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Equitable Coloring of IC-Planar Graphs with Girth $g \geq 7$

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Abstract: An equitable k -coloring of a graph G is a proper vertex coloring such that the size of any two color classes differ at most 1. If there is an equitable k -coloring of G , then the graph G is said to be equitably k -colorable. A 1-planar graph is a graph that can be embedded in the Euclidean plane such that each edge can be crossed by other edges at most once. An IC-planar graph is a 1-planar graph with distinct end vertices of any two crossings. In this paper, we will prove that every IC-planar graph with girth $g \geq 7$ is equitably $\Delta(G)$ -colorable, where $\Delta(G)$ is the maximum degree of G .

Keywords: equitable k -coloring; IC-planar graph; discharging method; girth

MSC: 05C15

1. Introduction

Chromatic graph theory originates from the Four Color Conjecture [1], is a focal problem in graph theory. As a special case of chromatic graph theory, equitable coloring has been widely used in industrial production, enterprise management and biology [2]. Especially, it plays a crucial role in the study of schedule [3], partition [4–7] and load balancing [8]. In this paper, we mainly discuss the equitable coloring of IC-planar graphs with girth $g \geq 7$. Some relevant definitions are as follows.

Only undirected, finite and simple graphs are considered in this paper. In a graph G , $V(G)$, $E(G)$, $|G|$, $\delta(G)$ and $\Delta(G)$ (or Δ in short) are used to denote the vertex set, edge set, order, minimum degree and maximum degree of G , respectively. The girth of a graph G is the length of shortest cycles of G , denoted by $g(G)$. For $v \in V(G)$, we use $d_G(v)$ to denote the *degree* of a vertex v in G . The set of all neighbors of v is denoted by $N_G(v)$. Trivially, $|N_G(v)| = d_G(v)$.

A graph that can be drawn in the Euclidean plane such that any two edges intersect only at their ends is called the *planar graph*. A planar graph with such a particular drawing is called a *plane graph*. The face set of a plane graph G is denoted by $F(G)$. For $f \in F(G)$, we use $d_G(f)$ to denote the *degree* of a face f in G . That is, the number of edges on the boundary of f . We denote $f = [v_1 v_2 \cdots v_k]$ when v_1, v_2, \cdots, v_k are the vertices on the boundary of f and arranged in clockwise order. In a graph G , a vertex is called as a k -vertex, a k^- -vertex or a k^+ -vertex if its degree is equal to k , at most k or at least k , respectively. Similarly, we can define a k -face, a k^- -face or a k^+ -face. For a k -face $f \in F(G)$, we denote f as (d_1, d_2, \cdots, d_k) -face if the vertices on the boundary of f are u_1, u_2, \cdots, u_k such that $d(u_i) = d_i$, $1 \leq i \leq k$.

A 1-planar graph is a graph that can be embedded in the Euclidean plane such that each edge can be crossed by other edges at most once. An IC-planar graph is a 1-planar graph with distinct end vertices of any two crossings, which is introduced by Alberson [9] in 2008.

A proper k -coloring of a graph G is a mapping $\phi: V(G) \rightarrow \{1, 2, \cdots, k\}$ such that $\phi(u) \neq \phi(v)$ for any two adjacent vertices u and v . The minimum positive integer k such that G has a proper k -coloring is called the *chromatic number* of G , denoted by $\chi(G)$. And we use V_i ($1 \leq i \leq k$) to denote the set of vertices colored with i . Notice that V_i ($1 \leq i \leq k$) is an independent set.



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Definition 1. For a proper k -coloring ϕ of G , if $||V_i| - |V_j|| \leq 1$ for any $i, j \in \{1, 2, \dots, k\}$, then ϕ is an equitable k -coloring of G , or G is equitably k -colorable. The minimum positive integer k such that G has an equitable k -coloring is called the equitable chromatic number of G , denoted by $\chi_e(G)$.

Obviously, $\chi_e(G) \geq \chi(G)$, and the inequality can be strictly held.

The rest of this article is organized as follows: In Section 2, we will introduce the history and recent progress on equitable coloring, and we also present our motivation and contribution of this paper. In Section 3, we provide some lemmas which are helpful to prove our main theorem. In Section 4, we discuss the structures of IC-planar graphs with girth $g \geq 7$ and get an important property, which is presented in Lemma 5. In Section 5, we use discharging method to prove Lemma 5, which is used to prove Lemma 6 and Theorem 1. In Section 6, to show Theorem 1, we first give proof of Lemma 6 and then get a corollary. Finally, we summarize our result and get the main theorem.

2. Related Work

In 1973, Meyer [10] introduced the concept of equitable coloring. At the same time, he proposed the following conjecture.

Conjecture 1 ([10]). If G is a connected graph except C_{2m+1} or K_m , then $\chi_e(G) \leq \Delta$.

In 1964, Erdős [11] conjectured that every graph is equitably k -colorable for any $k \geq \Delta + 1$, which was confirmed by Hajnal and Szemerédi [12] in 1970. In 2010, by applying algorithm analysis, Kierstead et al. [13] gave a new and short proof of Erdős's conjecture. So we are interested in the sufficient conditions for graphs to be equitably Δ -colorable. In 1994, Chen, Lih and Wu [14] put forth the following conjecture.

Conjecture 2 ([14]). If G is a connected graph except C_{2m+1} , K_m or $K_{2m+1, 2m+1}$ for all $m \geq 1$, then G is equitably Δ -colorable.

The authors [14] also confirmed Conjecture 2 for all connected graphs with $\Delta \leq 3$, which was strengthened to graphs with $\Delta \leq 4$ by Kierstead and Kostochka [15]. Chen and Lih [16], Lih and Wu [17] showed Conjecture 2 holds for trees and bipartite graphs, respectively. Kostochka [18] proved that Conjecture 2 holds for outerplanar graphs. Kostochka and Nakprasit [19] proved further that Conjecture 2 holds for d -degenerate graphs with $\Delta \geq 14d + 1$. Wang and Zhang [20] showed Conjecture 2 holds for line graphs. In 1998, Zhang and Yap [21] confirmed Conjecture 2 for planar graphs with $\Delta \geq 13$. In 2012, Nakprasit [22] improved the above result by showing that Conjecture 2 holds for planar graphs with $\Delta \geq 9$. Recently, this result has been extended to planar graphs with $\Delta \geq 8$ by Kostochka, Lin and Xiang [23]. Therefore, for planar graphs, only the case of $5 \leq \Delta \leq 7$ remains unsolved. In 2008, Zhu and Bu [24] proved that every planar graph without 4-cycles and 5-cycles is equitably k -colorable for $k \geq \max\{\Delta, 7\}$. Later, the above result was extended to all planar graphs without 4-cycles and 5-cycles by Wang and Gui [25]. Thus, we have every planar graph with girth $g \geq 6$ is equitably Δ -colorable. In 2016, Zhang [26] first considered the equitable coloring of 1-planar graphs and obtained that Conjecture 2 holds when $\Delta \geq 17$, which was improved to $\Delta \geq 15$ by Zhang, Wang and Xu [27] in 2018. At the same time, Zhang, Wang and Xu [27] also confirmed that Conjecture 2 holds for IC-planar graphs with $\Delta \geq 12$.

