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Application of the Triple Sumudu Decomposition Method for Solving 1 + 1 and 2 + 1-Dimensional Boussinesq Equations

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Abstract: The triple Sumudu transform decomposition method (TSTDM) is a combination of the Adomian decomposition method (ADM) and the triple Sumudu transform. It is a computational method that can be appropriate for solving linear and nonlinear partial differential equations. The existence analysis of the method and partial derivatives theorems are proven. Finally, we solve the $1+1$ and $2+1$-dimensional Boussinesq equations by applying the (TSTDM)technique, which gives the approximate solution with quick convergence. It is more precise than using ADM alone. In addition, three examples are offered to examine the performance and precision of our method.

Keywords: triple Sumud transform; inverse triple Sumud transform; singular Boussinesq equation; double Sumudu transform; decomposition methods and partial derivative

MSC: 35A44; 65M44; 35A22

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1. Introduction

The study of wave propagation in fluid mechanics became very significant many years ago, with much research in this field. The many remaining mathematicians studying this subject using Whitham's shallow water equations immediately use a coupled form of the Boussinesq equation. The Boussinesq equations are named after the French scientist J, who originated an interpretation of the equations to find solutions for solitary waves on a water surface. Since then, different versions of Boussinesq equations have been introduced. The authors in [1] modified the residual power series and applied it to obtain the fractal solution of the Newell–Whitehead–Segel (NWS) model with fractal derivatives. In [2], Nadeem et.al. presented a new plan, which is known as the Aboodh homotopy integral transform method (AHITM), in order to find the approximate solution of wave problems in multidimensional orders. Several strong methods have been modified and advanced to obtain numerical and analytical solutions of linear and nonlinear partial differential equations. Instances include the double natural and Laplace decomposition method [3,4], the modified double Laplace decomposition method, a singular generalized modified linear Boussinesq equation, and a singular nonlinear Boussinesq equation [5]. The coupled Boussinesq–Burgers equations appear in the diffusion of shallow water waves [6]. The unidirectional expansion of long waves in diffusive media [7] and the fractional variational principles aside from the semi-inverse method are applied to deduce the space–time fractional Boussinesq equation [8]. The space–time fractional Boussinesq equations in Caputo sense derivatives are discussed by applying the homotopy perturbation technique [9]. The authors in [10] discussed the partial differential equations using the double Laplace–Sumudu transform method. Numerical solutions of partial differential equations with variable coefficients have been examined by the Sumudu transform method (STM) [11]. The authors have developed a method with the approximate solutions of the nonlinear systems of partial differential equations with the help of the Sumudu decomposition method (SDM) [12]. In this paper, a new approach is suggested that uses the Sumudu transform decomposition method to obtain the exact solution of several types of Boussinesq equations. This technique is a combination of the decomposition method and the Sumudu transform method. The new double and triple Sumudu transform decomposition method is used to develop the solutions of $1+1$ and $2+1$-dimensional Boussinesq equations. The rest of the work is arranged as follows: Section 2 covers important definitions, the existing condition of the triple Sumudu transform (TST), and theorems of partial derivates with (TST). In Section 3, the $1+1$-dimensional Boussinesq equation is studied by using the double Sumudu transform, and one example is given to support our method. In Section 4, the triple Sumudu transform decomposition method is applied to solve the singular $2+1$-dimensional Boussinesq equation, and one example is given. In Section 5, we study the solution of the singular $2+1$-dimensional coupled system Boussinesq equation by utilizing the triple Sumudu transform decomposition method. Finally, Section 6 outlines the concluding observations.

2. Properties of the Triple Sumudu Transform

In this section, the definitions and existence condition of the triple Sumudu transform are presented. Here, we work with the double and triple Sumudu transform, which is defined by

$$S_xS_t(f(x,t)) = F(u_1,v) = \frac{1}{u_1 v}\int_0^\infty\int_0^\infty e^{-\frac{x}{u_1}-\frac{t}{v}} f(x,t)dtdx, \tag{1}$$

where S_xS_t indicates the double Sumudu transform and $u_1, v \in \mathbb{C}$.

$$S_xS_yS_t(f(x,y,t)) = F(u_1,u_2,v) = \frac{1}{u_1u_2v}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\frac{x}{u_1}-\frac{y}{u_2}-\frac{t}{v}} f(x,y,t)dtdydx \tag{2}$$

where $S_xS_yS_t$ indicates the triple Sumudu transform and $u_1, u_2, v \in \mathbb{C}$.

Next, the conditions for the existence of the triple Sumudu transform are given.

If $f(x,y,t)$ is an exponential order a, c, and b as $x \to \infty$, $y \to \infty$, and $t \to \infty$, and if $\exists K > 0$ such that for all $x > X$, $y > Y$ and $t > T$

$$|f(x,y,t)| \leq Ke^{ax+by+ct}, \tag{3}$$

for some X, Y, and T, then we write

$$f(x,y,t) = O\left(e^{ax+by+ct}\right) \text{ as } y \to \infty,\ y \to \infty,\ t \to \infty;$$

equivalently,

$$\lim_{\substack{x\to\infty\\ y\to\infty\\ t\to\infty}} e^{-\frac{1}{\mu}x-\frac{1}{\eta}y-\frac{1}{\varepsilon}t}|f(x,y,t)| = K\lim_{\substack{x\to\infty\\ y\to\infty\\ t\to\infty}} e^{-\left(\frac{1}{\mu}-a\right)x-\left(\frac{1}{\varepsilon}-b\right)y-\left(\frac{1}{\delta}-c\right)t} = 0, \tag{4}$$

whenever $\frac{1}{\mu} > a$, $\frac{1}{\varepsilon} > c$, and $\frac{1}{\delta} > b$. The function $f(x,y,t)$ does not grow faster than $K(x,y,t)$ as $x \to \infty$, $y \to \infty$, and $t \to \infty$.

Theorem 1. *The function $f(x,y,t)$ is defined on $(0,X)$, $(0,Y)$, and $(0,T)$ and on the exponential order (x,y,t). Then, the triple Sumudu transform of $f(x,y,t)$ exists for all $Re\frac{1}{u_1} > \frac{1}{\mu}$, $Re\frac{1}{u_2} > \frac{1}{\varepsilon}$, and $Re\frac{1}{v} > \frac{1}{\delta}$.*

Proof. By using Equations (2) and (3), we obtain

$$\begin{aligned} |F(u_1,u_2,v)| &= \left|\frac{1}{u_1u_2v}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(\frac{1}{u_1}x+\frac{1}{u_2}y+\frac{1}{v}t)}f(x,y,t)dxdydt\right| \\ &\leq K\left|\frac{1}{u}\int_0^\infty\int_0^\infty\int_0^\infty e^{\left(\frac{1}{u_1}-a\right)x-\left(\frac{1}{u_2}-b\right)y-(\frac{1}{v}-c)t}dxdy\,dt\right| \\ &= \frac{K}{(1-au_1)(1-cu_2)(1-bv)}. \end{aligned} \tag{5}$$

From the condition $Re\frac{1}{u_1} > \frac{1}{\mu}$, $Re\frac{1}{u_2} > \frac{1}{\varepsilon}$, $Re\frac{1}{v} > \frac{1}{\delta}$, and Equation (4), we have

$$\lim_{\substack{u_1\to\infty\\ u_2\to\infty\\ v\to\infty}}|F(u_1,u_2,v)| = 0 \text{ or } \lim_{\substack{u_1\to\infty\\ u_2\to\infty\\ v\to\infty}} F(u_1,u_2,v) = 0.$$

This result can be considered the limiting property of the triple Sumudu transform. □

The next theorem discusses the convergence of the triple Sumudu transform.

