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Barotropic-Baroclinic Coherent-Structure Rossby Waves in Two-Layer Cylindrical Fluids

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Abstract: In this paper, the propagation of Rossby waves under barotropic-baroclinic interaction in polar co-ordinates is studied. By starting from the two-layer quasi-geotropic potential vorticity equation (of equal depth) with the β effect, the coupled KdV equations describing barotropic-baroclinic waves are derived using multi-scale analysis and the perturbation expansion method. Furthermore, in order to more accurately describe the propagation characteristics of barotropic-baroclinic waves, fifth-order coupled KdV-mKdV equations were obtained for the first time. On this basis, the Lie symmetry and conservation laws of the fifth-order coupled KdV-mKdV equations are analyzed in terms of their properties. Then, the elliptic function expansion method is applied to find the soliton solutions of the fifth-order coupled KdV-mKdV equations. Based on the solutions, we further simulate the evolution of Rossby wave amplitudes and investigate the influence of the high-order terms—time and wave number—on the propagation of barotropic waves and baroclinic waves. The results show that the appearance of the higher-order effect makes the amplitude of the wave lower, the width of the wave larger, and the whole wave flatter, which is obviously closer to actual Rossby wave propagation. The time and wave number will also influence wave amplitude and wave width.

Keywords: rossby waves; fifth-order coupled KdV-mKdV equations; barotropic-baroclinic coherent structures; two-layer cylindrical fluids

MSC: 35B20; 35C07; 35C08; 35Q53; 35Q86

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1. Introduction

Due to the movement of the atmosphere and the ocean, various nonlinear fluctuations are generated, such as Rossby waves, ocean internal waves, gravity waves, and so on. Among them, the stable Rossby waves generated by the ocean atmosphere are the most classical waves and have always been studied by people. Rossby waves are closely related to various natural phenomena, such as weather changes and ocean currents [1]. For example, the Gulf Stream, Kuroshio, and El *Niño* phenomenon in the ocean [2–4], atmospheric blockages, the zonal winds in climate change [5], the red dots in the atmosphere of Jupiter, etc. [6]. In recent years, Rossby waves have been discovered in sundial highlights [7], which has greatly aroused interest in Rossby waves in the solar atmosphere. Therefore, the study of Rossby waves still has great theoretical significance and potential application value.

Nonlinear equations can be used to describe various nonlinear fluctuations. Typically, the classical solitary wave theory is used to characterize the Korteweg-de Vries (KdV) equation, and this has long been used for the KdV equation to describe Rossby waves in a single-layer barotropic fluid; this laid a solid foundation for future research [8]. Next, many scholars tried to describe the solitary waves by using different equations, such as the modified Korteweg-de Vries (mKdV) equation, the Zakharov-Kuznetsov (ZK) equation, the Kadomtsev-Petviashvili(KP) equation, etc. [9–11], which led to the rapid development of classical solitary wave theory. Luo studied nonlinear baroclinic Rossby waves based on the KdV equation and the mKdV equation, and the relationship between the baroclinic Rossby wave and the baroclinic elliptic cosine wave was also obtained [12]. Both barotropic waves and baroclinic waves may cause a change in sea surface height and weather. In order to judge and predict the complex and changeable atmosphere or ocean more accurately, baroclinic waves need to be considered, so many multilayer models for studying wave propagation have been established. In this paper, we mainly consider the two-layer model.