Similar with planar graphs, we also want to see whether Conjecture 2 holds for IC-planar graphs with girth limitation. In this paper, we will concern on the equitable coloring of IC-planar graphs with large girth g and will prove the result in the following:

Theorem 1. Every IC-planar graph with girth $g \geq 7$ is equitably Δ -colorable.

3. Preliminaries

In this section, we present some helpful lemmas that will be used to prove our main result.

Lemma 1 ([24]). Let $S = \{v_1, v_2, \dots, v_k\}$ be a subset of $V(G)$, where v_1, v_2, \dots, v_k are distinct vertices. If $G - S$ is equitably k -colorable, and $|N_G(v_i) - S| \leq k - i$ for $1 \leq i \leq k$, then G is equitably k -colorable.

Lemma 2 ([12,13]). For $k \geq \Delta + 1$, every graph is equitably k -colorable.

Lemma 3 ([28]). If G is a planar graph of order n and with no cycles of length 3 and 4, then $e(G) \leq \frac{5}{3}n - \frac{10}{3}$.

Lemma 4. Let G be an IC-planar graph of order n and with no cycles of length 3 and 4, then $e(G) \leq \frac{23}{12}n - \frac{10}{3}$ and G is 3-degenerate.

Proof. We denote the graph obtained by deleting one of the edges from each crossing as G' . By the definition of IC-planar graph, we delete at most $\frac{n}{4}$ edges. Obviously, G' is a planar graph of order n and with no cycles of length 3 and 4. So $e(G') \leq \frac{5}{3}n - \frac{10}{3}$ by Lemma 3. Thus, $e(G) \leq \frac{5}{3}n - \frac{10}{3} + \frac{n}{4} = \frac{23}{12}n - \frac{10}{3}$.

By Handshaking Theorem, $\delta(G) \cdot n \leq \sum_{v \in V(G)} d(v) = 2e(G) \leq \frac{23}{6}n - \frac{20}{3}$, which implies that $\delta(G) \leq 3$. Thus, G is 3-degenerate. \square

4. Properties of IC-Planar Graphs with Girth $g \geq 7$

Every IC-planar graph with girth $g \geq 7$ in the following is assumed to be embedded in the Euclidean plane with the number of crossings as small as possible. The set of all crossing vertices and the edges with no crossings are denoted by $C(G)$ and $E_0(G)$, respectively. Let $E_1(G) = \{xz, zy \mid xy \in E(G) \setminus E_0(G), \text{ and } z \text{ is the crossing vertex on edge } xy\}$. The associated plane graph of G , denoted by G^\times , is a plane graph such that $V(G^\times) = V(G) \cup C(G)$, and $E(G^\times) = E_0(G) \cup E_1(G)$. In G^\times , a vertex v is called *true* if $v \in V(G)$, and *false* if $v \in C(G)$. Obviously, a false vertex v satisfies $d_{G^\times}(v) = 4$. A face f in G^\times is *false* if f is incident with at least one false vertex; otherwise, f is *true*. Let f be a false i -face and v be a true vertex, where $i \in \{3, 4\}$, f is called the pendant false i -face of v if v is not incident with f but is adjacent to the false vertex incident with f . Finally, we use $f_i(v)$, $n_i(G)$ and $n_i^G(v)$ to denote the number of i -faces incident with the vertex v in the associated plane graph G^\times , the number of i -vertices in the graph G and the number of i -vertices adjacent to the vertex v in the graph G , respectively.

It is easy to check that the following two properties hold by the fact that G is the IC-planar graph with girth $g \geq 7$.

Fact 1 (1) Any two false vertices are not adjacent.

(2) A true vertex is adjacent to at most one false vertex.

Fact 2 A true vertex is incident with at most one false 3-face.

Next, let's introduce some facts, which is also involved in the proofs of the next lemma. Since the IC-planar graph has been embedded with minimal number of crossings, the following fact holds.

Fact 3 A false 3-face is not incident with any 2-vertex.

Suppose that v is a false vertex. Let v_i ($1 \leq i \leq 4$) be the vertices adjacent to v in G^\times and arranged in clockwise order. Let $f_i = [v_{i+1}vv_i \dots]$ be the false face with vv_i and vv_{i+1} as boundary edges, where $1 \leq i \leq 4$ and $v_5 = v_1$. Note that $v_1v_3 \in E(G)$ and $v_2v_4 \in E(G)$.

Fact 4 Let v be a false vertex. If $f_3(v) = 1$, say $d_{G^\times}(f_1) = 3$, then $d_{G^\times}(f_i) \geq 6$, where $i \in \{2, 3, 4\}$. Hence, v is incident with at most one false 3-face.

Proof. Let $f_1 = [v_2vv_1]$. By Fact 2, we have $d_{G^\times}(v_1) \geq 3$ and $d_{G^\times}(v_2) \geq 3$. Since $v_1v_3 \in E(G)$, $v_2v_4 \in E(G)$ and $v_1v_2 \in E(G)$, we have $v_2v_3 \notin E(G)$, $v_3v_4 \notin E(G)$ and $v_1v_4 \notin E(G)$ by the fact that $g \geq 7$. That is, $d_{G^\times}(f_i) \geq 4$ for $i \in \{2, 3, 4\}$.

Now we are going to show that $d_{G^\times}(f_2) \geq 6$. Assume to the contrary that $d_{G^\times}(f_2) \leq 5$. If $d_{G^\times}(f_2) = 4$, say $f_2 = [v_3vv_2x]$, then x is a true vertex, $x \notin \{v_1, v_4\}$, and G contains a 4-cycle $v_1v_2xv_3v_1$, a contradiction. If $d_{G^\times}(f_2) = 5$, say $f_2 = [v_3vv_2xy]$, then x and y are true vertices, $x, y \notin \{v_1, v_4\}$, and G contains a 5-cycle $v_1v_2xyv_3v_1$, a contradiction. Hence, $d_{G^\times}(f_2) \geq 6$. By symmetry, we have $d_{G^\times}(f_4) \geq 6$.

Next we will show that $d_{G^\times}(f_3) \geq 6$. Assume to the contrary that $d_{G^\times}(f_3) \leq 5$. If $d_{G^\times}(f_3) = 4$, say $f_3 = [v_4vv_3x]$, then x is a true vertex, $x \notin \{v_1, v_2\}$, and G contains a 5-cycle $v_1v_2v_4xv_3v_1$, a contradiction. If $d_{G^\times}(f_3) = 5$, say $f_3 = [v_4vv_3xy]$, then x and y are true vertices, $x, y \notin \{v_1, v_2\}$, and G contains a 6-cycle $v_1v_2v_4yxxv_3v_1$, a contradiction. Hence, $d_{G^\times}(f_3) \geq 6$. \square