Theorem 2. *Let $\varphi(x,y,t)$ be a function of three variables continuous in the x, y, and t-plane. If the integral*

$$\frac{1}{pqv}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{x}{up}+\frac{y}{q}+\frac{t}{v}\right)}\varphi(x,y,t)dxdydt$$

converges at $p = p_0$, $q = q_0$, and $v = v_0$, then the integral converges for $p < p_0$, $q < q_0$, and $v < v_0$.

The proof of this theorem is similar to the proof given by Theorem (2.3) in [13].

Theorem 3. *If the triple Sumudu transform of the function $f(x,y,t)$ is presented by $F(u_1,u_2,v) = S_xS_yS_t[f(x,y,t)]$, then the triple Sumudu transforms of the functions*

$$xyf(x,y,t)$$

are given by

$$S_xS_yS_t[xyf(x,y,t)] = u_1u_2\frac{\partial^2}{\partial u_1\partial u_2}(u_1u_2F(u_1,u_2,v)). \tag{6}$$

Proof. By applying the partial derivative with respect to u_1 for Equation (2), we obtain

$$\begin{aligned} \frac{\partial F(u_1,u_2,v)}{\partial u_1} &= \frac{\partial}{\partial u_1}\int_0^\infty\int_0^\infty\int_0^\infty \frac{1}{u_1u_2v}e^{-\left(\frac{1}{u_1}x+\frac{1}{u_2}y+\frac{1}{v}t\right)}f(x,y,t)dxdydt, \\ &= \int_0^\infty\int_0^\infty \frac{1}{vu_2}e^{-\left(\frac{1}{u_2}y+\frac{1}{v}t\right)}\left(\int_0^\infty \frac{\partial}{\partial u_1}\frac{1}{u_1}e^{-\frac{1}{u_1}x}f(x,y,t)dx\right)dydt. \end{aligned} \tag{7}$$

By computing the partial derivative into brackets, we obtain

$$\begin{aligned} \int_0^\infty \frac{\partial}{\partial u_1}\frac{1}{u_1}e^{-\frac{1}{u_1}x}f(x,y,t)dx &= \int_0^\infty\left(\frac{1}{u_1^3}x-\frac{1}{u_1^2}\right)e^{-\frac{1}{u_1}x}f(x,y,t)dx \\ &= \int_0^\infty \frac{1}{u_1^3}xe^{-\frac{1}{u_1}x}f(x,y,t)dx \\ &\quad -\int_0^\infty \frac{1}{u_1^2}e^{-\frac{1}{u_1}(x,y,t)}f(x,y,t)dx. \end{aligned} \tag{8}$$

By substituting Equation (8) into Equation (7), we obtain

$$\begin{aligned}\frac{\partial F(u_1,u_2,v)}{\partial u_1} &= \int_0^\infty \int_0^\infty \frac{1}{vu_2} e^{-\left(\frac{1}{u_2}y+\frac{1}{v}t\right)} \left(\int_0^\infty \frac{1}{u_1^3} x e^{-\frac{1}{u_1}x} f(x,y,t)dx\right)dydt \\ &- \int_0^\infty \int_0^\infty \frac{1}{vu_2} e^{-\left(\frac{1}{u_2}y+\frac{1}{v}t\right)} \left(\int_0^\infty \frac{1}{u_1^2} e^{-\frac{1}{u_1}x} f(x,y,t)dx\right)dydt.\end{aligned} \quad (9)$$

By taking the derivative with respect to u_2 for Equation (9), we have achieved

$$\begin{aligned}\frac{\partial^2 F(u_1,u_2,v)}{\partial u_1 \partial u_2} &= \frac{1}{u_1^3 v} \int_0^\infty \int_0^\infty x e^{-\left(\frac{1}{u_1}x+\frac{1}{v}t\right)} \left(\int_0^\infty e^{-\frac{1}{u_2}y} \left(\frac{1}{u_2^3}y - \frac{1}{u_2^2}\right) f(x,y,t)\right)dxdydt \\ &- \frac{1}{u_1^2 v} \int_0^\infty \int_0^\infty e^{-\left(\frac{1}{u_1}x+\frac{1}{v}t\right)} \left(\int_0^\infty e^{-\frac{1}{u_2}y} \left(\frac{1}{u_2^3}y - \frac{1}{u_2^2}\right) f(x,y,t)\right)dxdydt.\end{aligned} \quad (10)$$

Equation (10) becomes

$$\begin{aligned}\frac{\partial^2 F(u_1,u_2,v)}{\partial u_1 \partial u_2} &= \frac{1}{u_1^2 u_2^2} S_x S_y S_t[xyf(x,y,t)] - \frac{1}{u_1^2 u_2} S_x S_y S_t[xf(x,y,t)] \\ &- \frac{1}{u_1 u_2^2} S_x S_y S_t[yf(x,y,t)] + \frac{1}{u_1 u_2} S_x S_y S_t[f(x,y,t)].\end{aligned} \quad (11)$$

By arranging the above equation, we obtain

$$\begin{aligned}S_x S_y S_t[xy(x,y,t)] &= u_1^2 u_2^2 \frac{\partial^2 F(u_1,u_2,v)}{\partial u_1 \partial u_2} + u_1^2 u_2 \frac{\partial F(u_1,u_2,v)}{\partial u_1} \\ &+ u_1 u_2^2 \frac{\partial F(u_1,u_2,v)}{\partial u_2} + u_1 u_2 F(u_1,u_2,v);\end{aligned}$$

hence,

$$S_x S_y S_t[xyf(x,y,t)] = u_1 u_2 \frac{\partial^2}{\partial u_1 \partial u_2}(u_1 u_2 F(u_1,u_2,v)).$$

The proof is complete. □

The next theorem provides the triple Sumudu transform of the partial derivatives $xy\frac{\partial \psi}{\partial t}$ and $xy\frac{\partial^2 \psi}{\partial t^2}$.

Theorem 4. *The triple Sumudu transform of the fractional partial derivatives* $xy\frac{\partial \psi}{\partial t}$ *and* $xy\frac{\partial^2 \psi}{\partial t^2}$ *are achieved by*

$$\begin{aligned}S_x S_y S_t\left[xy\frac{\partial \psi}{\partial t}\right] &= \frac{u_1 u_2}{v} \frac{\partial^2}{\partial u_1 \partial u_2}(u_1 u_2 \Psi(u_1,u_2,v)) \\ &- \frac{u_1 u_2}{v} \frac{\partial^2}{\partial u_1 \partial u_2}(u_1 u_2 \Psi(u_1,u_2,0)),\end{aligned} \quad (12)$$

and

$$\begin{aligned}S_x S_y S_t\left[xy\frac{\partial^2 \psi}{\partial t^2}\right] &= \frac{u_1 u_2}{v} \frac{\partial^2}{\partial u_1 \partial u_2}(u_1 u_2 \Psi(u_1,u_2,v)) \\ &- \frac{u_1 u_2}{v} \frac{\partial^2}{\partial u_1 \partial u_2}(u_1 u_2 \Psi(u_1,u_2,0)) \\ &- \frac{u_1 u_2}{v} \frac{\partial^2}{\partial u_1 \partial u_2} u_1 u_2 \left[S_x S_y \left(\frac{\partial \psi(x,y,0)}{\partial t}\right)\right],\end{aligned} \quad (13)$$

respectively.

