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Inertial Iterative Algorithms for Split Variational Inclusion and Fixed Point Problems

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Abstract: This paper aims to present two inertial iterative algorithms for estimating the solution of split variational inclusion (S_pVI_sP) and its extended version for estimating the common solution of (S_pVI_sP) and fixed point problem (FPP) of a nonexpansive mapping in the setting of real Hilbert spaces. We establish the weak convergence of the proposed algorithms and strong convergence of the extended version without using the pre-estimated norm of a bounded linear operator. We also exhibit the reliability and behavior of the proposed algorithms using appropriate assumptions in a numerical example.

Keywords: split variational inclusion; fixed point problem; inertial algorithms; weak convergence; strong convergence

MSC: 47H05; 47H06; 49J53

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1. Introduction

The split feasibility problems (S_pFP), due to Censor and Elfving [1], have ample applications in medical science. Therefore, (S_pFP) has been widely used over the past twenty years in the design of intensity-modulation therapy treatments and other areas of applied sciences, see, e.g., [2–5]. Censor et al. [6,7] merged the variational inequality problem (VI_tP) and (S_pFP), and a different kind of problem came to existence known as split variational inequality problem (S_pVI_tP) defined as:

$$\text{Find out } l^* \in Q_1 \text{ such that } l^* \in VI_tP(F_1; Q_1) \text{ and } B(l^*) \in VI_tP(F_2; Q_2), \tag{1}$$

where Q_1 and Q_2 are subsets of Hilbert spaces X_1 and X_2, respectively, $B : X_1 \to X_2$ is a bounded linear operator, $F_1 : X_1 \to X_1$ and $F_2 : X_2 \to X_2$ are two operators, $VI_tP(F_1; Q_1) = \{q \in C : \langle F_1(q), p - q\rangle \geq 0, \forall p \in Q_1\}$ and $VI_tP(F_2; Q_2) = \{r \in Q_2 : \langle F_2(r), s - r\rangle \geq 0, \forall s \in Q_2\}$.

Moudafi [8] extended S_pVI_tP into a split monotone variational inclusion problem (S_pMVI_sP) defined as:

$$\text{Find out } l^* \in X_1 \text{ such that } l^* \in VI_sP(F_1; A_1; X_1) \text{ and } B(l^*) \in VI_sP(A_2; F_2; X_2), \tag{2}$$

where $A_1 : X_1 \to 2^{X_1}$ and $A_2 : X_2 \to 2^{X_2}$ are set-valued mappings on Hilbert spaces X_1 and X_2, respectively, $VI_sP(F_1, A_1; X_1) = \{p \in X_1 : 0 \in F_1(p) + A_1(p)\}$ and $VI_sP(F_2, A_2; X_2) = \{q \in X_2 : 0 \in F_2(q) + A_2(q)\}$. Moudafi [8] proposed the following iterative scheme for (S_pMVI_sP). Let $\mu > 0$, choose any starting point $z_0 \in X_1$ and compute

$$z_{n+1} = V[z_n + \lambda B^*(W - I)Bz_n], \tag{3}$$

where B^* is an adjoint operator of B, $\lambda \in (0, 1/R)$ with R being the spectral radius of the operator B^*B, $V = R_\mu^{A_1}(I - \mu F_1) = (I + \mu A_1)^{-1}(I - \mu F_1)$ and $W = R_\mu^{A_2}(I - \mu F_2) = (I + \mu A_2)^{-1}(I - \mu F_2)$.

If $F_1 = F_2 = 0$, then ($\mathrm{S_pMVI_sP}$) turns into the split inclusion problem (in short, ($\mathrm{S_pVI_sP}$) suggested and discussed by Byrne et al. [9]:

$$\text{Find out } l^* \in X_1 \text{ such that } l^* \in \mathrm{VI_sP}(A_1; X_1) \text{ and } B(l^*) \in \mathrm{VI_sP}(A_2; X_2), \tag{4}$$

where $\mathrm{VI_sP}(A_1; X_1) = \{p \in X_1 : 0 \in A_1(p)\}$ and $\mathrm{VI_sP}(A_2; X_2) = \{q \in X_2 : 0 \in A_2(q)\}$, A_1, A_2 are the same as in (2). Moreover, Byrne et al. [9] suggested the following iterative scheme for ($\mathrm{S_pVI_sP}$). Let $\mu > 0$ and select a starting point $z_0 \in X_1$; then, compute

$$z_{n+1} = R_\mu^{A_1}[z_n + \lambda B^*(I - R_\mu^{A_2})Bz_n], \tag{5}$$

where B^* is the adjoint operator of B, $R = \|B^*B\| = \|B\|^2$, $\lambda \in (0, 2/R)$ and $R_\mu^{A_1}$, $R_\mu^{A_2}$ are the resolvents of monotone mappings A_1, A_2, respectively . It is obvious to see that l^* solves ($\mathrm{S_pVI_sP}$) if and only if $l^* = R_\mu^{A_1}[l^* + \lambda B^*(I - R_\mu^{A_2})Bl^*]$. Kazmi and Rizvi [10] studied the following iterative scheme for calculating the common solutions of ($\mathrm{S_pVI_sP}$) and (FPP) of a nonexpansive mapping S. For $z_0 \in X_1$, compute

$$\begin{aligned} y_n &= R_\mu^{A_1}[z_n + \lambda B^*(R_\mu^{A_2} - I)Bz_n], \\ z_{n+1} &= \zeta_n f(z_n) + (1 - \zeta_n)Sy_n, \end{aligned} \tag{6}$$

where f is contraction and $\lambda \in (0, \frac{2}{\|B\|^2})$. By extending the work of Kazmi and Rizvi [10], Dilshad et al. [11] discussed the common solution of ($\mathrm{S_pVI_sP}$) and the fixed point of a finite collection of nonexpansive mappings. Sitthithakerngkiet et al. [12] investigated the common solutions of ($\mathrm{S_pVI_sP}$) and a fixed point of a countably infinite collection of nonexpansive mappings and proposed and discussed the following method. For $z_0 \in X_1$, compute

$$\begin{aligned} y_n &= R_\mu^{A_1}[z_n + \lambda B^*(R_\mu^{A_2} - I)Bz_n], \\ z_{n+1} &= \zeta_n u + \xi_n z_n + [(1 - \xi_n)I - \zeta_n D]W_n y_n, \ \forall n \geq 1, \end{aligned} \tag{7}$$

where $u \in X_1$ is arbitrary, and W_n is W-mapping, which is created by an infinite collection of nonexpansive mappings. Furthermore, Akram et al. [13] modify the method discussed in [10] and investigate the common solution of ($\mathrm{S_pVI_sP}$) and (FPP):

$$\begin{aligned} y_n &= z_n - \lambda\big[(I - R_{\mu_1}^{A_1})z_n + B^*(I - R_{\mu_2}^{A_2})Bz_n\big], \\ z_{n+1} &= \zeta_n f(z_n) + (1 - \zeta_n)S(y_n), \end{aligned} \tag{8}$$

where $\lambda = \frac{1}{1+\|B\|^2}$, $\zeta_n \in (0,1)$ satisfying $\lim_{n\to\infty} \zeta_n = 0$, $\sum_{n=1}^{\infty} \zeta_n = \infty$ and $\sum_{n=1}^{\infty} |\zeta_n - \zeta_{n-1}| < \infty$. Some results related to ($\mathrm{S_pVI_sP}$) and (FPP) can be found in [14–19] and the references therein.

It is noted that the step size depending upon the norm $\|B^*B\|$ is commonly used in the above-mentioned iterative schemes. To skip this restruction, a new type of iterative method with a self-adaptive step size has been invented. López et al. [20] composed a relaxed iterative method for ($\mathrm{S_pFP}$) with a self-adaptive step size. Dilshad et al. [21] studied the ($\mathrm{S_pVI_sP}$) without using a pre-calculated norm $\|B\|$. Some useful related work can be found in [22–26] and the references therein.