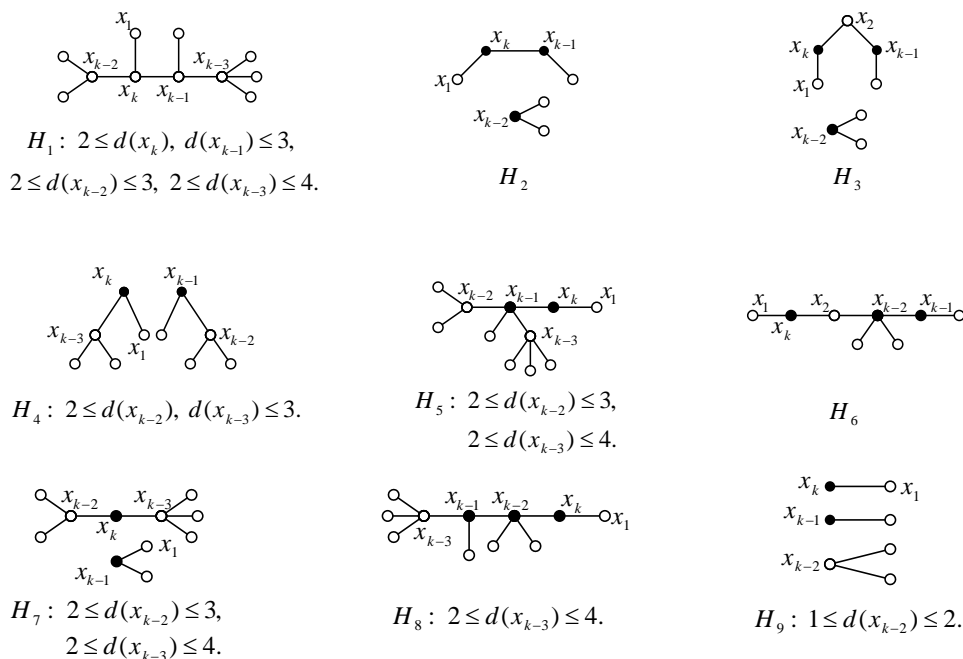
Fact 5 Let v be a false vertex. If $f_4(v) = 1$, say $d_{G^\times}(f_1) = 4$, then $d_{G^\times}(f_3) \geq 5$, $d_{G^\times}(f_2) \geq 6$, and $d_{G^\times}(f_4) \geq 6$. Hence, v is incident with at most one false 4-face.

Proof. Let $f_1 = [v_2vv_1x_1]$, then x_1 is a true vertex, $x_1 \notin \{v_3, v_4\}$. By Fact 4, we have $d_{G^\times}(f_i) \geq 4$ for $i \in \{2, 3, 4\}$. By $g \geq 7$, we have $v_3x_1 \notin E(G)$, $v_3v_2 \notin E(G)$, $v_4x_1 \notin E(G)$, $v_4v_1 \notin E(G)$, $v_4v_3 \notin E(G)$ and $v_2v_1 \notin E(G)$.

Now we are going to show that $d_{G^\times}(f_2) \geq 6$. Assume to the contrary that $d_{G^\times}(f_2) \leq 5$. If $d_{G^\times}(f_2) = 4$, say $f_2 = [v_3vv_2x_2]$, then x_2 is a true vertex, $x_2 \notin \{v_1, x_1, v_4\}$ and G contains a 5-cycle $v_1x_1v_2x_2v_3v_1$, a contradiction. If $d_{G^\times}(f_2) = 5$, say $f_2 = [v_3vv_2x_2y_1]$, then x_2 and y_1 are true vertices, $x_2, y_1 \notin \{v_1, v_4\}$ and $y_1 \neq x_1$. If $x_2 = x_1$, then G contains a 4-cycle $v_1x_1y_1v_3v_1$, a contradiction. If $x_2 \neq x_1$, then G contains a 6-cycle $v_1x_1v_2x_2y_1v_3v_1$, a contradiction. Hence, $d_{G^\times}(f_2) \geq 6$. By symmetry, we have $d_{G^\times}(f_4) \geq 6$.

Next we will show that $d_{G^\times}(f_3) \geq 5$. Assume to the contrary that $d_{G^\times}(f_3) = 4$, say $f_3 = [v_4vv_3x_3]$, then x_3 is a true vertex, $x_3 \notin \{v_1, x_1, v_2\}$, and G contains a 6-cycle $v_1x_1v_2v_4x_3v_3v_1$, a contradiction. Hence, $d_{G^\times}(f_3) \geq 5$. \square

Lemma 5. Let G be a connected IC-planar graph with $g \geq 7$ and $|G| \geq 6$, then G contains one of the following configurations $H_1 \sim H_9$.



Remark 1. Each configuration depicted above is such that: (1) the vertices which are labelled as x_k, x_{k-1}, x_{k-2} in every configuration are different while the other vertices may be not distinct; (2) the degree of solid vertices are exactly shown; (3) the degree of hollow vertices may be larger than or equal to the degree shown in the figure, except for specially pointed.

5. Proof of Lemma 5

On the contrary, assume that Lemma 5 is not true. Let G be a counterexample. Then G is a connected IC-planar graph with $g \geq 7$ and $|G| \geq 6$, but without configurations $H_1 \sim H_9$. Now we consider G^\times , the associated plane graph of G .

In the following, we use Euler's formula and discharging analysis on G^\times to derive a contradiction. We define the initial weight function w such that $w(v) = 2d_{G^\times}(v) - 6$ for each $v \in V(G^\times)$, and $w(f) = d_{G^\times}(f) - 6$ for each $f \in F(G^\times)$. By Euler's formula $|V(G^\times)| - |E(G^\times)| + |F(G^\times)| = 2$ and Handshaking Theorem, we can deduce that

$$\sum_{x \in V(G^\times) \cup F(G^\times)} w(x) = \sum_{v \in V(G^\times)} (2d_{G^\times}(v) - 6) + \sum_{f \in F(G^\times)} (d_{G^\times}(f) - 6) = -12.$$

Next, we will design the appropriate discharging rules to redistribute the weights on $V(G^\times) \cup F(G^\times)$ depending on the value of $\delta(G^\times)$. Once the discharging process is finished, a new weight function $w'(x)$ is produced while the total sum of weights is kept fixed. Then we will show that $\sum_{x \in V(G^\times) \cup F(G^\times)} w'(x) > -12 = \sum_{x \in V(G^\times) \cup F(G^\times)} w(x)$ to derive the

contradiction. For $x, y \in V(G^\times) \cup F(G^\times)$, let $\tau(x \rightarrow y)$ be the weight transferred from x to y .

By Lemma 4, we have $\delta(G) \leq 3$. So $\delta(G^\times) \leq 3$. Thus, the proof will be divided into the following cases depending on the value of $\delta(G^\times)$.

Case 1. $\delta(G^\times) = 3$.

Our discharging rules are defined as follows.

R1. Suppose that v is a false 4-vertex and f is the face incident with v . If $d_{G^\times}(f) = 3$, then $\tau(v \rightarrow f) = 2$; if $d_{G^\times}(f) = 4$, then $\tau(v \rightarrow f) = \frac{3}{2}$; if $d_{G^\times}(f) = 5$, then $\tau(v \rightarrow f) = \frac{1}{2}$.

R2. Suppose that v is a true 4-vertex.

R2.1 Let f be a false face incident with v . If $d_{G^\times}(f) = 3$, then $\tau(v \rightarrow f) = 1$; if $d_{G^\times}(f) = 4$, then $\tau(v \rightarrow f) = \frac{1}{2}$; if $d_{G^\times}(f) = 5$, then $\tau(v \rightarrow f) = \frac{1}{2}$.

