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Metrical Boundedness and Compactness of a New Operator between Some Spaces of Analytic Functions

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Abstract: The metrical boundedness and metrical compactness of a new operator from the weighted Bergman-Orlicz spaces to the weighted-type spaces and little weighted-type spaces of analytic functions are characterized.

Keywords: metrical boundedness; metrical compactness; differentiation operator; Bergman-Orlicz space; weighted-type space

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1. Introduction

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{R}$ be the set or real numbers, $\mathbb{R}_+ = [0, +\infty)$, and $\mathbb{C}$ be the set of all complex numbers. We use the expression $k = \overline{r, s}$, where $r, s \in \mathbb{N}_0$, instead of the following one: $r \le k \le s$, $k \in \mathbb{N}_0$. We also understand that $\sum_{k=p}^{q} b_k = 0$, when $q < p$, and $\prod_{k=p}^{p-1} b_k = 1$, for every $p, q \in \mathbb{N}_0$, where b_k are some complex numbers. A nonzero function Ψ is called a growth function if it is continuous, nondecreasing and $\Psi(\mathbb{R}_+) = \mathbb{R}_+$ (the functions appear in defining the Orlicz-type spaces; see, e.g., [1–3]). By $G(\mathbb{R}_+)$ we denote the set of all growth functions.

Let $\mathbb{B}^n = \mathbb{B} := \{z \in \mathbb{C}^n : |z| < 1\}$, where $z = (z_1, z_2, \dots, z_n)$, $|z| = \langle z, z\rangle^{1/2}$ and

$$\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j,$$

and let $\mathbb{S} = \partial\mathbb{B}$.

Let $H(\Omega)$ be the family of analytic function on a domain $\Omega \subseteq \mathbb{C}^n$ [4,5], $S(\Omega)$ the class of analytic self-maps of Ω, $\mathcal{P}_n$ the set of polynomials in $\mathbb{C}^n$, $D_j f = \frac{\partial f}{\partial z_j}$, $j = \overline{1, n}$, and

$$\Re f(z) = \sum_{j=1}^{n} z_j D_j f(z),$$

the so-called radial derivative.

By $dv(z)$ we denote the Lebesgue measure on $\mathbb{B}$, whereas $dv_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dv(z)$, $\alpha > -1$, is the normalized weighted Lebesgue measure on $\mathbb{B}$ (i.e., $v_\alpha(\mathbb{B}) = 1$). By $d\sigma$ we denote the normalized surface measure on $\mathbb{S}$ (i.e., $\sigma(\mathbb{S}) = 1$). Positive and continuous functions on $\mathbb{B}$ are called weights. The set of all such functions we denote by $W(\mathbb{B})$.

Recall that each $u \in H(\Omega)$ induces the multiplication operator $M_u f = uf$ on $H(\Omega)$, whereas each $\varphi \in S(\Omega)$ induces the composition operator $C_\varphi f = f \circ \varphi$ on $H(\Omega)$. If $n = 1$, then $Df = f'$ is the differentiation operator. These three operators are linear and have been studied a lot, as well as many of their products.

Let us briefly mention a part of the investigations preceding the one in the paper. Of the product-type operators containing the differential one, first were studied the operators DC_φ and $C_\varphi D$ (see, e.g., [6–11] and the related references therein). In [12] was studied the product of the multiplication operator followed by the differentiation one on the Bloch-type spaces. In [13,14] were studied some operators containing each of the three operators C_φ, M_u and D.

Soon after the first investigations of the operators DC_φ and $C_\varphi D$, the following operator

$$D^m_{\varphi,u} := M_u C_\varphi D^m$$

containing also the operators C_φ, M_u and D, was introduced and considerably studied (see, e.g., [15–28] and the related references therein).

The following operator

$$\mathfrak{R}^m_{\varphi,u} := M_u C_\varphi \mathfrak{R}^m,$$

which was defined in [29] and investigated also in [30] (see also the related references therein), can be regarded as an n-dimensional counterpart of the operator $D^m_{\varphi,u}$.

Investigation of the sum

$$M_{u_1} C_\varphi + M_{u_2} C_\varphi D \tag{1}$$

where $u_1, u_2 \in H(\mathbb{B}^1)$ and $\varphi \in S(\mathbb{B}^1)$, was initiated by the author of the paper and A. K. Sharma. The operator was first studied on the weighted Bergman spaces in [31], and later on many other spaces. For example, in [32] it was studied from weighted Bergman spaces to weighted-type spaces, in [33] from Hardy spaces to Stević weighted spaces, whereas in [34] from the mixed-norm spaces to Zygmund-type spaces.

The generalization of the operator in (1)

$$M_{u_1} C_\varphi D^m + M_{u_2} C_\varphi D^{m+1},$$

where $m \in \mathbb{N}_0$, $u_1, u_2 \in H(\mathbb{B}^1)$ and $\varphi \in S(\mathbb{B}^1)$, was considered for the first time in [35], where the boundedness and compactness of the operator from a general space to the Bloch-type spaces were characterized.

After the publication of [35], we proposed studying finite sums of the operators $D^m_{\varphi,u}$ and $\mathfrak{R}^m_{\varphi,u}$, as well as the following sum operator

$$P_{D,m} f := \sum_{j=0}^{m} u_j C_\varphi D_{l_j} \cdots D_{l_1} f, \tag{2}$$

where $m \in \mathbb{N}_0$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, and $\varphi \in S(\mathbb{B})$, on normed subspaces of $H(\mathbb{B})$, which is a polynomial differentiation composition operator. The first results on operator (2) between some spaces of analytic functions were presented in [36], where we gave some necessary and sufficient conditions for the boundedness and compactness of the operator from the logarithmic Bloch spaces to weighted-type spaces of holomorphic functions. The investigation was continued in [37].

In [38–45] can be found several other product-type operators, some of which include integral-type ones.

If there are $q > 0$ and $C > 0$ such that $\Psi(st) \le Ct^q \Psi(s)$, for $s > 0$ and $t \ge 1$, we say that Ψ is of positive upper type q. The family of growth functions Ψ of positive upper type $q \ge 1$ such that $\Psi(t)/t$ is nondecreasing on $(0, +\infty)$ is denoted by $\mathfrak{U}^q$. If there are $p > 0$ and $C > 0$ such that $\Psi(st) \le Ct^p \Psi(s)$, for each $s > 0$ and $0 < t \le 1$, we say that Ψ is of positive lower type. The family of growth functions Ψ of positive lower type $p \in (0,1)$ such that $\Psi(t)/t$ is nonincreasing on $(0, +\infty)$ is denoted by $\mathfrak{L}_p$. It is not difficult to prove that the functions in $\mathfrak{U}^q \cup \mathfrak{L}_p$ are increasing.