Proof. By taking a partial derivative with respect to u_1 for Equation (26), we have

$$\begin{aligned}\frac{\partial}{\partial u_1}\left(S_x S_y S_t\left[\frac{\partial \psi}{\partial t}\right]\right) &= \frac{\partial}{\partial u_1} \int_0^\infty \int_0^\infty \int_0^\infty \frac{1}{u_1 u_2 v} e^{-\left(\frac{1}{u_1}x+\frac{1}{u_2}y+\frac{1}{v}t\right)} \frac{\partial \psi}{\partial t} dxdydt, \\ &= \int_0^\infty \int_0^\infty \frac{1}{vu_2} e^{-\left(\frac{1}{u_2}y+\frac{1}{v}t\right)} \left(\int_0^\infty \frac{\partial}{\partial u_1} \frac{1}{u_1} e^{-\frac{1}{u_1}x} \frac{\partial \psi}{\partial t} dx\right)dydt.\end{aligned} \quad (14)$$

We calculate the partial derivative inside brackets as follows:

$$\begin{aligned}\int_0^\infty \frac{\partial}{\partial u_1}\frac{1}{u_1}e^{-\frac{1}{u_1}x}\frac{\partial\psi}{\partial t}dx &= \int_0^\infty \left(\frac{1}{u_1^3}x-\frac{1}{u_1^2}\right)e^{-\frac{1}{u_1}x}\frac{\partial\psi}{\partial t}dx \\ &= \int_0^\infty \frac{1}{u_1^3}xe^{-\frac{1}{u_1}x}\frac{\partial\psi}{\partial t}dx \\ &\quad -\int_0^\infty \frac{1}{u_1^2}e^{-\frac{1}{u_1}x}\frac{\partial\psi}{\partial t}dx.\end{aligned} \tag{15}$$

Substituting Equation (15) into Equation (14), we obtain

$$\begin{aligned}\frac{\partial}{\partial u_1}\left(S_xS_yS_t\left[\frac{\partial\psi}{\partial t}\right]\right) &= \int_0^\infty\int_0^\infty \frac{1}{vu_2}e^{-\left(\frac{1}{u_2}x+\frac{1}{v}t\right)}\left(\int_0^\infty \frac{1}{u_1^3}xe^{-\frac{1}{u_1}x}\frac{\partial\psi}{\partial t}dx\right)dydt \\ &\quad -\int_0^\infty\int_0^\infty \frac{1}{vu_2}e^{-\left(\frac{1}{u_2}y+\frac{1}{v}t\right)}\left(\int_0^\infty \frac{1}{u_1^2}e^{-\frac{1}{u_1}x}\frac{\partial\psi}{\partial t}dx\right)dydt.\end{aligned} \tag{16}$$

By taking the partial derivative of expression Equation (16) with respect to u_2, we obtain the formula

$$\begin{aligned}\frac{\partial^2}{\partial u_1\partial u_2}\left(S_xS_yS_t\left[\frac{\partial\psi}{\partial t}\right]\right) &= \frac{\partial}{\partial u_2}\left(\int_0^\infty\int_0^\infty \frac{1}{vu_2}e^{-\left(\frac{1}{u_2}y+\frac{1}{v}t\right)}\left(\int_0^\infty \frac{1}{u_1^3}xe^{-\frac{1}{u_1}x}\frac{\partial\psi}{\partial t}dx\right)dydt\right) \\ &\quad -\frac{\partial}{\partial u_2}\left(\int_0^\infty\int_0^\infty \frac{1}{vu_2}e^{-\left(\frac{1}{u_2}y+\frac{1}{v}t\right)}\left(\int_0^\infty \frac{1}{u_1^2}e^{-\frac{1}{u_1}x}\frac{\partial\psi}{\partial t}dx\right)dydt\right).\end{aligned} \tag{17}$$

Therefore, Equation (17) becomes

$$\begin{aligned}\frac{\partial^2}{\partial u_1\partial u_2}\left(S_xS_yS_t\left[\frac{\partial\psi}{\partial t}\right]\right) &= \frac{1}{u_1^2u_2^2}\left(\frac{1}{u_1u_2v}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{1}{u_2}y+\frac{1}{v}t+\frac{1}{u_1}x\right)}xy\frac{\partial\psi}{\partial t}dxdydt\right) \\ &\quad +\frac{1}{u_1u_2}\left(\frac{1}{u_1u_2v}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{1}{u_2}y+\frac{1}{v}t+\frac{1}{u_1}x\right)}\frac{\partial\psi}{\partial t}dxdydt\right) \\ &\quad -\frac{1}{u_1u_2^2}\left(\frac{1}{u_1u_2v}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{1}{u_2}y+\frac{1}{v}t+\frac{1}{u_1}x\right)}y\frac{\partial\psi}{\partial t}dxdydt\right) \\ &\quad -\frac{1}{u_1^2u_2}\left(\frac{1}{u_1u_2v}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{1}{u_2}y+\frac{1}{v}t+\frac{1}{u_1}x\right)}x\frac{\partial\psi}{\partial t}dxdydt\right);\end{aligned} \tag{18}$$

hence,

$$\begin{aligned}\frac{\partial^2}{\partial u_1\partial u_2}\left(S_xS_yS_t\left[\frac{\partial\psi}{\partial t}\right]\right) &= \frac{1}{u_1^2u_2^2}S_xS_yS_t\left[xy\frac{\partial\psi}{\partial t}\right]+\frac{1}{u_1u_2}S_xS_yS_t\left[\frac{\partial\psi}{\partial t}\right] \\ &\quad -\frac{1}{u_1u_2^2}S_xS_yS_t\left[y\frac{\partial\psi}{\partial t}\right]\frac{1}{u_1^2u_2}S_xS_yS_t\left[x\frac{\partial\psi}{\partial t}\right],\end{aligned} \tag{19}$$

By rearranging Equation (19), we proved Equation (12)

$$S_xS_yS_t\left[xy\frac{\partial\psi}{\partial t}\right]=\frac{u_1u_2}{v}\frac{\partial^2}{\partial u_1\partial u_2}(u_1u_2\Psi(u_1,u_2,v))-\frac{u_1u_2}{v}\frac{\partial^2}{\partial u_1\partial u_2}(u_1u_2\Psi(u_1,u_2,0)).$$

In a similar way, one can prove Equation (13). □

The double Sumudu transform of the function $\psi(x,t)$ is given by $S_xS_t[\psi(x,t)]=\psi(u_1,v)$. Then, the triple Sumudu transform of $\frac{\partial\psi}{\partial x}$, $\frac{\partial^2\psi}{\partial x^2}$ and $\frac{\partial\psi(x,t)}{\partial t}$ are given by

$$\begin{aligned}S_xS_t\left[\frac{\partial\psi}{\partial x}\right] &= \frac{\psi(u_1,v)-\psi(0,v)}{u_1}, \\ S_xS_t\left(\frac{\partial^2\psi}{\partial x^2}\right) &= \frac{\psi(u_1,v)}{u_1^2}-\frac{\psi(0,v)}{u_1^2}-\frac{\psi_t(0,v)}{u_1}\end{aligned}$$

and

$$S_xS_t\left[\frac{\partial\psi}{\partial t}\right] = \frac{\psi(u_1,v)-\psi(u_1,0)}{v},$$
$$S_xS_t\left(\frac{\partial^2\psi}{\partial t^2}\right) = \frac{\psi(u_1,v)}{v^2}-\frac{\psi(u_1,0)}{v^2}-\frac{\psi_t(u_1,0)}{v}.$$