Pedlosky first established the quasi-geostrophic vorticity conservation equation in two-layer fluid motion and discussed the influence of physical factors, such as the β effect and baroclinic instability on wave propagation [13]. Based on the study of Pedlosky, Steinsaltz continued to consider the influence of topography on wave height [14,15]. Lou obtained the exact solution and found the existence of multiple soliton solutions for this equation based on the above equation [16,17]. The coupled nonlinear mKdV equations and other coupled equations were used to describe Rossby waves, which led to the further development of solitary wave theory [18]. It is noted that the above equations are all based on the original model of upper-lower coherent structures. The influence of baroclinic waves can not be ignored in the two-layer model. Luo analyzed this and stated that a barotropic wave and two barotropic waves would have a three-wave resonance, and noted the total energy conservation in the interaction process by introducing the barotropic wave flow function into the above quasi-geostroic vorticity conservation equation [19]. Zhang established the coupled KdV equations for barotropic-baroclinic coherent structures by using this method and analyzed the influence of various physical factors on the amplitude of the baroclinic flow function [20,21]. As is known to all, the higher-order models can describe various nonlinear phenomena more accurately under the same conditions. In this paper, a high-order coupling equation is deduced based on the above-mentioned two-layer quasi-geostrophic model using a barotropic and baroclinic coherent structure, which is unprecedented as far as we know. All the above models are based on the zonal area, which obviously ignores the objective factor that the Earth is a constantly rotating sphere. Therefore, the establishment of a barotropic-baroclinic coherent structure model under a rotating fluid in this paper is extremely reasonable and novel.

In recent years, scholars have pay more and more attention to the study of higher-order models. Ichikawa studied the role of higher-order terms in the simplified perturbation method [22,23]. Ito proposed the extension of KdV (mKdV)-type nonlinear equations to several higher-order methods and obtained multi-soliton solutions for higher-order equations [24]. Grimshaw derived the higher-order KdV equation to describe internal solitary waves, noting that all the coefficients of the equation were in the integral form and were clearly obtained according to the parameters of the model [25]. Aly and Biswas conducted much work on the analytical solutions of nonlinear partial differential equations [26,27]. However, most problems in the past have described wave propagation in terms of a single higher-order equation or a system of lower-order equations, and systems of higher-order equations have rarely been addressed. In this paper, the propagation of Rossby waves with a barotropic-barotropic coherent structure in the polar co-ordinate system will be described by higher-order KdV equations.

The study of nonlinear problems has concerned many scholars. One of the most important tasks is to find the solutions to nonlinear equations (systems). The common solution methods are the exp function method, the homogeneous balance method, the Jacobi elliptic function expansion method, Darboux transformation, etc. [28–31]. Lie symmetry and conservation laws also occupy an important place in the study of nonlinear equations [32–34]. Lie symmetry plays an important role in obtaining the solution of the equation and can transform the known solution into other solutions, which is of great significance for solving more complex partial differential problems. The nature of conservation laws is derived from symmetry. Noether's theorem states that there is some important correspondence between Lie symmetry and conservation laws; that is, every symmetry corresponds to a conservation law. Conversely, for every conservation law, there must be a symmetry. The physical properties of the equation can be better described by conservation laws, and conservation laws are equally important to the integrity of the equation and the linearization of nonlinear problems.

In this paper, the amplitude evolution of Rossby waves with barotropic-baroclinic coherent structures is studied by deriving the fifth-order coupled KdV-mKdV equations in polar co-ordinates. In Section 2, a quasi-geotropic potential vorticity equation with barotropic-baroclinic interaction in a polar co-ordinate system is derived by introducing the barotropic flow function and the baroclinic flow function. In the following, based on the method of scale analysis and perturbation expansion, the coupled fifth-order KdV-mKdV equations for describing barotropic and barotropic Rossby waves in polar co-ordinates are derived for the first time. In Section 3, the Lie symmetry and conservation laws of the above newly deduced equations are discussed. In Section 4, the soliton and periodic solutions of the newly obtained equation are obtained, combining the Jacobian elliptic function expansion method. In Section 5, we discuss the influence of the higher-order effect, time, and wave number on the amplitude of barotropic and baroclinic Rosby waves. In addition, we also discuss the changes in barotropic wave amplitude and baroclinic wave amplitude with time. Finally, the conclusions of this paper are given.

2. Establishment of the Coherent Structure Model and the Fifth-Order Coupled KdV-mKdV Equations in Polar Co-Ordinates

In the past, the models have basically concentrated on the single-layer positive pressure quasi-geostrophic model. In recent years, many scholars have paid more attention to the two-layer or multilayer model to study the propagation and evolution of Rossby waves. It should be noted that baroclinic fluid also has important research value for the ocean atmosphere, so it is necessary to study the barotropic and baroclinic interaction model. Next, we will establish a two-layer quasi-geostrophic model with a barotropic and baroclinic coherent structure in rotating fluids.