In recent years, great efforts have been made to speed up various algorithms. The inertia term as one of the speed-up techniques has been studied by many scientists because of its simple form and good speed-up effect. Recall that using the concepts of implicit descritization for the derivatives, Alvarez and Attouch [27] have developed the inertial proximal point method, which can be expressed as

$$z_{n+1} = R_{\mu}^{A}[z_n + \phi_n(z_n - z_{n-1})],$$

where A is monotone mapping, R_{μ}^{A} is the resolvent of A and $\mu > 0$. Such types of schemes have a better convergence rate, and hence, this scheme was modified and applied to solve numerous nonlinear problems; see [28–34] and the references therein.

Following the above-mentioned inertial method, we consider two inertial iterative algorithms for approximating the solution of ($\mathrm{S_pVI_sP}$) and common solutions of ($\mathrm{S_pVI_sP}$) and (FPP) of a nonexpansive mapping.

The next section contains some theory and auxiliary results which are helpful in the proof of the main results. In Section 3, we explain two self-adaptive inertial iterative methods. Section 4 is focused on the proof of the main results discussing the solution of ($\mathrm{S_pVI_sP}$) and a common solution of ($\mathrm{S_pVI_sP}$) and (FPP). At last, we illustrate a numerical example in favor of the proposed iterative algorithms showing their behavior and reliability.

2. Preliminaries

Assume that $(X, \|\cdot\|)$ is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. The strong convergence of the real sequence $\{z_n\}$ to z is indicated by $z_n \to z$ and the weak convergence is indicated by $z_n \rightharpoonup z$. If $\{z_n\}$ is a sequence in X, $\omega_w(z_n)$ indicates the weak ω-limit set of $\{z_n\}$, that is

$$\omega_w(z_n) = \{z \in H : z_{n_j} \rightharpoonup z \text{ as} j \to \infty \text{ where } z_{n_j} \text{ is a subsequnce of } z_n\}.$$

We know that for some $z \in X$, there exists a unique nearest point in Q denoted by $P_Q z$ such that

$$\|z - P_Q z\| \leq \|z - v\|, \ \forall v \in Q.$$

$P_Q z$ is called the projection of z onto $Q \subset X$, which satisfies

$$\langle z - v, P_Q z - P_C v \rangle \geq \|P_Q z - P_Q v\|^2, \forall z, v \in X.$$

Moreover, $P_Q z$ is identified by the fact

$$P_Q z = x \Leftrightarrow \langle z - v, v - x \rangle \geq 0, \ v \in Q.$$

For all p, q, r in Hilbert space X, $\phi, \varphi, \psi \in [0,1]$ such that $\phi + \varphi + \psi = 1$; then, we have the following equality and inequality

$$\|\phi p + \varphi q + \psi r\|^2 = \phi\|p\|^2 + \varphi\|q\|^2 + \psi\|r\|^2 - \phi\varphi\|p - q\|^2 - \varphi\psi\|q - r - w\|^2 - \psi\phi\|p - r\|^2, \tag{9}$$

and

$$\|p + q\|^2 \leq \|p\|^2 + 2\langle q, p + q \rangle. \tag{10}$$

Definition 1. *A mapping $F : X \to X$ is called*

(i) Contraction, if $\|F(p) - F(q)\| \leq \kappa\|p - q\|, \ \forall \ p, q \in X, \kappa \in (0,1)$;
(ii) Nonexpansive, if $\|F(p) - F(q)\| \leq \|p - q\|, \forall \ p, q \in X$;
(iii) Firmly nonexpansive, if $\|F(p) - F(q)\|^2 \leq \langle p - q, F(p) - F(q) \rangle, \forall \ p, q \in X$;
(iv) τ-inverse strongly monotone, if there exists $\tau > 0$ such that

$$\langle F(p) - F(q), p - q \rangle \geq \tau\|F(p) - F(q)\|^2, \forall \ p, q \in X.$$

Definition 2. *Let $A : X \to 2^X$ be a set valued mapping. Then*

(i) The mapping A is called monotone if $\langle u - v, p - q \rangle \geq 0, \forall \ u, v \in X, u \in A(p), v \in A(q)$;
(ii) $Graph(A) = \{(u, p) \in X \times X : u \in A(p)\}$;

(iii) *The mapping A is called maximal monotone if $Graph(A)$ is not properly contained in the graph of any other monotone operator.*

Lemma 1 ([35])**.** *If $\{s_n\}$ is a sequence of non-negative real numbers such that*

$$s_{n+1} \leq (1-\xi_n)s_n + \delta_n, \quad n \geq 0,$$

where $\{\xi_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence of real numbers such that

(i) $\sum_{n=1}^{\infty} \xi_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\xi_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2 ([36])**.** *In a Hilbert space X,*

(i) *a mapping $A : X \to X$ is τ-inverse strongly monotone if and only if $I - \tau A$ is firmly nonexpansive for $\tau > 0$.*

(ii) *If $A : X \to 2^X$ is monotone and R_μ^A is the resolvent of A, then R_μ^A and $I - R_\mu^A$ are firmly nonexpansive for $\mu > 0$.*

(iii) *If $A : X \to X$ is nonexpansive, then $I - A$ is demiclosed at zero and if A is firmly nonexpansive, then $I - A$ is firmly nonexpansive.*

Lemma 3 ([37])**.** *Let $\{\psi_n\}$ be a bounded sequence in Hilbert space X. Assume there exists a subset $Q \neq \emptyset$ and $Q \subset X$ satisfying the properties*

(i) $\lim_{n\to\infty} \|\psi_n - z\|$ *exists,* $\forall\, z \in Q$,

(ii) $\omega_w(\psi_n) \subset Q$.

Then, there exists $z^* \in C$ such that $\psi_n \rightharpoonup z^*$.

Lemma 4 ([38])**.** *Let Γ_n be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence Γ_{n_k} of Γ_n such that $\Gamma_{n_k} < \Gamma_{n_k+1}$ for all $k \geq 0$. In addition, consider the sequence of integers $\{\sigma(n)\}_{n\geq n_0}$ defined by*

$$\sigma(n) = \max\{k \leq n : \Gamma_k \leq \Gamma_{k+1}\}.$$

Then, $\{\sigma(n)\}_{n\geq n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty} \sigma(n) = \infty$ and $\forall\, n \geq n_0$,

$$\max\{\Gamma_{\sigma(n)}, \Gamma_{(n)}\} \leq \Gamma_{\sigma(n)+1}.$$

Lemma 5 ([38])**.** *Assume that $\{s_n\}$ is a non-negative sequence of real numbers satisfying*

(i) $s_{n+1} - s_n \leq \delta_n(s_n - s_{n-1}) + \theta_n$;

(ii) $\sum_{n=1}^{\infty} \theta_n < \infty$;

(iii) $\delta_n \in [0, \kappa]$, *where* $\kappa \in [0,1)$.

Then, $\{s_n\}$ is convergent and $\sum_{n=1}^{\infty} (s_{n+1} - s_n) < \infty$, where $[h]_+ = \max\{h, 0\}$ for any $h \in \mathbb{R}$.

3. Inertial Iterative Methods

Suppose that X_1 and X_2 are real Hilbert spaces and $A_1 : X_1 \to 2^{X_1}$, $A_2 : X_2 \to 2^{X_2}$ are monotone mappings; $R_{\mu_1}^{A_1}$, $R_{\mu_2}^{A_2}$ are the resolvents of A_1 and A_2, respectively. We assume that $\Lambda \cap \mathrm{Fix(F)} \neq \emptyset$, where Λ denotes the solution set of $\mathrm{S_pVI_sP}$ and $\mathrm{Fix(F)}$ denotes the fixed point set of FPP. First, we suggest the following iterative algorithm for $\mathrm{S_pVI_sP}$.

