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Existence and Asymptotic Stability of the Solution for the Timoshenko Transmission System with Distributed Delay

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Abstract: In the present paper, a transmission problem of the Timoshenko beam in the presence of distributed delay is considered. Under appropriate assumptions, we prove the well-posedness by using the semi-group theory. Furthermore, we study the asymptotic behavior of solutions using the multiplier method. We investigate the techniques and ideas used by the second author to extend the recent results.

Keywords: Timoshenko system; transmission problem; distributed delay; asymptotic behavior; multiplier method

MSC: 35B40; 35L55; 74D05; 93D15

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1. Introduction and Position of Problem

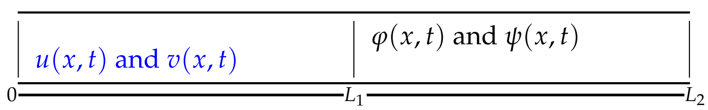
The need for fundamental research on the Timoshenko transmission problem became apparent when many physical processes were found to lead to initial boundary values and mixed problems involving partial derivatives of fractional order. Moreover, these systems belong to the class of modern differential equations, which, as a rule, are not self-adjoint. The main purpose of this work is to study the structural and qualitative properties of the model (see [1–5]).

In this paper, we study the following transmission problem with distributed delay:

$$\begin{cases} (u_{tt} - au_{xx} - \alpha(u_x+v)_x + \mu_1 u_t + u + v)(x,t) & \\ \qquad -\mu_2 \int_{t-\tau_1}^{t-\tau_2} \sigma_1(t-s)u_t(x,s)ds = 0, & \text{in } \mathcal{Q}_1 \\ (v_{tt} - av_{xx} + \alpha(u_x+v) + u + v)(x,t) = 0, & \text{in } \mathcal{Q}_1 \\ (\varphi_{tt} - b\varphi_{xx} - \beta(\varphi_x+\psi)_x + \mu_3\varphi_t)(x,t) & \\ \qquad -\mu_4 \int_{t-\tau_1}^{t-\tau_2} \sigma_2(t-s)\varphi_t(x,s)ds = 0, & \text{in } \mathcal{Q}_2 \\ (\psi_{tt} - b\psi_{xx} + \mu_5\psi_t + \beta(\varphi_x+\psi))(x,t) = 0, & \text{in } \mathcal{Q}_2, \end{cases} \tag{1}$$

where $\mathcal{Q}_j = ((j-1)L_{j-1}, L_j) \times \mathbb{R}_+,\ \ j = 1;2.$ with $0 < L_0 < L_1 < L_2$ and $a, b, \mu_{2i-1}|_{i=1;2;3}$, τ_2, α, β are positive constants, $\mu_{2j}|_{j=1;2} \in \mathbb{R}$, τ_1 is a nonnegative constant with $\tau_1 < \tau_2$ and $\sigma_j : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$, $j = 1;2$ is a bounded function.

L_2 is the length of the beam. Functions u and φ are the transverse displacements of a beam with reference configuration $(0, L_2) \subseteq \mathbb{R}$ and functions v and ψ are the rotation angles of a filament of the beam.



The well-posedness and asymptotic behavior of the solution of system (1) is studied $\forall t \geq 0$, under the following boundary and transmission conditions:

$$\begin{array}{l} u(0,t) = v(0,t) = \varphi(L_2,t) = \psi(L_2,t) = 0, \\ u(L_1,t) = \varphi(L_1,t), \quad au_x(L_1,t) = b\varphi_x(L_1,t), \\ v(L_1,t) = \psi(L_1,t), \quad av_x(L_1,t) = b\psi_x(L_1,t), \\ \alpha(u_x+v)(L_1,t) = \beta(\varphi_x+\psi)(L_1,t), \end{array} \tag{2}$$

and the initial conditions

$$\begin{array}{l} u(0) = u^0, v(0) = v^0, u_t(0) = u^1, v_t(0) = v^1, \quad x \in (0,L_1), \\ \varphi(0) = \varphi^0, \psi(0) = \psi^0, \varphi_t(0) = \varphi^1, \psi_t(0) = \psi^1, \quad x \in (L_1,L_2), \\ u_t(x,-t) = f_0(x,t), \quad x \in (0,L_1), t \in [0,\tau_2], \\ \varphi_t(x,-t) = h_0(x,t), \quad x \in (L_1,L_2), t \in [0,\tau_2]. \end{array} \tag{3}$$

Time delays are used in many applications, such as physical, chemical, biological, thermal, and economic phenomena; these phenomena do not naturally depend on the current state, but on some past events.

The presence of a delay in the system can turn the system into an unstable state or a well-behaved system into a wild system. It has been shown that adding a slight delay to a uniformly asymptotic system can destabilize that system unless additional control conditions are used; for example, see [6–10].

By the beginning of this century, the study of transmission problems—such as the vibration propagation over objects consisting of two different types of materials—gained significant importance; as can be seen in [1,2,11].

Transmission problems frequently occur in scenarios where the field encompasses multiple materials whose properties have different elasticities and are interconnected over the entire surface. Mathematically, the transmission problem for wave propagation is governed by a hyperbolic problem. Green and Naghdi in [12,13] discussed two models of thermal elasticity: a type II class of thermoelasticity that does not conserve energy dissipation, and a third class that is dissipative in nature.

In the absence of delay, there are many works around transmission problems. Interested readers are referred to [14–16]. We mention here some results on the relation between the delay term and the source term [10,17–21].

Benaissa et al. [17] considered the following system in $(0,L) \times \mathbb{R}_+$, with delay terms in the internal feedback:

$$\begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + lw + \psi)_x - Ehl(w_x + l\varphi) + \mu_1\varphi_t + \mu_2\varphi_t(x,t-\tau_1) = 0 \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + lw + \psi) + \tilde{\mu}_1\psi_t + \tilde{\mu}_2\psi_t(x,t-\tau_2) = 0 \\ \rho_1 w_{tt} - Eh(w_x + l\varphi)_x + lGh(\varphi_x + lw + \psi) + \tilde{\tilde{\mu}}_1 w_t + \tilde{\tilde{\mu}}_2 w_t(x,t-\tau_2) = 0, \end{cases} \tag{4}$$

with initial and Dirichlet boundary conditions. The authors demonstrated that the well-posedness using the semi-group theory and the decay of the solution via the multiplier method, under the assumption that

$$|\mu_2| < \mu_1, \ |\tilde{\mu}_2| < \tilde{\mu}_1 \ \text{ and } \ \left|\tilde{\tilde{\mu}}_2\right| < \tilde{\tilde{\mu}}_1.$$

In [8], the authors explored the wave equation with linear frictional damping and internal distributed delay:

$$u_{tt} - \Delta_x u + \mu_1 u_t + a(x)\int_{\tau_1}^{\tau_2} \mu_2(s)u_t(t-s)ds = 0,$$

in $\Omega \times \mathbb{R}_+$. The exponential decay of the solution is obtained under the following assumption:

$$\|a(x)\|_\infty \int_{\tau_1}^{\tau_2} \mu_2(s)ds < \mu_1.$$

The author of [22] investigated a uniform stability result for the thermoelasticity of type III with boundary-distributed delay and found exponential stability under an appropriate condition.

In [23], Lamine Bouzettouta et al. considered a Bresse system in $(0, L) \times \mathbb{R}_+$, with delay terms in the internal feedback acting in the first and third equations, and the distributed delay term in the second equation.

$$\begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + lw + \psi)_x - Ehl(w_x + l\varphi) + \mu_1\varphi_t + \mu_2\varphi_t(x, t - \tau_1) = 0 \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + lw + \psi) + \mu_0\psi_t + \int_{\tau_2}^{\tau_1} \mu(s)\psi_t(x, t - s)ds = 0 \\ \rho_1 w_{tt} - Eh(w_x + l\varphi)_x + lGh(\varphi_x + lw + \psi) + \widetilde{\mu}_1 w_t + \widetilde{\mu}_2 w_t(x, t - \tau_2) = 0, \end{cases} \quad (5)$$

with initial and Dirichlet boundary conditions. The authors demonstrated the existence of solutions by the semi-group theory and studied the stability of solutions using the multiplier method, under the assumption that

$$|\mu_2| < \mu_1, \ \ |\widetilde{\mu}_2| < \widetilde{\mu}_1 \ \ \text{and} \ \ \mu_0 < \int_{\tau_2}^{\tau_1} \mu(s)ds.$$

Waves with frictional damping were studied in [24]. The transmission problem for the Timoshenko beam with frictional damping was studied in [25]. The transmission problem for the Timoshenko beam with unique memory was studied by Raposo [26]. In this case, there is no uniform decay.