2.1. The Coherent Structure Model

In this paper, the two-layer quasi-geotropic potential vorticity equations (of equal depth) under the β effect in polar co-ordinates are considered as

$$\begin{cases} \left[\frac{\partial}{\partial t}+\left(\frac{1}{r}\frac{\partial\psi_1}{\partial r}\frac{\partial}{\partial\theta}-\frac{1}{r}\frac{\partial\psi_1}{\partial\theta}\frac{\partial}{\partial r}\right)\right]\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_1}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2}+F(\psi_2-\psi_1)\right]+\frac{\beta}{r}\frac{\partial\psi_1}{\partial\theta}=0, \\ \left[\frac{\partial}{\partial t}+\left(\frac{1}{r}\frac{\partial\psi_2}{\partial r}\frac{\partial}{\partial\theta}-\frac{1}{r}\frac{\partial\psi_2}{\partial\theta}\frac{\partial}{\partial r}\right)\right]\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_2}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi_2}{\partial\theta^2}+F(\psi_1-\psi_2)\right]+\frac{\beta}{r}\frac{\partial\psi_2}{\partial\theta}=0, \end{cases} \tag{1}$$

with β effect

$$\beta=-\frac{df}{dz}=-\frac{2\omega}{a}\cos\varphi_0, \tag{2}$$

where a represents the Earth's radius, ω is the rotational angular velocity of the Earth, and φ_0 represents latitude, respectively. In polar co-ordinates (r,θ), r pointing to the lower dimension is positive and θ counterclockwise is positive. ψ_1 and ψ_2 represent the flow function of the upper and lower fluids, respectively. F is the weak coupling coefficient between two layers of the fluid.

The boundary conditions of Equation (1) are

$$\psi_1(r)=\psi_2(r)=0, r=r_1,r_2. \tag{3}$$

In order to have a better understanding of the nonlinear barotropic-baroclinic coherent structure, the barotropic flow function $\overline{\psi}_B$ and the baroclinic flow function $\overline{\psi}_T$ are here introduced. The specific forms of $\overline{\psi}_B$ and $\overline{\psi}_T$ are as follows:

$$\begin{cases} \overline{\psi}_B=\frac{1}{2}(\psi_1+\psi_2), \\ \overline{\psi}_T=\psi_1-\psi_2. \end{cases} \tag{4}$$

Thus, the new representation of the upper and lower laminar flow functions $\psi_i (i = 1, 2)$ is as follows:

$$\begin{cases} \psi_1 = \overline{\psi}_B + \dfrac{1}{2}\overline{\psi}_T, \\ \psi_2 = \overline{\psi}_B - \dfrac{1}{2}\overline{\psi}_T. \end{cases} \tag{5}$$