If $\Psi \in \mathfrak{U}^q \cup \mathfrak{L}_p$, we may assume that $\Psi \in C^1$ and that there are positive numbers c_1 and c_2 such that

$$c_1\frac{\Psi(t)}{t}\le\Psi'(t)\le c_2\frac{\Psi(t)}{t} \tag{3}$$

for $t\in(0,+\infty)$. If $\Psi\in\mathfrak{U}^q$ we also assume that it is convex [44].

If $\Psi\in\mathfrak{L}_p$ is a C^1 function satisfying (3), then for sufficiently large r the function $\Psi(t^r)$ is comparable to a convex C^1 function G satisfying the inequalities in (3) with some constants $c_1(G)$ and $c_2(G)$ [46].

If $\Psi\in G(\mathbb{R}_+)$, then the set of all $f\in H(\mathbb{B})$ such that

$$\|f\|_{A^\Psi_\alpha(\mathbb{B})}:=\int_{\mathbb{B}}\Psi(|f(z)|)dv_\alpha(z)<+\infty,$$

is called the weighted Bergman-Orlicz space and is denoted by $A^\Psi_\alpha(\mathbb{B})$. The space generalizes the weighted Bergman space $A^p_\alpha(\mathbb{B})$. A quasi-norm on the space is given by

$$\|f\|^{lux}_{A^\Psi_\alpha(\mathbb{B})}:=\inf\left\{\lambda>0:\int_{\mathbb{B}}\Psi\left(\frac{|f(z)|}{\lambda}\right)dv_\alpha(z)\le1\right\},$$

and if $\Psi\in\mathfrak{U}^q\cup\mathfrak{L}_p$, it is finite for every $f\in A^\Psi_\alpha(\mathbb{B})$. This is the so-called Luxembourg quasi-norm.

The set of all $f\in H(\mathbb{B})$ such that

$$\|f\|_{H^\Psi(\mathbb{B})}:=\sup_{0<r<1}\int_{\mathbb{S}}\Psi(|f(r\xi)|)d\sigma(\xi)<\infty,$$

is called the Hardy-Orlicz space and is denoted by $H^\Psi(\mathbb{B})$. It generalizes the Hardy space $H^p(\mathbb{B})$. A quasi-norm on the space is given by

$$\|f\|^{lux}_{H^\Psi(\mathbb{B})}:=\sup_{0<r<1}\|f_r\|^{lux}_{L^\Psi(\mathbb{B})},$$

where $f_r(\xi)=f(r\xi)$, $0\le r<1$, $\xi\in\mathbb{S}$, and $\|\cdot\|^{lux}_{L^\Psi(\mathbb{B})}$ is the Luxembourg quasi-norm

$$\|g\|^{lux}_{L^\Psi(\mathbb{B})}:=\inf\left\{\lambda>0:\int_{\mathbb{S}}\Psi\left(\frac{|g(\xi)|}{\lambda}\right)d\sigma(\xi)\le1\right\}.$$

The quasi-norm is finite for every $f\in H^\Psi(\mathbb{B})$. The Hardy-Orlicz space is a kind of a limit of the space $A^\Psi_\alpha(\mathbb{B})$ as $\alpha\to-1+0$. Hence, we also denote the space by $A^\Psi_{-1}(\mathbb{B})$.

Let $\omega\in W(\mathbb{B})$. Then, the weighted-type space is defined by

$$H^\infty_\omega(\mathbb{B}):=\{f\in H(\mathbb{B}):\|f\|_{H^\infty_\omega}:=\sup_{z\in\mathbb{B}}\omega(z)|f(z)|<+\infty\},$$

whereas the little weighted-type space $H^\infty_{\omega,0}(\mathbb{B})$ contains all $f\in H^\infty_\omega(\mathbb{B})$ such that

$$\lim_{|z|\to1}\omega(z)|f(z)|=0.$$

The quantity $\|\cdot\|_{H^\infty_\omega}$ is a norm on the spaces, and with the norm they both are Banach spaces. There is a huge literature on the spaces and operators on them (see, e.g., [13,17,20,22,27,30,42,47–53]). If $\omega(z)\equiv1$, then the norm $\|\cdot\|_{H^\infty_\omega}$ we denote by $\|\cdot\|_\infty$.

Let X and Y be metric spaces with the translation invariant metrics d_X and d_Y, respectively. For a linear operator $A:X\to Y$ is said that it is metrically bounded if there is $C\in\mathbb{R}_+$ such that

$$d_Y(Af,0)\le Cd_X(f,0),$$

for every $f\in X$. For the operator is said that it is metrically compact if it maps bounded balls into relatively compact sets [54,55]. There is also a huge literature on the topics (see, e.g., [6–45,53,56–58]).

Here we characterize the metrical boundedness and compactness of the operator $P_{D,m} : A^{\Psi}_{\alpha}(\mathbb{B}) \to H^{\infty}_{\omega}(\mathbb{B})$ (or $H^{\infty}_{\omega,0}(\mathbb{B})$), for $\alpha \geq -1$.

By C we denote some constants, which may be different from one appearance to another. If we write $a \lesssim b$ (resp. $a \gtrsim b$), then $a \leq Cb$ (resp. $a \geq Cb$) for some $C > 0$. We write $a \asymp b$, if $a \lesssim b$ and $b \gtrsim a$.

2. Some Lemmas

Our first lemma was proved in [44].

Lemma 1. *Let $\alpha \geq -1$ and $\Psi \in \mathfrak{U}^q \cup \mathfrak{L}_p$. Then for each $t \in \mathbb{R}_+$, $C > 0$ and $w \in \mathbb{B}$, the function*

$$f_{w,t}(z) = \Psi^{-1}\left(\frac{C}{(1-|w|^2)^{n+1+\alpha}}\right)\left(\frac{1-|w|^2}{1-\langle z,w\rangle}\right)^{2(n+1+\alpha)+t}, \tag{4}$$

belongs to $A^{\Psi}_{\alpha}(\mathbb{B})$, and

$$\sup_{w\in\mathbb{B}} \|f_{w,t}\|^{lux}_{A^{\Psi}_{\alpha}(\mathbb{B})} \lesssim 1.$$

The following lemma is well known (see, for example, Proposition 1.4.10 in [5]).

Lemma 2. *Let*

$$I_c(z) = \int_{\mathbb{S}} \frac{d\sigma(\zeta)}{|1-\langle z,\zeta\rangle|^{n+c}}$$

and

$$J_{c,t}(z) = \int_{\mathbb{B}} \frac{(1-|w|^2)^t dv(w)}{|1-\langle z,w\rangle|^{n+1+t+c}},$$

where $z \in \mathbb{B}$, $t > -1$ and $c \in \mathbb{R}$. If $c > 0$, then the following asymptotic relations hold

$$I_c(z) \asymp \frac{1}{(1-|z|^2)^c} \asymp J_{c,t}(z).$$

The following lemma was proved in [58].

