ h}Q\mathcal{D}\Gamma\right]_{\otimes B}\mathbf{e}_{v,k}^{[\iota]}\right\}}_{\mathcal{U}_{5,k}} \\
&\quad + \underbrace{\mathbb{E}\sum_{\iota=1}^{\ell-1} \Delta_{\hbar}^{2}\mathbf{e}_{u,k}^{[\iota-1]T}\left[M_\varepsilon^T\mathcal{D}Q\mathcal{D}M_\varepsilon\right]_{\otimes}\Delta_{\hbar}^{2}\mathbf{e}_{u,k}^{[\iota-1]}}_{\mathcal{U}_{6,k}} \\
&\quad + \underbrace{\mathbb{E}\sum_{\iota=1}^{\ell-1} sym\left\{\Delta_{\hbar}^{2}\mathbf{e}_{u,k}^{[\iota-1]T}\left[M_\varepsilon^T\mathcal{D}Q\mathcal{D}C_\varepsilon\right]_{\otimes}\mathbf{e}_{u,k}^{[\iota]}\right\}}_{\mathcal{U}_{7,k}} \\
&\quad + \underbrace{\mathbb{E}\sum_{\iota=1}^{\ell-1} sym\left\{\Delta_{\hbar}^{2}\mathbf{e}_{u,k}^{[\iota-1]T}\left[M_\varepsilon^T\mathcal{D}Q\mathcal{D}A_\varepsilon\right]_{\otimes}\mathbf{F}(\mathbf{e}_{u,k}^{[\iota]})\right\}}_{\mathcal{U}_{8,k}} \\
&\quad + \underbrace{\alpha\mathbb{E}\sum_{\iota=1}^{\ell-1} sym\left\{\Delta_{\hbar}^{2}\mathbf{e}_{u,k}^{[\iota-1]T}\left[M_\varepsilon^T\mathcal{D}Q\mathcal{D}\Gamma\right]_{\otimes B}\mathbf{e}_{v,k}^{[\iota]}\right\}}_{\mathcal{U}_{9,k}} \\
&\quad + \underbrace{\mathbb{E}\sum_{\iota=1}^{\ell-1} \mathbf{e}_{u,k}^{[\iota]T}\left[C_\varepsilon\mathcal{D}Q\mathcal{D}C_\varepsilon\right]_{\otimes}\mathbf{e}_{u,k}^{[\iota]}}_{\mathcal{U}_{10,k}} \\
&\quad + \underbrace{\mathbb{E}\sum_{\iota=1}^{\ell-1} sym\left\{\mathbf{e}_{u,k}^{[\iota]T}\left[C_\varepsilon\mathcal{D}Q\mathcal{D}A_\varepsilon\right]_{\otimes}\mathbf{F}(\mathbf{e}_{u,k}^{[\iota]})\right\}}_{\mathcal{U}_{11,k}} \\
&\quad + \underbrace{\alpha\mathbb{E}\sum_{\iota=1}^{\ell-1} sym\left\{\mathbf{e}_{u,k}^{[\iota]T}\left[C_\varepsilon\mathcal{D}Q\mathcal{D}\Gamma\right]_{\otimes B}\mathbf{e}_{v,k}^{[\iota]}\right\}}_{\mathcal{U}_{12,k}} \\
&\quad + \underbrace{\mathbb{E}\sum_{\iota=1}^{\ell-1} \mathbf{F}^T(\mathbf{e}_{u,k}^{[\iota]})\left[A_\varepsilon^T\mathcal{D}Q\mathcal{D}A_\varepsilon\right]_{\otimes}\mathbf{F}(\mathbf{e}_{u,k}^{[\iota]})}_{\mathcal{U}_{13,k}}
\end{aligned}
$$

$$
\begin{aligned}
&+\underbrace{\alpha\mathbb{E}\sum_{\iota=1}^{\ell-1} sym\Big\{\mathbf{F}^T(\mathbf{e}_{u,k}^{[\iota]})\Big[A_\varepsilon^T\mathcal{D}Q\mathcal{D}\Gamma\Big]_{\otimes B}\mathbf{e}_{v,k}^{[\iota]}\Big\}}_{\mathcal{U}_{14,k}} \\
&+\underbrace{\alpha^2\mathbb{E}\sum_{\iota=1}^{\ell-1}\mathbf{e}_{v,k}^{[\iota]T}\Big[\Gamma^T\mathcal{D}Q\mathcal{D}\Gamma\Big]_{\otimes_{B^TB}}\mathbf{e}_{v,k}^{[\iota]}}_{\mathcal{U}_{15,k}} \\
&+\underbrace{\mathbb{E}\sum_{\iota=1}^{\ell-1}\mathbf{G}^T(\mathbf{e}_{u,k}^{[\iota]})\Big[\Xi_\varepsilon^T\mathcal{D}Q\mathcal{D}\Xi_\varepsilon\Big]_{\otimes_{w_k^2}}\mathbf{G}(\mathbf{e}_{u,k}^{[\iota]})}_{\mathcal{U}_{16,k}}, \qquad (17)
\end{aligned}
$$

where $k \in \mathbf{Z}_0$.