On the other hand, Margareth S. Alves et al. [3] studied and proved the uniform stabilization for the following transmission problem of the Timoshenko system with two memories:

$$\begin{array}{l} \rho_1^1\phi_{tt}^1 - k_1(\phi_x^1 + \psi^1)_x = 0 \text{ in } (0, L_0) \times (0, +\infty), \\ \rho_1^2\phi_{tt}^2 - k_2(\phi_x^2 + \psi^2)_x = 0 \text{ in } (L_0, L) \times (0, +\infty), \\ \rho_2^1\psi_{tt}^1 - b_1\psi_{xx}^1 + k_1(\phi_x^1 + \psi^1) + g_1 * \psi_{xx}^1 = 0 \text{ in } (0, L_0) \times (0, +\infty), \\ \rho_2^2\psi_{tt}^2 - b_2\psi_{xx}^2 + k_2(\phi_x^2 + \psi^2) + g_2 * \psi_{xx}^2 = 0 \text{ in } (L_0, L) \times (0, +\infty), \end{array} \quad (6)$$

subject to initial boundary and transmission conditions.

Now, we will consider and prove the existence and uniqueness of solutions and uniform stabilization for the transmission problem for a partially viscoelastic beam of the Timoshenko system with distributed delay. This beam comprises two components: elastic and viscoelastic.

The rest of our paper is organized as follows. In Section 2, we present the main results and some preliminaries necessary for proving these results. In Section 3, we prove the well-posedness of our problem using semi-group methods. In Section 4, we prove the exponential decay of the energy by the multiplier method.

2. Preliminaries and the Main Results

In this section, we present the main assumptions of the parameters in (1) and functions σ_j, $j = 1, 2$. We also provide some essential preliminaries and present the key findings of this paper.

We present some lemmas, which will be needed later. Let us first recall Sobolev–Poincaré's inequality.

Lemma 1 ([3]). *Let p be a number with $2 \leq p < +\infty$. We will use the same embedding constants denoted by c_{P_1} and c_{P_2}, such that*

$$\|w\|_{L^p(0,L_1)} \leq c_{P_1}\|w_x\|_{L^2(0,L_1)} \quad \text{for} \quad w \in H_0^1(0, L_1),$$

and

$$\|w\|_{L^p(L_1,L_2)} \leq c_{P_2}\|w_x\|_{L^2(L_1,L_2)} \quad \text{for} \quad w \in H_0^1(L_1, L_2).$$

Lemma 2 ([27,28]). *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and assume that there are two constants, $\sigma > -1$ and $\omega > 0$, such that*

$$\int_S^{+\infty} E^{1+\sigma}(t)\,dt \leq \frac{1}{\omega} E^{\sigma}(0)E(S). \quad 0 \leq S < +\infty, \tag{7}$$

then we have

$$E(t) = 0 \quad \forall t \geq \frac{E(0)^{\sigma}}{\omega|\sigma|} \qquad \text{if} \quad -1 < \sigma < 0, \tag{8}$$

$$E(t) \leq E(0)\left(\frac{1+\sigma}{1+\omega\sigma t}\right)^{\frac{1}{\sigma}} \quad \forall t \geq 0, \quad \textit{if} \quad \sigma > 0, \tag{9}$$

and

$$E(t) \leq E(0)e^{1-\omega t} \qquad \forall t \geq 0, \quad \textit{if} \quad \sigma = 0. \tag{10}$$

We list all necessary assumptions for our claimed results.
Regarding the weight of distributed delay, we assume

$$\textbf{[A1]:} \quad \mu_{2i-1} - |\mu_{2i}| \int_{\tau_1}^{\tau_2} \sigma_i(s)ds > 0, \qquad i = 1;2.$$

and

$$\textbf{[A2]:} \quad \sum_{i=1}^{2} \frac{|\mu_{2i}|}{2} \int_{\tau_1}^{\tau_2} e^{2s}|\sigma_i(s)|ds < 2\min\{1, 2e^{-2s}\}.$$

As [10], let us introduce the following new variables:

$$\begin{cases} y_1(x,\rho,s,t) = u_t(x,t-s\rho) & (x,\rho,s,t) \in (0,L_1)\times(0,1)\times(\tau_1,\tau_2)\times\mathbb{R}_+ \\ y_2(x,\rho,s,t) = \varphi_t(x,t-s\rho) & (x,\rho,s,t) \in (L_1,L_2)\times(0,1)\times(\tau_1,\tau_2)\times\mathbb{R}_+. \end{cases} \tag{11}$$

The variables y_1 and y_2 satisfy

$$\begin{cases} sy_{1,t}(x,\rho,s,t) + y_{1,\rho}(x,\rho,s,t) = 0 & (x,\rho,s,t) \in \Sigma_1 \\ sy_{2,t}(x,\rho,s,t) + y_{2,\rho}(x,\rho,s,t) = 0 & (x,\rho,s,t) \in \Sigma_2 \end{cases} \tag{12}$$

where

$$\Sigma_i = ((i-1)L_{i-1}, L_i)\times(0,1)\times(\tau_1,\tau_2)\times\mathbb{R}_+, \quad i = 1;2.$$

Then, system (1) is equivalent to

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) - \alpha(u_x+v)_x(x,t) + \mu_1 u_t(x,t) \\ \qquad +(u+v)(x,t) + \mu_2\int_{\tau_1}^{\tau_2}\sigma_1(s)y_1(x,1,s,t)ds = 0 \text{ in } \mathcal{Q}_1 \\ v_{tt}(x,t) - av_{xx}(x,t) + \alpha(u_x+v)(x,t) + (u+v)(x,t) = 0 \text{ in } \mathcal{Q}_1 \\ \varphi_{tt}(x,t) - b\varphi_{xx}(x,t) - \beta(\varphi_x+\psi)_x(x,t) + \mu_3\varphi_t(x,t) \\ \qquad +\mu_4\int_{\tau_2}^{\tau_1}\sigma_2(s)y_2(x,1,s,t)ds = 0 \text{ in } \mathcal{Q}_2 \\ \psi_{tt}(x,t) - b\psi_{xx}(x,t) + \mu_5\psi_t + \beta(\varphi_x+\psi)(x,t) = 0 \text{ in } \mathcal{Q}_2 \\ sy_{1,t}(x,\rho,s,t) + y_{1,\rho}(x,\rho,s,t) = 0 \text{ in } \Sigma_1 \\ sy_{2,t}(x,\rho,s,t) + y_{2,\rho}(x,\rho,s,t) = 0 \text{ in } \Sigma_2, \end{cases} \tag{13}$$

where

$$\mathcal{Q}_i = ((i-1)L_{i-1}, L_i)\times\mathbb{R}_+, \quad y_{i,\rho} := \frac{\partial y_i}{\partial\rho} \text{ and } y_{i,t} := \frac{\partial y_i}{\partial t}, \quad i = 1;2.$$

From now on, we use the following notations

$$w := w(x,t) \ / \ w \in \{u, v, \varphi, \psi\}, \quad y_i(x,\rho,s,t) := y_i(\rho,s), \quad i = 1;2.$$

For any regular solution of (13), we define the energy as

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^{L_1}\left[u_t^2+v_t^2+a\left(u_x^2+v_x^2\right)+\alpha(u_x+v)^2+(u+v)^2\right]dx \\ &+\frac{1}{2}\int_{L_1}^{L_2}\left[\varphi_t^2+\psi_t^2+b\left(\varphi_x^2+\psi_x^2\right)+\beta(\varphi_x+\psi)^2\right]dx \\ &+\sum_{i=1}^{2}\frac{|\mu_{2i}|}{2}\int_{(i-1)L_{i-1}}^{L_i}\int_0^1\int_{\tau_1}^{\tau_2} s|\sigma_i(s)|y_i^2(\rho,s)dsd\rho dx. \end{aligned} \tag{14}$$

The main result of the present work is as follows:

Theorem 1. *Let (u,v,φ,ψ) be the solution of (1). Assume that* ***[A1]*** *and* ***[A2]*** *hold. Then there exist two positive constants, c and ω, such that*

$$E(t) \le cE(0)e^{-\omega t}, \qquad t \ge 0. \tag{15}$$

3. Well-Posedness of the Problem

Owing to the semi-group theory, we prove the existence and uniqueness of a local solution of system (1).