R2.2 Let f be a pendant false 3-face of v . Then $\tau(v \rightarrow f) = \frac{1}{2}$.

R3. Suppose that v is a true 5^+ -vertex.

R3.1 Let f be a false face incident with v . If $d_{G^\times}(f) = 3$, then $\tau(v \rightarrow f) = 1$; if $d_{G^\times}(f) = 4$, then $\tau(v \rightarrow f) = \frac{1}{2}$; if $d_{G^\times}(f) = 5$, then $\tau(v \rightarrow f) = \frac{1}{2}$.

R3.2 Let f be a pendant false face of v . If $d_{G^\times}(f) = 3$, then $\tau(v \rightarrow f) = 1$; if $d_{G^\times}(f) = 4$, then $\tau(v \rightarrow f) = \frac{1}{2}$.

Next, we are going to show that for each element $x \in V(G^\times) \cup F(G^\times)$, $w'(x) \geq 0$.

Suppose that $v \in V(G^\times)$ with $d_{G^\times}(v) = k$. Let v_1, \dots, v_k be the vertices adjacent to v in G^\times and arranged in clockwise order. Let f_i be the face with vv_i and vv_{i+1} as boundary edges, where $1 \leq i \leq k$ and $v_{k+1} = v_1$.

Claim 1.1 $w'(v) \geq \frac{3}{2}k - 6$ for each true k -vertex v with $k \geq 4$.

Proof. Let v be a true k -vertex with $k \geq 4$. Then $w(v) = 2k - 6$. By Fact 2 and $g \geq 7$, $f_3(v) \leq 1$.

If $f_3(v) = 1$, say f_1 , then by Fact 4, v does not have pendant false i -faces, $i \in \{3, 4\}$ and $\max\{d_{G^\times}(f_2), d_{G^\times}(f_k)\} \geq 6$. Thus, $f_4(v) + f_5(v) \leq k - 2$. Hence, $w'(v) \geq 2k - 6 - 1 - \frac{1}{2}(k - 2) = \frac{3}{2}k - 6$ by R2–R3.

If $f_3(v) = 0$, then by Fact 1, Fact 4 and Fact 5, v has at most one pendant false i -face, $i \in \{3, 4\}$. If v has one pendant false 3-face f , say v_1 is the false vertex incident with f , then $\min\{d_{G^\times}(f_1), d_{G^\times}(f_k)\} \geq 6$ by Fact 4. Thus, $f_4(v) + f_5(v) \leq k - 2$. Hence, $w'(v) \geq 2k - 6 - 1 - \frac{1}{2}(k - 2) = \frac{3}{2}k - 6$ by R2–R3. If v has a pendant false 4-face f , say v_1 is the false vertex incident with f , then $\max\{d_{G^\times}(f_1), d_{G^\times}(f_k)\} \geq 6$ by Fact 5. Thus, $f_4(v) + f_5(v) \leq k - 1$. Hence, $w'(v) \geq 2k - 6 - \frac{1}{2} - \frac{1}{2}(k - 1) = \frac{3}{2}k - 6$ by R2–R3. If v does not have any pendant false i -face, $i \in \{3, 4\}$, then $w'(v) \geq 2k - 6 - \frac{1}{2}k = \frac{3}{2}k - 6$ by R2–R3. \square

Claim 1.2 $w'(v) \geq 0$ for each $v \in V(G^\times)$.

Proof. Suppose that v is a true 3-vertex. Then $w'(v) = w(v) = 0$. Suppose that v is a false 4-vertex. Then $w(v) = 2$. By Fact 4, $f_3(v) \leq 1$. If $f_3(v) = 1$, then $f_4(v) = f_5(v) = 0$ by Fact 4. Thus, $w'(v) = 2 - 2 = 0$ by R1. So suppose that $f_3(v) = 0$. Then $f_4(v) \leq 1$ by Fact 5. If $f_4(v) = 1$, then $f_5(v) \leq 1$ by Fact 5. Thus, $w'(v) \geq 2 - \frac{3}{2} - \frac{1}{2} = 0$ by R1. If $f_4(v) = 0$, $w'(v) \geq 2 - 4 \times \frac{1}{2} = 0$ by R1. Suppose that v is a true k -vertex with $k \geq 4$. By Claim 1.1, we have $w'(v) \geq \frac{3}{2}k - 6 \geq \frac{3}{2} \times 4 - 6 = 0$. \square

Claim 1.3 $w'(f) \geq 0$ for each $f \in F(G^\times)$.

Proof. Suppose that $d_{G^\times}(f) \geq 6$. Then $w'(f) = w(f) \geq 0$. Suppose that $d_{G^\times}(f) = k$ ($3 \leq k \leq 5$). Since G is an IC-planar graph with $g \geq 7$, then f is a false face. Let $f = [u_1 u_2 \cdots u_k]$. By Fact 1, f is incident with one false vertex, say u_1 . Let v_1, v_2 be the neighbors of u_1 in G^\times such that $v_1 u_2 \in E(G)$ and $v_2 u_k \in E(G)$.

Suppose that $k = 3$. Then $w(f) = -3$, and $\tau(u_1 \rightarrow f) = 2$ by R1. If f is a $(3, 3, 4)$ -face, then $\max\{d_G(v_1), d_G(v_2)\} \geq 5$ or $d_G(v_1) = d_G(v_2) = 4$ by G contains no configuration H_1 . Note that f is the pendant false 3-face of v_1 and v_2 . If $\max\{d_G(v_1), d_G(v_2)\} \geq 5$, say $d_G(v_1) \geq 5$, then $\tau(v_1 \rightarrow f) = 1$ by R3. Hence, $w'(f) \geq -3 + 2 + 1 = 0$. If $d_G(v_i) = 4$, where $i = 1, 2$, then $\tau(v_i \rightarrow f) = \frac{1}{2}$ by R2. Hence, $w'(f) = -3 + 2 + 2 \times \frac{1}{2} = 0$. If f is a $(3^+, 4^+, 4)$ -face, say $d_G(u_i) \geq 4$ for some $i \in \{2, 3\}$, then $\tau(u_i \rightarrow f) = 1$ by R2–R3. Hence, $w'(f) \geq -3 + 2 + 1 = 0$.

Suppose that $k = 4$. Then $w(f) = -2$, and $\tau(u_1 \rightarrow f) = \frac{3}{2}$ by R1. If f is a $(3, 3, 3, 4)$ -face, then $\min\{d_G(v_1), d_G(v_2)\} \geq 5$ by G contains no configuration H_1 . Note that f is the pendant false 4-face of v_1 and v_2 . So $\tau(v_i \rightarrow f) = \frac{1}{2}$ by R3, where $i = 1, 2$. Hence, $w'(f) = -2 + \frac{3}{2} + 2 \times \frac{1}{2} = \frac{1}{2}$. If f is a $(3^+, 3^+, 4^+, 4)$ -face, say $d_G(u_i) \geq 4$ for some $i \in \{2, 3, 4\}$, then $\tau(u_i \rightarrow f) = \frac{1}{2}$ by R2–R3. Hence, $w'(f) \geq -2 + \frac{3}{2} + \frac{1}{2} = 0$.