According to Lemmas 1–3 and boundary conditions in Equation (11), we calculate

$$
\begin{aligned}
\mathcal{U}_{2,k} &\le \mathbb{E}\sum_{\iota=1}^{\ell-1}\mathbf{e}_{v,k}^{[\iota]T}\Big[e^{-D_\circ h}Qe^{-D_\circ h}\Big]_{\otimes}\mathbf{e}_{v,k}^{[\iota]} + \mathbb{E}\sum_{\iota=1}^{\ell-1}\Delta_\hbar^2\mathbf{e}_{u,k}^{[\iota-1]T}\Big[M_\varepsilon^T\mathcal{D}Q\mathcal{D}M_\varepsilon\Big]_{\otimes}\Delta_\hbar^2\mathbf{e}_{u,k}^{[\iota-1]} \\
&\le \mathbb{E}\sum_{\iota=1}^{\ell-1}\mathbf{e}_{v,k}^{[\iota]T}\Big[e^{-D_\circ h}Qe^{-D_\circ h}\Big]_{\otimes}\mathbf{e}_{v,k}^{[\iota]} + \frac{\mu_{\ell-1}}{\hbar^2}\mathbb{E}\sum_{\iota=1}^{\ell-1}\Delta_\hbar\mathbf{e}_{u,k}^{[\iota]T}\Big[M_\varepsilon^T\mathcal{D}Q\mathcal{D}M_\varepsilon\Big]_{\otimes}\Delta_\hbar\mathbf{e}_{u,k}^{[\iota]}, \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{U}_{6,k} &= \mathbb{E}\sum_{\iota=0}^{\ell-2}\Delta_\hbar^2\mathbf{e}_z^T(x_I,t_k)\Big[M_\varepsilon^T\mathcal{D}Q\mathcal{D}M_\varepsilon\Big]_{\otimes}\Delta_\hbar^2\mathbf{e}_{u,k}^{[\iota]} \\
&\le \frac{\mu_{\ell-1}}{\hbar^2}\mathbb{E}\sum_{\iota=1}^{\ell-1}\Delta_\hbar\mathbf{e}_{u,k}^{[\iota]T}\Big[M_\varepsilon^T\mathcal{D}Q\mathcal{D}M_\varepsilon\Big]_{\otimes}\Delta_\hbar\mathbf{e}_{u,k}^{[\iota]}, \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{U}_{7,k} &= \frac{1}{\hbar}\mathbb{E}sym\Big\{\mathbf{e}_{u,k}^{[\iota]T}\Big[C_\varepsilon\mathcal{D}Q\mathcal{D}M_\varepsilon\Big]_{\otimes}\Delta_\hbar\mathbf{e}_{u,k}^{[\iota-1]}\Big\}\Big|_1^\ell - \mathbb{E}\sum_{\iota=1}^{\ell-1}sym\Big\{\Delta_\hbar\mathbf{e}_{u,k}^{[\iota]T}\Big[C_\varepsilon\mathcal{D}Q\mathcal{D}M_\varepsilon\Big]_{\otimes}\Delta_\hbar\mathbf{e}_{u,k}^{[\iota]}\Big\} \\
&= \frac{1}{\hbar}\mathbb{E}sym\Big\{\mathbf{e}_{u,k}^{[\ell]T}\Big[C_\varepsilon\mathcal{D}Q\mathcal{D}M_\varepsilon\Big]_{\otimes}\rho_k\Big\} - \mathbb{E}\sum_{\iota=1}^{\ell-1}\Delta_\hbar\mathbf{e}_{u,k}^{[\iota]T}sym\Big[C_\varepsilon\mathcal{D}Q\mathcal{D}M_\varepsilon\Big]_{\otimes}\Delta_\hbar\mathbf{e}_{u,k}^{[\iota]}, \qquad (20)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{U}_{8,k} &\le \mathbb{E}\sum_{\iota=1}^{\ell-1}\Delta_\hbar^2\mathbf{e}_{u,k}^{[\iota-1]T}\Big[M_\varepsilon^T\mathcal{D}Q\mathcal{D}M_\varepsilon\Big]_{\otimes}\Delta_\hbar^2\mathbf{e}_{u,k}^{[\iota-1]} + \mathbb{E}\sum_{\iota=1}^{\ell-1}\mathbf{F}^T(\mathbf{e}_{u,k}^{[\iota]})\Big[A_\varepsilon^T\mathcal{D}Q\mathcal{D}A_\varepsilon\Big]_{\otimes}\mathbf{F}(\mathbf{e}_{u,k}^{[\iota]}) \\
&\le \frac{\mu_{\ell-1}}{\hbar^2}\mathbb{E}\sum_{\iota=1}^{\ell-1}\Delta_\hbar\mathbf{e}_{u,k}^{[\iota]T}\Big[M_\varepsilon^T\mathcal{D}Q\mathcal{D}M_\varepsilon\Big]_{\otimes}\Delta_\hbar\mathbf{e}_{u,k}^{[\iota]} + \mathbb{E}\sum_{\iota=1}^{\ell-1}\mathbf{F}^T(\mathbf{e}_{u,k}^{[\iota]})\Big[A_\varepsilon^T\mathcal{D}Q\mathcal{D}A_\varepsilon\Big]_{\otimes}\mathbf{F}(\mathbf{e}_{u,k}^{[\iota]}), \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{U}_{9,k} &\le \alpha^2\mathbb{E}\sum_{\iota=1}^{\ell-1}\mathbf{e}_{v,k}^{[\iota]T}\Big[\Gamma^T\mathcal{D}Q\mathcal{D}\Gamma\Big]_{\otimes_{B^TB}}\mathbf{e}_{v,k}^{[\iota]} + \mathbb{E}\sum_{\iota=1}^{\ell-1}\Delta_\hbar^2\mathbf{e}_{u,k}^{[\iota-1]T}\Big[M_\varepsilon^T\mathcal{D}Q\mathcal{D}M_\varepsilon\Big]_{\otimes}\Delta_\hbar^2\mathbf{e}_{u,k}^{[\iota-1]} \\
&\le \alpha^2\mathbb{E}\sum_{\iota=1}^{\ell-1}\mathbf{e}_{v,k}^{[\iota]T}\Big[\Gamma^T\mathcal{D}Q\mathcal{D}\Gamma\Big]_{\otimes_{B^TB}}\mathbf{e}_{v,k}^{[\iota]} + \frac{\mu_{\ell-1}}{\hbar^2}\mathbb{E}\sum_{\iota=1}^{\ell-1}\Delta_\hbar\mathbf{e}_{u,k}^{[\iota]T}\Big[M_\varepsilon^T\mathcal{D}Q\mathcal{D}M_\varepsilon\Big]_{\otimes}\Delta_\hbar\mathbf{e}_{u,k}^{[\iota]}, \qquad (22)
\end{aligned}
$$

$$
\mathcal{U}_{16,k} = \mathbb{E}\sum_{\iota=1}^{\ell-1}\mathbf{G}^T(\mathbf{e}_{u,k}^{[\iota]})\Big[\Xi_\varepsilon^T\mathcal{D}Q\mathcal{D}\Xi_\varepsilon\Big]_{\otimes}\mathbf{G}(\mathbf{e}_{u,k}^{[\iota]}), \quad \forall k \in \mathbf{Z}_0. \qquad (23)
$$

With the help of **(F)**, we have

$$\sum_{\iota=1}^{\ell-1}\mathbf{F}^T(\mathbf{e}_{u,k}^{[\iota]})\mathbf{F}(\mathbf{e}_{u,k}^{[\iota]})\le\sum_{\iota=1}^{\ell-1}\mathbf{e}_{u,k}^{[\iota]T}(L_f)_\otimes\mathbf{e}_{u,k}^{[\iota]},\quad\sum_{\iota=1}^{\ell-1}\mathbf{G}^T(\mathbf{e}_{u,k}^{[\iota]})\mathbf{G}(\mathbf{e}_{u,k}^{[\iota]})\le\sum_{\iota=1}^{\ell-1}\mathbf{e}_{u,k}^{[\iota]T}(L_g)_\otimes\mathbf{e}_{u,k}^{[\iota]},\tag{24}$$

and by using $\hat{\mathbf{e}}_{u,\cdot}^{[\cdot]}:=\mathbf{e}_{u,\cdot}^{[\ell]}-\mathbf{e}_{u,\cdot}^{[\cdot]}$ and Lemma 4, it gets