Algorithm 1. *Choose ϕ such that $0 \leq \phi < 1$ and let δ_n be a positive sequence satisfying $\sum_{n=1}^{\infty} \delta_n < \infty$. Iterative Step: Given arbitrary x_0, and x_1, for $n \geq 1$, choose $0 < \phi_n < \tilde{\phi}_n$, where*

$$\tilde{\phi}_n = \begin{cases} \min\{\frac{\delta_n}{\|x_n - x_{n-1}\|}, \quad \phi\}, & \text{if } x_n \neq x_{n-1}, \\ \phi, & \text{otherwise.} \end{cases} \tag{11}$$

Compute

$$\begin{aligned} v_n &= x_n + \phi_n(x_n - x_{n-1}), \\ u_n &= v_n - \sigma_n(I - R_{\mu_1}^{A_1})(v_n), \\ x_{n+1} &= u_n - \varrho_n B^*(I - R_{\mu_2}^{A_2})(Bu_n), \end{aligned}$$

where σ_n and ϱ_n are defined as

$$\sigma_n = \begin{cases} \frac{\tau_n \|(I - R_{\mu_1}^{A_1})(v_n)\|^2}{\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2}, & \text{if } \|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2 \neq 0 \\ 0, & \textit{otherwise} \end{cases} \tag{12}$$

and

$$\varrho_n = \begin{cases} \frac{\tau_n \|(I - R_{\mu_2}^{A_2})(Bu_n)\|^2}{\|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2}, & \text{if } \|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2 \neq 0 \\ 0, & \textit{otherwise} \end{cases} \tag{13}$$

where $\mu_1 > 0$, $\mu_2 > 0$ and $\tau_n \in (0, 2)$.

Algorithm 2. *Choose ϕ such that $0 \leq \phi < 1$ and let δ_n be a positive sequence satisfying $\sum_{n=1}^{\infty} \delta_n < \infty$. Iterative Step: Given arbitrary x_0, and x_1, for $n \geq 1$, choose $0 < \phi_n < \tilde{\phi}_n$, where*

$$\tilde{\phi}_n = \begin{cases} \min\{\frac{\delta_n}{\|x_n - x_{n-1}\|}, \quad \phi\}, & \text{if } x_n \neq x_{n-1}, \\ \phi, & \text{otherwise.} \end{cases} \tag{14}$$

Compute

$$\begin{aligned} v_n &= x_n + \phi_n(x_n - x_{n-1}), \\ u_n &= v_n - \sigma_n(I - R_{\mu_1}^{A_1})(v_n), \\ w_n &= u_n - \varrho_n B^*(I - R_{\mu_2}^{A_2})(Bu_n), \\ x_{n+1} &= (1 - \zeta_n - \xi_n)u_n + \zeta_n F(w_n). \end{aligned}$$

where σ_n and ϱ_n are defined as

$$\sigma_n = \begin{cases} \frac{\tau_n \|(I - R_{\mu_1}^{A_1})(v_n)\|^2}{\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2}, & \text{if } \|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2 \neq 0 \\ 0, & \textit{otherwise} \end{cases} \tag{15}$$

and

$$\varrho_n = \begin{cases} \frac{\tau_n \|(I - R_{\mu_2}^{A_2})(Bu_n)\|^2}{\|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2}, & \text{if } \|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2 \neq 0 \\ 0, & \textit{otherwise} \end{cases} \tag{16}$$

where $\zeta_n, \xi_n \in (0, 1)$, $\mu_1 > 0$, $\mu_2 > 0$, and $\tau_n \in (0, 2)$.

Remark 1. *It is not difficult to show that if* $\|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2 = 0$ *or* $\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2 = 0$ *for some* $n \geq 0$*, then* $x_n \in \Lambda$*. In this case, the iteration process ended after a finite number of iterations. We suppose that the proposed algorithms generate infinite sequences which do not end in a finite number of terms.*

Remark 2. *From the selection of* δ_n*, such that* $\sum_{n=1}^{\infty} \delta_n < \infty$*, we can conclude that* $\lim_{n\to\infty} \phi_n \|x_n - x_{n-1}\| = 0$.

Remark 3. *By using the definitions of resolvent of monotone mappings* A_1 *and* A_2*, we can easily obtain that* $l^\star \in \Lambda$ *if and only if* $R_{\mu_1}^{A_1}(l^\star) = l^*$ *and* $R_{\mu_2}^{A_2}(Bl^\star) = B(l^\star)$.

4. Main Results

Theorem 1. *Let* X_1*,* X_2 *be real Hilbert spaces;* $A_1 : X_1 \to 2^{X_1}$*,* $A_2 : X_2 \to 2^{X_2}$ *be maximal monotone mappings and* $B : X_1 \to X_2$ *be a bounded linear operator. If* $\tau_n \in (0, 2)$ *and* $\zeta_n, \xi_n \in (0, 1)$ *such that*

$$\lim_{n\to\infty} \xi_n = 0, \quad \sum_{n=0}^{\infty} \xi_n = \infty, \quad \lim_{n\to\infty} (1 - \zeta_n - \xi_n)\zeta_n > 0, \quad \inf_n \tau_n(2 - \tau_n) > 0. \tag{17}$$

Then, the sequence $\{x_n\}$ *generated from Algorithm 1 converges weakly to a point* $z \in \Lambda$.

Proof. Let $l \in \Lambda$, then $(I - R_{\mu_1}^{A_1})(l) = 0$. Since resolvent operator $R_{\mu_1}^{A_1}$ is firmly nonexpansive, hence, so is $(I - R_{\mu_1}^{A_1})$ for $\mu_1 > 0$, then by Algorithm 1 and (10), we have

$$\begin{aligned}
\|u_n - l\|^2 &= \|v_n - \sigma_n(I - R_{\mu_1}^{A_1})(v_n) - l\|^2 \\
&\leq \|v_n - l\|^2 + \sigma_n^2\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 - 2\sigma_n\langle (I - R_{\mu_1}^{A_1})(v_n), v_n - l\rangle \\
&= \|v_n - l\|^2 + \sigma_n^2\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 - 2\sigma_n\langle (I - R_{\mu_1}^{A_1})(v_n) - (I - R_{\mu_1}^{A_1})(l), v_n - l\rangle \\
&\leq \|v_n - l\|^2 + \sigma_n^2\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 - 2\sigma_n\|(I - R_{\mu_1}^{A_1})(v_n) - (I - R_{\mu_1}^{A_1})(l)\|^2 \\
&= \|v_n - l\|^2 + (\sigma_n^2 - 2\sigma_n)\|(I - R_{\mu_1}^{A_1})(v_n)\|^2.
\end{aligned} \tag{18}$$

Now, using (12), we estimate that

$$\begin{aligned}
& (\sigma_n^2 - 2\sigma_n)\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 \\
= {} & \|(I - R_{\mu_1}^{A_1})(v_n)\|^2 \Bigg[\frac{{\tau_n}^2 \, \|(I - R_{\mu_1}^{A_1})(v_n)\|^4}{\left(\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2\right)^2} \\
& - \frac{2\tau_n \, \|(I - R_{\mu_1}^{A_1})(v_n)\|^2}{\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2} \Bigg]
\end{aligned}$$

$$\begin{aligned}
&= \|(I - R_{\mu_1}^{A_1})(v_n)\|^4 \left[\frac{{\tau_n}^2 \, \|(I - R_{\mu_1}^{A_1})(v_n)\|^2 - 2\tau_n(\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2)}{\left(\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2\right)^2} \right] \\
&\leq \|(I - R_{\mu_1}^{A_1})(v_n)\|^4 \left[\frac{({\tau_n}^2 - 2\tau_n)(\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2)}{\left(\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2\right)^2} \right] \\
&= \frac{({\tau_n}^2 - 2\tau_n)\|(I - R_{\mu_1}^{A_1})(v_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2}.
\end{aligned} \tag{19}$$