Suppose that $k = 5$. Then $w(f) = -1$, and $\tau(u_1 \rightarrow f) = \frac{1}{2}$ by R1. Since G contains no configuration H_1 , f must be a $(3^+, 3^+, 3^+, 4^+, 4)$ -face. So $d_G(u_i) \geq 4$ for some $i \in \{2, 3, 4, 5\}$ and $\tau(u_i \rightarrow f) = \frac{1}{2}$ by R2–R3. Hence, $w'(f) \geq -1 + \frac{1}{2} + \frac{1}{2} = 0$. \square

Thus, $-12 = \sum_{x \in V(G^\times) \cup F(G^\times)} w(x) = \sum_{x \in V(G^\times) \cup F(G^\times)} w'(x) \geq 0$, which is a contradiction.

Hence, the counterexample graph G does not exist and Lemma 5 holds.

Case 2. $\delta(G^\times) = 2$ and $n_2(G^\times) \leq 2$.

In Case 2, the discharging rules are the same as those in Case 1. With the same arguments as Case 1, we can prove that $w'(x) \geq 0$ if $x \in V(G^\times)$ with $d_{G^\times}(x) \geq 3$ or $x \in F(G^\times)$ with $d_{G^\times}(x) \geq 6$, or $x \in F(G^\times)$ with $d_{G^\times}(x) \leq 5$ and x is not incident with any 2-vertex.

Claim 2.1 $w'(f) \geq 0$, where $f \in F(G^\times)$ is a k -face incident with 2-vertices and $k \leq 5$.

Proof. By $g \geq 7$, we have f is a false face. By Fact 3, we have $4 \leq k \leq 5$. Let $f = [u_1 u_2 \cdots u_k]$. By Fact 1, f is incident with one false vertex, say u_1 . Let v_1, v_2 be the neighbors of u_1 in G^\times such that $v_1 u_2 \in E(G)$ and $v_2 u_k \in E(G)$.

Suppose that $k = 4$. Then $w(f) = -2$, and $\tau(u_1 \rightarrow f) = \frac{3}{2}$ by R1. If f is a $(2, 3^-, 3, 4)$ -face, then $\min\{d_G(v_1), d_G(v_2)\} \geq 5$ by G contains no configuration H_1 . Note that f is the pendant false 4-face of v_1 and v_2 . So $\tau(v_i \rightarrow f) = \frac{1}{2}$ by R3, where $i = 1, 2$. Hence, $w'(f) = -2 + \frac{3}{2} + 2 \times \frac{1}{2} = \frac{1}{2}$. If f is a $(2, 2^+, 4^+, 4)$ -face, say $d_G(u_i) \geq 4$ for some $i \in \{2, 3, 4\}$, then $\tau(u_i \rightarrow f) = \frac{1}{2}$ by R2–R3. Hence, $w'(f) \geq -2 + \frac{3}{2} + \frac{1}{2} = 0$.

Suppose that $k = 5$. Then $w(f) = -1$, and $\tau(u_1 \rightarrow f) = \frac{1}{2}$ by R1. Since G contains no configuration H_1 , f is a $(2, 2^+, 3^+, 4^+, 4)$ -face. So $d_G(u_i) \geq 4$ for some $i \in \{2, 3, 4, 5\}$ and $\tau(u_i \rightarrow f) = \frac{1}{2}$ by R2–R3. Hence, $w'(f) \geq -1 + \frac{1}{2} + \frac{1}{2} = 0$. \square

Note that $w'(v) = w(v) = -2$ for each 2-vertex v . Thus, $-12 = \sum_{x \in V(G^\times) \cup F(G^\times)} w(x) = \sum_{x \in V(G^\times) \cup F(G^\times)} w'(x) \geq 2 \times (-2) = -4$, which is a contradiction. Hence, the counterexample graph G does not exist and Lemma 5 holds.

Case 3. $\delta(G^\times) = 2$ and $n_2(G^\times) \geq 3$.

Since G does not contain H_2 and H_3 as subgraphs, G satisfies the following properties.

Claim 3.1 2-vertex is not adjacent to any 2-vertex.

Claim 3.2 Any 3^+ -vertex is adjacent to at most one 2-vertex.

A 2-vertex $u \in V(G^\times)$ is called a *special 2-vertex* if $uv \in E(G)$ with $d_G(v) = 3$. Since G does not contain H_4 as a subgraph, the following claim holds.

Claim 3.3 G^\times has at most one special 2-vertex.

A 4-vertex $v \in V(G)$ is called a *special 4-vertex* if v is adjacent to a 2-vertex in G ; a *bad 4-vertex* if v is special and adjacent to a 3-vertex in G ; a *poor 4-vertex* if v is special and not adjacent to any 3-vertex. Since G does not contain H_3 , H_5 and H_6 as subgraphs, the following three claims hold.

Claim 3.4 Suppose that v is a poor 4-vertex. Then $n_2^G(v) = 1$ and $n_{4^+}^G(v) = 3$.

Claim 3.5 Suppose that v is a bad 4-vertex. Then $n_2^G(v) = 1$, $n_3^G(v) = 1$ and $n_{5^+}^G(v) = 2$.

Claim 3.6 A special 4-vertex is not adjacent to any special 4-vertex in G .

Claim 3.7 In G^\times , if v is a special 4-vertex incident with a 4-face, then $f_{6^+}(v) \geq 1$.

Proof. Suppose that v is a special 4-vertex and v_i ($1 \leq i \leq 4$) are the neighbors of v in G^\times and arranged in clockwise order. Let f_i be the face with vv_i and vv_{i+1} as boundary edges, where $1 \leq i \leq 4$ and $v_5 = v_1$.

Without loss of generality, assume that $f_1 = [vv_1x_1v_2]$ is a 4-face. By $g \geq 7$, f_1 is a false face. If v_1 or v_2 is a false vertex, then $d_{G^\times}(f_4) \geq 6$ or $d_{G^\times}(f_2) \geq 6$ by Fact 5. So $f_{6^+}(v) \geq 1$. If x_1 is a false vertex, then the vertices in $(N_{G^\times}(v_1) \cup N_{G^\times}(v_2)) - \{x_1\}$ are true. If $d_{G^\times}(v_1) = 2$ or $d_{G^\times}(v_2) = 2$, then $d_{G^\times}(f_4) \geq 6$ or $d_{G^\times}(f_2) \geq 6$ by Fact 5. So $f_{6^+}(v) \geq 1$. So suppose that $d_{G^\times}(v_1) \geq 3$ and $d_{G^\times}(v_2) \geq 3$. Without loss of generality, assume that $d_{G^\times}(v_3) = 2$ since v is a special 4-vertex. If $d_{G^\times}(f_2) \geq 6$ or $d_{G^\times}(f_4) \geq 6$, then Claim 3.7 holds. So suppose that $d_{G^\times}(f_i) \leq 5$, where $i = 2, 4$. That is, f_2 and f_4 are false faces. Since the vertices in $(N_{G^\times}(v_1) \cup N_{G^\times}(v_2)) - \{x_1\}$ are true and $g \geq 7$, we have $d_{G^\times}(f_i) = 5$, where $i = 2, 4$. Let $f_2 = [vv_2y_1y_2v_3]$. Then y_2 is a false vertex. Let $f_3 = [vv_3y_2z_1 \cdots z_tv_4]$, where $t \geq 0$. If $t = 0$, then $y_2v_4 \in E(G^\times)$. Since y_2 is a false vertex, we have $y_1v_4 \in E(G)$. Thus, G contains a 4-cycle $vv_2y_1v_4v$, a contradiction. If $t = 1$, then $y_2z_1 \in E(G^\times)$, and z_1 is a true vertex. Since y_2 is a false vertex, we have $y_1z_1 \in E(G)$. Thus, G contains a 5-cycle $vv_2y_1z_1v_4v$, a contradiction. So $t \geq 2$. That is, $d_{G^\times}(f_3) \geq 6$. So $f_{6^+}(v) \geq 1$. \square

Claim 3.8 If G contains a path $P_3 = xyz$ and x is a special 4-vertex, then $d_G(y) \geq 4$ or $d_G(z) \geq 4$.