The Triple Sumudu transform of the function $\psi(x,y,t)$ is given by $S_xS_yS_t[\psi(x,y,t)] = \psi(u_1,u_2,v)$. Then, the triple Sumudu transform of $\frac{\partial\psi}{\partial x}$, $\frac{\partial^2\psi}{\partial x^2}$, $\frac{\partial\psi(x,y,t)}{\partial t}$, and $\frac{\partial^2\psi(x,y,t)}{\partial t^2}$ is given by

$$S_xS_yS_t\left[\frac{\partial\psi}{\partial x}\right] = \frac{\psi(u_1,u_2,v)-\psi(0,u_2,v)}{u_1}, \tag{20}$$
$$S_xS_yS_t\left(\frac{\partial^2\psi}{\partial x^2}\right) = \frac{\psi(u_1,u_2,v)}{u_1^2}-\frac{\psi(0,u_2,v)}{u_1^2}-\frac{\psi_t(0,u_2,v)}{u_1}, \tag{21}$$

$$S_xS_yS_t\left[\frac{\partial\psi}{\partial y}\right] = \frac{\psi(u_1,u_2,v)-\psi(u_1,0,v)}{u_2},$$
$$S_xS_yS_t\left(\frac{\partial^2\psi}{\partial y^2}\right) = \frac{\psi(u_1,u_2,v)}{u_2^2}-\frac{\psi(u_1,0,v)}{u_2^2}-\frac{\psi_t(u_1,0,v)}{u_2}, \tag{22}$$

and

$$S_xS_yS_t\left[\frac{\partial\psi}{\partial t}\right] = \frac{\psi(u_1,u_2,v)-\psi(u_1,u_2,0)}{v}, \tag{23}$$
$$S_xS_yS_t\left(\frac{\partial^2\psi}{\partial t^2}\right) = \frac{\psi(u_1,u_2,v)}{v^2}-\frac{\psi(u_1,u_2,0)}{v^2}-\frac{\psi_t(u_1,u_2,0)}{v}. \tag{24}$$

Next, we generalized the triple Sumudu transform of partial derivatives.

Theorem 5. *The triple transforms of the functions* $\psi(x,y,t)$, $\frac{\partial^m\psi}{\partial x^m}$ *and* $\frac{\partial^n\psi}{\partial t^n}$ *are*

$$S_xS_yS_t\left(\frac{\partial^m}{\partial x^m}\psi(x,y,t)\right) = \frac{\psi(u_1,u_2,v)}{u_1^m}-\frac{\psi(0,u_2,v)}{u_1^m}-\sum_{i=0}^{m-1}\frac{1}{u_1^{m-i}}S_xS_yS_t\left(\frac{\partial^i}{\partial x^i}\psi(0,y,t)\right),$$
$$S_xS_yS_t\left(\frac{\partial^m}{\partial y^m}\psi(x,y,t)\right) = \frac{\psi(u_1,u_2,v)}{u_2^m}-\frac{\psi(u_1,0,v)}{u_2^m}-\sum_{i=0}^{m-1}\frac{1}{u_2^{m-i}}S_xS_yS_t\left(\frac{\partial^i}{\partial y^i}\psi(x,0,t)\right),$$
$$S_xS_yS_t\left(\frac{\partial^n}{\partial t^n}\psi(x,y,t)\right) = \frac{\psi(u_1,u_2,v)}{v^n}-\frac{\psi(u_1,u_2,0)}{v^n}-\sum_{i=1}^{n-1}\frac{1}{v^{n-i}}S_xS_yS_t\left(\frac{\partial^i}{\partial t^i}\psi(x,y,0)\right).$$

3. Double Sumudu Transform Decomposition Method and 1 + 1-Dimensional Boussinesq Equation

The solution of the 1+1-dimensional Boussinesq equation is reviewed by using the double Sumudu transform decomposition method (DSTDM). In this paper, we indicated the double Sumudu transform of the function $\psi(x,t)$ by $\Psi(u_1,v)$.

We consider the general form of the linear Boussinesq equation in one dimension with the initial conditions given below.

$$\frac{\partial^2\psi}{\partial t^2} = a\frac{\partial^2\psi}{\partial x^2}+b\frac{\partial^2\ln\psi}{\partial x^2}+c\frac{\partial^4\psi}{\partial x^4}, \tag{25}$$

subject to

$$\psi(x,0) = f_1(x), \quad \frac{\partial\psi(x,0)}{\partial t} = f_2(x), \tag{26}$$

where the functions $f_1(x)$, and $f_2(x)$ are given, and a, b and c are constants. First, applying the double Sumudu transform on both sides of Equation (25) and the single Sumudu transform for Equation (26), we obtain

$$\begin{aligned}\frac{\Psi(u_1,v)}{v^2} &= \frac{\Psi(u_1,0)}{v^2}+\frac{\Psi_t(u_1,0)}{v}\\ &+S_xS_t\left[a\frac{\partial^2\psi}{\partial x^2}+b\frac{\partial^2\ln\psi}{\partial x^2}+c\frac{\partial^4\psi}{\partial x^4}\right],\end{aligned} \tag{27}$$

which by arranging Equation (27) becomes

$$\begin{aligned}\Psi(u_1,v) &= F_1(u_1,0)+vF_1(u_1,0)\\ &+v^2S_xS_t\left[a\frac{\partial^2\psi}{\partial x^2}+b\frac{\partial^2\ln\psi}{\partial x^2}+c\frac{\partial^4\psi}{\partial x^4}\right].\end{aligned} \tag{28}$$

The solution is received by using the inverse double Sumudu to transform for Equation (28),

$$\psi(x,t)=f_1(x)+tf_2(x)+S_{u_1}^{-1}S_v^{-1}\left[v^2S_xS_t\left[a\frac{\partial^2\psi}{\partial x^2}+b\frac{\partial^2\ln\psi}{\partial x^2}+c\frac{\partial^4\psi}{\partial x^4}\right]\right], \tag{29}$$

where $S_{u_1}^{-1}S_v^{-1}$ indicates the double inverse Sumudu transform. The double Sumudu transform decomposition method (DSDM) defines the solutions $\psi(x,t)$ with the support of infinite series as:

$$\psi(x,t)=\sum_{n=0}^{\infty}\psi_n(x,t). \tag{30}$$

By substituting Equation (30) into Equation (29), we receive

$$\begin{aligned}\sum_{n=0}^{\infty}\psi_n(x,t) &= f_1(x)+tf_2(x)\\ &+S_{u_1}^{-1}S_v^{-1}\left[v^2S_xS_t\left[a\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty}\psi_n(x,t)\right)+b\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty}\ln\psi_n(x,t)\right]\right]\\ &+S_{u_1}^{-1}S_v^{-1}\left[v^2S_xS_t\left[c\frac{\partial^4}{\partial x^4}\left(\sum_{n=0}^{\infty}\psi_n(x,t)\right)\right]\right].\end{aligned} \tag{31}$$

By matching both sides of the Equation (31), we obtain

$$\psi_0(x,t)=f_1(x)+tf_2(x). \tag{32}$$

In general, the rest terms are given by

$$\begin{aligned}\psi_{n+1}(x,t) &= S_{u_1}^{-1}S_v^{-1}\left[v^2S_xS_t\left[a\frac{\partial^2}{\partial x^2}(\psi_n(x,t))+b\frac{\partial^2}{\partial x^2}(\ln\psi_n(x,t))\right]\right]\\ &+S_{u_1}^{-1}S_v^{-1}\left[v^2S_xS_t\left[c\frac{\partial^4}{\partial x^4}(\psi_n(x,t))\right]\right],\end{aligned} \tag{33}$$

where the inverse double Sumudu transform is given by $S_{u_1}^{-1}S_v^{-1}$. Here, we offered that the inverse exists for Equations (32) and (33). In order to explain the advantages and the precision of the DSTDM for solving Boussinesq equations, we used the method described in Example 1.