$$\begin{aligned}\sum_{\iota=1}^{\ell}\hat{\mathbf{e}}_{u,k}^{[\iota]T}H_\otimes\hat{\mathbf{e}}_{u,k}^{[\iota]}\le{}&\frac{\hbar^2}{\kappa_\ell}\sum_{\iota=1}^{\ell-1}\Delta_\hbar\hat{\mathbf{e}}_{u,k}^{[\iota]T}H_\otimes\Delta_\hbar\hat{\mathbf{e}}_{u,k}^{[\iota]}+\frac{1}{\kappa_\ell}\Big[\mathbf{e}_{u,k}^{[\ell]}-\mathbf{e}_{u,k}^{[1]}\Big]^TH_\otimes\Big[\mathbf{e}_{u,k}^{[\ell]}-\mathbf{e}_{u,k}^{[1]}\Big]\\={}&\frac{\hbar^2}{\kappa_\ell}\sum_{\iota=1}^{\ell-1}\Delta_\hbar\hat{\mathbf{e}}_{u,k}^{[\iota]T}H_\otimes\Delta_\hbar\hat{\mathbf{e}}_{u,k}^{[\iota]}+\frac{\hbar^2}{\kappa_\ell}\Big[\sum_{\iota=1}^{\ell-1}\Delta_\hbar\hat{\mathbf{e}}_{u,k}^{[\iota]}\Big]^TH_\otimes\Big[\sum_{\iota=1}^{\ell-1}\Delta_\hbar\hat{\mathbf{e}}_{u,k}^{[\iota]}\Big]\\\le{}&\frac{\hbar^2\ell}{\kappa_\ell}\sum_{\iota=1}^{\ell-1}\Delta_\hbar\hat{\mathbf{e}}_{u,k}^{[\iota]T}H_\otimes\Delta_\hbar\hat{\mathbf{e}}_{u,k}^{[\iota]},\quad\forall k\in\mathbf{Z}_0.\end{aligned}\tag{25}$$

Considering Equation (20), we have

$$\begin{aligned}\varrho_k:={}&\frac{1}{\hbar}sym\Big\{\mathbf{e}_{u,k}^{[\ell]T}\Big[C_\varepsilon\mathcal{D}Q\mathcal{D}M_\varepsilon\Big]_\otimes\rho_k\Big\}\\\le{}&-\frac{1}{\hbar}\sum_{\iota=1}^{\ell-1}sym\Big\{\mathbf{e}_{u,k}^{[\ell]T}\Big[C_\varepsilon\mathcal{D}Q\mathcal{D}M_\varepsilon\Theta\Big]_\otimes\mathbf{e}_{u,k}^{[\iota]}\Big\}\\={}&-\frac{1}{\hbar}\sum_{\iota=1}^{\ell-1}sym\Big\{\hat{\mathbf{e}}_{u,k}^{[\iota]T}\Big[C_\varepsilon\mathcal{D}Q\mathcal{D}M_\varepsilon\Theta\Big]_\otimes\mathbf{e}_{u,k}^{[\iota]}\Big\}\\&-\frac{1}{\hbar}\sum_{\iota=1}^{\ell-1}sym\Big\{\mathbf{e}_{u,k}^{[\iota]T}\Big[C_\varepsilon\mathcal{D}Q\mathcal{D}M_\varepsilon\Theta\Big]_\otimes\mathbf{e}_{u,k}^{[\iota]}\Big\},\end{aligned}\tag{26}$$

for all $k\in\mathbf{Z}_0$.

Taking into account Equations (16)–(26), we obtain

$$\mathbb{E}[\Delta V_k]=\mathbb{E}[\Delta V_{1,k}]+\mathbb{E}[\Delta V_{2,k}]\le\mathbb{E}\sum_{\iota=1}^{\ell-1}\xi_k^{[\iota]T}\mathbb{O}\xi_k^{[\iota]},\quad\forall k\in\mathbf{Z}_0,\tag{27}$$

where $\xi_k^{[\iota]}:=\Big(\mathbf{e}_{u,k}^{[\iota]},\mathbf{e}_{v,k}^{[\iota]},\hat{\mathbf{e}}_{u,k}^{[\iota]},\Delta_\hbar\mathbf{e}_{u,k}^{[\iota]},\mathbf{F}(\mathbf{e}_{u,k}^{[\iota]}),\mathbf{G}(\mathbf{e}_{u,k}^{[\iota]})\Big)^T$ for $k\in\mathbf{Z}_0$, $\iota=1,2,\ldots,\ell$.

Based on Equation (27), we get

$$\mathbb{E}[\Delta V_k]\le\lambda_{\max}(\mathbb{O})\sum_{\iota=1}^{\ell-1}\Big[\mathbb{E}\Big\|\mathbf{e}_{u,k}^{[\iota]}\Big\|^2+\mathbb{E}\Big\|\mathbf{e}_{v,k}^{[\iota]}\Big\|^2\Big],\quad\forall k\in\mathbf{Z}_0.\tag{28}$$

With the help of Equation (13), we get

$$\mathbb{E}V_0\le\max\Big\{\lambda_{\max}(P_\otimes),\lambda_{\max}(Q_\otimes)\Big\}\mathbb{E}\sum_{\iota=1}^{\ell-1}\Big[\Big\|\mathbf{e}_{u,0}^{\langle\iota\rangle}\Big\|^2+\Big\|\mathbf{e}_{v,0}^{\langle\iota\rangle}\Big\|^2\Big]<\infty.\tag{29}$$

Noting that $\lambda_{\max}(\mathbb{O})<0$ owing to the assumption $\mathbb{O}<0$ in Theorem 1, we can use Equations (28) and (29) to arrive at

$$\lambda_{\max}(\mathbb{O})\sum_{k=1}^{k'-1}\sum_{\iota=1}^{\ell-1}\Big[\mathbb{E}\Big\|\mathbf{e}_{u,k}^{[\iota]}\Big\|^2+\mathbb{E}\Big\|\mathbf{e}_{v,k}^{[\iota]}\Big\|^2\Big]\ge\mathbb{E}V_{k'}-\mathbb{E}V_0\ge-\mathbb{E}V_0,$$

which is equal to

$$\sum_{k=1}^{k'-1}\sum_{\iota=1}^{\ell-1}\left[\mathbb{E}\left\|\mathbf{e}_{u,k}^{[\iota]}\right\|^2+\mathbb{E}\left\|\mathbf{e}_{v,k}^{[\iota]}\right\|^2\right] \leq -\frac{\mathbb{E}V_0}{\lambda_{\max}(\mathbb{O})}<\infty$$
$$\xrightarrow{k'\to\infty} \sum_{k=1}^{\infty}\sum_{\iota=1}^{\ell-1}\left[\mathbb{E}\left\|\mathbf{e}_{u,k}^{[\iota]}\right\|^2+\mathbb{E}\left\|\mathbf{e}_{v,k}^{[\iota]}\right\|^2\right]<\infty. \tag{30}$$

Then,

$$\lim_{k\to\infty}\sum_{\iota=1}^{\ell-1}\mathbb{E}\left\|\mathbf{e}_{u,k}^{[\iota]}\right\|^2=0=\lim_{k\to\infty}\sum_{\iota=1}^{\ell-1}\mathbb{E}\left\|\mathbf{e}_{v,k}^{[\iota]}\right\|^2,$$

which implies that model Equation (11) achieves global mean-squared asymptotic stability. This completes the proof. □