Here, we denote the following function space:

$$\begin{aligned} \mathbf{H}_*^1 &= \{(w,\widetilde{w}) \in H^1(0,L_1)\times H^1(L_1,L_2) \;/\; w(0,t)=\widetilde{w}(L_2,t)=0, \\ &\quad w(L_1,t)=\widetilde{w}(L_1,t),\; aw_x(L_1,t)=b\widetilde{w}_x(L_1,t), \\ &\quad \alpha w_x(L_1,t)=\beta\widetilde{w}_x(L_1,t),\}. \end{aligned} \tag{16}$$

and

$$\begin{aligned} \mathbf{H}_*^2 &= \{(w,\widetilde{w}) \in H^1(0,L_1)\times H^1(L_1,L_2) \;/\; w(0,t)=\widetilde{w}(L_2,t)=0, \\ &\quad w(L_1,t)=\widetilde{w}(L_1,t),\; aw_x(L_1,t)=b\widetilde{w}_x(L_1,t), \\ &\quad \alpha w(L_1,t)=\beta\widetilde{w}(L_1,t)\}. \end{aligned} \tag{17}$$

The phase space of our problem is the Hilbert space:

$$\begin{aligned} \mathcal{H} = \mathbf{H}_*^1\times\mathbf{H}_*^2\times\left(L^2(0,L_1)\times L^2(L_1,L_2)\right)^2\times L^2((0,L_1)\times(0,1)\times(\tau_1,\tau_2)) \\ \times L^2((L_1,L_2)\times(0,1)\times(\tau_1,\tau_2)), \end{aligned}$$

provided by the inner product, defined by:

for all vectors $\mathcal{U}=(w_1,\cdots,w_{10})^T$ and $\widetilde{\mathcal{U}}=(\widetilde{w}_1,\cdots,\widetilde{w}_{10})^T$ in $\mathcal{H}$,

$$\begin{aligned} \left\langle \mathcal{U},\widetilde{\mathcal{U}}\right\rangle_{\mathcal{H}} &= a\left[\langle w_{1,x},\widetilde{w}_{1,x}\rangle_{L^2(0,L_1)}+\langle w_{3,x},\widetilde{w}_{3,x}\rangle_{L^2(0,L_1)}\right] \\ &+b\left[\langle w_{2,x},\widetilde{w}_{2,x}\rangle_{L^2(L_1,L_2)}+\langle w_{4,x},\widetilde{w}_{4,x}\rangle_{L^2(L_1,L_2)}\right] \\ &+\alpha\langle w_{1,x}+w_3,\widetilde{w}_{1,x}+\widetilde{w}_3\rangle_{L^2(0,L_1)} \\ &+\langle w_1+w_3,\widetilde{w}_1+\widetilde{w}_3\rangle_{L^2(0,L_1)} \\ &+\beta\langle w_{2,x}+w_4,\widetilde{w}_{2,x}+\widetilde{w}_4\rangle_{L^2(L_1,L_2)}+\langle w_5',\widetilde{w}_5\rangle_{L^2(0,L_1)} \\ &+\langle w_6',\widetilde{w}_6'\rangle_{L^2(L_1,L_2)}+\langle w_7',\widetilde{w}_7'\rangle_{L^2(0,L_1)}+\langle w_8',\widetilde{w}_8'\rangle_{L^2(L_1,L_2)} \\ &+|\mu_2|\int_0^1\int_{\tau_1}^{\tau_2} s|\sigma_1(s)|\langle w_9,\widetilde{w}_9\rangle_{L^2(0,L_1)}dsd\rho \\ &+|\mu_4|\int_0^1\int_{\tau_1}^{\tau_2} s|\sigma_2(s)|\langle w_{10},\widetilde{w}_{10}\rangle_{L^2(L_1,L_2)}dsd\rho. \end{aligned}$$

Let $\mathcal{U}=(u,\varphi,v,\psi,u',\varphi',v',\psi',y_1,y_2)^T\in\mathcal{H}$ and with (3) can be rewritten (13) as an abstract Cauchy problem.

$$\begin{cases} \mathcal{A}\mathcal{U}=\mathcal{U}', \\ \mathcal{U}(0)=(u^0,\varphi^0,v^0,\psi^0,u^1,\varphi^1,v^1,\psi^1,f_0,h_0)^T, \end{cases} \tag{18}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A} = \begin{pmatrix} \frac{\partial}{\partial t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \begin{pmatrix} (a+\alpha)\frac{\partial^2}{\partial x^2} \\ -I \end{pmatrix} & 0 & \frac{\partial}{\partial x} - I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\alpha\frac{\partial}{\partial x} - I & 0 & \begin{pmatrix} a\frac{\partial^2}{\partial x^2} \\ -(\alpha+1)I \end{pmatrix} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (b+\beta)\frac{\partial^2}{\partial x^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial t} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu_1 I & 0 & 0 & 0 & -\mu_2\mathcal{T}_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta\frac{\partial}{\partial x} & 0 & -\mu_3 I & 0 & 0 & 0 & -\mu_4\mathcal{T}_2 \\ 0 & 0 & 0 & b\frac{\partial^2}{\partial x^2} - \beta I & 0 & -\mu_5 I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{s}\frac{\partial}{\partial \rho} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{s}\frac{\partial}{\partial \rho} \end{pmatrix}$$

where

$$\mathcal{T}_i(w_i) = \int_{\tau_1}^{\tau_2} \sigma_i(s) w_i(1,s)ds, \quad i = 1;2.$$

The domain of the operator $\mathcal{A}$ is defined by

$$\begin{aligned} \mathcal{D}(\mathcal{A}) \quad &= \Big\{(u,\varphi,v,\psi,u',\varphi',v',\psi',y_1,y_2) \in \mathcal{H} : \ (u',v') \in \big(H^1(0,L_1)\big)^2, \\ &(\varphi',\psi') \in \big(H^1(L_1,L_2)\big)^2, \\ &(u,\varphi) \in \big(\big(H^2(0,L_1) \times H^2(L_1,L_2)\big) \cap \mathbf{H}^{\mathbf{1}}_*\big), \\ &(v,\psi) \in \big(\big(H^2(0,L_1) \times H^2(L_1,L_2)\big) \cap \mathbf{H}^{\mathbf{2}}_*\big), \\ &y_1(\rho,s), y_{1,\rho}(\rho,s) \in L^2((0,L_1)\times(0,1)\times]\tau_1,\tau_2[), y_1(0,s) = u', \\ &y_2(\rho,s), y_{2,\rho}(\rho,s) \in L^2((L_1,L_2)\times(0,1)\times]\tau_1,\tau_2[), y_2(0,s) = \varphi'\}. \end{aligned}$$

Clearly, $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$.

The existence and uniqueness of a solution to the system (13) with (2) and (3) is stated by the following theorem:

Theorem 2. *Under the assumption **[A1]**, for any $\mathcal{U}_0 \in \mathcal{H}$, there exists a unique weak solution $\mathcal{U} \in C([0,+\infty[,\mathcal{H})$ of problem (18). Moreover, if $\mathcal{U}_0 \in D(\mathcal{A})$, then*

$$\mathcal{U} \in C([0,+\infty[,D(\mathcal{A})) \cap C^1([0,+\infty[,\mathcal{H}).$$

Proof. To prove the result from Theorem 2, we use the theory of semi-groups, i.e., we show that the operator $\mathcal{A}$ generates a C_0-semi-group in $\mathcal{H}$. In this step, we prove that the operator $\mathcal{A}$ is dissipative. Indeed, for $\mathcal{U} = (u,\varphi,v,\psi,u',\varphi',v',\psi',y_1,y_2)^T \in \mathcal{D}(\mathcal{A})$, where $u(L_1) = \varphi(L_1)$ and $v(L_1) = \psi(L_1)$, we have

$$
\begin{aligned}
< \mathcal{AU},\mathcal{U} >_{\mathcal{H}} &= a\Big[\big\langle u'_x, u_x\big\rangle_{L^2(0,L_1)} + \big\langle v'_x, v_x\big\rangle_{L^2(0,L_1)}\Big] \\
&+ b\Big[\big\langle \varphi'_x, \varphi_x\big\rangle_{L^2(L_1,L_2)} + \big\langle \psi'_x, \psi_x\big\rangle_{L^2(L_1,L_2)}\Big] \\
&+ \alpha\big\langle u'_x + v', u_x + v\big\rangle_{L^2(0,L_1)} + \big\langle u' + v', u + v\big\rangle_{L^2(0,L_1)} \\
&+ \beta\big\langle \varphi'_x + \psi, \varphi_x + \psi\big\rangle_{L^2(L_1,L_2)} + \big\langle u'', u'\big\rangle_{L^2(0,L_1)} \\
&+ \big\langle \varphi'', \varphi'\big\rangle_{L^2(L_1,L_2)} + \big\langle v'', v'\big\rangle_{L^2(0,L_1)} + \big\langle \psi'', \psi'\big\rangle_{L^2(L_1,L_2)} \\
&+ |\mu_2|\int_0^1\int_{\tau_1}^{\tau_2}|\sigma_1(s)|\big\langle -y_{1,\rho}(\rho,s), y_1(\rho,s)\big\rangle_{L^2(0,L_1)}dsd\rho, \\
&+ |\mu_4|\int_0^1\int_{\tau_1}^{\tau_2}|\sigma_2(s)|\big\langle -y_{2,\rho}(\rho,s), y_2(\rho,s)\big\rangle_{L^2(L_1,L_2)}dsd\rho.
\end{aligned} \tag{19}
$$