From (18) and (19), we obtain

$$\|u_n - l\|^2 \le \|v_n - l\|^2 + \frac{({\tau_n}^2 - 2\tau_n)\|(I - R_{\mu_1}^{A_1})(v_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2}. \tag{20}$$

Since $(I - R_{\mu_2}^{A_2})$ is firmly nonexpasive and using $(I - R_{\mu_2}^{A_2})(Bl) = 0$ and (10), we estimate

$$\begin{aligned}
\|x_{n+1} - l\|^2 &= \|u_n - \varrho_n(I - R_{\mu_2}^{A_2})(Bu_n) - l\|^2 \\
&\le \|u_n - l\|^2 + \varrho_n^2\|(I - R_{\mu_2}^{A_2})(Bu_n)\|^2 - 2\varrho_n\langle(I - R_{\mu_2}^{A_2})(Bu_n), u_n - l\rangle \\
&= \|u_n - l\|^2 + \varrho_n^2\|(I - R_{\mu_2}^{A_2})(Bu_n)\|^2 \\
&\quad - 2\varrho_n\langle(I - R_{\mu_2}^{A_2})(Bu_n) - (I - R_{\mu_2}^{A_2})(Bl), u_n - l\rangle \\
&= \|u_n - l\|^2 + \varrho_n^2\|(I - R_{\mu_2}^{A_2})(Bu_n)\|^2 - 2\varrho_n\|(I - R_{\mu_1}^{A_1})(Bu_n)\|^2 \\
&= \|u_n - l\|^2 + (\varrho_n^2 - 2\varrho_n)\|J_{\lambda_1}^{A_2}(Bu_n)\|^2.
\end{aligned} \tag{21}$$

By (13), it turns out that

$$\begin{aligned}
&(\varrho_n^2 - 2\varrho_n)\|(I - R_{\mu_2}^{A_2})(Bu_n)\|^2 \\
=\ & \|(I - R_{\mu_2}^{A_2})(Bu_n)\|^2\Bigg[\frac{{\tau_n}^2\ \|(I - R_{\mu_2}^{A_2})(Bu_n)\|^4}{\big(\|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2\big)^2} \\
&\qquad - \frac{2\tau_n\ \|(I - R_{\mu_2}^{A_2})(Bu_n)\|^2}{\|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2}\Bigg] \\
=\ & \|(I - R_{\mu_2}^{A_2})(Bu_n)\|^4 \times \\
&\left[\frac{{\tau_n}^2\ \|(I - R_{\mu_2}^{A_2})(Bu_n)\|^2 - 2\tau_n(\|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2)}{\big(\|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2\big)^2}\right] \\
\le\ & \|(I - R_{\mu_2}^{A_2})(Bu_n)\|^4\left[\frac{({\tau_n}^2 - 2\tau_n)\big(\|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2\big)}{\big(\|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2\big)^2}\right] \\
=\ & \frac{({\tau_n}^2 - 2\tau_n)\|(I - R_{\mu_2}^{A_2})(Bu_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2}.
\end{aligned} \tag{22}$$

It follows from (21) and (22) that

$$\|x_{n+1} - l\|^2 \le \|u_n - l\|^2 + \frac{({\tau_n}^2 - 2\tau_n)\|(I - R_{\mu_2}^{A_2})(Bu_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2}. \tag{23}$$

Combining (20) and (23) , we obtain

$$\begin{aligned}
\|x_{n+1} - l\|^2 \quad \le \quad & \|v_n - l\|^2 + \frac{\tau_n(\tau_n - 2)\|(I - R_{\mu_1}^{A_1})(v_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2} \\
& + \frac{\tau_n(\tau_n - 2)\|(I - R_{\mu_2}^{A_2})(Bu_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2}
\end{aligned} \tag{24}$$

By using the Cauchy–Schwartz inequality, we observe that

$$\begin{aligned}\|v_n - l\|^2 &= \|x_n - l + \phi_n(x_n - x_{n-1})\|^2 \\ &= \|x_n - l\|^2 + 2\phi_n\langle x_n - x_{n-1}, x_n - l\rangle + \phi_n^2\|x_n - x_{n-1}\|^2. \\ &\leq \|x_n - l\|^2 + 2\phi_n\|x_n - x_{n-1}\|\|x_n - l\| + \phi_n\|x_n - x_{n-1}\|^2.\end{aligned}$$

Since $2\|x_n - x_{n-1}\|\|x_n - l\| = \|x_n - x_{n-1}\|^2 + \|x_n - l\|^2 - \|(x_n - x_{n-1}) - (x_n - l)\|^2$, we get

$$\|v_n - l\|^2 \leq \|x_n - l\|^2 + 2\phi_n\|x_n - x_{n-1}\|^2 + \phi_n\{\|x_n - l\|^2 - \|x_{n-1} - l\|^2\}. \quad (25)$$

From (25) and (24), we obtain

$$\begin{aligned}\|x_{n+1} - l\|^2 &\leq \|x_n - l\|^2 + 2\phi_n\|x_n - x_{n-1}\|^2 + \phi_n\{\|x_n - l\|^2 - \|x_{n-1} - l\|^2\} \\ &+ \frac{\tau_n(\tau_n - 2)\|(I - R_{\mu_1}^{A_1})(v_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2} \\ &\quad + \frac{\tau_n(\tau_n - 2)\|(I - R_{\mu_2}^{A_2})(Bu_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2}.\end{aligned} \quad (26)$$

Since, $\tau_n \in (0, 2)$, that is $\tau_n - 2 < 0$, we obtain

$$(\|x_{n+1} - l\|^2 - \|x_n - l\|^2) \leq \phi_n\{\|x_n - l\|^2 - \|x_{n-1} - l\|^2\} + 2\phi_n\|x_n - x_{n-1}\|^2$$

Applying Lemma 5, we deduce that the limit $\|x_n - l\|$ exists, which guarantees the boundednesss of sequence $\{x_n\}$ and hence $\{u_n\}$ and $\{v_n\}$. From (26), it follows that $\sum_{n=1}^{\infty} \phi_n(\|x_n - l\|^2 - \|x_{n-1} - l\|^2) < \infty$ and

$$\sum_{n=1}^{\infty}\left[\frac{\|(I - R_{\mu_1}^{A_1})(v_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2} + \frac{\|(I - R_{\mu_2}^{A_2})(Bu_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2}\right] < \infty,$$

which concludes

$$\lim_{n\to\infty}\left[\frac{\|(I - R_{\mu_1}^{A_1})(v_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2} + \frac{\|(I - R_{\mu_2}^{A_2})(Bu_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2}\right] = 0,$$

hence

$$\lim_{n\to\infty}\frac{\|(I - R_{\mu_1}^{A_1})(v_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2} = 0,$$

$$\lim_{n\to\infty}\frac{\|(I - R_{\mu_2}^{A_2})(Bu_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2} = 0,$$

which concludes that

$$\lim_{n\to\infty}\|(I - R_{\mu_1}^{A_1})(v_n)\| = \lim_{n\to\infty}\|(I - R_{\mu_2}^{A_2})(Bu_n)\| = 0. \quad (27)$$