Upon substituting Equation (5) into Equation (1), we obtain

$$\begin{cases} \Big(\dfrac{1}{r}\dfrac{\partial^2}{\partial t\partial r} + \dfrac{\partial^3}{\partial t\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^3}{\partial t\partial \theta^2} + \dfrac{\beta}{r}\dfrac{\partial}{\partial \theta}\Big)\overline{\psi}_B + \dfrac{1}{r}\dfrac{\partial}{\partial r}\Big(\dfrac{1}{r}\dfrac{\partial^2}{\partial \theta\partial r} + \dfrac{\partial^3}{\partial \theta\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^3}{\partial \theta^3}\Big) \\ \overline{\psi}_B^2 - \dfrac{1}{r}\dfrac{\partial}{\partial \theta}\Big(-\dfrac{1}{r^2}\dfrac{\partial}{\partial r} + \dfrac{1}{r}\dfrac{\partial^2}{\partial r^2} + \dfrac{\partial^3}{\partial r^3} - \dfrac{2}{r^3}\dfrac{\partial^2}{\partial \theta^2} + \dfrac{1}{r^2}\dfrac{\partial^3}{\partial \theta^2\partial r}\Big)\overline{\psi}_B^2 + \dfrac{1}{4r}\dfrac{\partial}{\partial r}\Big(\dfrac{1}{r}\dfrac{\partial^2}{\partial \theta\partial r} \\ + \dfrac{\partial^3}{\partial \theta\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^3}{\partial \theta^3} - 2F\dfrac{\partial}{\partial \theta}\Big)\overline{\psi}_T^2 - \dfrac{1}{4r}\dfrac{\partial}{\partial \theta}\Big(-\dfrac{1}{r^2}\dfrac{\partial}{\partial r} + \dfrac{1}{r}\dfrac{\partial^2}{\partial r^2} + \dfrac{\partial^3}{\partial r^3} - \dfrac{2}{r^3}\dfrac{\partial^2}{\partial \theta^2} + \\ \dfrac{1}{r^2}\dfrac{\partial^3}{\partial \theta^2\partial r} - 2F\dfrac{\partial}{\partial r}\Big)\overline{\psi}_T^2 = 0, \\ \Big(\dfrac{1}{r}\dfrac{\partial^2}{\partial t\partial r} + \dfrac{\partial^3}{\partial t\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^3}{\partial t\partial \theta^2} - 2F\dfrac{\partial}{\partial t} + \dfrac{\beta}{r}\dfrac{\partial}{\partial \theta}\Big)\overline{\psi}_T + \Big[\dfrac{1}{r}\dfrac{\partial\overline{\psi}_B}{\partial r}\Big(\dfrac{1}{r}\dfrac{\partial^2}{\partial \theta\partial r} + \dfrac{\partial^3}{\partial \theta\partial r^2} \\ + \dfrac{1}{r^2}\dfrac{\partial^3}{\partial \theta^3} - 2F\dfrac{\partial}{\partial \theta}\Big) - \dfrac{1}{r}\dfrac{\partial\overline{\psi}_B}{\partial \theta}\Big(-\dfrac{1}{r^2}\dfrac{\partial}{\partial r} + \dfrac{1}{r}\dfrac{\partial^2}{\partial r^2} + \dfrac{\partial^3}{\partial r^3} - \dfrac{2}{r^3}\dfrac{\partial^2}{\partial \theta^2} + \dfrac{1}{r^2}\dfrac{\partial^3}{\partial \theta^2\partial r} - \\ 2F\dfrac{\partial}{\partial r}\Big)\Big]\overline{\psi}_T + \Big[\dfrac{1}{r}\dfrac{\partial\overline{\psi}_T}{\partial r}\Big(\dfrac{1}{r}\dfrac{\partial^2}{\partial \theta\partial r} + \dfrac{\partial^3}{\partial \theta\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^3}{\partial \theta^3}\Big) - \dfrac{1}{r}\dfrac{\partial\overline{\psi}_T}{\partial \theta}\Big(-\dfrac{1}{r^2}\dfrac{\partial}{\partial r} + \dfrac{1}{r}\dfrac{\partial^2}{\partial r^2} + \\ \dfrac{\partial^3}{\partial r^3} - \dfrac{2}{r^3}\dfrac{\partial^2}{\partial \theta^2} + \dfrac{1}{r^2}\dfrac{\partial^3}{\partial \theta^2\partial r}\Big)\Big]\overline{\psi}_B = 0. \end{cases} \tag{6}$$

When compared with previous models, this model can describe Rossby waves more accurately, which is more suitable for the actual complex and changeable marine environment. Based on Equation (6), we will deduce the fifth-order coupled KdV-mKdV equations with barotropic-baroclinic coherent structures in the polar co-ordinate system, and we consider the effect of higher-order terms.

2.2. Derivation of the Third-Order Coupled KdV Equations

We suppose the total flow function satisfies

$$\begin{cases} \overline{\psi}_B = \varphi_{B_0}(r) + \epsilon\varphi_B(r, \theta, t), \\ \overline{\psi}_T = \epsilon\varphi_{T_0}(t) + \epsilon\varphi_T(r, \theta, t), \end{cases} \tag{7}$$

where ϵ is a small parameter, φ_{B_0} and φ_{T_0} are functions of r only. In particular, φ_{B_0} represents the basic background fluid of the barotropic flow, and φ_{T_0} represents the basic background fluid of the baroclinic flow. φ_B and φ_T are the perturbed flow functions, describing barotropic and baroclinic fluids.