For the two last terms on the right-hand side of the above equality, we have

$$
\begin{aligned}
& |\mu_{2i}|\int_0^1\int_{\tau_1}^{\tau_2}|\sigma_i(s)|\big\langle -y_{yi,\rho}(\rho,s), y_i(\rho,s)\big\rangle_{L^2((i-1)L_{i-1},L_i)}dsd\rho,\ i = 1;2. \\
=\ & \frac{|\mu_{2i}|}{2}\int_{(i-1)L_{i-1}}^{L_i}\int_{\tau_1}^{\tau_2}\int_0^1|\sigma_i(s)|\frac{d}{d\rho}|y_i(\rho,s)|^2 d\rho dsdx,\ i = 1;2. \\
=\ & \frac{|\mu_{2i}|}{2}\int_{(i-1)L_{i-1}}^{L_i}\int_{\tau_1}^{\tau_2}|\sigma_i(s)|\Big[y_i^2(1,s) - y_i^2(0,s)\Big]dsdx,\ i = 1;2. \\
=\ & \frac{|\mu_{2i}|}{2}\int_{(i-1)L_{i-1}}^{L_i}\int_{\tau_1}^{\tau_2}|\sigma_i(s)|y_i^2(1,s)dsdx \\
-\ & \frac{|\mu_{2i}|}{2}\int_{\tau_1}^{\tau_2}|\sigma_i(s)|ds\int_{(i-1)L_{i-1}}^{L_i}y_i^2(0,s)dx,\ i = 1;2.
\end{aligned} \tag{20}
$$

Using $(13)_1$, and applying integration by parts along with (2), we have

$$
\begin{aligned}
\big\langle u'', u'\big\rangle_{L^2(0,L_1)} &= -a\big\langle u_x, u'_x\big\rangle_{L^2(0,L_1)} - \mu_1\|u'\|^2_{L^2(0,L_1)} \\
&\quad -\alpha\big\langle u_x + v, u'_x\big\rangle_{L^2(0,L_1)} - \big\langle u + v, u'\big\rangle_{L^2(0,L_1)} \\
&\quad -\mu_2\Big\langle \int_{\tau_1}^{\tau_2}\sigma_1(s)y_1(1,s)ds, u'\Big\rangle_{L^2(0,L_1)}.
\end{aligned} \tag{21}
$$

For the last right-hand side term of the above equality, using Young's inequality, we estimate

$$
\begin{aligned}
-\mu_2\Big\langle \int_{\tau_1}^{\tau_2}\sigma_1(s)y_1(1,s)ds, u'\Big\rangle_{L^2(0,L_1)} &\leq \frac{|\mu_2|}{2}\int_{\tau_1}^{\tau_2}|\sigma_1(s)|ds\|u'\|^2_{L^2(0,L_1)} \\
&+ \frac{|\mu_2|}{2}\int_0^{L_1}\int_{\tau_1}^{\tau_2}|\sigma_1(s)|y_1^2(1,s)dsdx.
\end{aligned} \tag{22}
$$

Similarly, using the second, third, and fourth equations of (13), integrating by parts and (2), we obtain

$$
\big\langle v'', v'\big\rangle_{L^2(0,L_1)} = -a\big\langle v_x, v'_x\big\rangle_{L^2(0,L_1)} - \alpha\big\langle u_x + v, v'\big\rangle_{L^2(0,L_1)} - \big\langle u + v, v'\big\rangle_{L^2(0,L_1)}. \tag{23}
$$

$$
\begin{aligned}
\big\langle \varphi'', \varphi'\big\rangle_{L^2(L_1,L_2)} &= -b\big\langle \varphi_x, \varphi'_x\big\rangle_{L^2(L_1,L_2)} - \mu_3\|\varphi'\|^2_{L^2(L_1,L_2)} \\
&\quad -\beta\big\langle \varphi_x + \psi, \varphi'_x\big\rangle_{L^2(L_1,L_2)} \\
&\quad -\mu_4\Big\langle \int_{\tau_1}^{\tau_2}\sigma_2(s)y_2(1,s)ds, \varphi'\Big\rangle_{L^2(L_1,L_2)}.
\end{aligned} \tag{24}
$$

For the last right-hand side term of (24), using Young's inequality, we have

$$-\mu_4\left\langle \int_{\tau_1}^{\tau_2} \sigma_2(s) y_2(1,s) ds, \varphi' \right\rangle_{L^2(L_1,L_2)} \leq \frac{|\mu_4|}{2} \int_{\tau_1}^{\tau_2} |\sigma_2(s)| ds \|\varphi'\|^2_{L^2(L_1,L_2)} + \frac{|\mu_4|}{2} \int_{L_1}^{L_2} \int_{\tau_1}^{\tau_2} |\sigma_2(s)| y_2^2(1,s) ds dx, \quad (25)$$

and

$$\langle \psi'', \psi' \rangle_{L^2(0,L_1)} = -b\langle \psi_x, \psi'_x \rangle_{L^2(0,L_1)} - \beta\langle \varphi_x + \psi, \psi' \rangle_{L^2(0,L_1)} - \mu_5 \|\psi'\|^2_{L^2(L_1,L_2)}. \quad (26)$$

Now, substituting (20)–(26) in (19), we have

$$< \mathcal{A}U, U >_{\mathcal{H}} \leq -\left(\mu_1 - |\mu_2| \int_{\tau_1}^{\tau_2} |\sigma_1(s)| ds\right) \|u'\|^2_{L^2(0,L_1)} - \left(\mu_3 - |\mu_4| \int_{\tau_1}^{\tau_2} |\sigma_2(s)| ds\right) \|\varphi'\|^2_{L^2(L_1,L_2)} - \mu_5 \|\psi'\|^2_{L^2(L_1,L_2)} \quad (27)$$

Thanks to assumption **[A1]**, we conclude that operator $\mathcal{A}$ is dissipative.

Now, we aim to show that operator $\mathcal{A}$ is maximally monotone;; thus, it is sufficient to show that the operator $\lambda I - \mathcal{A}$ is surjective for a fixed $0 < \lambda$. That is, we prove that for all $F = (f_1, \cdots, f_{10})$ in $\mathcal{H}$, there exists at least one solution $\mathcal{U} = (u, \varphi, v, \psi, u', \varphi', v', \psi', y_1, y_2)^T \in \mathcal{D}(\mathcal{A})$ of the equation

$$(\lambda I - \mathcal{A})\mathcal{U} = F. \quad (28)$$

The above equation is equivalent to

$$\begin{cases} u' = \lambda u - f_1 \\ \varphi' = \lambda \varphi - f_2 \\ v' = \lambda v - f_3 \\ \psi' = \lambda \psi - f_4 \\ (\lambda + \mu_1) u' - (a + \alpha) u_{xx} + u + v - \alpha v_x + \mu_2 \int_{\tau_1}^{\tau_2} \sigma_1(s) y_1(1,s) ds = f_5 \\ (\lambda + \mu_3) \varphi' - (b + \beta) \varphi_{xx} - \beta \psi_x + \mu_4 \int_{\tau_1}^{\tau_2} \sigma_2(s) y_2(1,s) ds = f_6 \\ \lambda v' - a v_{xx} + (\alpha + 1) v + \alpha u_x + u = f_7 \\ (\lambda + \mu_5) \psi' - b \psi_{xx} + \beta \psi + \beta \psi_x = f_8 \\ \lambda y_1 + \frac{1}{s} y_{1,\rho}(\rho, s) = f_9 \\ \lambda y_2 + \frac{1}{s} y_{2,\rho}(\rho, s) = f_{10}. \end{cases} \quad (29)$$

As Nicaise and Pignotti [10] show, each of the last two equations of (29) has a unique solution:

$$y_i(x, \rho, s) = y_i(x, 0, s) e^{-\lambda \rho s} + s e^{-\lambda \rho s} \int_0^{\rho} e^{\lambda \sigma s} f_{i+8}(x, \sigma, s) d\sigma, \ i = 1;2.$$

According to the first two equations of (29), we have

$$y_1(x, \rho) = \lambda u(x) e^{-\lambda \rho s} - f_1 e^{-\lambda \rho s} + s e^{-\lambda \rho s} \int_0^{\rho} e^{\lambda \sigma s} f_9(x, \sigma, s) d\sigma,$$

and

$$y_2(x, \rho) = \lambda \varphi(x) e^{-\lambda \rho s} - f_2 e^{-\lambda \rho s} + s e^{-\lambda \rho s} \int_0^{\rho} e^{\lambda \sigma s} f_{10}(x, \sigma, s) d\sigma.$$

In particular,

$$\begin{cases} y_1(x,1) = \lambda u(x)e^{-\lambda s} + y_1^0(x,s) \\ y_2(x,1) = \lambda \varphi(x)e^{-\lambda s} + y_2^0(x,s), \end{cases} \tag{30}$$

where

$$\begin{cases} y_i^0(x,s) = e^{-\lambda s}\left(-f_i + s\int_0^\rho e^{\lambda\sigma s} f_{i+8}(x,\sigma,s)d\sigma\right), \\ y_i^0(x,s) \in L^2(((i-1)L_{i-1}, L_i) \times (\tau_1,\tau_2)), \ \ i = 1;2. \end{cases} \tag{31}$$