It remains to show that $\omega_w(x_n) \in \Lambda$. Let $l^\star \in \omega_w(x_n)$ and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ so that $x_{n_k} \rightharpoonup l^\star$, as $k \to \infty$. Applying (27) and Remark 2, in Algorithm 1, it follows that

$$\|x_n - v_n\| = \phi_n\|x_n - x_{n-1}\| \to 0, n \to \infty$$

$$\begin{aligned}\|x_n - u_n\| &= \|x_n - [v_n - \sigma_n(I - R_{\mu_1}^{A_1})(v_n)]\| \\ &\le \|x_n - v_n\| + \sigma_n\|(I - R_{\mu_1}^{A_1})(v_n)\| \\ &\le \|x_n - v_n\| + \frac{\tau_n\|(I - R_{\mu_1}^{A_1})(v_n)\|^3}{\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2} \to 0, n \to \infty\end{aligned}$$

and

$$\|v_n - u_n\| \le \|x_n - u_n\| + \|v_n - x_n\| \to 0, n \to \infty.$$

Hence, there exist subsequences $\{u_{n_k}\}$ and $\{v_{n_k}\}$ of $\{u_n\}$ and $\{v_n\}$, respectively, which converge to $l^\star$. From (27), we obtain

$$\lim_{k\to\infty}\|(I - R_{\mu_1}^{A_1})(v_{n_k})\| = \lim_{k\to\infty}\|(I - R_{\mu_1}^{A_1})(l^\star)\| = 0,$$
$$\lim_{k\to\infty}\|(I - R_{\mu_2}^{A_2})(Bu_{n_k})\| = \lim_{k\to\infty}\|(I - R_{\mu_2}^{A_2})(Bl^\star)\| = 0,$$

which imply that $l^\star \in A_1^{-1}(0)$ and $B(l^\star) \in A_2^{-1}(0)$. □

Theorem 2. *Let X_1, X_2 be real Hilbert spaces; and let $A_1 : X_1 \to 2^{X_1}$, $A_2 : X_2 \to 2^{X_2}$ be set-valued maximal monotone mappings. If $\{\zeta_n\}$, $\{\xi_n\}$ are real sequences in $(0,1)$, $\tau_n \in (0,2)$ and*

$$\lim_{n\to\infty}\xi_n = 0,\ \sum_{n=0}^{\infty}\xi_n = \infty,\ \lim_{n\to\infty}(1 - \zeta_n - \xi_n)\zeta_n > 0,\ \inf_n \tau_n(2 - \tau_n) > 0. \tag{28}$$

Then, the sequence $\{x_n\}$ obtained from Algorithm 2 converges strongly to $l = P_{\Lambda\cap \mathrm{Fix}(F)}(0)$.

Proof. Let $l = P_{\Lambda\cap \mathrm{Fix}(F)}(0)$. From Algorithm 2, we have

$$\begin{aligned}\|v_n - l\| &= \|x_n + \phi_n(x_n - x_{n-1}) - l\| \\ &\le (1 - \phi_n)\|x_n - l\| + \phi_n\|x_{n-1} - l\| \\ &\le \max\{\|x_n - l\|, \|x_{n-1} - l\|\},\end{aligned} \tag{29}$$

and

$$\begin{aligned}\|x_{n+1} - l\| &= \|(1 - \zeta_n - \xi_n)u_n + \zeta_n F(w_n) - l\| \\ &\le (1 - \zeta_n - \xi_n)\|u_n - l\| + \| + \zeta_n\|F(w_n) - l\| + \xi_n\| - l\| \\ &\le (1 - \zeta_n - \xi_n)\|u_n - l\| + \| + \zeta_n\|w_n - l\| + \xi_n\| - l\| \\ &\le (1 - \xi_n)\|v_n - l\| + \xi_n\|l\| \\ &\le (1 - \xi_n)\big[(1 - \phi_n)\|x_n - l\| + \phi_n\|x_{n-1} - l\|\big] + \xi_n\|l\| \\ &\le \max\{\|x_n - l\|, \|x_{n-1} - l\|, \|l\|\} \\ &\le \vdots \\ &\le \max\{\|x_0 - l\|, \|x_1 - l\|, \|l\|\},\end{aligned}$$

which shows that $\{x_n\}$ is bounded and hence the $\{v_n\}$, $\{u_n\}$, and $\{w_n\}$ are bounded. From (20) and (23) of the proof of Theorem 1, we have

$$\|u_n - l\|^2 \le \|v_n - l\|^2 + \frac{(\tau_n{}^2 - 2\tau_n)\|(I - R_{\mu_1}^{A_1})(v_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2} \le \|v_n - l\|^2. \tag{30}$$

$$\|w_n - l\|^2 \le \|u_n - l\|^2 + \frac{({\tau_n}^2 - 2\tau_n)\|(I - R_{\mu_2}^{A_2})(Bu_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2} \le \|u_n - l\|^2. \quad (31)$$

Now,

$$\begin{aligned}
\|x_{n+1} - l\|^2 &= \|(1 - \zeta_n - \xi_n)u_n + \zeta_n F(w_n) - l\|^2 \\
&= \|(1 - \zeta_n - \xi_n)(u_n - l) + \zeta_n(F(w_n) - l) + \xi_n(-l)\|^2 \\
&\le (1 - \zeta_n - \xi_n)\|u_n - l\|^2 + \zeta_n\|F(w_n) - l\|^2 + \xi_n\|l\|^2 \\
&\quad - \zeta_n(1 - \zeta_n - \xi_n)\|u_n - F(w_n)\|^2 \\
&= (1 - \zeta_n - \xi_n)\|u_n - l\|^2 + \zeta_n\|w_n - l\|^2 + \xi_n\|l\|^2 \\
&\quad - \zeta_n(1 - \zeta_n - \xi_n)\|u_n - F(w_n)\|^2.
\end{aligned} \quad (32)$$

Combining (30)–(32), we obtain

$$\begin{aligned}
\|x_{n+1} - l\|^2 \le\ & (1 - \zeta_n - \xi_n)\left[\|v_n - l\|^2 + \frac{({\tau_n}^2 - 2\tau_n)\|(I - R_{\mu_1}^{A_1})(v_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2}\right] \\
& + \zeta_n\left[\|u_n - l\|^2 + \frac{({\tau_n}^2 - 2\tau_n)\|(I - R_{\mu_2}^{A_2})(Bu_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2}\right] + \xi_n\|l\|^2 \\
& - \zeta_n(1 - \zeta_n - \xi_n)\|u_n - F(w_n)\|^2 \\
\le\ & (1 - \zeta_n - \xi_n)\|v_n - l\|^2 + \zeta_n\|u_n - l\|^2 - \zeta_n(1 - \zeta_n - \xi_n)\|u_) - F(w_n)\|^2 \\
& - \frac{(1 - \zeta_n - \xi_n)({\tau_n}^2 - 2\tau_n)\|(I - R_{\mu_1}^{A_1})(v_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2} \\
& - \frac{\zeta_n({\tau_n}^2 - 2\tau_n)\|(I - R_{\mu_2}^{A_2})(Bu_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2} + \xi_n\|l\|^2 \\
\le\ & (1 - \xi_n)\Big[\|x_n - l\|^2 + 2\phi_n\|x_n - x_{n-1}\|^2 + \phi_n\{\|x_n - l\|^2 - \|x_{n-1} - l\|^2\}\Big] \\
& - \zeta_n(1 - \zeta_n - \xi_n)\|u_n - F(w_n)\|^2 + \frac{(1 - \zeta_n - \xi_n)({\tau_n}^2 - 2\tau_n)\|(I - R_{\mu_1}^{A_1})(v_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2} \\
& + \frac{\zeta_n({\tau_n}^2 - 2\tau_n)\|(I - R_{\mu_2}^{A_2})(Bu_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2} + \xi_n\|l\|^2 \\
\le\ & \|x_n - l\|^2 + 2\phi_n\|x_n - x_{n-1}\|^2 + \phi_n\{\|x_n - l\|^2 - \|x_{n-1} - l\|^2\} \\
& - \zeta_n(1 - \zeta_n - \xi_n)\|u_n - F(w_n)\|^2 + \frac{(1 - \zeta_n - \xi_n)\tau_n(2 - \tau_n)\|(I - R_{\mu_1}^{A_1})(v_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(v_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bv_n)\|^2} \\
& + \frac{\zeta_n\tau_n(2 - \tau_n)\|(I - R_{\mu_2}^{A_2})(Bu_n)\|^4}{\|(I - R_{\mu_1}^{A_1})(u_n)\|^2 + \|B^*(I - R_{\mu_2}^{A_2})(Bu_n)\|^2} + \xi_n\|l\|^2.
\end{aligned} \quad (33)$$

Two possible cases occur.