Example 1. *Consider a Boussinesq equation in one dimension*

$$\frac{\partial^2\psi}{\partial t^2}=a\frac{\partial^2\psi}{\partial x^2}+b\frac{\partial^2\ln\psi}{\partial x^2}+c\frac{\partial^4\psi}{\partial x^4} \tag{34}$$

subject to the initial condition

$$\psi(x,0)=e^x,\quad \frac{\partial\psi(x,0)}{\partial t}=2e^x. \tag{35}$$

By applying the above method, Equation (25) becomes

$$\begin{aligned} \psi(x,t) &= e^x + 2te^x + S_{u_1}^{-1}S_v^{-1}\left[v^2 S_x S_t\left[\tfrac{\partial^2}{\partial x^2}(\psi_n(x,t)) + \tfrac{\partial^2}{\partial x^2}(\ln\psi_n(x,t))\right]\right] \\ &\quad + S_{u_1}^{-1}S_v^{-1}\left[v^2 S_x S_t\left[\tfrac{\partial^4}{\partial x^4}(\psi_n(x,t))\right]\right]. \end{aligned} \tag{36}$$

Our wanted recursive relation is given by

$$\psi_0 = e^x + 2te^x,$$

and

$$\begin{aligned} \psi_{n+1}(x,t) &= S_{u_1}^{-1}S_v^{-1}\left[v^2 S_x S_t\left[\frac{\partial^2}{\partial x^2}(\psi_n(x,t)) + \frac{\partial^2}{\partial x^2}(\ln\psi_n(x,t))\right]\right] \\ &\quad + S_{u_1}^{-1}S_v^{-1}\left[v^2 S_x S_t\left[\frac{\partial^4}{\partial x^4}(\psi_n(x,t))\right]\right], \end{aligned}$$

for $n = 0, 1, 2, \ldots$. *Hence, at* $n = 0$,

$$\begin{aligned} \psi_1(x,t) &= S_{u_1}^{-1}S_v^{-1}\left[v^2 S_x S_t\left[\frac{\partial^2}{\partial x^2}(\psi_0(x,t)) + \frac{\partial^2}{\partial x^2}(\ln\psi_0(x,t)) + \frac{\partial^4}{\partial x^4}(\psi_0(x,t))\right]\right] \\ &= S_{u_1}^{-1}S_v^{-1}\left[v^2 S_x S_t[4e^x + 8te^x]\right] = 2t^2e^x + \frac{8}{3!}t^3e^x; \end{aligned}$$

at $n = 1$,

$$\begin{aligned} \psi_2(x,t) &= S_{u_1}^{-1}S_v^{-1}\left[v^2 S_x S_t\left[\frac{\partial^2}{\partial x^2}(\psi_1(x,t)) + \frac{\partial^2}{\partial x^2}(\ln\psi_1(x,t)) + \frac{\partial^4}{\partial x^4}(\psi_1(x,t))\right]\right] \\ &= S_{u_1}^{-1}S_v^{-1}\left[v^2 S_x S_t\left[8t^2e^x + \frac{24}{3!}t^3e^x\right]\right] = \frac{16}{4!}t^4e^x + \frac{32}{5!}t^5e^x; \end{aligned}$$

and at $n = 2$,

$$\psi_3(x,t) = \frac{64}{6!}t^6e^x + \frac{128}{7!}t^7e^x.$$

By applying Equation (30), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty}\psi_n(x,t) &= \psi_0 + \psi_1 + \psi_2 + \ldots. \\ &= e^x + 2te^x + \frac{(2t)^2}{2!}e^x + \frac{(2t)^3}{3!}e^x + \frac{(2t)^4}{4!}e^x + \frac{(2t)^5}{5!}e^x \\ &\quad \frac{(2t)^6}{6!}e^x + \frac{(2t)^7}{7!}e^x. \end{aligned}$$

Therefore, the solution to Equation (34) is given by

$$\psi(x,t) = e^{x+2t}.$$

The surface in Figure 1 shows the approximate solution of function $\psi(x,t) = e^{x+2t}$.

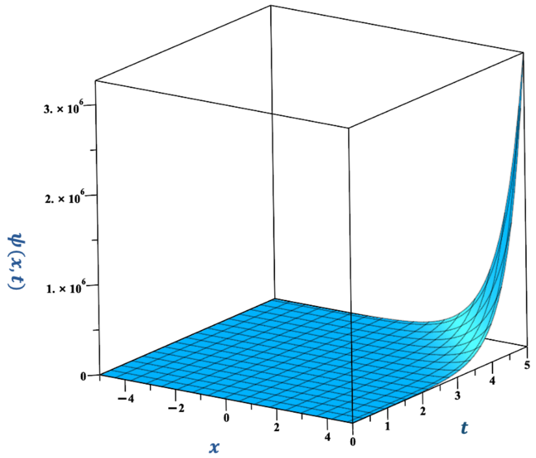


Figure 1. $\psi(x,t) = e^{x+2t}$.

4. Triple Sumudu Transform Decomposition Method and Singular $2+1$-Dimensional Boussinesq Equation

Now, we explain the triple Sumudu transform decomposition method to solve the singular $2+1$-dimensional Boussinesq equation:

Consider the following general form of the singular $2+1$-dimensional Boussinesq equation of the form:

$$\begin{aligned} &\psi_{tt} - \frac{1}{x}\frac{\partial}{\partial x}(x\psi_x) - \frac{1}{y}\frac{\partial}{\partial y}(y\psi_y) + a(x,y)\psi_{xxxx} + b(x,y)\psi_{yyyy} \\ &+c(x,y)\psi_{xxtt} + d(x,y)\psi_{yytt} = f(x,y,t), \end{aligned} \tag{37}$$

with the initial condition

$$\psi(x,y,0) = g_1(x,y), \quad \frac{\partial\psi(x,y,0)}{\partial t} = g_2(x,y), \tag{38}$$

where the functions $a(x,y)$, $b(x,y)$, $c(x,y)$, and $d(x,y)$ are arbitrary. In order to obtain the solution of Equation (37), we first take the product of both sides of Equation (37) by xy, and applying the triple Sumudu transform, we obtain Equation (39)

$$\begin{aligned} S_xS_yS_t[xy\psi_{tt}] &= S_xS_yS_t\left[y\frac{\partial}{\partial x}(x\psi_x) + x\frac{\partial}{\partial y}(y\psi_y) - axy\psi_{xxxx} - b(x)xy\psi_{yyyy}\right] \\ &+ S_xS_yS_t\left[-cxy\psi_{xxtt} - dxy\psi_{yytt} + xyf(x,y,t)\right]. \end{aligned} \tag{39}$$

Second, applying Equation (13), one obtains Equation (40) by arranging

$$\begin{aligned} u_1u_2\frac{\partial^2}{\partial u_1\partial u_2}S_xS_yS_t[\psi_{tt}] &= S_xS_yS_t\left[y\frac{\partial}{\partial x}(x\psi_x) + x\frac{\partial}{\partial y}(y\psi_y) - axy\psi_{xxxx} - b(x)xy\psi_{yyyy}\right] \\ &+S_xS_yS_t\left[-cxy\psi_{xxtt} - dxy\psi_{yytt} + xyf(x,y,t)\right], \end{aligned} \tag{40}$$

and by arranging Equation (40)

$$\begin{aligned} \frac{\partial^2}{\partial u_1\partial u_2}S_xS_yS_t[\psi_{tt}] &= \frac{1}{u_1u_2}S_xS_yS_t\left[y\frac{\partial}{\partial x}(x\psi_x) + x\frac{\partial}{\partial y}(y\psi_y) - axy\psi_{xxxx} - b(x)xy\psi_{yyyy}\right] \\ &+\frac{1}{u_1u_2}S_xS_yS_t\left[-cxy\psi_{xxtt} - dxy\psi_{yytt} + xyf(x,y,t)\right]. \end{aligned} \tag{41}$$