By (29) and (30), the functions u, φ, v, ψ satisfy the following system:

$$\begin{cases} \widetilde{K}_1 u - a u_{xx} - \alpha(u_x + v)_x + u + v = \widetilde{g}_1 \\ \widetilde{K}_2 \varphi - b\varphi_{xx} - \beta(\varphi_x + \psi)_x = \widetilde{g}_2 \\ \lambda^2 v - a v_{xx} + \alpha(u_x + v) + u + v = \lambda f_3 + f_7 \\ \lambda(\lambda + \mu_5)\psi - b\psi_{xx} + \beta(\varphi_x + \psi) = (\lambda + \mu_5) f_4 + f_8, \end{cases} \tag{32}$$

where

$$\begin{cases} \widetilde{g}_i = (\lambda + \mu_{2i-1}) f_i + f_{i+4} - \mu_{2i}\int_{\tau_1}^{\tau_2} \sigma_i(s) y_i^0(x,s)ds \in L^2((i-1)L_{i-1}, L_i), \\ \widetilde{K}_i = \lambda\left(\lambda + \mu_{2i-1} + \mu_{2i}\int_{\tau_1}^{\tau_2} e^{-\lambda s}\sigma_i(s)ds\right) > 0, \ \ i = 1;2. \end{cases} \tag{33}$$

For any $(\omega_1, \omega_2, \omega_3, \omega_4) \in \mathbf{H}^1_* \times \mathbf{H}^2_*$, we can reformulate (32) as

$$\begin{cases} \int_0^{L_1}(\widetilde{K}_1 u - a u_{xx} - \alpha(u_x + v)_x + u + v)\omega_1 dx = \int_0^{L_1} \widetilde{g}_1 \omega_1 dx \\ \int_{L_1}^{L_2}(\widetilde{K}_2 \varphi - b\varphi_{xx} - \beta(\varphi_x + \psi)_x)\omega_2 dx = \int_{L_1}^{L_2} \widetilde{g}_2 \omega_2 dx \\ \int_0^{L_1}(\lambda^2 v - a v_{xx} + \alpha(u_x + v) + u + v)\omega_3 dx = \int_0^{L_1}(\lambda f_3 + f_7)\omega_3 dx \\ \int_{L_1}^{L_2}(\lambda(\lambda + \mu_5)\psi - b\psi_{xx} + \beta(\varphi_x + \psi))\omega_4 dx = \int_{L_1}^{L_2}((\lambda + \mu_5) f_4 + f_8)\omega_4 dx. \end{cases} \tag{34}$$

Integrating by parts in (34), we obtain the following variational formulation of (32):

$$\Phi((u,\varphi,v,\psi),(\omega_1,\omega_2,\omega_3,\omega_4)) = l(\omega_1,\omega_2,\omega_3,\omega_4), \tag{35}$$

where the bilinear form $\Phi : \left(\mathbf{H}^1_* \times \mathbf{H}^2_*\right)^2 \to \mathbb{R}$ and the linear form $l : \mathbf{H}^1_* \times \mathbf{H}^2_* \to \mathbb{R}$ are defined by

$$\begin{aligned} & \Phi((u,\varphi,v,\psi),(\omega_1,\omega_2,\omega_3,\omega_4)) \\ = & \int_0^{L_1}\left(\widetilde{K}_1 u + u + v\right)\omega_1 + (a u_x + \alpha(u_x + v))\omega_{1,x} dx \\ + & \int_{L_1}^{L_2} \widetilde{K}_2 \varphi\omega_2 + (b\varphi_x + \beta(\varphi_x + \psi))\omega_{2,x} dx \\ + & \int_0^{L_1}\left(\lambda^2 v + u + v + \alpha(u_x + v)\right)\omega_3 + a v_x \omega_{3,x} dx \\ + & \int_{L_1}^{L_2}(\lambda(\lambda + \mu_5)\psi + \beta(\varphi_x + \psi))\omega_4 + b\psi_x\omega_{4,x} dx \\ - & [a u_x \omega_1]_0^{L_1} - [b\varphi_x\omega_2]_{L_1}^{L_2} - [a v_x \omega_3]_0^{L_1} - [b\psi_x\omega_4]_{L_1}^{L_2} \\ - & [\alpha(u_x + v)\omega_1]_0^{L_1} - [\beta(\varphi_x + \psi)\omega_2]_{L_1}^{L_2}, \end{aligned}$$

and

$$\begin{aligned} l(\omega_1,\omega_2,\omega_3,\omega_4) = & \int_0^{L_1} \widetilde{g}_1\omega_1 + (\lambda f_3 + f_7)\omega_3 dx \\ & + \int_{L_1}^{L_2} \widetilde{g}_2\omega_2 + ((\lambda + \mu_5) f_4 + f_8)\omega_4 dx. \end{aligned}$$

By the properties of the space $\mathbf{H}_*^1$, $\mathbf{H}_*^2$ it is easy to see that Φ and l are continuous, and for all $V = (u, \varphi, v, \psi) \in \mathbf{H}_*^1 \times \mathbf{H}_*^2$, the bilinear form Φ checks the condition of coercivity:

$$\begin{aligned}\Phi(V,V) &= \widetilde{K}_1\|u\|^2_{L^2(0,L_1)} + \|u+v\|^2_{L^2(0,L_1)} + a\|u_x\|^2_{L^2(0,L_1)} \\ &\quad +\alpha\|u_x+v\|^2_{L^2(0,L_1)} + \lambda^2\|v\|^2_{L^2(0,L_1)} + a\|v_x\|^2_{L^2(0,L_1)} \\ &\quad +\widetilde{K}_2\|\varphi\|^2_{L^2(L_1,L_2)} + b\|\varphi_x\|^2_{L^2(L_1,L_2)} + \beta\|\varphi_x+\psi\|^2_{L^2(L_1,L_2)} \\ &\quad +\lambda(\lambda+\mu_5)\|\psi\|^2_{L^2(L_1,L_2)} + b\|\psi_x\|^2_{L^2(L_1,L_2)} \\ &\geq c\|V\|_{\mathbf{H}_*^1\times\mathbf{H}_*^2}.\end{aligned}$$

By applying the Lax–Milgram theorem, we deduce that the problem (35) has a unique solution $(u, \varphi, v, \psi) \in \mathbf{H}_*^1 \times \mathbf{H}_*^2$ for all $(\omega_1, \omega_2, \omega_3, \omega_4) \in \mathbf{H}_*^1 \times \mathbf{H}_*^2$. From (32), this implies that $(u, \varphi, v, \psi) \in \left((H^2(0, L_1) \times H^2(L_1, L_2))\right)^2 \cap \mathbf{H}_*^1 \times \mathbf{H}_*^2$. Therefore, the operator $\lambda I - \mathcal{A}$ is surjective. We use the Hille–Yosida theorem to guarantee the existence and uniqueness of the solution of problem (28). □

4. Asymptotic Behavior

In this section, we study the asymptotic behavior of the system (1).
For the proof of Theorem 1, we use Lemma 2.

Lemma 3. *Let $(u, v, \varphi, \psi, y_1, y_2)$ be the solution of (13), and assumption **[A1]** holds. Then the functional E defined by (14), satisfies the following inequality:*

$$\begin{aligned}\frac{d}{dt}E(t) &\leq -\left(\mu_1 - |\mu_2|\int_{\tau_1}^{\tau_2}|\sigma_1(s)|ds\right)\int_0^{L_1} u_t^2(x,t)dx \\ &\quad -\left(\mu_3 - |\mu_4|\int_{\tau_1}^{\tau_2}|\sigma_2(s)|ds\right)\int_{L_1}^{L_2} \varphi_t^2(x,t)dx \\ &\quad -\mu_5\int_{L_1}^{L_2}\psi_t^2(x,t)dx.\end{aligned} \tag{36}$$

Proof. By differentiating (14), using (13), and integrating by parts, we find

$$\begin{aligned}\frac{d}{dt}E(t) &= a[u_xu_t + v_xv_t]_0^{L_1} + \alpha[(u_x+v)u_t]_0^{L_1} - \mu_1\|u_t\|^2_{L^2(0,L_1)} \\ &\quad b[\varphi_x\varphi_t + \psi_x\psi_t]_{L_1}^{L_2} + \beta[(\varphi_x+\psi)\varphi_t]_{L_1}^{L_2} - \mu_3\|\varphi_t\|^2_{L^2(L_1,L_2)} \\ &\quad -\sum_{i=1}^{2}\frac{|\mu_{2i}|}{2}\int_{(i-1)L_{i-1}}^{L_i}\int_{\tau_1}^{\tau_2}|\sigma_i(s)|\left[y_i^2(\rho,s)\right]_0^1 dsdx \\ &\quad -\sum_{i=1}^{2}\mu_{2i}\int_{(i-1)L_{i-1}}^{L_i}\left(\int_{\tau_1}^{\tau_2}\sigma_i(s)y_i(1,s)ds\right)y_i(0,s)dx.\end{aligned} \tag{37}$$