Case I. Suppose the sequence $\{\|x_n - l\|\}$ is nonincreasing; then, there exists $m \ge 0$ such that $\|x_{n+1} - l\| \le \|x_n - l\|$, for each $n \ge m$. Then, $\lim_{n\to\infty} \|x_n - l\|$ exists and hence $\lim_{n\to\infty}(\|x_{n+1} - l\| - \|x_n - l\|) = 0$. Since $\xi_n \to 0$, $\tau_n \in (0, 2)$, and $\inf \zeta_n(1 - \zeta_n - \xi_n) > 0$, hence from (33), we have

$$\frac{\|(I - R^{A_1}_{\mu_1})(v_n)\|^4}{\|(I - R^{A_1}_{\mu_1})(v_n)\|^2 + \|B^*(I - R^{A_2}_{\mu_2})(Bv_n)\|^2} \to 0, \tag{34}$$

$$\frac{\|(I - R^{A_2}_{\mu_2})(Bu_n)\|^4}{\|(I - R^{A_1}_{\mu_1})(u_n)\|^2 + \|B^*(I - R^{A_2}_{\mu_2})(Bu_n)\|^2} \to 0, \tag{35}$$

$$\|u_n - F(w_n)\| \to 0. \tag{36}$$

From (34) and (35), we obtain

$$\lim_{n\to\infty} \|(I - R^{A_1}_{\mu_1})(u_n)\| = 0 \text{ and } \lim_{n\to\infty} \|(I - R^{A_2}_{\mu_2})(Bv_n)\| = 0. \tag{37}$$

From Algorithm 2, using Remark 2, we obtain

$$\lim_{n\to\infty} \|v_n - x_n\| = 0. \tag{38}$$

From Algorithm 2, using (34) and (35), we obtain as $n \to \infty$

$$\|u_n - v_n\| \to 0, \tag{39}$$

$$\|w_n - u_n\| \to 0. \tag{40}$$

By using (38)–(40), we obtain

$$\|u_n - x_n\| \leq \|u_n - v_n\| + \|v_n - x_n\| \to 0, \text{ as } n \to \infty \tag{41}$$

$$\|w_n - x_n\| \leq \|w_n - u_n\| + \|u_n - x_n\| \to 0, \text{ as } n \to \infty. \tag{42}$$

Thus, since $\xi_n \to 0$, and using (36), (40) and (41), we obtain

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|(1 - \zeta_n - \xi_n)u_n + \zeta_n F(w_n) - x_n\| \\ &\leq \|u_n - x_n\| + \zeta_n\|F(w_n) - u_n\| + \xi_n\| - u_n\| \to 0 \; as \; n \to \infty,\end{aligned} \tag{43}$$

and

$$\begin{aligned}\|F(w_n) - w_n\| &\leq \|F(w_n) - u_n\| + \|u_n - w_n\| \to 0 \; as \; n \to \infty. \\ \|F(u_n) - u_n\| &\leq \|F(u_n) - F(w_n)\| + \|F(w_n) - u_n\| \to 0 \; as \; n \to \infty. \\ &\leq \|u_n - w_n\| + \|F(w_n) - u_n\| \to 0 \; as \; n \to \infty.\end{aligned} \tag{44}$$

Hence, there exists a subsequence $\{u_{nk}\}$ of $\{u_n\}$ which converges weakly to l. By using Lemma 3, we conclude that $l \in \text{Fix}(F)$. By Theorem 1, we have that $\omega_w(x_n) \subset \Lambda$. So, we obtain $l \in \text{Fix}(F) \cap \Lambda$. Setting $s_n = (1 - \zeta_n)u_n + \zeta_n F(w_n)$ and rewrite $x_{n+1} = (1 - \xi_n)s_n + \zeta_n \xi_n (F(w_n) - u_n)$, we have

$$\begin{aligned}\|s_n - l\| &= \|(1 - \zeta_n)u_n + \zeta_n F(w_n) - l\| \\ &\leq (1 - \zeta_n)\|u_n - l\| + \zeta_n\|F(w_n) - l\| \\ &\leq (1 - \zeta_n)\|v_n - l\| + \zeta_n\|w_n - l\| \\ &\leq \|v_n - l\|.\end{aligned}$$

From (45) and Algorithm 2, we obtain

$$\begin{aligned}
\|x_{n+1}-l\|^2 &= \|(1-\xi_n)(s_n-l)+\xi_n\zeta_n(F(w_n)-u_n)-l\|^2 \\
&\leq (1-\xi_n)\|s_n-l\|^2+2\xi_n\langle\zeta_n(F(w_n)-u_n)-l,x_{n+1}-l\rangle \\
&\leq (1-\xi_n)\Big[\|x_n-l\|^2+2\phi_n\|x_n-x_{n-1}\|^2+\phi_n\{\|x_n-l\|^2-\|x_{n-1}-l\|^2\}\Big] \\
&\quad +2\xi_n\{\zeta_n\langle F(w_n)-u_n,x_{n+1}-l\rangle+\langle -l,x_{n+1}-l\rangle\}, \\
&\leq (1-\xi_n)\|x_n-l\|^2+2\phi_n\|x_n-x_{n-1}\|^2+\phi_n\{\|x_n-l\|^2-\|x_{n-1}-l\|^2\} \\
&\quad +2\xi_n\{\zeta_n\langle F(w_n)-u_n,x_{n+1}-l\rangle+\langle -l,x_{n+1}-l\rangle\}.
\end{aligned} \tag{45}$$

Since $\omega_w(x_n)\subset \mathrm{Fix}(F)\cap\Lambda$ and $l=P_{\mathrm{Fix}(F)\cap\Lambda}(0)$, then using (35), we obtain

$$\begin{aligned}
\limsup_{n\to\infty}\frac{b_n}{\xi_n} &= \limsup_{n\to\infty}\{2\zeta_n\langle F(w_n)-u_n,x_{n+1}-l\rangle+\langle -l,x_{n+1}-l\rangle\} \\
&= \limsup_{n\to\infty}\langle -l,x_{n+1}-l\rangle\leq 0.
\end{aligned}$$

Thus, by Lemma 1 in (45), we obtain $x_n\to l$.