From Lemma 4, the following inequality is valid:

$$\frac{\mu_{\ell-1}}{\hbar^2}\mathbb{E}\sum_{\iota=1}^{\ell-1}\Delta_\hbar\mathbf{e}_{u,k}^{[\iota]T}\left[M_\varepsilon^T\mathcal{D}\mathcal{Q}\mathcal{D}M_\varepsilon\right]_\otimes\Delta_\hbar\mathbf{e}_{u,k}^{[\iota]} \leq \frac{4-\kappa_\ell}{\hbar^2}\frac{\mu_{\ell-1}}{\hbar^2}\mathbb{E}\sum_{\iota=1}^{\ell-1}\hat{\mathbf{e}}_{u,k}^{[\iota]T}\left[M_\varepsilon^T\mathcal{D}\mathcal{Q}\mathcal{D}M_\varepsilon\right]_\otimes\hat{\mathbf{e}}_{u,k}^{[\iota]},$$

where $k \in \mathbf{Z}_0$. Further,

$$\mathbb{E}[\Delta V_k] \leq \mathbb{E}\sum_{\iota=1}^{\ell-1}\xi_k^{[\iota]T}\tilde{\mathbb{O}}\xi_k^{[\iota]}, \quad \forall k\in\mathbf{Z}_0, \tag{31}$$

here $\tilde{\mathbb{O}}=(\tilde{\mathbb{O}}_{ij})_{1\leq i,j\leq 6}$ is defined as $\mathbb{O}$ defined in Theorem 1, except that

$$\tilde{\mathbb{O}}_{33}=-H_\otimes+\frac{4(1-\beta)(4-\kappa_\ell)\mu_{\ell-1}}{\hbar^4}\left[M_\varepsilon^T\mathcal{D}\mathcal{Q}\mathcal{D}M_\varepsilon\right]_\otimes,$$

$$\tilde{\mathbb{O}}_{44}=-sym\left[C_\varepsilon\mathcal{D}\mathcal{Q}\mathcal{D}M_\varepsilon\right]_\otimes+\frac{4\mu_{\ell-1}\beta}{\hbar^2}\left[M_\varepsilon^T\mathcal{D}\mathcal{Q}\mathcal{D}M_\varepsilon\right]_\otimes+\frac{\hbar^2\ell}{\kappa_\ell}H_\otimes.$$

So, we have the following:

Corollary 1. *Assuming that* **(F)** *is valid, we pre-give values of* $\varepsilon>0$ *and* $\beta\in[0,1]$. *Additionally, we assume that* $\mathcal{D}$ *and* M_ε *are nonsingular, and we define* Θ *as indicated in Theorem 1. Under these conditions, the slave INNs Equation (1) stochastically synchronize with the master INNs Equation (5), meaning that the model Equation (11) achieves global mean-squared asymptotic stability. This holds true if the model has positive constants* λ_f,λ_g, *and positive definite n-order matrices P, Q, H, and K such that the* $\tilde{\mathbb{O}}$ *matrix defined in Equation (31) is negative definite.*

Remark 1. *Reports [22,24] addressed the issues of synchronization for inertial neural networks with reaction-diffusion terms. However, the networks in reports [22,24] were involved in the Dirichlet boundary condition and the controller is embedded in the model of the networks. In this article, the controller does not exist in the model of the networks, but it is designed in the boundary.*

3.2. Passivity-Based Control

The error vector networks described by Equation (11) with respect to a supply rate can be represented as

$$\varpi(\mathcal{Y},\gamma):=\sum_{\iota=1}^{\ell-1}\mathcal{Y}^{[\iota]T}\gamma^{[\iota]} \text{ for some } \mathcal{Y}\in\mathbb{R}^{Nn}. \tag{32}$$

This system is stochastically passive if there exists a nonnegative mapping θ that satisfies

$$\mathbb{E}\sum_{k=s_1}^{s_2-1}\sum_{\iota=1}^{\ell-1}\mathcal{Y}_k^{[\iota]T}\gamma_k^{[\iota]} \geq \theta(s_2)-\theta(s_1), \quad \forall s_1 < s_2, s_1, s_2 \in \mathbf{Z}_0.$$

Theorem 2. *Let Hypothesis* **(F)** *be satisfied, $\varepsilon > 0$ be given, and $\mathcal{D}$, M_ε be nonsingular. Additionally, let the controller gain Θ be as provided in Theorem 1. The error networks Equation (11) are stochastically passive if there exist positive constants λ_f, λ_g and n-order positive definite matrices P, Q, H, K, $\Re_1$, $\Re_2$, $\Re_3$ such that*

$$\mathcal{O} := \begin{bmatrix} \mathcal{O}_{11} & \mathcal{O}_{12} & \mathcal{O}_{13} & \mathcal{O}_{14} & \mathcal{O}_{15} & \mathcal{O}_{16} & \mathcal{O}_{17} \\ * & \mathcal{O}_{22} & \mathcal{O}_{23} & \mathcal{O}_{24} & \mathcal{O}_{25} & \mathcal{O}_{26} & \mathcal{O}_{27} \\ * & * & \mathcal{O}_{33} & \mathcal{O}_{34} & \mathcal{O}_{35} & \mathcal{O}_{36} & \mathcal{O}_{37} \\ * & * & * & \mathcal{O}_{44} & \mathcal{O}_{45} & \mathcal{O}_{46} & \mathcal{O}_{47} \\ * & * & * & * & \mathcal{O}_{55} & \mathcal{O}_{56} & \mathcal{O}_{57} \\ * & * & * & * & * & \mathcal{O}_{66} & \mathcal{O}_{67} \\ * & * & * & * & * & * & \mathcal{O}_{77} \end{bmatrix} < 0,$$

where

$$\mathcal{O}_{44} = \mathbb{O}_{44} + \frac{\mu_{\ell-1}}{\hbar^2}\left[M_\varepsilon^T\mathcal{D}Q\mathcal{D}M_\varepsilon\right]_\otimes, \quad \mathcal{O}_{17} = -(\Re_1)_\otimes + \left[C_\varepsilon\mathcal{D}Q\mathcal{D}\Lambda_\varepsilon\right]_\otimes,$$

$$\mathcal{O}_{27} = -(\Re_2)_\otimes + \left[e^{-D_\circ h}Q\mathcal{D}\Lambda_\varepsilon\right]_\otimes + \alpha\left[\Gamma^T\mathcal{D}Q\mathcal{D}\Lambda_\varepsilon\right]_{\otimes B^T}, \quad \mathcal{O}_{57} = \left[A_\varepsilon^T\mathcal{D}Q\mathcal{D}\Lambda_\varepsilon\right]_\otimes,$$

$$\mathcal{O}_{77} = -2(\Re_3)_\otimes + 2\left[\Lambda_\varepsilon^T\mathcal{D}Q\mathcal{D}\Lambda_\varepsilon\right]_\otimes, \quad \mathcal{O}_{37} = \mathcal{O}_{47} = \mathcal{O}_{67} = 0,$$

and the other unmentioned block matrices $\mathcal{O}_{ij}$ in $\mathcal{O}$ are equal to $\mathbb{O}_{ij}$ in $\mathbb{O}$ for $i, j = 1, 2, \ldots, 6$.