By taking the integral for Equation (41) from 0 to u_1 and 0 to u_2 with respect to u_1 and u_2, we have

$$\begin{aligned} S_xS_yS_t[\psi_{tt}] &= \int_0^{u_1}\int_0^{u_1}\frac{1}{u_1u_2}S_xS_yS_t\left[y\frac{\partial}{\partial x}(x\psi_x) + x\frac{\partial}{\partial y}(y\psi_y) - axy\psi_{xxxx} - bxy\psi_{yyyy}\right]du_1du_2 \\ &+\int_0^{u_1}\int_0^{u_1}\frac{1}{u_1u_2}S_xS_yS_t\left[-cxy\psi_{xxtt} - dxy\psi_{yytt} + xyf(x,y,t)\right]du_1du_2. \end{aligned} \tag{42}$$

For the double Sumudu transform for the initial condition given in Equation (38), we obtain

$$\begin{aligned}
&\frac{\psi(u_1,u_2,v)}{v^2}-\frac{\psi(u_1,u_2,0)}{v^2}-\frac{\psi_t(u_1,u_2,0)}{v} \\
= &\int_0^{u_1}\int_0^{u_1}\frac{1}{u_1u_2}S_xS_yS_t\left[y\frac{\partial}{\partial x}(x\psi_x)+x\frac{\partial}{\partial y}(y\psi_y)-axy\psi_{xxxx}-bxy\psi_{yyyy}\right]du_1du_2 \\
&+\int_0^{u_1}\int_0^{u_1}\frac{1}{u_1u_2}S_xS_yS_t\left[-cxy\psi_{xxtt}-dxy\psi_{yytt}+xyf(x,y,t)\right]du_1du_2.
\end{aligned} \tag{43}$$

For the third step, using the triple inverse Sumudu transform for both sides of Equation (43), the solution to Equation (37) can be written as

$$\begin{aligned}
\psi(x,y,t) = \; & g_1(x.y)+tg_2(x,y) \\
&+S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[\int_0^{u_1}\int_0^{u_1}\frac{1}{u_1u_2}S_xS_yS_t\left[y\frac{\partial}{\partial x}(x\psi_x)+x\frac{\partial}{\partial y}(y\psi_y)\right]du_1du_2\right] \\
&-S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[\int_0^{u_1}\int_0^{u_1}\frac{1}{u_1u_2}S_xS_yS_t\left[axy\psi_{xxxx}+bxy\psi_{yyyy}\right]du_1du_2\right] \\
&-S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[\int_0^{u_1}\int_0^{u_1}\frac{1}{u_1u_2}S_xS_yS_t\left[cxy\psi_{xxtt}+dxy\psi_{yytt}-xyf(x,y,t)\right]du_1du_2\right].
\end{aligned} \tag{44}$$

By substituting Equation (30) into Equation (44), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}\psi_n(x,y,t) = \; & g_1(x)+tg_2(x)+S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[\int_0^{u_1}\int_0^{u_1}\frac{1}{u_1u_2}S_xS_yS_t[xyf(x,y,t)]\right] \\
&+S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[\int_0^{u_1}\int_0^{u_1}\frac{1}{u_1u_2}S_xS_yS_t\left[y\frac{\partial}{\partial x}\left(x\left(\sum_{n=0}^{\infty}\psi_{nx}\right)\right)\right]du_1du_2\right] \\
&+S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[\int_0^{u_1}\int_0^{u_1}\frac{1}{u_1u_2}S_xS_yS_t\left[x\frac{\partial}{\partial y}\left(y\left(\sum_{n=0}^{\infty}\psi_{ny}\right)\right)\right]du_1du_2\right] \\
&-S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[\int_0^{u_1}\int_0^{u_1}\frac{1}{u_1u_2}\left[xy\left(\sum_{n=0}^{\infty}a\psi_{nxxxx}+b\psi_{nyyyy}\right)\right]du_1du_2\right] \\
&-S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[\int_0^{u_1}\int_0^{u_1}\frac{1}{u_1u_2}\left[xy\left(\sum_{n=0}^{\infty}cxy\psi_{nxxtt}+dxy\psi_{nyytt}\right)\right]du_1du_2\right],
\end{aligned} \tag{45}$$

where $n=0,1,2,\ldots$. Hence, from Equation (45) above, we have

$$\psi_0(x,y,t)=g_1(x,y)+tg_2(x,y)+S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[\int_0^{u_1}\int_0^{u_1}\frac{1}{u_1u_2}S_xS_yS_t[xyf(x,y,t)]\right],$$

and

$$\begin{aligned}
\psi_{n+1}(x,y,t) = \; & S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[\int_0^{u_1}\int_0^{u_1}\frac{1}{u_1u_2}S_xS_yS_t\left[y\frac{\partial}{\partial x}(x(\psi_{nx}))+x\frac{\partial}{\partial y}(y(\psi_{ny}))\right]du_1du_2\right] \\
&-S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[\int_0^{u_1}\int_0^{u_1}\frac{1}{u_1u_2}\left[xy((a(x)\psi_{nxxxx}+b\psi_{nyyyy}))\right]du_1du_2\right] \\
&-S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[\int_0^{u_1}\int_0^{u_1}\frac{1}{u_1u_2}\left[xy((cxy\psi_{nxxtt}+dxy\psi_{nyytt}))\right]du_1du_2\right].
\end{aligned} \tag{46}$$

To clarify this method for the linear singular Boussinesq equation, we give Example 2. We let $a=b=c=d=1$ and $f(x,t)=-(x^2-y^2)\sin t$.

Example 2. *The linear singular Boussinesq equation in one dimension is given by*

$$\begin{aligned}
&\psi_{tt}-\frac{1}{x}\frac{\partial}{\partial x}(x\psi_x)-\frac{1}{y}\frac{\partial}{\partial y}(y\psi_y)+\psi_{xxxx}+\psi_{yyyy} \\
&+\psi_{xxtt}+\psi_{yytt}=-\left(x^2-y^2\right)\sin t,
\end{aligned} \tag{47}$$

with the initial conditions

$$\psi(x,y,0)=0,\quad \frac{\partial\psi(x,y,0)}{\partial t}=\left(x^2-y^2\right). \tag{48}$$

In order to proceed with our method for Equation (47), we obtain

$$\psi_0(x,y,t) = \left(x^2 - y^2\right)\sin t,$$

and

$$\begin{aligned}\psi_{n+1}(x,y,t) &= S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[\int_0^{u_1}\int_0^{u_1}\frac{1}{u_1u_2}S_xS_yS_t\left[y\frac{\partial}{\partial x}(x(\psi_{nx})) + x\frac{\partial}{\partial y}(y(\psi))\right]du_1du_2\right] \\ &\quad -S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[\int_0^{u_1}\int_0^{u_1}\frac{1}{u_1u_2}[xy((\psi + \psi_{nyyyy}))]du_1du_2\right] \\ &\quad -S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[\int_0^{u_1}\int_0^{u_1}\frac{1}{u_1u_2}[xy((xy\psi_{nxxtt} + xy\psi_{nyytt}))]du_1du_2\right].\end{aligned}$$

The first repetition at $n = 0$ is denoted by

$$\begin{aligned}\psi_1(x,y,t) &= S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[\int_0^{u_1}\int_0^{u_1}\frac{1}{u_1u_2}S_xS_yS_t\left[y\frac{\partial}{\partial x}(x(\psi_{0x})) + x\frac{\partial}{\partial y}(y(\psi_{0y}))\right]du_1du_2\right] \\ &\quad -S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[\int_0^{u_1}\int_0^{u_1}\frac{1}{u_1u_2}[xy((\psi_{0xxxx} + \psi_{0yyyy}))]du_1du_2\right] \\ &\quad -S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[\int_0^{u_1}\int_0^{u_1}\frac{1}{u_1u_2}[xy((xy\psi_{0xxtt} + xy\psi_{0yytt}))]du_1du_2\right], \\ &= 0,\end{aligned}$$

at $n = 1$. We have

$$\psi_2(x,y,t) = 0.$$

Likewise, let $n = 2$. We obtain

$$\psi_3(x,y,t) = 0.$$

Hence, by using Equation (30), the series solutions are denoted by

$$\begin{aligned}\sum_{n=0}^{\infty}\psi_n(x,y,t) &= \psi_0 + \psi_1 + \psi_2 + \ldots. \\ &= \left(x^2 - y^2\right)\sin t.\end{aligned}$$

Figure 2A,B show the approximate solutions of Example 2 for the function $\psi(x,y,t) = (x^2 - y^2)\sin t$ at $y = 0$ and $x = 0$, respectively.