Case II. If the sequence $\{\|x_n-l\|\}$ is increasing, we can construct a subsequence $\{\|x_{n_k}-l\|\}$ of $\{\|x_n-l\|\}$ such that $\|x_{n_k}-l\|\leq\|x_n-l\|$ for all $k\in\mathbb{N}$. In this case, we define a subsequence of positive integers $\gamma(n)$

$$\gamma(n)=\max\{k\leq n:\|x_k-l\|\leq\|x_{k+1}-l\|\},$$

then $\gamma(n)\to\infty$ and $n\to\infty$ and $\|x_{\gamma(n)}-l\|\leq\|x_{\gamma(n)+1}-l\|$, it follows from (33) that

$$\begin{aligned}
\|x_{\gamma(n)}-l\|^2 \leq\ & (1-\xi_{\gamma(n)})\|x_{\gamma(n)}-l\|^2+2\phi_{\gamma(n)}\|x_n-x_{\gamma(n)-1}\|^2+\phi_{\gamma(n)}\{\|x_{\gamma(n)}-l\|^2 \\
&-\|x_{\gamma(n)-1}-l\|^2\}-\zeta_{\gamma(n)}(1-\zeta_{\gamma(n)}-\xi_{\gamma(n)})\|u_{\gamma(n)}-F(w_{\gamma(n)})\|^2 \\
&-\frac{(1-\zeta_{\gamma(n)}-\xi_{\gamma(n)})\tau_{\gamma(n)}(\tau_{\gamma(n)}-2)\|(I-R_{\mu_1}^{A_1})(v_{\gamma(n)})\|^4}{\|(I-R_{\mu_1}^{A_1})(v_{\gamma(n)})\|^2+\|B^*(I-R_{\mu_2}^{A_2})(Bv_{\gamma(n)})\|^2} \\
&-\frac{\zeta_{\gamma(n)}\tau_{\gamma(n)}(\tau_{\gamma(n)}-2)\|(I-R_{\mu_2}^{A_2})(Bu_{\gamma(n)})\|^4}{\|(I-R_{\mu_1}^{A_1})(u_{\gamma(n)})\|^2+\|B^*(I-R_{\mu_2}^{A_2})(Bu_{\gamma(n)})\|^2}+\xi_{\gamma(n)}\|l\|^2
\end{aligned}$$

that is

$$\begin{aligned}
&\xi_{\gamma(n)}(\|l\|^2-\|x_{\gamma(n)}-l\|^2)+2\phi_{\gamma(n)}\|x_{\gamma(n)}-x_{\gamma(n)-1}\|^2+\phi_n\{\|x_{\gamma(n)}-l\|^2-\|x_{\gamma(n)-1}-l\|^2\} \\
&+\frac{(1-\zeta_{\gamma(n)}-\xi_{\gamma(n)})\tau_{\gamma(n)}(\tau_{\gamma(n)}-2)\|(I-R_{\mu_1}^{A_1})(v_{\gamma(n)})\|^4}{\|(I-R_{\mu_1}^{A_1})(v_{\gamma(n)})\|^2+\|B^*(I-R_{\mu_2}^{A_2})(Bv_{\gamma(n)})\|^2} \\
&+\frac{\zeta_{\gamma(n)}\tau_{\gamma(n)}(\tau_{\gamma(n)}-2)\|(I-R_{\mu_2}^{A_2})(Bu_{\gamma(n)})\|^4}{\|(I-R_{\mu_1}^{A_1})(u_{\gamma(n)})\|^2+\|B^*(I-R_{\mu_2}^{A_2})(Bu_{\gamma(n)})\|^2} \\
&\geq\zeta_{\gamma(n)}(1-\zeta_{\gamma(n)}-\xi_{\gamma(n)})\|u_{\gamma(n)}-F(w_{\gamma(n)})\|^2.
\end{aligned}$$

Since $\xi_{\gamma(n)}\to 0$ and $\phi_{\gamma(n)}\to 0$ and $\gamma(n)\to 0$, then for subsequences $\{x_{\gamma(n)}\}$, $\{u_{\gamma(n)}\}$ and $\{w_{\sigma(n)}\}$, we obtain

$$\lim_{n\to\infty}\|u_{\gamma(n)}-F(w_{\gamma(n)})\|=0,\ \lim_{n\to\infty}\|(I-R_{\mu_1}^{A_1})(v_{\gamma(n)})\|=0 \text{ and } \lim_{n\to\infty}\|(I-R_{\mu_2}^{A_2})(Bu_{\gamma(n)})\|=0.$$

Similarly, we can show that $\|x_{\gamma(n+1)}-x_{\gamma(n)}\|\to 0$, as $n\to\infty$ and $\omega_w(x_{\gamma(n)})\subset\mathrm{Fix}(F)\cap\Lambda$. It is remaining to show that $x_n\to l$.

By using $\|x_{\gamma(n)} - l\| < \|x_{\gamma(n)+1} - l\|$ and the boundedness of $\|x_n - l\|$, we have

$$\begin{aligned} \|x_{\gamma(n)} - l\|^2 &\leq 2\alpha_{\gamma(n)}\langle F(w_{\gamma(n)}) - u_{\gamma(n)}, x_{\gamma(n+1)} - l\rangle + 2\langle -l, x_{\gamma(n+1)} - l\rangle, \\ &\leq M\|F(w_{\gamma(n)}) - u_{\gamma(n)}\| - 2\langle l, x_{\gamma(n)+1} - l\rangle. \end{aligned}$$

Since $\|x_{\gamma(n)+1} - x_{\gamma(n)}\| \to 0$, we obtain

$$\begin{aligned} \limsup_{n\to\infty} \langle -l, x_{\gamma(n)+1} - l\rangle &= -\limsup_{n\to\infty} \langle l, x_{\gamma(n)} - l\rangle \\ &= -\max_{r\in\omega_w(x_{\gamma(n)})} \langle l, r - l\rangle \leq 0, \end{aligned} \tag{46}$$

due to $l = P_{\text{Fix}(F)\cap\Lambda}(0)$, $\omega(x_{\gamma(n)}) \subset \text{Fix}(F) \cap \Lambda$ and using $\|F(w_{\gamma(n)}) - u_{\gamma(n)}\| \to 0$, using Lemma 1 in (46), we obtain that $x_{\gamma(n)} \to l$, and

$$\|x_n - l\| \leq \|x_{\gamma(n)+1} - l\| \leq \|x_{\gamma(n)+1} - x_{\gamma(n)}\| + \|x_{\gamma(n)} - l\| \to 0,$$

that is, $x_n \to l$. Hence, the theorem is proved. □

For $\tau_n = 1$, we obtain the following corollary of Theorem 2.

Corollary 1. *Let X_1, X_2, A_1, A_2, B, B^* and ϕ_n be identical as in Theorem 2. Let $\{\zeta_n\}$, $\{\xi_n\}$ be sequences in $(0,1)$ such that*

$$\lim_{n\to\infty} \xi_n = 0, \ \sum_{n=0}^{\infty} \xi_n = \infty, \ \lim_{n\to\infty} (1 - \zeta_n - \xi_n)\zeta_n > 0,$$

hold. Then, the sequence $\{x_n\}$ obtained by Algorithm 2 (with $\tau_n = 1$), converges strongly to $l = P_{\text{Fix}(F)\cap\Lambda}(0)$.

For $\xi_n = 0$, we obtain the following corollary of Theorem 2.

Corollary 2. *Let X_1, X_2, A_1, A_2, B, B^* and ϕ_n be identical as in Theorem 2. If $\{\zeta_n\}$ is a sequence in $(0,1)$ so that*

$$\lim_{n\to\infty} (1 - \zeta_n)\zeta_n > 0, \ \inf_n \tau_n(2 - \tau_n) > 0$$

holds, then the sequence $\{x_n\}$ obtained by the following scheme

$$\begin{aligned} v_n &= x_n + \phi_n(x_n - x_{n-1}), \\ u_n &= v_n - \sigma_n(I - R_{\mu_1}^{A_1})(v_n), \\ w_n &= u_n - \varrho_n B^*(I - R_{\mu_2}^{A_2})(Bu_n), \\ x_{n+1} &= (1 - \zeta_n)w_n + \zeta_n F(w_n), \end{aligned}$$

where σ_n and ϱ_n are defined by (15) and (16), respectively, converges strongly to $l \in \text{Fix}(F) \cap \Lambda$.

For $\tau_n = 1$ and $\xi_n = 0$, we obtain the following corollary of Theorem 2.