Lemma 3. *Assume that $a > 0$ and*

$$D_n(a) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & a+1 & \cdots & a+n-1 \\ a(a+1) & (a+1)(a+2) & \cdots & (a+n-1)(a+n) \\ & & \cdots & \\ \prod_{j=0}^{n-2}(a+j) & \prod_{j=0}^{n-2}(a+j+1) & \cdots & \prod_{j=0}^{n-2}(a+j+n-1) \end{vmatrix}.$$

Then

$$D_n(a) = \prod_{j=1}^{n-1} j!.$$

The following lemma gives an important family of test functions, which is used in the proofs of the main results in the paper.

Lemma 4. *Let $\alpha \geq -1$ and $\Psi \in \mathfrak{U}^q \cup \mathfrak{L}_p$, $m \in \mathbb{N}$, $w \in \mathbb{B}$ and $C > 0$. Then for each $s \in \{0,1,\ldots,m\}$ there exist $c^{(s)}_j$, $j = \overline{0,m}$, such that the function $g^{(s)}_w(z) = \sum_{k=0}^m c^{(s)}_k f_{w,k}(z)$ satisfies the conditions*

$$D_{l_s}\cdots D_{l_1} g^{(s)}_w(w) = \frac{\overline{w}_{l_1}\overline{w}_{l_2}\cdots\overline{w}_{l_s}}{(1-|w|^2)^s}\Psi^{-1}\left(\frac{C}{(1-|w|^2)^{n+1+\alpha}}\right) \tag{5}$$

and

$$D_{l_t} \cdots D_{l_1} g_w^{(s)}(w) = 0, \quad t \in \{0, 1, \dots, m\} \setminus \{s\}. \tag{6}$$

Besides,

$$\sup_{w \in \mathbb{B}} \|g_w^{(s)}\|^{lux}_{A^{\Psi}_{\alpha}(\mathbb{B})} \lesssim 1. \tag{7}$$

Proof. Let

$$g_w(z) = \sum_{k=0}^{m} y_k f_{w,k}(z)$$

and $a_k = 2(n+1+\alpha) + k$, $k \in \mathbb{N}_0$. It is easy to see that for each $t \in \mathbb{N}_0$ we have

$$D_{l_t} \cdots D_{l_1} g_w(w) = \frac{\overline{w}_{l_1}\overline{w}_{l_2}\cdots\overline{w}_{l_t}}{(1-|w|^2)^t}\Psi^{-1}\left(\frac{C}{(1-|w|^2)^{n+1+\alpha}}\right)\sum_{k=0}^{m} y_k \prod_{l=0}^{t-1} a_{k+l}. \tag{8}$$

Lemma 3 shows that the determinant of the system

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_m \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{k=0}^{s-1} a_k & \prod_{k=0}^{s-1} a_{k+1} & \cdots & \prod_{k=0}^{s-1} a_{k+m} \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{k=0}^{m-1} a_k & \prod_{k=0}^{m-1} a_{k+1} & \cdots & \prod_{k=0}^{m-1} a_{k+m} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_s \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \tag{9}$$

is not equal to zero.

Therefore, for each $s \in \{0, 1, \dots, m\}$, there is a unique solution

$$y_k := c_k^{(s)}, \quad k = \overline{0, m}$$

to system (9).

Then, the function

$$g_w^{(s)}(z) := \sum_{k=0}^{m} c_k^{(s)} f_{w,k}(z)$$

satisfies (5) and (6), whereas the relation in (7) follows from the relations

$$\sup_{w \in \mathbb{B}} \|f_{w,t}\|^{lux}_{A^{\Psi}_{\alpha}(\mathbb{B})} \lesssim 1, \quad t = \overline{0, m},$$

which are direct consequences of Lemma 1. □

The folowing lemma is a Schwartz-type characterization for the compactness [57]. The proof is standard, so we do not present it here.

Lemma 5. *Let $\alpha \geq -1$, $\Psi \in \mathfrak{U}^q \cup \mathfrak{L}_p$, $u_j \in H(\mathbb{B})$, $j = \overline{0, m}$, $\varphi \in S(\mathbb{B})$ and $\omega \in W(\mathbb{B})$. Then the metrically bounded operator $P_{D,m} : A^{\Psi}_{\alpha}(\mathbb{B}) \to H^{\infty}_{\omega}(\mathbb{B})$ is metrically compact if and only if for any bounded sequence $(f_k)_{k \in \mathbb{N}} \subset A^{\Psi}_{\alpha}(\mathbb{B})$ such that $f_k \to 0$ uniformly on compacts of $\mathbb{B}$ as $k \to +\infty$,*

$$\lim_{k \to +\infty} \|P_{D,m} f_k\|_{H^{\infty}_{\omega}} = 0.$$

The following lemma is a known extension of Lemma 1 in [56] (see, for example, [29]).

Lemma 6. *A closed set K in $H^\infty_{\omega,0}$ is compact if and only if it is bounded and*

$$\lim_{|z|\to 1}\sup_{f\in K}\omega(z)|f(z)| = 0.$$

Lemma 7. *Let $\alpha \geq -1$, $\Psi \in \mathfrak{U}^q \cup \mathfrak{L}_p$ and $N \in \mathbb{N}_0$. Then for any $\vec{l} = (l_1, l_2, \dots, l_j)$ such that $|\vec{l}| = N$, there are $C_{\vec{l}} > 0$ and $\widehat{C}_{\vec{l}} > 0$ such that*

$$\left|\frac{\partial^N f(z)}{\partial z_{k_1}^{l_1}\partial z_{k_2}^{l_2}\cdots\partial z_{k_j}^{l_j}}\right| \leq \frac{\widehat{C}_{\vec{l}}}{(1-|z|^2)^N}\Psi^{-1}\left(\frac{C_{\vec{l}}}{(1-|z|^2)^{n+1+\alpha}}\right)\|f\|^{lux}_{A^\Psi_\alpha(\mathbb{B})}, \tag{10}$$

for $f \in A^\Psi_\alpha(\mathbb{B})$ and $z \in \mathbb{B}$.

Proof. The estimate (10) in the case $\alpha > -1$ was proved in [30]. Hence, from now on we consider only the case $\alpha = -1$.

Suppose $\Psi \in \mathfrak{U}^q$. Then the space $H^\Psi(\mathbb{B})$ embeds into $H^1(\mathbb{B})$ continuously (see Lemma 2.1 in [44]). Hence

$$f(z) = \int_{\mathbb{S}}\frac{f^*(\zeta)d\sigma(\zeta)}{(1-\langle z,\zeta\rangle)^n}, \tag{11}$$

for every $f \in H^\Psi(\mathbb{B})$ and $z \in \mathbb{B}$, where f^* is the K-limit [5] (limit in the Koranyi domain).