Proof. Define the Lyapunov-Krasovskii function V for the error vector networks Equation (11), following the approach described in Section 3.1. Additionally, introduce an output vector $\mathcal{Y} \in \mathbb{R}^{Nn}$ to the error vector networks Equation (11) using the expression

$$\mathcal{Y} = (I_N \otimes \Re_1)\mathbf{e}_u + (I_N \otimes \Re_2)\mathbf{e}_v + (I_N \otimes \Re_3)\gamma.$$

Similar to the argument in Equation (17), we get

$$\begin{aligned}\mathbb{E}[V_{2,k+1}] = \sum_{i=1}^{16}\mathcal{U}_{i,k} &+ \underbrace{\mathbb{E}\sum_{\iota=1}^{\ell-1} sym\left\{\mathbf{e}_{v,k}^{[\iota]T}\left[e^{-D_\circ h}Q\mathcal{D}\Lambda_\varepsilon\right]_\otimes\gamma_k^{[\iota]}\right\}}_{\mathcal{U}_{17,k}} \\ &+ \underbrace{\mathbb{E}\sum_{\iota=1}^{\ell-1} sym\left\{\Delta_\hbar^2\mathbf{e}_{u,k}^{[\iota-1]T}\left[M_\varepsilon^T\mathcal{D}Q\mathcal{D}\Lambda_\varepsilon\right]_\otimes\gamma_k^{[\iota]}\right\}}_{\mathcal{U}_{18,k}} \\ &+ \underbrace{\mathbb{E}\sum_{\iota=1}^{\ell-1} sym\left\{\mathbf{e}_{u,k}^{[\iota]T}\left[C_\varepsilon\mathcal{D}Q\mathcal{D}\Lambda_\varepsilon\right]_\otimes\gamma_k^{[\iota]}\right\}}_{\mathcal{U}_{19,k}}\end{aligned}$$

$$+\underbrace{\mathbb{E}\sum_{\iota=1}^{\ell-1} sym\Big\{\mathbf{F}^T(\mathbf{e}_{u,k}^{[\iota]})\Big[A_\varepsilon^T\mathcal{D}Q\mathcal{D}\Lambda_\varepsilon\Big]_\otimes\gamma_k^{[\iota]}\Big\}}_{\mathcal{U}_{20,k}}$$

$$+\underbrace{\alpha\mathbb{E}\sum_{\iota=1}^{\ell-1} sym\Big\{\mathbf{e}_{v,k}^{[\iota]T}\Big[\Gamma^T\mathcal{D}Q\mathcal{D}\Lambda_\varepsilon\Big]_{\otimes B^T}\gamma_k^{[\iota]}\Big\}}_{\mathcal{U}_{21,k}}$$

$$+\underbrace{\mathbb{E}\sum_{\iota=1}^{\ell-1}\gamma_k^{[\iota]T}\Big[\Lambda_\varepsilon^T\mathcal{D}Q\mathcal{D}\Lambda_\varepsilon\Big]_\otimes\gamma_k^{[\iota]}}_{\mathcal{U}_{22,k}},\quad \forall k\in\mathbf{Z}_0. \tag{33}$$

Meanwhile, similar to the estimates in inequalities Equations (18)–(23), we obtain from Equation (33) the following:

$$\begin{aligned}\mathcal{U}_{18,k} &\le \mathbb{E}\sum_{\iota=1}^{\ell-1}\Delta_\hbar^2\mathbf{e}_{u,k}^{[\iota-1]T}\Big[M_\varepsilon^T\mathcal{D}Q\mathcal{D}M_\varepsilon\Big]_\otimes\Delta_\hbar^2\mathbf{e}_{u,k}^{[\iota-1]}+\mathbb{E}\sum_{\iota=1}^{\ell-1}\gamma_k^{[\iota]T}\Big[\Lambda_\varepsilon^T\mathcal{D}Q\mathcal{D}\Lambda_\varepsilon\Big]_\otimes\gamma_k^{[\iota]}\\ &\le \frac{\mu_{\ell-1}}{\hbar^2}\mathbb{E}\sum_{\iota=1}^{\ell-1}\Delta_\hbar\mathbf{e}_{u,k}^{[\iota]T}\Big[M_\varepsilon^T\mathcal{D}Q\mathcal{D}M_\varepsilon\Big]_\otimes\Delta_\hbar\mathbf{e}_{u,k}^{[\iota]}+\mathbb{E}\sum_{\iota=1}^{\ell-1}\gamma_k^{[\iota]T}\Big[\Lambda_\varepsilon^T\mathcal{D}Q\mathcal{D}\Lambda_\varepsilon\Big]_\otimes\gamma_k^{[\iota]},\end{aligned} \tag{34}$$

$\forall k\in\mathbf{Z}_0$.

By employing Equations (16)–(26) and (33) and (34), we can compute

$$\mathbb{E}[\Delta V_k]-2\mathbb{E}\sum_{\iota=1}^{\ell-1}\mathcal{Y}_k^{[\iota]T}\gamma_k^{[\iota]}\le\mathbb{E}\sum_{\iota=1}^{\ell-1}\eta_k^{[\iota]T}\mathcal{O}\eta_k^{[\iota]},\quad\forall k\in\mathbf{Z}_0, \tag{35}$$

where $\eta_k^{[\iota]}:=\Big[\mathbf{e}_{u,k}^{[\iota]},\mathbf{e}_{v,k}^{[\iota]},\hat{\mathbf{e}}_{u,k}^{[\iota]},\Delta_\hbar\mathbf{e}_{u,k}^{[\iota]},\mathbf{F}(\mathbf{e}_{u,k}^{[\iota]}),\mathbf{G}(\mathbf{e}_{u,k}^{[\iota]}),\gamma_k^{[\iota]}\Big]^T$ for $k\in\mathbf{Z}_0$, $\iota=1,2,\ldots,\ell$.

In accordance with Equation (35), we get

$$2\mathbb{E}\sum_{\iota=1}^{\ell-1}\mathcal{Y}_k^{[\iota]T}\gamma_k^{[\iota]}\ge\mathbb{E}[\Delta V_k],$$

which is equal to

$$2\mathbb{E}\sum_{k=s_1}^{s_2-1}\sum_{\iota=1}^{\ell-1}\mathcal{Y}_k^{[\iota]T}\gamma_k^{[\iota]}\ge\mathbb{E}V_{s_2}-\mathbb{E}V_{s_1},\quad\forall s_1<s_2,s_1,s_2\in\mathbf{Z}_0.$$

Accordingly, INNs Equation (11) is stochastic passive. This completes the proof. □

So, we have the following:

Corollary 2. *Assuming that* **(F)** *is satisfied,* $\varepsilon>0$ *and* $\beta\in[0,1]$ *are pre-given,* $\mathcal{D}$ *and* M_ε *are nonsingular, and the controller gain* Θ *is provided in Theorem 1, the error network Equation (11) is stochastically passive if there exist positive constants* λ_f, λ_g, *and n-order positive definite matrices* P, Q, H, K, $\Re_1$, $\Re_2$, *and* $\Re_3$ *such that* $\tilde{\mathcal{O}}<0$. *Here,* $\tilde{\mathcal{O}}=(\tilde{\mathcal{O}}ij)1\le i,j\le 7$ *is defined as* $\mathcal{O}$ *in Theorem 2 except for the following modifications:*

$$\tilde{\mathcal{O}}_{33}=-H_\otimes+\frac{5(1-\beta)(4-\kappa_\ell)\mu_{\ell-1}}{\hbar^4}\Big[M_\varepsilon^T\mathcal{D}Q\mathcal{D}M_\varepsilon\Big]_\otimes,$$

$$\tilde{\mathcal{O}}_{44}=-sym\Big[C_\varepsilon\mathcal{D}Q\mathcal{D}M_\varepsilon\Big]_\otimes+\frac{5\mu_{\ell-1}\beta}{\hbar^2}\Big[M_\varepsilon^T\mathcal{D}Q\mathcal{D}M_\varepsilon\Big]_\otimes+\frac{\hbar^2\ell}{\kappa_\ell}H_\otimes.$$

According to Theorems 1 and 2, a realizable algorithm for stochastic synchronization or passivity of INNs Equations (1) and (5) is designed as Algorithm 1, and its O-chart is described in Figure 1.