The substitution of Equation (7) into Equation (6) gives

$$
\begin{cases}
(\frac{1}{r}\frac{\partial^2}{\partial t\partial r} + \frac{\partial^3}{\partial t\partial r^2} + \frac{1}{r^2}\frac{\partial^3}{\partial t\partial\theta^2} + \frac{\beta}{r}\frac{\partial}{\partial\theta})\varphi_B + \frac{1}{r}(\frac{\partial\varphi_{B_0}}{\partial r} + \epsilon\frac{\partial\varphi_B}{\partial r})(\frac{1}{r}\frac{\partial^2}{\partial\theta\partial r} + \\
\frac{\partial^3}{\partial\theta\partial r^2} + \frac{1}{r^2}\frac{\partial^3}{\partial\theta^3})\varphi_B + \frac{1}{4r}\epsilon(\frac{\partial\varphi_{T_0}}{\partial r} + \frac{\partial\varphi_T}{\partial r})(\frac{1}{r}\frac{\partial^2}{\partial\theta\partial r} + \frac{\partial^3}{\partial\theta\partial r^2} + \frac{1}{r^2}\frac{\partial^3}{\partial\theta^3} - \\
2F\frac{\partial}{\partial\theta})\varphi_T - \frac{1}{r}\frac{\partial\varphi_B}{\partial\theta}[-\frac{1}{r^2}(\frac{\partial\varphi_{B_0}}{\partial r} + \epsilon\frac{\partial\varphi_B}{\partial r}) + \frac{1}{r}(\frac{\partial^2\varphi_{B_0}}{\partial r^2} + \epsilon\frac{\partial^2\varphi_B}{\partial r^2}) + \\
(\frac{\partial^3\varphi_{B_0}}{\partial r^3} + \epsilon\frac{\partial^3\varphi_B}{\partial r^3}) - \epsilon\frac{2}{r^3}\frac{\partial^2\varphi_B}{\partial\theta^2} + \epsilon\frac{1}{r^2}\frac{\partial}{\partial r}(\frac{\partial^2\varphi_B}{\partial\theta^2})] - \frac{1}{4r}\epsilon\frac{\partial\varphi_T}{\partial\theta}[-\frac{1}{r^2}(\frac{\partial\varphi_{T_0}}{\partial r} + \\
\frac{\partial\varphi_T}{\partial r}) + \frac{1}{r}(\frac{\partial^2\varphi_{T_0}}{\partial r^2} + \frac{\partial^2\varphi_T}{\partial r^2}) + (\frac{\partial^3\varphi_{T_0}}{\partial r^3} + \frac{\partial^3\varphi_T}{\partial r^3}) - \frac{2}{r^3}\frac{\partial^2\varphi_T}{\partial\theta^2} + \frac{1}{r^2}\frac{\partial}{\partial r}(\frac{\partial^2\varphi_T}{\partial\theta^2}) - \\
2F(\frac{\partial\varphi_{T_0}}{\partial r} + \epsilon\frac{\partial\varphi_T}{\partial r})] = 0, \\
(\frac{1}{r}\frac{\partial^2}{\partial t\partial r} + \frac{\partial^3}{\partial t\partial r^2} + \frac{1}{r^2}\frac{\partial^3}{\partial t\partial\theta^2} + \frac{\beta}{r}\frac{\partial}{\partial\theta} - 2F\frac{\partial}{\partial t})\varphi_T + \frac{1}{r}(\frac{\partial\varphi_{B_0}}{\partial r} + \epsilon\frac{\partial\varphi_B}{\partial r}) \\
(\frac{1}{r}\frac{\partial^2}{\partial\theta\partial r} + \frac{\partial^3}{\partial\theta\partial r^2} + \frac{1}{r^2}\frac{\partial^3}{\partial\theta^3} - 2F\frac{\partial}{\partial\theta})\varphi_T + \frac{1}{r}\epsilon(\frac{\partial\varphi_{T_0}}{\partial r} + \epsilon\frac{\partial\varphi_T}{\partial r})(\frac{1}{r}\frac{\partial^2}{\partial\theta\partial r} + \\
\frac{\partial^3}{\partial\theta\partial r^2} + \frac{1}{r^2}\frac{\partial^3}{\partial\theta^3})\varphi_B - \frac{1}{r}\epsilon\frac{\partial\varphi_B}{\partial\theta}[-\frac{1}{r^2}(\frac{\partial\varphi_{T_0}}{\partial r} + \frac{\partial\varphi_T}{\partial r}) + \frac{1}{r}(\frac{\partial^2\varphi_{T_0}}{\partial r^2} + \frac{\partial^2\varphi_T}{\partial r^2}) + \\
(\frac{\partial^3\varphi_{T_0}}{\partial r^3} + \frac{\partial^3\varphi_T}{\partial r^3}) - \frac{2}{r^3}\frac{\partial^2\varphi_T}{\partial\theta^2} + \frac{1}{r^2}\frac{\partial}{\partial r}(\frac{\partial^2\varphi_T}{\partial\theta^2}) - 2F(\frac{\partial\varphi_{T_0}}{\partial r} + \epsilon\frac{\partial\varphi_T}{\partial r})] - \frac{1}{r}\frac{\partial\varphi_T}{\partial\theta} \\
[-\frac{1}{r^2}(\frac{\partial\varphi_{B_0}}{\partial r} + \epsilon\frac{\partial\varphi_B}{\partial r}) + \frac{1}{r}(\frac{\partial^2\varphi_{B_0}}{\partial r^2} + \epsilon\frac{\partial^2\varphi_B}{\partial r^2}) + (\frac{\partial^3\varphi_{B_0}}{\partial r^3} + \epsilon\frac{\partial^3\varphi_B}{\partial r^3}) - \\
\epsilon\frac{2}{r^3}\frac{\partial^2\varphi_B}{\partial\theta^2} + \epsilon\frac{1}{r^2}\frac{\partial}{\partial r}(\frac{\partial^2\varphi_B}{\partial\theta^2})] = 0.
\end{cases} \tag{8}
$$