By differentiating both sides of the relation in (11) we get

$$\frac{\partial^N f(z)}{\partial z_{k_1}^{l_1}\partial z_{k_2}^{l_2}\cdots\partial z_{k_j}^{l_j}} = \frac{\Gamma(n+N)}{\Gamma(n)}\int_{\mathbb{S}}\frac{\overline{\zeta}_{k_1}^{l_1}\overline{\zeta}_{k_2}^{l_2}\cdots\overline{\zeta}_{k_j}^{l_j}f^*(\zeta)}{(1-\langle z,\zeta\rangle)^{n+N}}d\sigma(\zeta). \tag{12}$$

Suppose

$$\int_{\mathbb{S}}\Psi\left(\frac{|f^*(\zeta)|}{\lambda}\right)d\sigma(\zeta) \leq 1, \tag{13}$$

for a $\lambda > 0$.

From (12), it follows that

$$\frac{(1-|z|^2)^N}{\lambda}\left|\frac{\partial^N f(z)}{\partial z_{k_1}^{l_1}\partial z_{k_2}^{l_2}\cdots\partial z_{k_j}^{l_j}}\right| \leq \frac{\Gamma(n+N)}{\Gamma(n)}\int_{\mathbb{S}}\frac{|f^*(\zeta)|}{\lambda}\frac{(1-|z|^2)^N}{|1-\langle z,\zeta\rangle|^{n+N}}d\sigma(\zeta). \tag{14}$$

Lemma 2 implies the finiteness of the measure

$$d\sigma_1(\zeta) := \frac{\Gamma(n+N)}{\Gamma(n)}\frac{(1-|z|^2)^N}{(1-\langle z,\zeta\rangle)^{n+N}}d\sigma(\zeta),$$

on $\mathbb{S}$. Note that the measure

$$\mu(\zeta) := \frac{d\sigma_1(\zeta)}{\sigma_1(\mathbb{S})}$$

is normalized (i.e., probability).

From (14), the condition $\Psi(st) \leq Ct^q\Psi(s)$, for $t \geq 1$ and $s > 0$, the monotonicity and convexity of Ψ, Jensen's inequality, and (13), we have

$$\Psi\left(\frac{(1-|z|^2)^N}{\lambda}\left|\frac{\partial^N f(z)}{\partial z_{k_1}^{l_1}\partial z_{k_2}^{l_2}\cdots\partial z_{k_j}^{l_j}}\right|\right)$$

$$
\begin{aligned}
&\leq C(\max\{1,\sigma_1(\mathbb{S})\})^q \frac{\Gamma(n+N)}{\sigma_1(\mathbb{B})\Gamma(n)} \int_{\mathbb{S}} \Psi\left(\frac{|f^*(\zeta)|}{\lambda}\right) \frac{(1-|z|^2)^N}{|1-\langle z,\zeta\rangle|^{n+N}} d\sigma(\zeta) \\
&\leq \frac{\widehat{C}}{(1-|z|^2)^n} \int_{\mathbb{S}} \Psi\left(\frac{|f^*(\zeta)|}{\lambda}\right) d\sigma(\zeta) \\
&\leq \frac{\widehat{C}}{(1-|z|^2)^n},
\end{aligned}
$$

for $z \in \mathbb{B}$, where

$$\widehat{C} = C2^{N+n}(\max\{1,\sigma_1(\mathbb{S})\})^q \frac{\Gamma(n+N)}{\sigma_1(\mathbb{S})\Gamma(n)},$$

and consequently by letting $\lambda \to \|f\|^{lux}_{H^\Psi(\mathbb{B})}$ we get

$$\left|\frac{\partial^N f(z)}{\partial z_{k_1}^{l_1} \partial z_{k_2}^{l_2} \cdots \partial z_{k_j}^{l_j}}\right| \leq \frac{1}{(1-|z|^2)^N} \Psi^{-1}\left(\frac{\widehat{C}}{(1-|z|^2)^n}\right) \|f\|^{lux}_{H^\Psi(\mathbb{B})}. \tag{15}$$

Now suppose that $\Psi \in \mathfrak{L}_s$, for an $s \in (0,1]$. If $p \in (0,1)$ is small enough, then $\Psi_{1/p}(t) := \Psi(t^{\frac{1}{p}})$ is convex [46].

Let

$$f_r(z) = f(rz), \quad z \in \mathbb{B},$$

where $r > 0$. Then, $f_r \in H^\infty(\mathbb{B})$, which implies $f_r \in A^1_\beta(\mathbb{B})$.

By a known theorem (see, for example, Theorem 2.2 in [59]), for each $\beta > -1$ we have

$$f_r(z) = \int_{\mathbb{B}} \frac{f_r(w)}{(1-\langle z,w\rangle)^{n+1+\beta}} dv_\beta(w).$$

Differentiating both sides of the last equality it follows that

$$r^N \frac{\partial^N f(rz)}{\partial z_{k_1}^{l_1} \partial z_{k_2}^{l_2} \cdots \partial z_{k_j}^{l_j}} = \frac{\Gamma(n+N+\beta+1)}{\Gamma(n+\beta+1)} \int_{\mathbb{B}} \frac{\overline{w}_{k_1}^{l_1} \overline{w}_{k_2}^{l_2} \cdots \overline{w}_{k_j}^{l_j} f_r(w)}{(1-\langle z,w\rangle)^{n+N+1+\beta}} dv_\beta(w),$$

and consequently

$$\left|r^N \frac{\partial^N f(rz)}{\partial z_{k_1}^{l_1} \partial z_{k_2}^{l_2} \cdots \partial z_{k_j}^{l_j}}\right| \leq \frac{\Gamma(n+N+\beta+1)}{\Gamma(n+\beta+1)} \int_{\mathbb{B}} \frac{|f_r(w)|}{|1-\langle z,w\rangle|^{n+N+1+\beta}} dv_\beta(w). \tag{16}$$

Let $\beta := \frac{n}{p} - n - 1$. Since $p \in (0,1)$, we have $\beta > -1$. From (16) and by Corollary 4.49 in [59], we have

$$\left|r^N \frac{\partial^N f(rz)}{\partial z_{k_1}^{l_1} \partial z_{k_2}^{l_2} \cdots \partial z_{k_j}^{l_j}}\right|^p \lesssim \int_{\mathbb{S}} \left|\frac{f_r(\zeta)}{(1-\langle z,\zeta\rangle)^{n+N+1+\beta}}\right|^p d\sigma(\zeta). \tag{17}$$

Then from (17) and the fact $(n+N+1+\beta)p = Np+n$, we have

$$\left(\frac{(1-|z|^2)^N}{\lambda}\left|r^N \frac{\partial^N f(rz)}{\partial z_{k_1}^{l_1} \partial z_{k_2}^{l_2} \cdots \partial z_{k_j}^{l_j}}\right|\right)^p \lesssim \int_{\mathbb{S}} \left(\frac{|f_r(\zeta)|}{\lambda}\right)^p \frac{(1-|z|^2)^{Np} d\sigma(\zeta)}{|1-\langle z,\zeta\rangle|^{Np+n}}. \tag{18}$$

Lemma 2 shows that

$$d\sigma_2(\zeta) := \frac{(1-|z|^2)^{Np}}{|1-\langle z,\zeta\rangle|^{Np+n}} d\sigma(\zeta)$$

is a finite measure, so that $d\sigma_2(\zeta)/\sigma_2(\mathbb{S})$ is a probability measure.