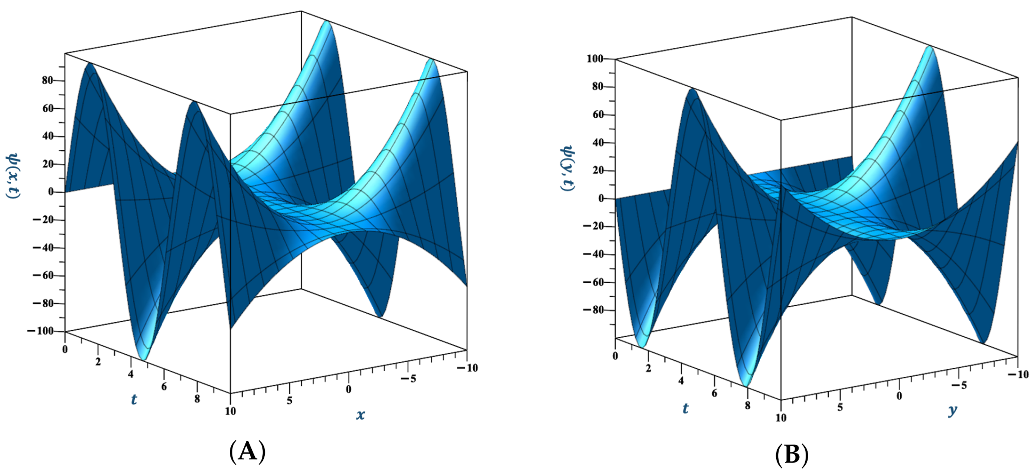


Figure 2. (**A**) $\psi(x,y,t)$ at $y = 0$; (**B**) $\psi(x,y,t)$ at $x = 0$.

5. The Triple Sumudu Transform Decomposition Method and the Singular $2+1$-Dimensional Coupled System Boussinesq Equation

In this part, the triple Sumudu transform decomposition is addressed for the solution of the singular $2+1$-dimensional coupled system Boussinesq equation. The general form of the singular $2+1$-dimensional coupled system Boussinesq equation is denoted by

$$\begin{aligned} w_t &= a(x,y)(w\psi)_x + b(x,y)\psi_{xxx} + c(x,y)\psi_{yyy} \\ \psi_t &= d(x,y)w_x + e(x,y)w_y - \psi\psi_x, \end{aligned} \tag{49}$$

with the initial condition

$$w(x,y,0) = f_1(x,y), \;\; \psi(x,y,0) = f_2(x,y), \tag{50}$$

where the functions $a(x,y), b(x,y), c(x,y), d(x,y)$, and $e(x,y)$ are arbitrary. For the purpose of obtaining the solution to Equation (49), we use the triple Sumudu transform Equation (49) and the double Sumudu transform for Equation (50). We have

$$\begin{aligned} \frac{W(u_1,u_2,v)}{v} &= \frac{F_1(u_1,u_2)}{v} + S_xS_yS_t\big[a(w\psi) + b\psi_{xxx} + c\psi_{yyy}\big] \\ \frac{\Psi(u_1,u_2,v)}{v} &= \frac{F_2(u_1,u_2)}{v} + S_xS_yS_t\big[dw_x + ew_y - \psi\psi_x\big]. \end{aligned} \tag{51}$$

By organizing Equation (51), we obtain

$$\begin{aligned} W(u_1,u_2,v) &= F_1(u_1,u_2) + vS_xS_yS_t\big[a(w\psi) + b\psi_{xxx} + c\psi_{yyy}\big] \\ \Psi(u_1,u_2,v) &= F_2(u_1,u_2) + vS_xS_yS_t\big[dw_x + ew_y - \psi\psi_x\big]. \end{aligned} \tag{52}$$

Applying the inverse transformation, we obtain

$$\begin{aligned} w(x,y,t) &= f_1(x,y) + S^{-1}_{u_1}S^{-1}_{u_2}S^{-1}_{v}\big[vS_xS_yS_t\big[a(w\psi) + b\psi_{xxx} + c\psi_{yyy}\big]\big] \\ \psi(x,y,t) &= f_2(x,y) + S^{-1}_{u_1}S^{-1}_{u_2}S^{-1}_{v}\big[vS_xS_yS_t\big[dw_x + ew_y - \psi\psi_x\big]\big]. \end{aligned} \tag{53}$$

By substituting Equation (30) into Equation (29), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} w_n(x,y,t) &= f_1(x,y) + S^{-1}_{u_1}S^{-1}_{u_2}S^{-1}_{v}\left[vS_xS_yS_t\left[a\left(\sum_{n=0}^{\infty} w_n\psi_n\right) + b\sum_{n=0}^{\infty}\psi_{nxxx} + c\sum_{n=0}^{\infty}\psi_{nyyy}\right]\right] \\ \sum_{n=0}^{\infty} \psi_n(x,y,t) &= f_2(x,y) + S^{-1}_{u_1}S^{-1}_{u_2}S^{-1}_{v}\left[vS_xS_yS_t\left[d\sum_{n=0}^{\infty} w_{nx} + e\sum_{n=0}^{\infty} w_{ny} - \sum_{n=0}^{\infty}\psi_n\psi_{nx}\right]\right]. \end{aligned} \tag{54}$$

The $w_0(x,y,t), \psi_0(x,y,t), w_{n+1}(x,y,t)$, and $\psi_{n+1}(x,y,t)$ are given by

$$w_0(x,y,t) = f_1(x,y), \; \psi_0(x,y,t) = f_2(x,y), \tag{55}$$

and

$$\begin{aligned} w_{n+1}(x,y,t) &= S^{-1}_{u_1}S^{-1}_{u_2}S^{-1}_{v}\big[vS_xS_yS_t\big[a(w_n\psi_n) + b\psi_{nxxx} + c\psi_{nyyy}\big]\big] \\ \psi_{n+1}(x,y,t) &= S^{-1}_{u_1}S^{-1}_{u_2}S^{-1}_{v}\big[vS_xS_yS_t\big[dw_{nx} + ew_{ny} - \psi_n\psi_{nx}\big]\big]. \end{aligned} \tag{56}$$

Now, we stipulate the triple inverse Sumudu transform with respect to u_1, u_2, and v, which exist for Equation (56).

To confirm the applicability of the method offered above, for a $2+1$-dimensional coupled system Boussinesq equation, we offer the following example, at $a = b = c = d = e = -1$.