For the last right-hand side term of the above equality, applying Young's inequality, we have

$$\begin{aligned}&-\mu_{2i}\int_{(i-1)L_{i-1}}^{L_i}\left(\int_{\tau_1}^{\tau_2}\sigma_i(s)y_i(\rho,s)ds\right)y_i(0,s)dx, \\ &\leq \frac{|\mu_{2i}|}{2}\left[\left(\int_{\tau_1}^{\tau_2}|\sigma_i(s)|ds\right)\|y_i(0,s)\|^2_{L^2((i-1)L_{i-1},L_i)}\right. \\ &\quad \left.+\int_{\tau_1}^{\tau_2}|\sigma_i(s)|\|y_i(1,s)\|^2_{L^2((i-1)L_{i-1},L_i)}ds\right], \quad i = 1;2.\end{aligned} \tag{38}$$

Inserting (38) into (37) and by (2) and the fact that $y_1(0,s) = u_t$, $y_2(0,s) = \varphi_t$, we show that (36) holds. □

Proof of Theorem 1. Now, we use c to denote various positive constants, which may be different at different occurrences. We multiply the first and second equations in (13) by u and v, respectively; we then integrate over $(0, L_1)$, multiply the third and fourth equations in (13) by φ and ψ, respectively, and integrate over (L_1, L_2), we have

$$\begin{cases} \int_0^{L_1} u(u_{tt} - au_{xx} - \alpha(u_x + v)_x + \mu_1 u_t + u + v \\ \qquad\qquad +\mu_2 \int_{\tau_1}^{\tau_2} \sigma_1(s) y_1(1, s) ds\Big) dx = 0 \\ \int_0^{L_1} v(v_{tt} - av_{xx} + \alpha(u_x + v) + u + v) dx = 0 \\ \int_{L_1}^{L_2} \varphi\Big(\varphi_{tt} - b\varphi_{xx} + \beta(\varphi_x + \psi)_x + \mu_3 \varphi_t + \mu_4 \int_{\tau_1}^{\tau_2} \sigma_2(s) y_2(1, s) ds\Big) dx = 0 \\ \int_{L_1}^{L_2} \psi(\psi_{tt} - b\psi_{xx} + \beta(\varphi_x + \psi) + \mu_5 \psi_t) dx = 0. \end{cases} \tag{39}$$

Next, multiplying the above equalities by E^q, integrating over (S, T), and using integration by parts, we have

$$\begin{aligned} &\int_S^T E^q \int_0^{L_1} u\left(u_{tt} - au_{xx} - \alpha(u_x + v)_x + \mu_1 u_t + u + v + \mu_2 \int_{\tau_1}^{\tau_2} \sigma_1(s) y_1(1, s) ds\right) dxdt \\ = &\left[E^q \int_0^{L_1} uu_t dx\right]_S^T - \int_S^T E^q \|u_t\|^2_{L^2(0,L_1)} dt - \int_S^T qE' E^{q-1} \int_0^{L_1} uu_t dxdt \qquad (40) \\ + &\ a \int_S^T E^q \|u_x\|^2_{L^2(0,L_1)} dt + \alpha \int_S^T E^q \int_0^{L_1} u_x(u_x + v) dxdt \\ + &\ \mu_1 \int_S^T E^q \int_0^{L_1} uu_t dxdt + \int_S^T E^q \int_0^{L_1} u(u + v) dxdt - a[uu_x]_0^{L_1} \\ - &\ \alpha[u(u_x + v)]_0^{L_1} + \mu_2 \int_S^T E^q \int_0^{L_1} u\left(\int_{\tau_1}^{\tau_2} \sigma_1(s) y_1(1, s) ds\right) dxdt = 0, \end{aligned}$$

$$\begin{aligned} &\int_S^T E^q \int_0^{L_1} v(v_{tt} - av_{xx} + \alpha(u_x + v) + u + v) dxdt \qquad (41) \\ = &\left[E^q \int_0^{L_1} vv_t dx\right]_S^T - \int_S^T E^q \|v_t\|^2_{L^2(0,L_1)} dt - \int_S^T qE' E^{q-1} \int_0^{L_1} vv_t dxdt \\ + &\ a \int_S^T E^q \|v_x\|^2_{L^2(0,L_1)} dt + \alpha \int_S^T E^q \int_0^{L_1} v(u_x + v) dxdt \\ + &\ \alpha \int_S^T E^q \int_0^{L_1} v(u + v) dxdt - a[vv_x]_0^{L_1} = 0, \qquad (42) \end{aligned}$$

$$\begin{aligned} &\int_S^T E^q \int_{L_1}^{L_2} \varphi\left(\varphi_{tt} - b\varphi_{xx} - \beta(\varphi_x + \psi)_x + \mu_3 \varphi_t + \mu_4 \int_{\tau_1}^{\tau_2} \sigma_2(s) y_2(1, s) ds\right) dxdt \\ = &\left[E^q \int_{L_1}^{L_2} \varphi\varphi_t dx\right]_S^T - \int_S^T E^q \|\varphi_t\|^2_{L^2(L_1,L_2)} dt - \int_S^T qE' E^{q-1} \int_{L_1}^{L_2} \varphi\varphi_t dxdt \qquad (43) \\ + &\ b \int_S^T E^q \|\varphi_x\|^2_{L^2(L_1,L_2)} dt + \beta \int_S^T E^q \int_{L_1}^{L_2} \varphi_x(\varphi_x + \psi) dxdt \\ + &\ \mu_3 \int_S^T E^q \int_{L_1}^{L_2} \varphi\varphi_t dxdt - b[\varphi\varphi_x]_{L_1}^{L_2} - \beta[\varphi(\varphi_x - \psi)]_{L_1}^{L_2} \\ + &\ \mu_4 \int_S^T E^q \int_{L_1}^{L_2} \varphi\left(\int_{\tau_1}^{\tau_2} \sigma_2(s) y_2(1, s) ds\right) dxdt = 0, \end{aligned}$$

and

$$\begin{aligned} & \int_S^T E^q \int_{L_1}^{L_2} \psi(\psi_{tt} - b\psi_{xx} + \beta(\varphi_x + \psi) + \mu_5\psi_t)dxdt \\ = & \left[E^q \int_{L_1}^{L_2} \psi\psi_t dx\right]_S^T - \int_S^T E^q \|\psi_t\|^2_{L^2(L_1,L_2)}dt - \int_S^T qE'E^{q-1} \int_{L_1}^{L_2} \psi\psi_t dxdt \\ + & b\int_S^T E^q \|\psi_x\|^2_{L^2(L_1,L_2)}dt + \beta\int_S^T E^q \int_{L_1}^{L_2} \psi(\varphi_x + \psi)dxdt \\ + & \mu_5 \int_S^T E^q \int_{L_1}^{L_2} \psi\psi_t dxdt - b[\psi\psi_x]_{L_1}^{L_2} = 0. \end{aligned} \tag{44}$$

Taking the sum, we obtain

$$\begin{aligned} & \left[E^q\left(\int_0^{L_1} uu_t + vv_t dx + \int_{L_1}^{L} \varphi\varphi_t + \psi\psi_t dx\right)\right]_S^T \\ - & \int_S^T qE'E^{q-1}\left(\int_0^{L_1} uu_t + vv_t dx + \int_{L_1}^{L_2} \varphi\varphi_t + \psi\psi_t dx\right)dt \\ - & 2\int_S^T E^q\left(\|u_t\|^2_{L^2(0,L_1)} + \|v_t\|^2_{L^2(0,L_1)} + \|\varphi_t\|^2_{L^2(L_1,L_2)} + \|\psi_t\|^2_{L^2(L_1,L_2)}\right)dt \\ + & \int_S^T E^q\Big[\|u_t\|^2_{L^2(0,L_1)} + \|v_t\|^2_{L^2(0,L_1)} + \|\varphi_t\|^2_{L^2(L_1,L_2)} + \|\psi_t\|^2_{L^2(L_1,L_2)} \\ + & a\left(\|u_x\|^2_{L^2(0,L_1)} + \|v_x\|^2_{L^2(0,L_1)}\right) \\ + & b\left(\|\psi_x\|^2_{L^2(L_1,L_2)} + \|\varphi_x\|^2_{L^2(L_1,L_2)}\right) + \alpha\|u_x + v\|^2_{L^2(0,L_1)} + \beta\|\varphi_x + \psi\|^2_{L^2(L_1,L_2)} \\ + & \|u + v\|^2_{L^2(0,L_1)}\Big]dt \\ + & \int_S^T E^q\left(\mu_1\int_0^{L_1} uu_t dx + \mu_3\int_{L_1}^{L_2} \varphi\varphi_t dx + \mu_5\int_{L_1}^{L_2} \psi\psi_t dx\right)dt \\ + & \mu_2\int_S^T E^q \int_0^{L_1} u\left(\int_{\tau_1}^{\tau_2} \sigma_1(s)y_1(1,s)ds\right)dxdt \\ + & \mu_4\int_S^T E^q \int_{L_1}^{L_2} \varphi\left(\int_{\tau_1}^{\tau_2} \sigma_2(s)y_2(1,s)ds\right)dxdt = 0. \end{aligned} \tag{45}$$