The monotonicity and convexity of the function $\Psi_{1/p}$, the fact $\Psi_{1/p} \in \mathfrak{U}^{1/ps}$, (18) and Jensen's inequality imply

$$\Psi_{1/p}\left(\left(\frac{(1-|z|^2)^N}{\lambda}\left|r^N \frac{\partial^N f(rz)}{\partial z_{k_1}^{l_1} \partial z_{k_2}^{l_2} \cdots \partial z_{k_j}^{l_j}}\right|\right)^p\right)$$
$$\lesssim \int_{\mathbb{S}} \Psi_{1/p}\left(\left(\frac{|f_r(\zeta)|}{\lambda}\right)^p\right) \frac{(1-|z|^2)^{Np}}{|1-\langle z,\zeta\rangle|^{Np+n}} d\sigma(\zeta). \tag{19}$$

From (19) we have

$$\Psi\left(r^N \frac{(1-|z|^2)^N}{\lambda}\left|\frac{\partial^N f(rz)}{\partial z_{k_1}^{l_1} \partial z_{k_2}^{l_2} \cdots \partial z_{k_j}^{l_j}}\right|\right) \leq \frac{\widetilde{C}}{(1-|z|^2)^n} \int_{\mathbb{S}} \Psi\left(\frac{|f(r\zeta)|}{\lambda}\right) d\sigma(\zeta), \tag{20}$$

for some $\widetilde{C} > 0$.

Since $\Psi_{1/p}$ is convex and increasing, and $|f|^p$ is subharmonic [5], the function $\Psi(c|f|)$ is subharmonic for each $c > 0$. Hence

$$M(f,r) := \int_{\mathbb{S}} \Psi\left(\frac{|f(r\zeta)|}{\lambda}\right) d\sigma(\zeta)$$

is nondecreasing in r (see, e.g., [60]). Thus, we have

$$\int_{\mathbb{S}} \Psi\left(\frac{|f(r\zeta)|}{\lambda}\right) d\sigma(\zeta) \leq \int_{\mathbb{S}} \Psi\left(\frac{|f^*(\zeta)|}{\lambda}\right) d\sigma(\zeta),$$

for $r \in (0,1)$.

From this and letting $r \to 1^-$ in (20) it follows that

$$\Psi\left(\frac{(1-|z|^2)^N}{\lambda}\left|\frac{\partial^N f(z)}{\partial z_{k_1}^{l_1} \partial z_{k_2}^{l_2} \cdots \partial z_{k_j}^{l_j}}\right|\right) \leq \frac{\widetilde{C}}{(1-|z|^2)^n} \int_{\mathbb{S}} \Psi\left(\frac{|f^*(\zeta)|}{\lambda}\right) d\sigma(\zeta). \tag{21}$$

Letting $\lambda \to \|f\|^{lux}_{H^\Psi(\mathbb{B})}$ it easily follows that

$$\left|\frac{\partial^N f(z)}{\partial z_{k_1}^{l_1} \partial z_{k_2}^{l_2} \cdots \partial z_{k_j}^{l_j}}\right| \leq \frac{1}{(1-|z|^2)^N} \Psi^{-1}\left(\frac{\widetilde{C}}{(1-|z|^2)^n}\right) \|f\|^{lux}_{H^\Psi(\mathbb{B})}. \tag{22}$$

From (15) and (22), estimate (10) follows for $C_{\vec{l}} := \max\{\widehat{C}, \widetilde{C}\}$. □

3. Main Results

Here we present our results on the metrical boundedness and compactness. Before we state our first theorem say that if $\varphi \in S(\mathbb{B})$, then we regard that $\varphi = (\varphi_1, \dots, \varphi_n)$.

Theorem 1. *Let $\alpha \geq -1$, $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, $\varphi \in S(\mathbb{B})$,*

$$\min_{j=\overline{1,n}} \inf_{z\in\mathbb{B}} |\varphi_j(z)| \geq \delta > 0, \tag{23}$$

$\Psi \in \mathfrak{U}^q \cup \mathfrak{L}_p$, and $\omega \in W(\mathbb{B})$. Then the following statements hold.

(a) The operator $P_{D,m} : A^\Psi_\alpha(\mathbb{B}) \to H^\infty_\omega(\mathbb{B})$ is metrically bounded if and only if

$$K_j := \sup_{z\in\mathbb{B}} \frac{\omega(z)|u_j(z)|}{(1-|\varphi(z)|^2)^j} \Psi^{-1}\left(\frac{\widetilde{C}_m}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right) < +\infty, \quad j = \overline{0,m}, \tag{24}$$

where

$$\widetilde{C}_m := \max\{C_{\vec{l}} : \vec{l} = (l_1, l_2, \ldots, l_j) \text{ such that } |\vec{l}| \le m\}, \tag{25}$$

and $C_{\vec{l}}$ is a constant in Lemma 7.

(b) *If the operator $P_{D,m} : A^{\Psi}_{\alpha}(\mathbb{B}) \to H^{\infty}_{\omega}(\mathbb{B})$ is metrically bounded, then*

$$\|P_{D,m}\|_{A^{\Psi}_{\alpha}(\mathbb{B}) \to H^{\infty}_{\omega}(\mathbb{B})} \asymp \sum_{j=0}^{m} K_j. \tag{26}$$

Proof.

(a) If (24) holds, then Lemma 7 implies

$$\begin{aligned} \omega(z)|P_{D,m}f(z)| =& \omega(z)\left|\sum_{j=0}^{m} u_j(z) D_{l_j} \cdots D_{l_1} f(\varphi(z))\right| \\ \le& C\sum_{j=0}^{m} \frac{\omega(z)|u_j(z)|}{(1-|\varphi(z)|^2)^j} \Psi^{-1}\left(\frac{\widetilde{C}_m}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right) \|f\|^{lux}_{A^{\Psi}_{\alpha}(\mathbb{B})}. \end{aligned} \tag{27}$$

From (24) and (27), the metrical boundedness of $P_{D,m} : A^{\Psi}_{\alpha}(\mathbb{B}) \to H^{\infty}_{\omega}(\mathbb{B})$ follows, and we have

$$\|P_{D,m}\|_{A^{\Psi}_{\alpha}(\mathbb{B}) \to H^{\infty}_{\omega}(\mathbb{B})} \lesssim \sum_{j=0}^{m} K_j. \tag{28}$$

If $P_{D,m} : A^{\Psi}_{\alpha}(\mathbb{B}) \to H^{\infty}_{\omega}(\mathbb{B})$ is metrically bounded, then $\|P_{D,m}f\|_{H^{\infty}_{\omega}(\mathbb{B})} \le C\|f\|^{lux}_{A^{\Psi}_{\alpha}(\mathbb{B})}$, for some $C \ge 0$ and every $f \in A^{\Psi}_{\alpha}(\mathbb{B})$.