Example 3. *The $2+1$-dimensional coupled system Boussinesq equation is given by*

$$\begin{aligned} w_t &= -\frac{1}{2}(w\psi)_x - \psi_{xxx} - \psi_{yyy} \\ \psi_t &= -w_x - w_y - \psi\psi_x, \end{aligned} \tag{57}$$

with the initial condition

$$w(x,y,0) = 2x - 2y, \;\; \psi(x,y,0) = 2x - 2y. \tag{58}$$

As indicated by the above method, the zeroth components w_0 and ψ_0 are proposed by the Adomian method,

$$w_0 = 2x - 2y, \;\; \psi_0 = 2x - 2y. \tag{59}$$

The remaining components w_{n+1}, ψ_{n+1}, $n \geq 0$ are given by using the relation

$$\begin{aligned} w_{n+1} &= -S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[vS_xS_yS_t\left[(w_n\psi_n)_x + \psi_{nxxx} + \psi_{nyyy}\right]\right] \\ \psi_{n+1} &= -S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[vS_xS_yS_t\left[w_{nx} + w_{ny} + \psi_n\psi_{nx}\right]\right]. \end{aligned} \tag{60}$$

By putting $n = 0$ into Equation (60), we have

$$\begin{aligned} w_1 &= -S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[vS_xS_yS_t\left[\frac{1}{2}(w_0\psi_0)_x + \psi_{0xxx} + \psi_{0yyy}\right]\right] \\ &= -S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[vS_xS_yS_t[4x-4y]\right] = -S_xS_yS_t[4u_1v - 4u_2v] \\ &= -(4xt - 4yt), \\ \psi_1 &= -S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[vS_xS_yS_t\left[w_{0x} + w_{0y} + \psi_0\psi_{0x}\right]\right] \\ &= -S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[vS_xS_yS_t[4x-4y]\right] = -S_xS_yS_t[4u_1v - 4u_2v] \\ &= -(4xt - 4yt), \end{aligned}$$

at $n = 1$,

$$\begin{aligned} w_2 &= -S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[vS_xS_yS_t\left[\frac{1}{2}(w_{0x}\psi_1 + w_{1x}\psi_0 + w_1\psi_{0x} + w_0\psi_{1x}) + \psi_{1xxx} + \psi_{1yyy}\right]\right], \\ w_2 &= 4t^2(2x-2y), \\ \psi_2 &= -S_{u_1}^{-1}S_{u_2}^{-1}S_v^{-1}\left[vS_xS_yS_t\left[w_{1x} + w_{1y} + \psi_0\psi_{1x} + \psi_1\psi_{0x}\right]\right] \\ &= 4t^2(2x-2y). \end{aligned}$$

In a similar manner, we have

$$\begin{aligned} w_3 &= -8t^3(2x-2y), \\ \psi_3 &= -8t^3(2x-2y). \end{aligned}$$

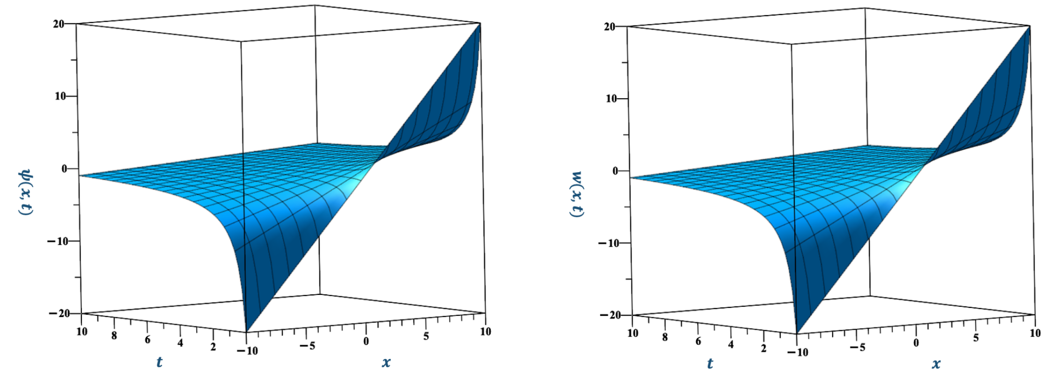
Hence, by using Equation (30), the series solutions are denoted by

$$\begin{aligned} \sum_{n=0}^{\infty} w_n(x,y,t) &= w_0 + w_1 + w_2 + \ldots. \\ &= (2x-2y)\left(1 - 2t + (2t)^2 - (2t)^3 + (2t)^4 - \ldots\right), \\ \sum_{n=0}^{\infty} \psi_n(x,y,t) &= \psi_0 + \psi_1 + \psi_2 + \ldots. \\ &= (2x-2y)\left(1 - 2t + (2t)^2 - (2t)^3 + (2t)^4 - \ldots\right); \end{aligned}$$

therefore, the solution to equation Equation (57) is given by

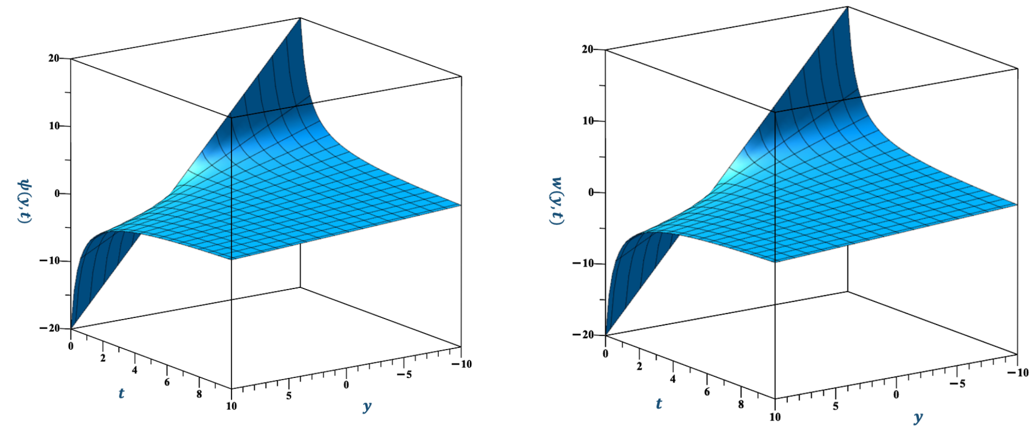
$$w(x,y,t) = \frac{2x-2y}{1+2t} \;\; and \;\; \psi(x,y,t) = \frac{2x-2y}{1+2t}.$$

The approximate solutions for the functions $\psi(x,y,t) = \frac{2x-2y}{1+2t}$ and $w(x,y,t) = \frac{2x-2y}{1+2t}$ at $y = 0$ were shown in Figure 3A,B, respectively. Moreover, in the Figure 4A,B, we show the approximate solutions of function $\psi(x,y,t) = \frac{2x-2y}{1+2t}$ and $w(x,y,t) = \frac{2x-2y}{1+2t}$ at $x = 0$, respectively.



(**A**) (**B**)

Figure 3. (**A**) $\psi(x, y, t)$ at y = 0; (**B**) $w(x, y, t)$ at y = 0.



(**A**) (**B**)

Figure 4. (**A**) $\psi(x, y, t)$ at x = 0; (**B**) $w(x, y, t)$ at x = 0.

6. Conclusions

In this work, we presented the triple Sumudu transform decomposition method (TSTDM) to find the approximate and series solutions of the Boussinesq equations. We examined three different types of examples connected to the one and two dimensional Boussinesq equations for systems of linear Boussinesq equations. By investigating the examples, we conclude that the TSTDM is a powerful tool for the solution of linear, nonlinear, and coupled systems of Boussinesq equations, compared with the Adomian decomposition method, homotopy analysis method (HAM), and variational iteration method (VAM). Nonetheless, there is still the open problem of investigating the rate of convergence to the exact solution for these types of problems. It is also possible to study the TSTDM by using an analytical solution to the other singular partial differential equations, which arise in applied science as well as engineering that may offer a better understanding of the real-world problems that represent singular partial differential equations. In later works, we plan to apply the TSTDM to several models related to engineering and physics.

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