Similarly, we multiply the fifth (resp. sixth) equation in (13) by $e^{-2s\rho}|\sigma_1(s)|y_1(\rho,s)$ (resp. $e^{-2s\rho}|\sigma_2(s)|y_2(\rho,s)$); integrating over $(0,L_1)\times(0,1)\times(\tau_1,\tau_2)$ (resp. $(L_1,L_2)\times(0,1)\times(\tau_1,\tau_2)$), we have

$$\int_{(i-1)L_{i-1}}^{L_i} \int_0^1 e^{-2s\rho}|\sigma_i(s)|y_i(\rho,s)\big(y_{i,\rho}(\rho,s) + sy_{i,t}(\rho,s)\big)d\rho dx = 0, \quad i = 1;2.$$

Then we multiply this equation by E^q; integrating over (S,T) and using integration by parts, we have

$$
\begin{aligned}
& \int_S^T E^q \int_{(i-1)L_{i-1}}^{L_i} \int_0^1 \int_{\tau_1}^{\tau_2} e^{-2s\rho} |\sigma_i(s)| y_i(\rho,s) \big(y_{i,\rho}(\rho,s) + s y_{i,t}(\rho,s) \big) ds d\rho dx dt \\
= & \left[E^q \int_{(i-1)L_{i-1}}^{L_i} \int_{\tau_1}^{\tau_2} \int_0^1 \frac{s}{2} e^{-2s\rho} |\sigma_i(s)| y_i^2(\rho,s) d\rho ds dx \right]_S^T \\
- & \int_S^T q E' E^{q-1} \int_{(i-1)L_{i-1}}^{L_i} \int_{\tau_1}^{\tau_2} \int_0^1 \frac{s}{2} e^{-2s\rho} |\sigma_i(s)| y_i^2(\rho,s) d\rho ds dx dt \\
+ & \int_S^T E^q \int_{(i-1)L_{i-1}}^{L_i} \int_{\tau_1}^{\tau_2} \int_0^1 \frac{e^{-2s\rho}}{2} |\sigma_i(s)| \frac{d}{d\rho} y_i^2(\rho,s) d\rho ds dx dt \\
= & \left[E^q \int_{(i-1)L_{i-1}}^{L_i} \int_{\tau_1}^{\tau_2} \int_0^1 \frac{s}{2} e^{-2s\rho} |\sigma_i(s)| y_i^2(\rho,s) d\rho ds dx \right]_S^T \\
- & \int_S^T q E' E^{q-1} \int_{(i-1)L_{i-1}}^{L_i} \int_{\tau_1}^{\tau_2} \int_0^1 \frac{s}{2} e^{-2s\rho} |\sigma_i(s)| y_i^2(\rho,s) d\rho ds dx dt \\
+ & \frac{1}{2} \int_S^T E^q \int_{(i-1)L_{i-1}}^{L_i} \int_{\tau_1}^{\tau_2} \int_0^1 \frac{d}{d\rho} \Big(e^{-2s\rho} |\sigma_i(s)| y_i^2(\rho,s) \Big) d\rho ds dx dt \\
+ & \frac{1}{2} \int_S^T E^q \int_{(i-1)L_{i-1}}^{L_i} \int_{\tau_1}^{\tau_2} \int_0^1 2s e^{-2s\rho} |\sigma_i(s)| y_i^2(\rho,s) d\rho ds dx dt \\
= & \left[E^q \int_{(i-1)L_{i-1}}^{L_i} \int_{\tau_1}^{\tau_2} \int_0^1 \frac{s}{2} e^{-2s\rho} |\sigma_i(s)| y_i^2(\rho,s) d\rho ds dx \right]_S^T \\
- & \int_S^T q E' E^{q-1} \int_{(i-1)L_{i-1}}^{L_i} \int_{\tau_1}^{\tau_2} \int_0^1 \frac{s}{2} e^{-2s\rho} |\sigma_i(s)| y_i^2(\rho,s) d\rho ds dx dt \\
+ & \frac{1}{2} \int_S^T E^q \int_{(i-1)L_{i-1}}^{L_i} \int_{\tau_1}^{\tau_2} |\sigma_i(s)| \Big(e^{-2s} y_i^2(1,s) - y_i^2(0,s) \Big) ds dx dt \\
+ & \int_S^T E^q \int_{(i-1)L_{i-1}}^{L_i} \int_{\tau_1}^{\tau_2} \int_0^1 s e^{-2s\rho} |\sigma_i(s)| y_i^2(\rho,s) d\rho ds dx dt = 0, \quad i = 1;2.
\end{aligned} \tag{46}
$$

Then, we multiply the above equation by $|\mu_{2i}|$ and by the sum with (45); by using the definition of E and the fact that $\forall \rho \in]0,1[; e^{-2s} \leq e^{-2s\rho}$, we have

$$
\begin{aligned}
C \int_S^T E^{q+1} dt \leq & -\Big[E^q \Big(\int_0^{L_1} u u_t + v v_t dx + \int_{L_1}^{L_2} \varphi \varphi_t + \psi \psi_t dx \\
+ & \sum_{i=1}^{2} \frac{|\mu_{2i}|}{2} \int_{(i-1)L_{i-1}}^{L_i} \int_{\tau_1}^{\tau_2} \int_0^1 s e^{-2s\rho} |\sigma_i(s)| y_i^2(\rho,s) d\rho ds dx \Big) \Big]_S^T \\
+ & \int_S^T q E' E^{q-1} \Big(\int_0^{L_1} u u_t + v v_t dx + \int_{L_1}^{L} \varphi \varphi_t + \psi \psi_t dx \\
+ & \sum_{i=1}^{2} \frac{|\mu_{2i}|}{2} \int_{(i-1)L_{i-1}}^{L_i} \int_{\tau_1}^{\tau_2} \int_0^1 s e^{-2s\rho} |\sigma_i(s)| y_i^2(\rho,s) d\rho ds dx \Big) dt \\
+ & 2 \int_S^T E^q \Big[\|u_t\|^2_{L^2(0,L_1)} + \|v_t\|^2_{L^2(0,L_1)} + \|\varphi_t\|^2_{L^2(L_1,L_2)} + \|\psi_t\|^2_{L^2(L_1,L_2)} \Big] dt \\
- & \int_S^T E^q \Big(\mu_1 \int_0^{L_1} u u_t dx + \mu_3 \int_{L_1}^{L_2} \varphi \varphi_t dx + \mu_5 \int_{L_1}^{L_2} \psi \psi_t dx \Big) dt \\
- & \sum_{i=1}^{2} \frac{|\mu_{2i}|}{2} \int_S^T E^q \int_{(i-1)L_{i-1}}^{L_i} \int_{\tau_1}^{\tau_2} |\sigma_i(s)| \Big(e^{-2s} y_i^2(1,s) - y_i^2(0,s) \Big) ds dx dt \\
- & \mu_2 \int_S^T E^q \int_0^{L_1} u \Big(\int_{\tau_1}^{\tau_2} \sigma_1(s) y_1(1,s) ds \Big) dx dt \\
- & \mu_4 \int_S^T E^q \int_{L_1}^{L_2} \varphi \Big(\int_{\tau_1}^{\tau_2} \sigma_2(s) y_2(1,s) ds \Big) dx dt.
\end{aligned} \tag{47}
$$

where $C = 2\min\{1, 2e^{-2s}\}$.