Proof. Suppose that x is a poor 4-vertex. By Claim 3.4, $n_2^G(x) = 1$ and $n_{4^+}^G(x) = 3$. That is, $d_G(y) \geq 4$ or $d_G(y) = 2$. If $d_G(y) = 2$, then $d_G(z) \geq 4$ since G contains no configuration H_7 .

Suppose that x is a bad 4-vertex. By Claim 3.5, $n_2^G(x) = 1$, $n_3^G(x) = 1$ and $n_{5^+}^G(x) = 2$. That is, $d_G(y) \geq 5$, or $d_G(y) = 2$, or $d_G(y) = 3$. If $d_G(y) = 2$, then $d_G(z) \geq 4$ since G contains no configuration H_7 . If $d_G(y) = 3$, then $d_G(z) \geq 5$ since G contains no configuration H_8 . \square

Corollary 1. If f is a false 4-face incident with a poor 4-vertex, then f is incident with at least two true 4^+ -vertices.

Proof. Suppose that $f = [v_1v_2v_3v_4]$ is a false 4-face, where v_1 is a false vertex. Then v_2, v_3 and v_4 are true vertices by Fact 1. If v_2 or v_4 is a poor 4-vertex, say v_2 is a poor 4-vertex, then $\max\{d_G(v_3), d_G(v_4)\} \geq 4$ by Claim 3.8. Therefore, f is incident with at least two true 4^+ -vertices. If v_3 is a poor 4-vertex, then $\max\{d_G(v_2), d_G(v_4)\} \geq 4$ by Claim 3.4. Therefore, f is incident with at least two true 4^+ -vertices. \square

Corollary 2. If f is a false 5-face incident with a special 4-vertex, then f is incident with at least two true 4^+ -vertices.

Proof. Suppose that $f = [v_1v_2v_3v_4v_5]$ is a false 5-face, where v_1 is a false vertex. Then v_2, v_3, v_4 and v_5 are true vertices by Fact 1. If v_2 or v_5 is a special 4-vertex, say v_2 is a special 4-vertex, then $\max\{d_G(v_3), d_G(v_4)\} \geq 4$ by Claim 3.8. Therefore, f is incident with at least two true 4^+ -vertices. If v_3 or v_4 is a special 4-vertex, say v_3 is a special 4-vertex, then $\max\{d_G(v_4), d_G(v_5)\} \geq 4$ by Claim 3.8. Therefore, f is incident with at least two true 4^+ -vertices. \square

In this Case, we define the following discharging rules R1–R6, where R1–R3 are the same as those in Case 1.

R4. Suppose that v is a true 4^+ -vertex. Then v gives 1 to each adjacent 2-vertex in G .

R5. Suppose that v is a true 5^+ -vertex. Then v gives $\frac{3}{10}$ to each adjacent bad 4-vertex in G .

R6. Suppose that f is a false face, and v is a special 4-vertex incident with f .

R6.1 Let f be a false 3-face. If v is a special 4-vertex, then $\tau(f \rightarrow v) = \frac{1}{2}$.

R6.2 Let f be a false 4-face. If v is a poor 4-vertex, then $\tau(f \rightarrow v) = \frac{1}{4}$; if v is a bad 4-vertex, then $\tau(f \rightarrow v) = 0$.

R6.3 Let f be a false 5-face. If v is a poor 4-vertex, then $\tau(f \rightarrow v) = \frac{1}{4}$; if v is a bad 4-vertex, then $\tau(f \rightarrow v) = \frac{1}{10}$.

Claim 3.9 Suppose that $v \in V(G^\times)$. If $d_{G^\times}(v) \geq 3$, or $d_{G^\times}(v) = 2$ and v is not a special 2-vertex, then $w'(v) \geq 0$.

Proof. Suppose that $d_{G^\times}(v) = k$. Let v_1, \dots, v_k be the vertices adjacent to v in G^\times and arranged in clockwise order. Let f_i be the face with vv_i and vv_{i+1} as boundary edges, where $1 \leq i \leq k$ and $v_{k+1} = v_1$.

Suppose that $k = 2$. Then $w(v) = -2$. If v is a special 2-vertex, then $w'(v) \geq w(v) = -2$. If v is not a special 2-vertex, then $n_{4^+}^G(v) = 2$ by Claim 3.1. Hence, $w'(v) = -2 + 2 \times 1 = 0$ by R4.

Suppose that $k = 3$. Then $w'(v) = w(v) = 0$.

Suppose that $k = 4$. If v is a false 4-vertex, then with the similar arguments as Case 1, we have $w'(v) \geq 0$. Assume that v is a true 4-vertex. Then $n_2^G(v) \leq 1$ by Claim 3.2. If $n_2^G(v) = 0$, then with the similar arguments as Case 1, we have $w'(v) \geq 0$. So suppose that $n_2^G(v) = 1$. That is, v is a special 4-vertex. By Fact 2 and $g \geq 7$, $f_3(v) \leq 1$.

If $f_3(v) = 1$, say $f_1 = [vv_1v_2]$ and v_1 is a false vertex, then by Fact 4, v does not have pendant false i -faces, $i \in \{3, 4\}$, and $d_{G^\times}(f_4) \geq 6$. Thus, $f_4(v) + f_5(v) \leq 2$. By R6.1, $\tau(f_1 \rightarrow v) = \frac{1}{2}$. If $f_4(v) + f_5(v) \leq 1$, then $w'(v) \geq 2 - 1 - 1 - \frac{1}{2} + \frac{1}{2} = 0$ by R2, R4 and R6. So suppose that $f_4(v) + f_5(v) = 2$. If v is a poor 4-vertex, then $\tau(f_i \rightarrow v) = \frac{1}{4}$ by R6, where $i = 2, 3$. Thus, $w'(v) = 2 - 1 - 1 - 2 \times \frac{1}{2} + \frac{1}{2} + 2 \times \frac{1}{4} = 0$ by R2, R4 and R6. If v is a bad 4-vertex, then by Claim 3.5, $n_{5^+}^G(v) = 2$. Thus, $w'(v) \geq 2 - 1 - 1 - 2 \times \frac{1}{2} + \frac{1}{2} + 2 \times \frac{3}{10} = \frac{1}{10}$ by R2, R4 and R5.