For each $s \in \{0, 1, \ldots, m\}$, $\varphi(w) \in \mathbb{B}$, and $C = \widetilde{C}_m$, Lemma 4 guaranty the existence of $g^{(s)}_{\varphi(w)} \in A^{\Psi}_{\alpha}(\mathbb{B})$ such that

$$D_{l_s} \cdots D_{l_1} g^{(s)}_{\varphi(w)}(\varphi(w)) = \frac{\overline{\varphi_{l_1}(w)\varphi_{l_2}(w) \cdots \varphi_{l_s}(w)}}{(1-|\varphi(w)|^2)^s} \Psi^{-1}\left(\frac{\widetilde{C}_m}{(1-|\varphi(w)|^2)^{n+1+\alpha}}\right), \tag{29}$$

$$D_{l_t} \cdots D_{l_1} g^{(s)}_{\varphi(w)}(\varphi(w)) = 0, \quad t \in \{0, 1, \ldots, m\} \setminus \{s\}, \tag{30}$$

and $\sup_{w \in \mathbb{B}} \|g^{(s)}_{\varphi(w)}\|^{lux}_{A^{\Psi}_{\alpha}(\mathbb{B})} \lesssim 1$.

This fact, (23), (29) and (30), yield

$$\begin{aligned} \|P_{D,m}\|_{A^{\Psi}_{\alpha}(\mathbb{B}) \to H^{\infty}_{\omega}(\mathbb{B})} \gtrsim& \|P_{D,m} g_{\varphi(w)}\|_{H^{\infty}_{\omega}(\mathbb{B})} \\ =& \sup_{z \in \mathbb{B}} \omega(z)\left|\sum_{j=0}^{m} u_j(z) D_{l_j} \cdots D_{l_1} g^{(s)}_{\varphi(w)}(\varphi(z))\right| \\ \ge& \omega(w)\left|\sum_{j=0}^{m} u_j(w) D_{l_j} \cdots D_{l_1} g^{(s)}_{\varphi(w)}(\varphi(w))\right| \\ =& \omega(w)|u_s(w)| \frac{\overline{|\varphi_{l_1}(w)| \cdots |\varphi_{l_s}(w)|}}{(1-|\varphi(w)|^2)^s} \Psi^{-1}\left(\frac{\widetilde{C}_m}{(1-|\varphi(w)|^2)^{n+1+\alpha}}\right) \\ \ge& \delta^s \frac{\omega(w)|u_s(w)|}{(1-|\varphi(w)|^2)^s} \Psi^{-1}\left(\frac{\widetilde{C}_m}{(1-|\varphi(w)|^2)^{n+1+\alpha}}\right), \end{aligned} \tag{31}$$

for every $w \in \mathbb{B}$, from which it follows that $K_s < +\infty$, for $s \in \{0, 1, \ldots, m\}$, and

$$K_s \lesssim \|P_{D,m}\|_{A^{\Psi}_{\alpha}(\mathbb{B}) \to H^{\infty}_{\omega}(\mathbb{B})}, \quad s = \overline{0, m},$$

which yields

$$\sum_{j=0}^{m} K_j \lesssim \|P_{D,m}\|_{A_\alpha^\Psi(\mathbb{B})\to H_\omega^\infty(\mathbb{B})}. \tag{32}$$

(*b*) If $P_{D,m} : A_\alpha^\Psi(\mathbb{B}) \to H_\omega^\infty(\mathbb{B})$ is metrically bounded, then relations (28) and (32) imply (26).

□

Theorem 2. *Let $\alpha \geq -1$, $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, $\varphi \in S(\mathbb{B})$, $\Psi \in \mathfrak{U}^q \cup \mathfrak{L}_p$, $\omega \in W(\mathbb{B})$, and $\mathcal{P}_n$ be dense in $A_\alpha^\Psi(\mathbb{B})$. Then $P_{D,m} : A_\alpha^\Psi(\mathbb{B}) \to H_{\omega,0}^\infty(\mathbb{B})$ is metrically bounded if and only if $P_{D,m} : A_\alpha^\Psi(\mathbb{B}) \to H_\omega^\infty(\mathbb{B})$ is metrically bounded and*

$$\lim_{|z|\to 1} \omega(z)|u_j(z)| = 0, \quad j = \overline{0,m}. \tag{33}$$

Proof. Suppose $P_{D,m} : A_\alpha^\Psi(\mathbb{B}) \to H_\omega^\infty(\mathbb{B})$ is metrically bounded and (33) holds. For each polynomial p, we have

$$\omega(z)\left|\sum_{j=0}^{m} u_j(z) D_{l_j}\cdots D_{l_1} p(\varphi(z))\right| \leq \sum_{j=0}^{m} \omega(z)|u_j(z)|\|D_{l_j}\cdots D_{l_1} p\|_\infty,$$

from which along with (33), we have $P_{D,m}p \in H_{\omega,0}^\infty(\mathbb{B})$.

Since $\overline{\mathcal{P}_n} = A_\alpha^\Psi(\mathbb{B})$, we have that for any $f \in A_\alpha^\Psi(\mathbb{B})$ there is $(p_k)_{k\in\mathbb{N}} \subset \mathcal{P}_n$ such that

$$\lim_{k\to+\infty} \|f - p_k\|_{A_\alpha^\Psi(\mathbb{B})} = 0.$$

So, from the metrical boundedness we have

$$\|P_{D,m}f - P_{D,m}p_k\|_{H_\omega^\infty(\mathbb{B})} \leq \|P_{D,m}\|_{A_\alpha^\Psi(\mathbb{B})\to H_\omega^\infty(\mathbb{B})}\|f - p_k\|_{A_\alpha^\Psi(\mathbb{B})} \to 0$$

as $k \to +\infty$, from which together with the fact that $H_{\omega,0}^\infty(\mathbb{B})$ is a closed subspace of $H_\omega^\infty(\mathbb{B})$, it follows that $P_{D,m}f \in H_{\omega,0}^\infty(\mathbb{B})$, that is, $P_{D,m}(A_\alpha^\Psi(\mathbb{B})) \subseteq H_{\omega,0}^\infty(\mathbb{B})$, which implies the metrical boundedness of the operator $P_{D,m} : A_\alpha^\Psi(\mathbb{B}) \to H_{\omega,0}^\infty(\mathbb{B})$.

If $P_{D,m} : A_\alpha^\Psi(\mathbb{B}) \to H_{\omega,0}^\infty(\mathbb{B})$ is metrically bounded, then $P_{D,m} : A_\alpha^\Psi(\mathbb{B}) \to H_\omega^\infty(\mathbb{B})$ is also metrically bounded. Let $f_0(z) \equiv 1$. Then $f_0 \in A_\alpha^\Psi(\mathbb{B})$, implying $P_{D,m}(f_0) \in H_{\omega,0}^\infty(\mathbb{B})$, that is, $u_0 \in H_{\omega,0}^\infty$.