Now, we estimate the two terms on the right-hand side of the above inequality. Using Young's inequality, we obtain

$$\begin{aligned}
- &\quad \mu_2 \int_S^T E^q \int_0^{L_1} u\left(\int_{\tau_1}^{\tau_2} \sigma_1(s) y_1(1,s) ds\right) dx dt \qquad (48)\\
\leq &\quad |\mu_2| \int_S^T E^q \int_0^{L_1} \int_{\tau_1}^{\tau_2} e^s \sqrt{|\sigma_1(s)|}|u| e^{-s} \sqrt{|\sigma_1(s)|}|y_1(1,s)| ds dx dt \\
\leq &\quad \frac{|\mu_2|}{2} \int_S^T E^q \int_0^{L_1} \int_{\tau_1}^{\tau_2} \left[e^{2s}|\sigma_1(s)|u^2 + e^{-2s}|\sigma_1(s)|y_1^2(1,s)\right] ds dx dt.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
- &\quad \mu_4 \int_S^T E^q \int_{L_1}^{L_2} \varphi\left(\int_{\tau_1}^{\tau_2} \sigma_2(s) y_2(1,s) ds\right) dx dt \qquad (49)\\
\leq &\quad \frac{|\mu_4|}{2} \int_S^T E^q \int_0^{L_1} \int_{\tau_1}^{\tau_2} \left[e^{2s}|\sigma_2(s)|\varphi^2 + e^{-2s}|\sigma_2(s)|y_2^2(1,s)\right] ds dx dt.
\end{aligned}$$

Combining the above estimates, we arrive at

$$\begin{aligned}
C\int_S^T E^{q+1} dt \quad \leq &\quad -\Big[E^q\Big(\int_0^{L_1} uu_t + vv_t dx + \int_{L_1}^{L_2} \varphi\varphi_t + \psi\psi_t dx \\
+ &\quad \sum_{i=1}^{2} \frac{|\mu_{2i}|}{2} \int_{(i-1)L_{i-1}}^{L_i} \int_{\tau_1}^{\tau_2} \int_0^1 s e^{-2s\rho} |\sigma_i(s)| y_i^2(\rho,s) d\rho ds dx\Big)\Big]_S^T \\
+ &\quad \int_S^T qE' E^{q-1}\Big(\int_0^{L_1} uu_t + vv_t dx + \int_{L_1}^{L} \varphi\varphi_t + \psi\psi_t dx \qquad (50)\\
+ &\quad \sum_{i=1}^{2} \frac{|\mu_{2i}|}{2} \int_{(i-1)L_{i-1}}^{L_i} \int_{\tau_1}^{\tau_2} \int_0^1 s e^{-2s\rho} |\sigma_i(s)| y_i^2(\rho,s) d\rho ds dx\Big) dt \\
+ &\quad 2\int_S^T E^q\Big[\|u_t\|^2_{L^2(0,L_1)} + \|v_t\|^2_{L^2(0,L_1)} + \|\varphi_t\|^2_{L^2(L_1,L_2)} + \|\psi_t\|^2_{L^2(L_1,L_2)}\Big] dt \\
- &\quad \int_S^T E^q\Big(\mu_1 \int_0^{L_1} uu_t dx + \mu_3 \int_{L_1}^{L_2} \varphi\varphi_t dx + \mu_5 \int_{L_1}^{L_2} \psi\psi_t dx\Big) dt \\
+ &\quad \frac{|\mu_2|}{2} \int_{\tau_1}^{\tau_2} e^{2s}|\sigma_1(s)| ds \int_S^T E^q \|u\|^2_{L^2(0,L_1)} dt \\
+ &\quad \frac{|\mu_4|}{2} \int_{\tau_1}^{\tau_2} e^{2s}|\sigma_2(s)| ds \int_S^T E^q \|\varphi\|^2_{L^2(L_1,L_2)} dt \\
+ &\quad \frac{|\mu_2|}{2} \int_{\tau_1}^{\tau_2} |\sigma_1(s)| ds \int_S^T E^q \|u_t\|^2_{L^2(0,L_1)} dt \\
+ &\quad \frac{|\mu_4|}{2} \int_{\tau_1}^{\tau_2} |\sigma_2(s)| ds \int_S^T E^q \|\varphi_t\|^2_{L^2(L_1,L_2)} dt.
\end{aligned}$$

By the Cauchy–Schwarz, Young, and Sobolev–Poincaré inequalities, and using definition of E, for

$$\begin{cases} w \in \{u, v\}, \text{ if } x \in [0, L_1] = I_1 \\ w \in \{\varphi, \psi\}, \text{ if } x \in [L_1, L_2] = I_2, \end{cases}$$

we have

$$\begin{aligned}
-\Big[E^q \int_{I_i} w w_t dx\Big]_S^T \quad = &\quad E^q(S)\int_{I_i} w(S) w_t(S) dx - E^q(T)\int_{I_i} w(T) w_t(T) dx \\
\leq &\quad cE^{q+1}(S), \quad i = 1; 2. \qquad (51)
\end{aligned}$$

$$\begin{aligned}\left|\int_S^T qE'E^{q-1}\int_{I_i} ww_t dxdt\right| &\leq c\int_S^T (-E')E^q dt, \ \ i=1;2. \\ &= -c\Big[E^{q+1}\Big]_S^T \\ &= c\Big[E^{q+1}(S)-E^{q+1}(T)\Big] \\ &\leq cE^{q+1}(S).\end{aligned} \tag{52}$$

$$\begin{aligned}\int_S^T E^q\|w_t\|^2_{L^2(I_i)}dt &\leq c\int_S^T(-E')E^q dt, \ \ i=1;2. \\ &\leq cE^{q+1}(S).\end{aligned} \tag{53}$$

$$\begin{aligned} &-\left[E^q\sum_{i=1}^2\frac{|\mu_{2i}|}{2}\int_{I_i}\int_{\tau_1}^{\tau_2}\int_0^1 se^{-2s\rho}|\sigma_i(s)|y_i^2(\rho,s)d\rho ds dx\right]_S^T \\ \leq\ & \Big[cE^{q+1}(t)\Big]_S^T \\ \leq\ & cE^{q+1}(S).\end{aligned}$$

$$\begin{aligned}\int_S^T qE'E^{q-1}\sum_{i=1}^2\frac{|\mu_{2i}|}{2}\int_{I_i}\int_{\tau_1}^{\tau_2}\int_0^1 se^{-2s\rho}|\sigma_i(s)|y_i^2(\rho,s)d\rho ds dx dt& \\ \leq c\int_S^T(-E')E^q dt& \\ \leq cE^{q+1}(S).&\end{aligned}$$

$$\begin{aligned}\mu_1\int_S^T E^q\int_0^{L_1} uu_t dxdt &\leq \varepsilon\int_S^T E^q\|u\|^2_{L^2(0,L_1)}dt + c(\varepsilon)\int_S^T E^q\|u_t\|^2_{L^2(0,L_1)}dt \\ &\leq \varepsilon c\int_S^T E^q\|u_x\|^2_{L^2(0,L_1)}dt + c(\varepsilon)\int_S^T E^q(-E')dt \\ &\leq \varepsilon c\int_S^T E^{q+1}dt + c(\varepsilon)E^{q+1}(S).\end{aligned} \tag{54}$$

Similarly, we have

$$\int_S^T E^q\int_{L_1}^{L_2}\mu_3\varphi\varphi_t+\mu_5\psi\psi_t dxdt \leq \varepsilon_1 c\int_S^T E^{q+1}dt + c(\varepsilon_1)E^{q+1}(S). \tag{55}$$

$$\begin{aligned} &\frac{|\mu_2|}{2}\int_{\tau_1}^{\tau_2}e^{2s}|\sigma_1(s)|ds\int_S^T E^q\|u\|^2_{L^2(0,L_1)}dt \\ +\ &\frac{|\mu_4|}{2}\int_{\tau_1}^{\tau_2}e^{2s}|\sigma_2(s)|ds\int_S^T E^q\|\varphi\|^2_{L^2(L_1,L_2)}dt \\ \leq\ &\sum_{i=1}^2\frac{|\mu_{2i}|}{2}\int_{\tau_1}^{\tau_2}e^{2s}|\sigma_i(s)|ds\int_S^T E^{q+1}dt \\ \leq\ & c_2\int_S^T E^{q+1}dt,\end{aligned} \tag{56}$$

where $c_2=\sum_{i=1}^2\frac{|\mu_{2i}|}{2}\int_{\tau_1}^{\tau_2}e^{2s}|\sigma_i(s)|ds$

Inserting the above estimates (51)–(56) in (50) and **[A2]**, we conclude that

$$(C - c_2 - c(\varepsilon + \varepsilon_1)) \int_S^T E^{q+1} dt \le c E^{q+1}(S). \tag{57}$$

Choosing ε and ε_1, which are small enough, such that

$$C - c_2 - c(\varepsilon + \varepsilon_1) > 0.$$

So, for the positive constant c, we arrive at

$$\int_S^T E^{q+1} dt \le c E^{q+1}(S). \tag{58}$$

From (58) and Lemma 2, we deduce that

$$E(t) \le c E(0) e^{-\omega t} \quad \forall t \ge 0,$$

where c is a positive constant that is independent of $E(0)$. This completes the proof of Theorem 1. □

5. Conclusions

The transmission problem of the Timoshenko system with distributed delay is important and has applications in various fields, such as physics, chemistry, biology, thermodynamics, and economics. In this paper, we investigated a transmission problem of the Timoshenko system in the presence of distributed delay. We established the well-posedness via the theory of semigroups; moreover, axiomatic questions can be asked about the qualitative studies. We investigated the asymptotic behaviors of solutions via several axiomatic methods; see [29,30].

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