If $f_3(v) = 0$, then by Fact 1, Fact 4 and Fact 5, v has at most one pendant false i -face, $i \in \{3, 4\}$.

- Suppose that v has a pendant false 3-face f . Then v is adjacent to a false vertex, say v_1 . So $d_{G^\times}(f_1) \geq 6$ and $d_{G^\times}(f_4) \geq 6$ by Fact 4. Thus, $f_4(v) + f_5(v) \leq 2$. Note that $\tau(v \rightarrow f) = \frac{1}{2}$ by R2. If $f_4(v) + f_5(v) \leq 1$, then $w'(v) \geq 2 - 1 - \frac{1}{2} - \frac{1}{2} = 0$ by R2 and R4. So suppose that $f_4(v) + f_5(v) = 2$. If v is a poor 4-vertex, then $\tau(f_i \rightarrow v) = \frac{1}{4}$ by R6, where $i = 2, 3$. Thus, $w'(v) = 2 - 1 - \frac{1}{2} - 2 \times \frac{1}{2} + 2 \times \frac{1}{4} = 0$ by R2, R4 and R6. If v is a bad 4-vertex, then by Claim 3.5, $n_{5^+}^G(v) = 2$. Thus, $w'(v) \geq 2 - 1 - \frac{1}{2} - 2 \times \frac{1}{2} + 2 \times \frac{3}{10} = \frac{1}{10}$ by R2, R4 and R5.
- Suppose that v has a pendant false 4-face f . Then v is adjacent to a false vertex, say v_1 . So $\max\{d_{G^\times}(f_1), d_{G^\times}(f_4)\} \geq 6$ by Fact 5. Thus, $f_4(v) + f_5(v) \leq 3$. Note that $\tau(v \rightarrow f) = 0$ by R2. If $f_4(v) + f_5(v) \leq 2$, then $w'(v) \geq 2 - 1 - 2 \times \frac{1}{2} = 0$ by R2 and R4. So suppose that $f_4(v) + f_5(v) = 3$ and $d_{G^\times}(f_1) \geq 6$. If v is a poor 4-vertex, then $\tau(f_i \rightarrow v) = \frac{1}{4}$ by R6, where $i = 2, 3, 4$. Thus, $w'(v) = 2 - 1 - 3 \times \frac{1}{2} + 3 \times \frac{1}{4} = \frac{1}{4}$ by R2, R4 and R6. If v is a bad 4-vertex, then by Claim 3.5, $n_{5^+}^G(v) = 2$. Thus, $w'(v) \geq 2 - 1 - 3 \times \frac{1}{2} + 2 \times \frac{3}{10} = \frac{1}{10}$ by R2, R4 and R5.

- Suppose that v does not have any pendant false i -face, $i \in \{3, 4\}$. Then $f_4(v) + f_5(v) \leq 4$. If $f_4(v) + f_5(v) \leq 2$, then $w'(v) \geq 2 - 1 - 2 \times \frac{1}{2} = 0$ by R2 and R4. Assume that $f_4(v) + f_5(v) = 3$ and $d_{G^\times}(f_1) \geq 6$. If v is a poor 4-vertex, then $\tau(f_i \rightarrow v) = \frac{1}{4}$ by R6, where $i = 2, 3, 4$. Thus, $w'(v) = 2 - 1 - 3 \times \frac{1}{2} + 3 \times \frac{1}{4} = \frac{1}{4}$ by R2, R4 and R6. If v is a bad 4-vertex, then by Claim 3.5, $n_{5^+}^G(v) = 2$. Thus, $w'(v) \geq 2 - 1 - 3 \times \frac{1}{2} + 2 \times \frac{3}{10} = \frac{1}{10}$ by R2, R4 and R5. Assume that $f_4(v) + f_5(v) = 4$. By Claim 3.7, we have $f_4(v) = 0$ and $f_5(v) = 4$. If v is a poor 4-vertex, then $\tau(f_i \rightarrow v) = \frac{1}{4}$ by R6, where $i = 1, 2, 3, 4$. Thus, $w'(v) = 2 - 1 - 4 \times \frac{1}{2} + 4 \times \frac{1}{4} = 0$ by R2, R4 and R6. If v is a bad 4-vertex, then by Claim 3.5, $n_{5^+}^G(v) = 2$. Thus, $w'(v) = 2 - 1 - 4 \times \frac{1}{2} + 2 \times \frac{3}{10} + 4 \times \frac{1}{10} = 0$ by R2, R4–R6.

Suppose that $k \geq 5$. By Claim 1.1, the remaining charge of 5^+ -vertex is at least $\frac{3}{2}k - 6$ after the discharging process R1–R3. By Claim 3.2, $n_2^G(v) \leq 1$. If $n_2^G(v) = 0$, then $w'(v) \geq \frac{3}{2}k - 6 - \frac{3}{10}k = \frac{6}{5}k - 6 \geq \frac{6}{5} \times 5 - 6 = 0$ by Claim 1.1 and R5. Now we consider that $n_2^G(v) = 1$. Since G contains no configuration H_6 , then v will not be adjacent to any special 4-vertex in G . Thus, $w'(v) \geq \frac{3}{2}k - 6 - 1 = \frac{3}{2}k - 7 \geq \frac{3}{2} \times 5 - 7 = \frac{1}{2}$ by Claim 1.1 and R4. \square

Claim 3.10 $w'(f) \geq 0$ for each $f \in F(G^\times)$.

Proof. Suppose that $d_{G^\times}(f) \geq 6$. Then $w'(f) = w(f) \geq 0$.

Suppose that $d_{G^\times}(f) = k$ ($3 \leq k \leq 5$). Since G is an IC-planar graph with $g \geq 7$, then f is a false face. Let $f = [u_1 u_2 \cdots u_k]$. By Fact 1, f is incident with one false vertex, say u_1 . Let v_1, v_2 be the neighbors of u_1 in G^\times such that $v_1 u_2 \in E(G)$ and $v_2 u_k \in E(G)$.

Suppose that $k = 3$. Then $w(f) = -3$, and $\tau(u_1 \rightarrow f) = 2$ by R1. By Fact 3, f is not incident with any 2-vertex. By Claim 3.6, f is incident with at most one special 4-vertex. If f is not incident with any special 4-vertex, then with the similar discussion as Case 1, we can deduce that $w'(f) \geq 0$. Now suppose that f is incident with a special 4-vertex, say u_2 , then $\tau(u_2 \rightarrow f) = 1$ by R2 and $\tau(f \rightarrow u_2) = \frac{1}{2}$ by R6. If f is a $(3, 4, 4)$ -face, then $d_G(v_2) \geq 5$ by G contains no configuration H_8 . Note that f is the pendant false 3-face of v_2 . Thus, $\tau(v_2 \rightarrow f) = 1$ by R3. Hence, $w'(f) \geq -3 + 2 + 1 + 1 - \frac{1}{2} = \frac{1}{2}$. If f is a $(4, 4^+, 4)$ -face, then $d_G(u_3) \geq 4$. Thus, $\tau(u_3 \rightarrow f) = 1$ by R2–R3. Hence, $w'(f) = -3 + 2 + 1 + 1 - \frac{1}{2} = \frac{1}{2}$.