Generally, the nonlinear equation cannot be treated analytically. Therefore, the multi-scale analysis and the perturbation expansion method are applicable. Firstly, we introduce the following slow stretch co-ordinates:

$$\xi = \epsilon(\theta - ct), \;\; \tau_1 = \epsilon^3 t, \;\; \tau_2 = \epsilon^4 t, \;\; \tau_3 = \epsilon^5 t, \;\; \ldots \tag{9}$$

where $\xi, \tau_1, \tau_2, \tau_3, \ldots$ are the slow stretch co-ordinates, and c is a constant.

Hence,

$$\frac{\partial}{\partial\theta} = \epsilon\frac{\partial}{\partial\xi}, \;\; \frac{\partial}{\partial t} = -c\epsilon\frac{\partial}{\partial\xi} + \epsilon^3\frac{\partial}{\partial\tau_1} + \epsilon^4\frac{\partial}{\partial\tau_2} + \epsilon^5\frac{\partial}{\partial\tau_3} + \ldots. \tag{10}$$

Next, we introduce the following perturbation expansions:

$$
\begin{cases}
\varphi_B = \epsilon\varphi_{B_1} + \epsilon^2\varphi_{B_2} + \epsilon^3\varphi_{B_3} + \ldots, \\
\varphi_T = \epsilon\varphi_{T_1} + \epsilon^2\varphi_{T_2} + \epsilon^3\varphi_{T_3} + \ldots,
\end{cases} \tag{11}
$$