Corollary 3. *Let X_1, X_2, A_1, A_2 and B, B^* be identical as in Algorithm 2. Let $\{\zeta_n\}$ be a sequence in $(0,1)$ so that*

$$\lim_{n\to\infty} (1 - \zeta_n)\zeta_n > 0.$$

Then, the sequence $\{x_n\}$ obtained by the following scheme

$$\begin{aligned} v_n &= x_n + \phi_n(x_n - x_{n-1}), \\ u_n &= v_n - \sigma_n(I - R^{A_1}_{\mu_1})(v_n), \\ w_n &= u_n - \varrho_n B^*(I - R^{A_2}_{\mu_2})(Bu_n), \\ x_{n+1} &= (1-\zeta_n)w_n + \zeta_n F(w_n), \end{aligned}$$

where σ_n and ϱ_n are defined by (15) *and* (16)*, respectively (with $\tau_n = 1$), converges strongly to $l \in \mathrm{Fix}(F) \cap \Lambda$.*

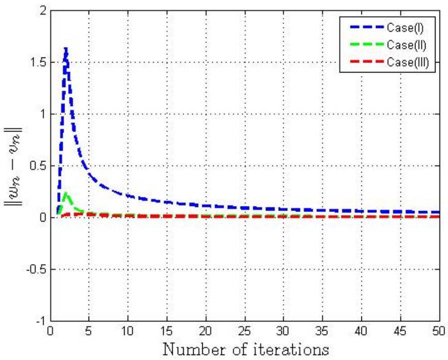
5. Numerical Experiments

Suppose $X_1 = X_2 = \mathbb{R}$. Let us consider the monotone mappings A_1 and A_2 defined as $A_1(x) = \frac{x}{2} + 2$ and $A_2(x) = x + 2$. The nonexpansive mapping $F : X_1 \to X_1$ is defined as $F(x) = \frac{x-4}{2}$ and bounded linear operator $B : X_1 \to X_2$ is defined as $B(x) = \frac{x}{2}$. It is not a difficult task to show that A_1 and A_2 are monotone mappings and B is nonexpansive mapping and $\mathrm{Fix}(F) \cap \Lambda = \{-4\}$. The resolvents of A_1 and A_2 with parameter $\mu_1 > 0, \mu_2 > 0$ are

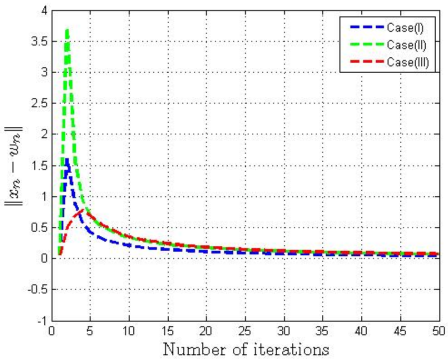
$$R^{A_1}_{\mu_1}(x) = [I + \mu_1 A_1]^{-1}(x) = \frac{2x - 4\mu_1}{2 + \mu_1}, \qquad R^{A_2}_{\mu_2}(x) = [I + \mu_2 A_2]^{-1}(x) = \frac{x - 2\mu_2}{1 + \mu_2}.$$

We choose $\tau_n = 1 - \frac{1}{n+1}$, $\Lambda_n = \frac{1}{n^2}$, $\xi = \frac{1}{n}$ and $\zeta_n = (1 - \frac{e^{\frac{1}{n}}}{3})$ satisfying the condition (28) in Algorithm 2. We fixed the maximum number of iterations 50 as a stopping criterion. The parameter ϕ_n is randomly generated in $(0, \tilde{\phi}_n)$, where $\tilde{\phi}_n$ is calculated by using (14). The behavior of the sequences $\{x_n\}, \{v_n\}$ and $\{u_n\}$ are plotted in Figure 1 by applying three distinct cases of parameters which are mentioned below:

Case (I): $x_0 = 0$, $x_1 = 5$, $\phi = 0.1$, $\mu_1 = 0.5$, $\mu_2 = 0.9$.
Case (II): $x_0 = -3$, $x_1 = 4$, $\phi = 0.5$, $\mu_1 = 5$, $\mu_2 = 8$.
Case (III): $x_0 = 5$, $x_1 = -5$, $\phi = 0.75$, $\mu_1 = 10$, $\mu_2 = 20$.



(**a**)



(**b**)

Figure 1. *Cont.*

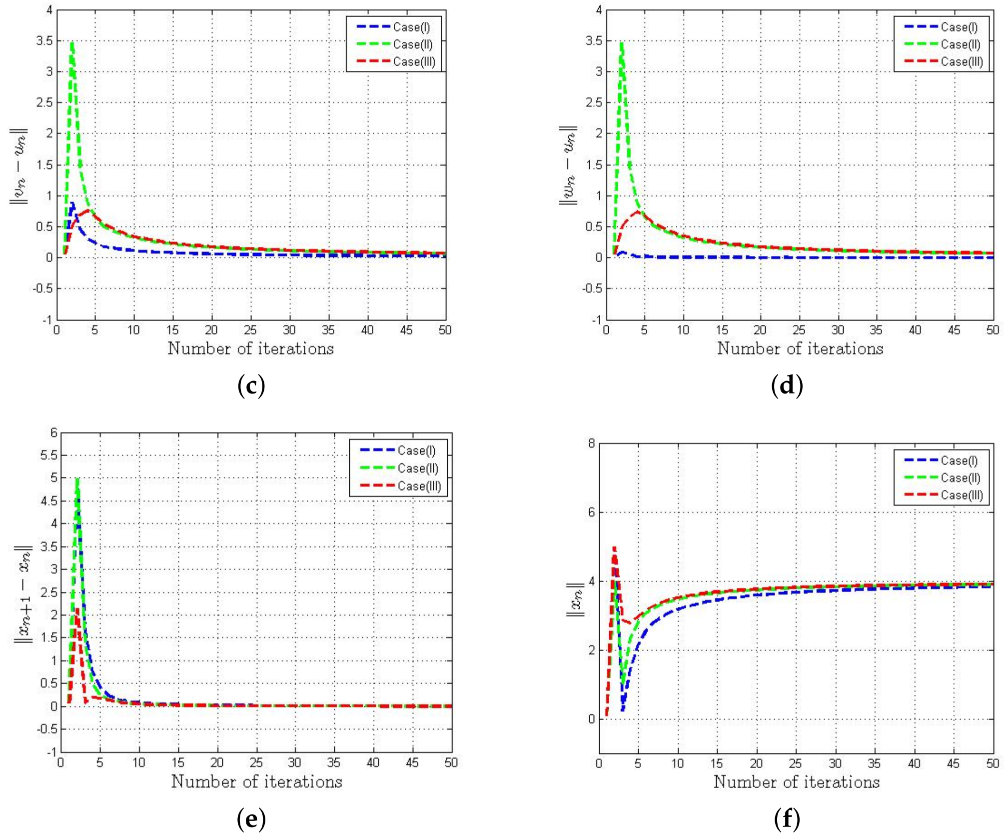


Figure 1. Numerical behavior of $\|w_n - u_n\|$, $\|x_n - u_n\|$, $\|v_n - u_n\|$, $\|w_n - u_n\|$, $\|x_{n+1} - x_n\|$ and $\|x_n\|$ choosing three cases of parameters.

Observations:

- In Figure 1a–d, we observed that the behavior of $\{w_n\}, \{v_n\}$ and $\{u_n\}$ is uniform irrespective of the selection of parameters.
- From Figure 1e–f, we notice that the sequence obtained from Algorithm 2 converges to the same limit with a suitable selection of parameters.
- It is worthwhile to mention that the estimation of $\|BB^*\|$ is not required to implement the algorithm, which is not so handy to calculate in general.

6. Conclusions

We have suggested and analyzed inertial methods to estimate the solution of (S_pVI_sP) and common solution of (S_pVI_sP) and (FPP). We proved the weak and strong convergence of algorithms to estimate the solution of (S_pVI_sP) and (FPP) with suitable assumptions in such a way that the estimation of the step size does not require a pre-estimated norm $\|BB^*\|$. Finally, we perform a numerical example to exhibit the behavior of the proposed algorithms using different cases of parameters.

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