Suppose that $k = 4$. Then $w(f) = -2$, and $\tau(u_1 \rightarrow f) = \frac{3}{2}$ by R1. Assume that f is incident with x poor 4-vertices, thus $w'(f) \geq -2 + \frac{3}{2} + \frac{1}{2}x - \frac{1}{4}x = \frac{1}{4}x - \frac{1}{2}$ by R1, R2 and R6. If $x \geq 2$, then $w'(f) \geq 0$. If $x = 1$, then f is incident with at least two true 4^+ -vertices by Corollary 1. Thus, $w'(f) \geq -2 + \frac{3}{2} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ by R1–R3 and R6. If $x = 0$, then with the similar discussion as Case 2, we can deduce that $w'(f) \geq 0$.

Suppose that $k = 5$. Then $w(f) = -1$, and $\tau(u_1 \rightarrow f) = \frac{1}{2}$ by R1. Assume that f is incident with x poor 4-vertices and y bad 4-vertices, thus $w'(f) \geq -1 + \frac{1}{2} + \frac{1}{2}(x + y) - \frac{1}{4}x - \frac{1}{10}y = -\frac{1}{2} + \frac{1}{4}x + \frac{2}{5}y$ by R1, R2 and R6. If $x \geq 2$, or $x = 1$ and $y \geq 1$, or $x = 0$ and $y \geq 2$, then we have $w'(f) \geq 0$. So suppose that $x = 1$ and $y = 0$, or $x = 0$ and $y \leq 1$. If $x = 1$ and $y = 0$, then f is incident with at least two true 4^+ -vertices by Corollary 2. Thus, $w'(f) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ by R1–R3 and R6. If $x = 0$ and $y = 1$, then f is incident with at least two true 4^+ -vertices by Corollary 2. Thus, $w'(f) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{2} - \frac{1}{10} = \frac{2}{5}$ by R1–R3 and R6. If $x = y = 0$, then with the similar discussion as Case 2, we can deduce that $w'(f) \geq 0$. \square

By Claim 3.3, there is at most one special 2-vertex in G^\times . Thus, $-12 = \sum_{x \in V(G^\times) \cup F(G^\times)} w(x) = \sum_{x \in V(G^\times) \cup F(G^\times)} w'(x) \geq -2$, which is a contradiction. Hence, the counterexample graph G does not exist and Lemma 5 holds.

Case 4. $\delta(G^\times) = 1$.

Subcase 4.1 $n_1(G^\times) = 1$ and $n_2(G^\times) \leq 2$.

In Subcase 4.1, the discharging rules are the same as those in Case 2. With the same arguments as Case 2, we can prove that $w'(x) \geq 0$ for any $x \in (V(G^\times) \cup F(G^\times)) - V'$, where V' is the set of 1-vertex and 2-vertices. Note that $w'(u) = w(u) = -4$ for the

1-vertex u and $w'(v) = w(v) = -2$ for each 2-vertex v . Thus, $-12 = \sum_{x \in V(G^\times) \cup F(G^\times)} w(x) = \sum_{x \in V(G^\times) \cup F(G^\times)} w'(x) \geq -4 + 2 \times (-2) = -8$, which is a contradiction. Hence, the counterexample graph G does not exist and Lemma 5 holds.

Subcase 4.2 $n_1(G^\times) = 1$ and $n_2(G^\times) \geq 3$.

In Subcase 4.2, the discharging rules are the same as those in Case 3. By Claim 3.3, there is at most one special 2-vertex in G^\times . With the same arguments as Case 3, we can prove that $w'(x) \geq 0$ for any $x \in (V(G^\times) \cup F(G^\times)) - V''$, where V'' is the set of 1-vertex and special 2-vertex. Note that $w'(u) = w(u) = -4$ for the 1-vertex u and $w'(v) \geq -2$ for the special 2-vertex v . Thus, $-12 = \sum_{x \in V(G^\times) \cup F(G^\times)} w(x) = \sum_{x \in V(G^\times) \cup F(G^\times)} w'(x) \geq -4 + (-2) = -6$, which is a contradiction. Hence, the counterexample graph G does not exist and Lemma 5 holds.

Subcase 4.3 $n_1(G^\times) \geq 2$.

Since G does not contain H_9 as a subgraph, G contains exactly two 1-vertices and no 2-vertex. In Subcase 4.3, the discharging rules are the same as those in Case 1. With the same arguments as Case 1, we can prove that $w'(x) \geq 0$ for any $x \in (V(G^\times) \cup F(G^\times)) - V'''$, where V''' is the set of 1-vertices. Note that $w'(u) = w(u) = -4$ for each 1-vertex u . Thus, $-12 = \sum_{x \in V(G^\times) \cup F(G^\times)} w(x) = \sum_{x \in V(G^\times) \cup F(G^\times)} w'(x) \geq 2 \times (-4) = -8$, which is a contradiction. Hence, the counterexample graph G does not exist and Lemma 5 holds. \square

6. Proof of Theorem 1

Before we prove Theorem 1, it is necessary to present the following result.

Lemma 6. *Let G be an IC-planar graph with $g \geq 7$. If $k \geq \max\{\Delta, 5\}$, then G is equitably k -colorable.*

Proof. Assume to the contrary that G is a counterexample with fewest vertices of Lemma 6. If the order of each component of G is at most five, then $\Delta \leq 4$. Thus, $k \geq \max\{\Delta, 5\} \geq \Delta + 1$, and G is equitably k -colorable by Lemma 2. Otherwise, there is a component of G with order at least six. By Lemma 5, G contains one of the configurations $H_1 \sim H_9$ in Lemma 6. By Lemma 1, we need to choose the subset S , and define the subset S' as follows.

If G contains configurations $H_i, i \in \{1, 4, 5, 7, 8\}$, then $S' = \{x_k, x_{k-1}, x_{k-2}, x_{k-3}, x_1\}$. If G contains configurations $H_i, i \in \{3, 6\}$, then $S' = \{x_k, x_{k-1}, x_{k-2}, x_2, x_1\}$. If G contains configurations $H_i, i \in \{2, 9\}$, then $S' = \{x_k, x_{k-1}, x_{k-2}, x_1\}$. By Lemma 4, G is 3-degenerate, thus the remaining unspecified vertices in S can be remarked by choosing the minimum degree vertex in a graph obtained from G by deleting the remarked vertices at each step from highest to lowest indices. It is easy to confirm that for every $1 \leq i \leq k$, the subset S defined above satisfies that $|N_G(x_i) - S| \leq k - i$.

Let $H = G - S \subseteq G$. Then $\Delta(H) \leq \Delta$. If $\Delta(H) < \Delta$, then H is equitably k -colorable by Lemma 2. Otherwise, H is equitably k -colorable since G is a counterexample with fewest vertices. Therefore, G is equitably k -colorable by Lemma 1. \square

By Lemma 2, we have the following corollary:

Corollary 3. *Let G be an IC-planar graph with $g \geq 7$. If $\Delta \geq 5$, then G is equitably Δ -colorable.*

Combining Corollary 3, and the conclusions in [14,15], we complete the proof of Theorem 1.

With this conclusion, we can solve the problem of equitable coloring of IC-planar graphs with large girth g ($g \geq 7$) limitation.

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