Algorithm 1 Stochastic synchronization or passivity of INNs Equations (1) and (5)

(1) Initialize the values of the coefficient matrices in INNs Equations (1) and (5)
(2) Compute LMIs in Theorems 1 or 2. When they are unviable, modify the values of coefficient matrices in INNs Equation (1); otherwise, switch to next step.
(3) Receive the values of matrices P, Q, K, etc. Calculate the controller gain $\Theta = \left[\mathcal{D}Q\mathcal{D}M_\varepsilon\right]^{-1}K$.
(4) Write iterative program based on INNs Equations (1) and (5) and plot the response trajectories.

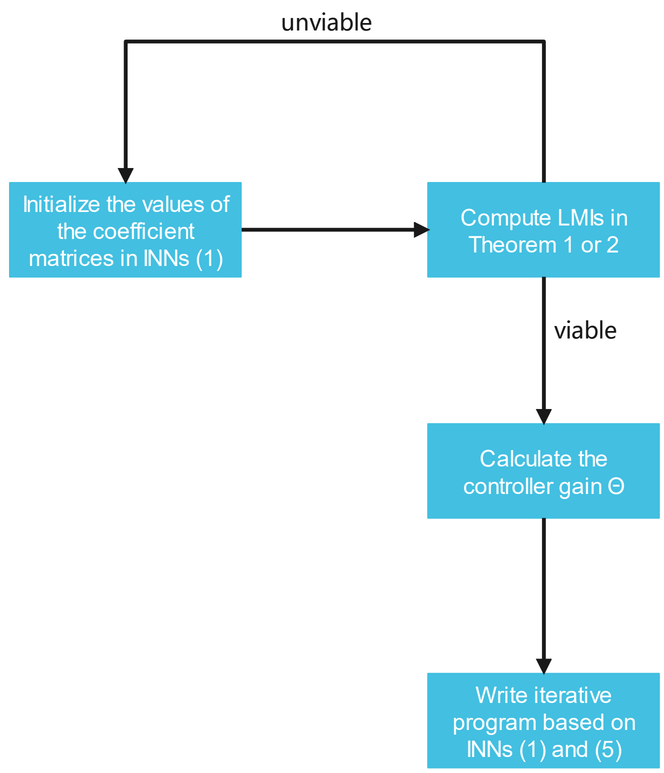

Figure 1. O-chart of Algorithm 1.

Remark 2. *Papers [44,45] investigated the passivity of inertial neural networks without reaction-diffusion terms. This paper considers the effects of the reaction diffusions, which complements the works in the literature [44,45].*

4. Numerical Example

In view of INNs Equation (1), we take $\alpha = 0.1$, $J = (10, 12)^T$,

$$D = 50\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = 45\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad M = 0.1\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad A = 0.1\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix},$$

$$B = 0.1\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}, \quad \Gamma = 0.01\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad \Xi = 0.01\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Taking $\varepsilon = 0.1$, $h = 0.01$, $\hbar = 0.2$, $\ell = 25$, $f(x) = (f_1(x), f_2(x))^T = 0.1(\sin x_1, |x_2|)^T = (g_1(x), g_2(x))^T = g(x)$ for any $x = (x_1, x_2)^T \in \mathbb{R}^2$. From Theorem 1, we can determine that $\lambda_f = 32693$, $\lambda_g = 32686$,

$$P = \begin{bmatrix} 1.2059 & -0.0142 \\ -0.0142 & 1.6238 \end{bmatrix} \times 10^5, \quad Q = \begin{bmatrix} 2.5836 & -0.0082 \\ -0.0082 & 1.5422 \end{bmatrix} \times 10^4,$$

$$H = \begin{bmatrix} 6.9393 & 3.4242 \\ 3.4242 & 5.5926 \end{bmatrix}, \quad K = \begin{bmatrix} 0.0197 & 0.0049 \\ 0.0049 & 0.0042 \end{bmatrix}.$$

In addition,

$$\Theta = \begin{bmatrix} 0.0164 & 0.0026 \\ 0.0026 & 0.0022 \end{bmatrix}.$$

By Theorem 1, INNs Equations (1) and (5) realize stochastic synchronization, see Figures 2–5.

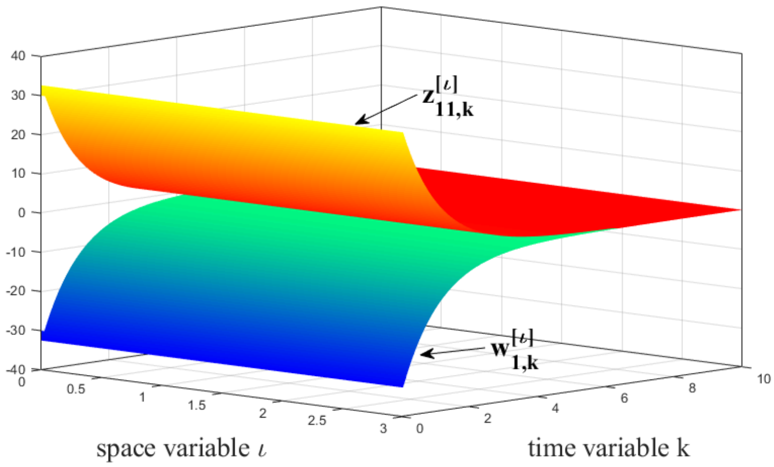

Figure 2. Stochastic synchronization to INNs Equations (1) and (5).

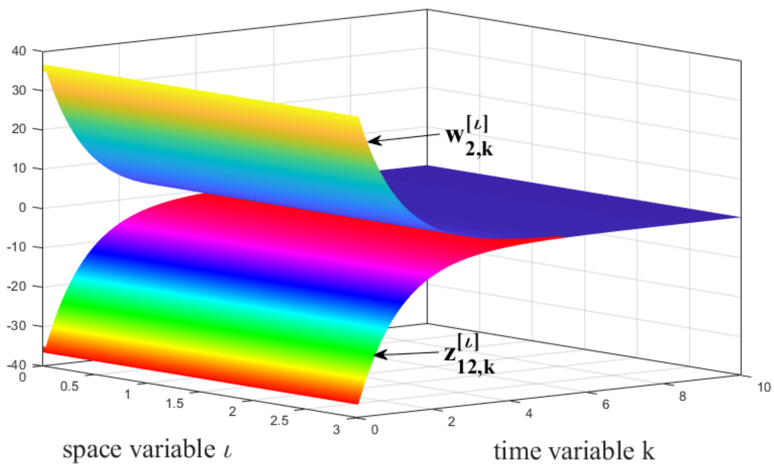

Figure 3. Stochastic synchronization to INNs Equations (1) and (5).