Suppose that (33) holds for $0 \leq j \leq s$, for some s, $2 \leq s < m$. Let

$$f_{s+1}(z) = z_{l_1} z_{l_2} \cdots z_{l_{s+1}}.$$

Then $f_{s+1} \in A_\alpha^\Psi(\mathbb{B})$, implying $P_{D,m}(f_{s+1}) \in H_{\omega,0}^\infty(\mathbb{B})$. It is easy to see that $f_{s+1}(z) = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, for some $\alpha_j \in \mathbb{N}_0$, $j = \overline{1,n}$, such that $\sum_{j=1}^n \alpha_j = s+1$. For each $t \in \mathbb{N}_0$, $0 \leq t \leq s+1$, we have

$$D_{j_t}\cdots D_{j_1} f_{s+1}(z) = c_t z_1^{\alpha_1 - k_1(t)} \cdots z_n^{\alpha_n - k_n(t)},$$

for some $c_t \in \mathbb{N}$, where $k_i(t)$ is the number of appearance the operators D_i in the product operator $D_{j_t}\cdots D_{j_1}$. We have $\sum_{j=1}^n k_i(t) = t$ and

$$D_{j_{s+1}}\cdots D_{j_1} f_{s+1}(z) \equiv c_{s+1} \in \mathbb{N}. \tag{34}$$

Thus

$$\lim_{|z|\to 1} \omega(z)|P_{D,m}f_{s+1}(z)| = \lim_{|z|\to 1} \omega(z)\left|\sum_{j=0}^{s+1} u_j(z) c_j \prod_{i=1}^{n} (\varphi_i(z))^{\alpha_i - k_i(j)}\right| = 0,$$

from which, the fact $|\varphi_i(z)| < 1$, $i = \overline{1,n}$, $\alpha_i \geq k_i(j)$, for $i = \overline{1,n}$, $j = \overline{0,s+1}$, the hypothesis $u_j \in H^\infty_{\omega,0}$, $j = \overline{0,s}$, and (34), we have

$$\lim_{|z|\to 1} c_{s+1}\omega(z)|u_{s+1}(z)| = 0.$$

This along with $c_{s+1} \neq 0$, imply $u_{s+1} \in H^\infty_{\omega,0}$. Thus, (33) holds for $j = \overline{0,m}$. □

Theorem 3. *Let $\alpha \geq -1$, $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, $\varphi \in S(\mathbb{B})$, $\Psi \in \mathfrak{U}^q \cup \mathfrak{L}_p$, $\omega \in W(\mathbb{B})$, and (23) holds. Then, $P_{D,m} : A^\Psi_\alpha(\mathbb{B}) \to H^\infty_\omega(\mathbb{B})$ is metrically compact if and only if the operator is metrically bounded and*

$$\lim_{|\varphi(z)|\to 1} \frac{\omega(z)|u_j(z)|}{(1-|\varphi(z)|^2)^j}\Psi^{-1}\left(\frac{\widetilde{C}_m}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right) = 0, \quad j = \overline{0,m}, \tag{35}$$

where $\widetilde{C}_m$ is defined in (25).

Proof. If $P_{D,m} : A^\Psi_\alpha(\mathbb{B}) \to H^\infty_\omega(\mathbb{B})$ is metrically compact, then it is metrically bounded. If $\|\varphi\|_\infty < 1$, then (35) vacuously holds. If $\|\varphi\|_\infty = 1$, then there is $(z_k)_{k\in\mathbb{N}} \subset \mathbb{B}$ such that $|\varphi(z_k)| \to 1$ as $k \to +\infty$.

Let

$$g^{(s)}_k := g^{(s)}_{\varphi(z_k)}, \quad s = \overline{0,m},$$

(see Lemma 4). Then $\sup_{k\in\mathbb{N}} \|g^{(s)}_k\|^{lux}_{A^\Psi_\alpha(\mathbb{B})} < +\infty$, $s = \overline{0,m}$. If $\Psi \in \mathfrak{U}^q$ or $\Psi \in \mathfrak{L}_p$, then as in [30] (Theorem 2), we get $g^{(s)}_k \to 0$ uniformly on compacts of $\mathbb{B}$ as $k \to +\infty$, $s = \overline{0,m}$. Lemma 5 implies

$$\lim_{k\to+\infty} \|P_{D,m}g^{(s)}_k\|_{H^\infty_\omega(\mathbb{B})} = 0, \quad s = \overline{0,m}. \tag{36}$$

From the proof of Theorem 1, we see that for $s = \overline{0,m}$ and large enough k

$$\frac{\omega(z_k)|u_s(z_k)|}{(1-|\varphi(z_k)|^2)^s}\Psi^{-1}\left(\frac{\widetilde{C}_m}{(1-|\varphi(z_k)|^2)^{n+1+\alpha}}\right) \lesssim \|P_{D,m}g^{(s)}_k\|_{H^\infty_\omega(\mathbb{B})}. \tag{37}$$

From (36) and (37), relation (35) follows.

If $P_{D,m} : A^\Psi_\alpha(\mathbb{B}) \to H^\infty_\omega(\mathbb{B})$ is metrically bounded and (35) holds, then for each $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\frac{\omega(z)|u_j(z)|}{(1-|\varphi(z)|^2)^j}\Psi^{-1}\left(\frac{\widetilde{C}_m}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right) < \varepsilon, \quad j = \overline{0,m}, \tag{38}$$

on $\mathcal{S}_\delta := \{z \in \mathbb{B} : |\varphi(z)| > \delta\}$.

Suppose $(f_k)_{k\in\mathbb{N}}$ is such that $\sup_{k\in\mathbb{N}} \|f_k\|^{lux}_{A^\Psi_\alpha(\mathbb{B})} \leq M$ and $f_k \to 0$ uniformly on compacts of $\mathbb{B}$ as $k \to +\infty$. Then Lemma 7 and (38) imply

$$\begin{aligned}\|P_{D,m}f_k\|_{H^\infty_\omega(\mathbb{B})} &= \sup_{z\in\mathbb{B}} \omega(z)\Big|\sum_{j=0}^m u_j(z)D_{l_j}\cdots D_{l_1}f_k(\varphi(z))\Big| \\ &= \sup_{z\in\mathcal{S}_\delta} \omega(z)\Big|\sum_{j=0}^m u_j(z)D_{l_j}\cdots D_{l_1}f_k(\varphi(z))\Big| \\ &\quad + \sup_{z\in\mathbb{B}\setminus\mathcal{S}_\delta} \omega(z)\Big|\sum_{j=0}^m u_j(z)D_{l_j}\cdots D_{l_1}f_k(\varphi(z))\Big|\end{aligned}$$