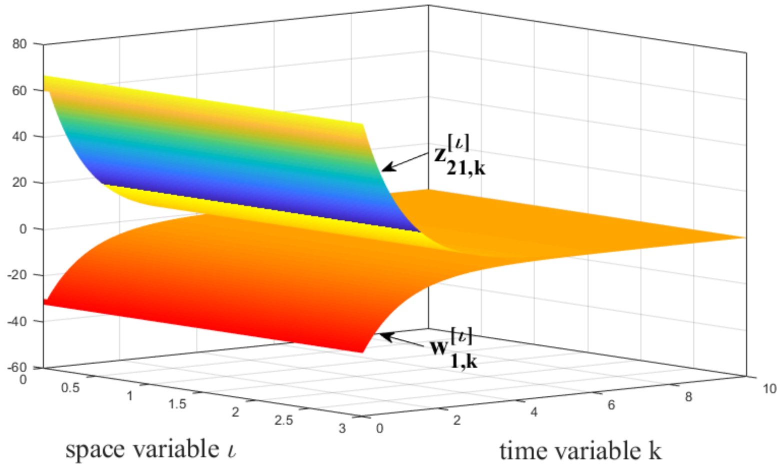

Figure 4. Stochastic synchronization to INNs Equations (1) and (5).

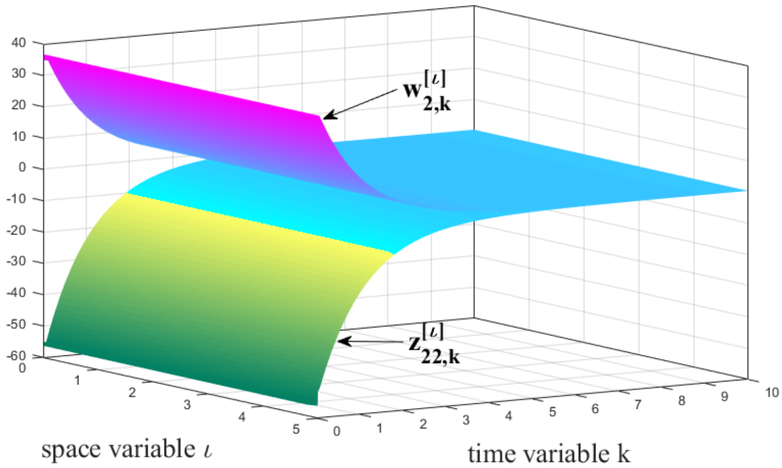

Figure 5. Stochastic synchronization to INNs Equations (1) and (5).

Furthermore, taking $\Lambda = 0.1\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $\gamma^{[\iota]}_{1,k} = (10+\sin(\iota+k), 8+\cos(\iota+k))^T$, $\gamma^{[\iota]}_{2,k} = (10+\sin(2\iota+k), 8+\cos(2\iota+k))^T$, $\forall k \in \mathbf{Z}_0$, $\iota = 1,2,\ldots,\ell$. The output vector $\mathcal{Y} \in \mathbb{R}^4$ for the network is defined as in Equation (32) with the following matrices:

$$\Re_1 = \begin{bmatrix} 183.9618 & -0.2256 \\ -0.2256 & 173.1908 \end{bmatrix}, \Re_2 = \begin{bmatrix} 376.9862 & 0.1127 \\ 0.1127 & 167.3857 \end{bmatrix}, \Re_3 = \begin{bmatrix} 1617.7 & -0.1 \\ -0.1 & 1721 \end{bmatrix}.$$

By Theorem 2, we have $\lambda_f = 1825.8$, $\lambda_g = 1825.7$,

$$P = \begin{bmatrix} 8041.2 & 71.1 \\ 71.1 & 8871.3 \end{bmatrix}, \quad Q = \begin{bmatrix} 1374.2 & 8.6 \\ 8.6 & 635.3 \end{bmatrix},$$

$$H = \begin{bmatrix} 0.4183 & 0.1974 \\ 0.1974 & 0.2709 \end{bmatrix}, \quad K = \begin{bmatrix} 0.0011 & 0.0005 \\ 0.0005 & 0.0011 \end{bmatrix}.$$

Now, the controller gain of the boundary controller is given by

$$\Theta = \begin{bmatrix} 0.0147 & -0.0058 \\ 0.0059 & 0.0142 \end{bmatrix}.$$

According to Theorem 2, INNs Equations (1) and (5) achieve stochastic passivity, as in Figures 6–11.

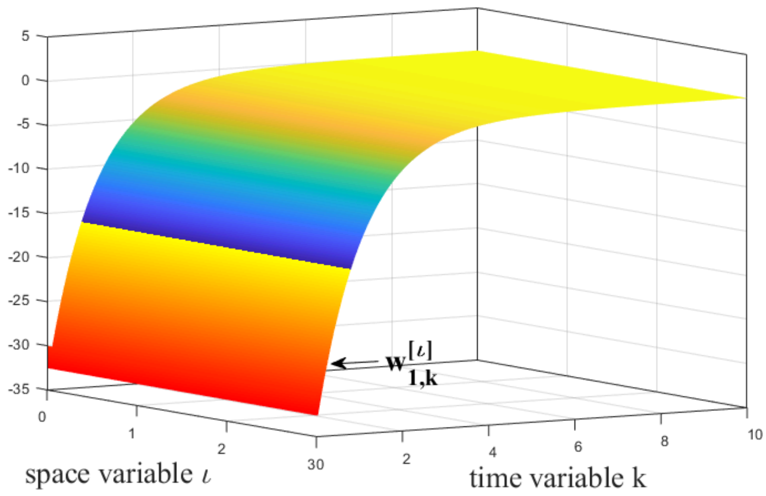

Figure 6. Trajectory of state variable $\mathbf{w}_1$ to INNs Equation (5).

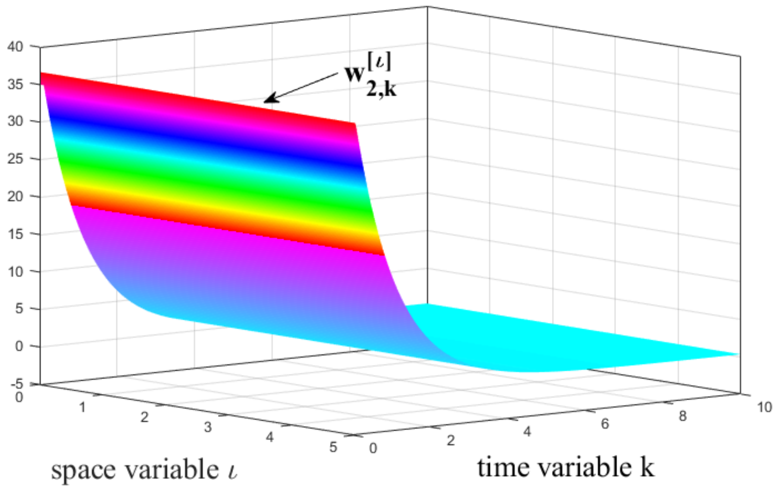

Figure 7. Trajectory of state variable $\mathbf{w}_2$ to INNs Equation (5).