$$\lesssim \sum_{j=0}^{m} \sup_{z\in\mathcal{S}_\delta} \frac{\omega(z)|u_j(z)|}{(1-|\varphi(z)|^2)^j} \Psi^{-1}\left(\frac{\widetilde{C}_m}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right) \|f_k\|^{lux}_{A^\Psi_\alpha(\mathbb{B})}$$
$$+ \sum_{j=0}^{m} \sup_{z\in\mathbb{B}\setminus\mathcal{S}_\delta} \omega(z)|u_j(z)||D_{l_j}\cdots D_{l_1} f_k(\varphi(z))|$$
$$\lesssim \varepsilon + \sum_{j=0}^{m} \sup_{z\in\mathbb{B}\setminus\mathcal{S}_\delta} \omega(z)|u_j(z)| \sup_{|\varphi(z)|\le\delta} |D_{l_j}\cdots D_{l_1} f_k(\varphi(z))|$$
$$\lesssim \varepsilon + \sum_{j=0}^{m} \|u_j\|_{H^\infty_\omega} \sup_{|\varphi(z)|\le\delta} |D_{l_j}\cdots D_{l_1} f_k(\varphi(z))|. \tag{39}$$

The assumption $f_k \to 0$ and the Cauchy estimate, imply

$$D_{l_j}\cdots D_{l_1} f_k \to 0, \quad j = \overline{0,m}, \tag{40}$$

uniformly on compacts of $\mathbb{B}$ as $k \to +\infty$.

Employing the functions $f_s(z) = \prod_{j=1}^{s} z_{l_j}$, $s = \overline{0,m}$, as in Theorem 2 we get $u_j \in H^\infty_\omega$, $j = \overline{0,m}$. From this, (39) and (40), the compactness of the ball $\delta\overline{\mathbb{B}}$, and the arbitrariness of $\varepsilon > 0$, we obtain

$$\lim_{k\to+\infty} \|P_{D,m} f_k\|_{H^\infty_\omega(\mathbb{B})} = 0,$$

from which by Lemma 5, the metrical compactness of $P_{D,m} : A^\Psi_\alpha(\mathbb{B}) \to H^\infty_\omega(\mathbb{B})$ follows. □

Theorem 4. *Let $\alpha \ge -1$, $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, $\varphi \in S(\mathbb{B})$, $\Psi \in \mathfrak{U}^q \cup \mathfrak{L}_p$, $\omega \in W(\mathbb{B})$, and (23) holds. Then, $P_{D,m} : A^\Psi_\alpha(\mathbb{B}) \to H^\infty_{\omega,0}(\mathbb{B})$ is metrically compact if and only if the operator is metrically bounded and*

$$\lim_{|z|\to1} \frac{\omega(z)|u_j(z)|}{(1-|\varphi(z)|^2)^j} \Psi^{-1}\left(\frac{\widetilde{C}_m}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right) = 0, \quad j = \overline{0,m}, \tag{41}$$

where $\widetilde{C}_m$ is defined in (25).

Proof. If (41) holds, then the relations in (24) also hold, so by Theorem 1 the metrical boundedness of the operator $P_{D,m} : A^\Psi_\alpha(\mathbb{B}) \to H^\infty_\omega(\mathbb{B})$ follows. From (27) and (41), we have

$$\lim_{|z|\to1} \omega(z)|P_{D,m} f(z)| = 0$$

for any $f \in A^\Psi_\alpha(\mathbb{B})$. Thus $P_{D,m}(A^\Psi_\alpha(\mathbb{B})) \subset H^\infty_{\omega,0}(\mathbb{B})$, implying the metrical boundedness of $P_{D,m} : A^\Psi_\alpha(\mathbb{B}) \to H^\infty_{\omega,0}(\mathbb{B})$.

Taking the supremum in (27) over $\mathbb{B}$ and $B_{A^\Psi_\alpha(\mathbb{B})}$ and using (24), we have

$$\sup_{f\in B_{A^\Psi_\alpha(\mathbb{B})}} \sup_{z\in\mathbb{B}} \omega(z)|P_{D,m} f(z)| \le C \sum_{j=0}^{m} K_j < +\infty. \tag{42}$$

Thus $K := \{P_{D,m} f : f \in B_{A^\Psi_\alpha(\mathbb{B})}\}$ is a bounded set in $H^\infty_{\omega,0}$. So, from (27) we have

$$\lim_{|z|\to1} \sup_{f\in B_{A^\Psi_\alpha(\mathbb{B})}} \omega(z)|P_{D,m} f(z)| = 0.$$

From this and by Lemma 6 the metrical compactness of $P_{D,m} : A^\Psi_\alpha(\mathbb{B}) \to H^\infty_{\omega,0}(\mathbb{B})$ follows.

If $P_{D,m} : A^{\Psi}_{\alpha}(\mathbb{B}) \to H^{\infty}_{\omega,0}(\mathbb{B})$ is metrically compact, then $P_{D,m} : A^{\Psi}_{\alpha}(\mathbb{B}) \to H^{\infty}_{\omega}(\mathbb{B})$ is metrically compact, from which and Theorem 3 we have that (38) holds. We also have

$$\lim_{|z|\to 1} \omega(z)|u_j(z)| = 0, \quad j = \overline{0,m},$$

so that there exist $\sigma \in (0,1)$ such that

$$\omega(z)|u_j(z)| < \varepsilon \frac{(1-\delta^2)^j}{\Psi^{-1}\Big(\frac{\widetilde{C}_m}{(1-\delta^2)^{n+1+\alpha}}\Big)}, \quad j = \overline{0,m}, \tag{43}$$

for $\sigma < |z| < 1$, where ε is from (38).

Relation (43) implies

$$\begin{aligned} &\frac{\omega(z)|u_j(z)|}{(1-|\varphi(z)|^2)^j}\Psi^{-1}\left(\frac{\widetilde{C}_m}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right) \\ \le &\frac{\omega(z)|u_j(z)|}{(1-\delta^2)^j}\Psi^{-1}\left(\frac{\widetilde{C}_m}{(1-\delta^2)^{n+1+\alpha}}\right) < \varepsilon, \end{aligned} \tag{44}$$

for $j = \overline{0,m}$, $|\varphi(z)| \le \delta$ and $\sigma < |z| < 1$.

Employing (38) and (44), we easily obtain (41). □

4. Conclusions

Here we characterize the metrical boundedness and metrical compactness of a recently introduced linear operator from the weighted Bergman-Orlicz spaces to the weighted-type spaces and little weighted-type spaces of analytic functions on the open unit ball in $\mathbb{C}^n$, continuing some of our previous investigations in the topic. We managed to estimate the point evaluation functional on the weighted Bergman-Orlicz spaces, which along with several other results enabled obtaining the characterizations. The methods, ideas and tricks in the paper should be useful for continuing the investigation of this, as well as related operators on spaces of holomorphic functions.

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