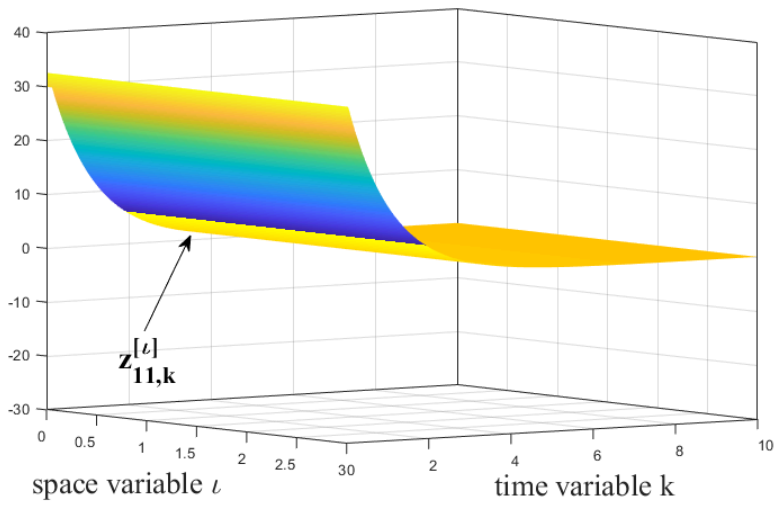

Figure 8. Trajectory of state variable $\mathbf{z}_{11}$ to INNs Equation (1).

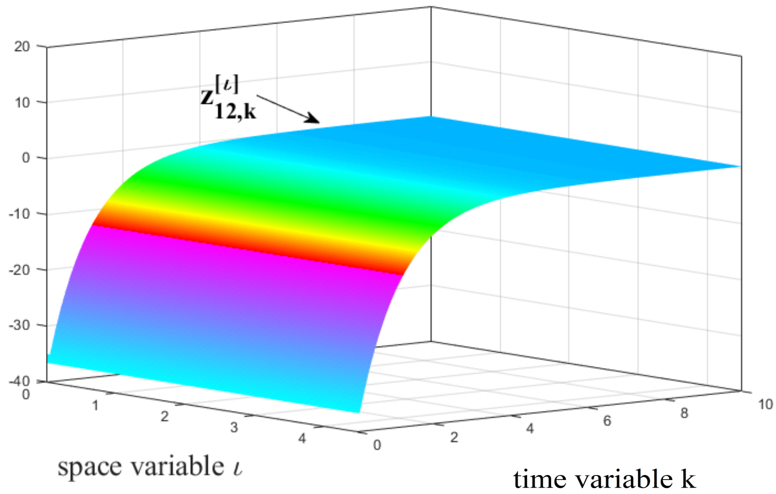

Figure 9. Trajectory of state variable $\mathbf{z}_{12}$ to INNs Equation (1).

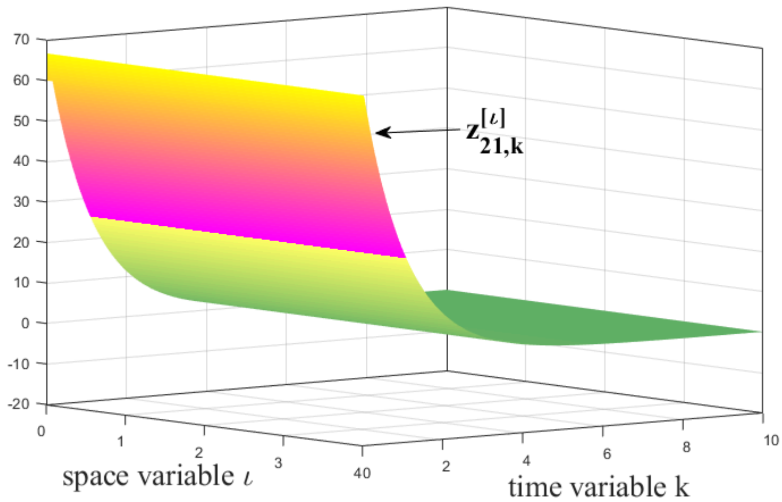

Figure 10. Trajectory of state variable $\mathbf{z}_{21}$ to INNs Equation (1).

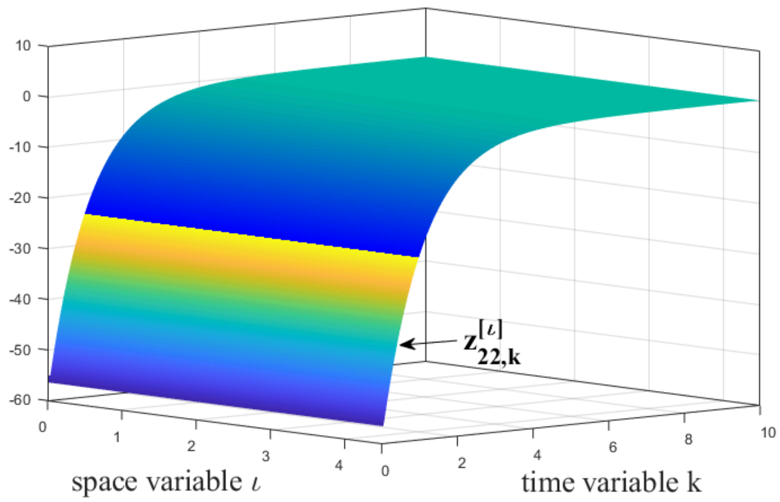

Figure 11. Trajectory of state variable $\mathbf{z}_{22}$ to INNs Equation (1).

Remark 3. *In the previous work in article [38], the authors discussed passivity of non-autonomous discrete-time inertial neural networks, overlooking discrete spatial diffusions. By contrast, the present literature addresses it, as can be seen in Figures 6–11.*

5. Conclusions and Future Works

For the first time, this discussion focuses on investigating discrete SINNs with the influence of spatial diffusions.

Firstly, we present the time and space difference model of SINNs with reaction diffusions using the time and space difference approaches, respectively.

Secondly, with the aid of a controller designed at the boundary, we address the issues of both stochastic synchronization and passivity-based control, employing the Lyapunov-Krasovskii function method.

As anticipated, we provide decision theorems for the aforementioned research topics concerning discrete SINNs. It is important to note that the method employed in this article predominantly considers homogeneous networks described by INNs Equations (1) and (5), making the study of heterogeneous networks challenging (see ref. [46]).

Moving forward, several aspects merit consideration in future work:

- Fractional dynamics has become a research hotspot in recent years, which could be discussed in the SINNs of this article.
- This paper only considers 1-dimensional space variables, which could be extended to higher dimensions.
- Exploration of alternative control techniques, such as impulsive controls and adaptive controls, holds promise for further investigation.

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