When substituting Equations (10) and (11) into Equation (8) and collecting the same order of ϵ , the system of equations are obtained as follows:

$$
\epsilon^2 : \begin{cases}
L_1\varphi_{B_1} = 0, \\
L_2\varphi_{T_1} = 0,
\end{cases} \tag{12}
$$

where

$$\begin{cases} L_1 = -\frac{c}{r}\frac{\partial^2}{\partial\xi\partial r} - c\frac{\partial^3}{\partial\xi\partial r^2} + \frac{1}{r^2}\frac{\partial\varphi_{B_0}}{\partial r}\frac{\partial^2}{\partial\xi\partial r} + \frac{1}{r}\frac{\partial\varphi_{B_0}}{\partial r}\frac{\partial^3}{\partial\xi\partial r^2} - \frac{1}{r}\frac{\partial}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{B_0}}{\partial r} \\ \quad + \frac{1}{r}\frac{\partial^2\varphi_{B_0}}{\partial r^2} + \frac{\partial^3\varphi_{B_0}}{\partial r^3}] + \frac{\beta}{r}\frac{\partial}{\partial\xi}, \\ L_2 = -\frac{c}{r}\frac{\partial^2}{\partial\xi\partial r} - c\frac{\partial^3}{\partial\xi\partial r^2} + \frac{1}{r^2}\frac{\partial\varphi_{B_0}}{\partial r}\frac{\partial^2}{\partial\xi\partial r} + \frac{1}{r}\frac{\partial\varphi_{B_0}}{\partial r}\frac{\partial^3}{\partial\xi\partial r^2} - \frac{1}{r}\frac{\partial}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{B_0}}{\partial r} \\ \quad + \frac{1}{r}\frac{\partial^2\varphi_{B_0}}{\partial r^2} + \frac{\partial^3\varphi_{B_0}}{\partial r^3}] + 2cF\frac{\partial}{\partial\xi} - \frac{2}{r}\frac{\partial\varphi_{B_0}}{\partial r}F\frac{\partial}{\partial\xi} + \frac{\beta}{r}\frac{\partial}{\partial\xi}. \end{cases} \tag{13}$$

$$\epsilon^3: \begin{cases} L_1\varphi_{B_2} = -\frac{1}{2r}\frac{\partial\varphi_{T_0}}{\partial r}[\frac{1}{2r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{T_1}}{\partial r}) + \frac{1}{2}\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{T_1}}{\partial r^2}) - F\frac{\partial\varphi_{T_1}}{\partial\xi}] + \frac{1}{2r}\frac{\partial\varphi_{T_1}}{\partial\xi}[\\ \qquad -\frac{1}{2r^2}\frac{\partial\varphi_{T_0}}{\partial r} + \frac{1}{2r}\frac{\partial^2\varphi_{T_0}}{\partial r^2} + \frac{1}{2}\frac{\partial^3\varphi_{T_0}}{\partial r^3} - F\frac{\partial\varphi_{T_0}}{\partial r}], \\ L_2\varphi_{T_2} = \frac{1}{r}\frac{\partial\varphi_{T_0}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{B_1}}{\partial r}) + \frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{B_1}}{\partial r^2})] + \frac{1}{r}\frac{\partial\varphi_{B_1}}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{T_0}}{\partial r} + \\ \qquad \frac{1}{r}\frac{\partial^2\varphi_{T_0}}{\partial r^2} + \frac{\partial^3\varphi_{T_0}}{\partial r^3} - 2F\frac{\partial\varphi_{T_0}}{\partial r}], \end{cases} \tag{14}$$

$$\epsilon^4: \begin{cases} L_1\varphi_{B_3} = \frac{\partial}{\partial\tau_1}(-\frac{1}{r}\frac{\partial\varphi_{B_1}}{\partial r} - \frac{\partial^2\varphi_{B_1}}{\partial r^2}) + (\frac{c}{r^2} - \frac{1}{r^3}\frac{\partial\varphi_{B_0}}{\partial r})\frac{\partial^3\varphi_{B_1}}{\partial\xi^3} - \frac{1}{r}\frac{\partial\varphi_{B_1}}{\partial r}[\\ \qquad \frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{B_1}}{\partial r}) + \frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{B_1}}{\partial r^2})] + \frac{1}{r}\frac{\partial\varphi_{B_1}}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{B_1}}{\partial r} + \frac{1}{r}\frac{\partial^2\varphi_{B_1}}{\partial r^2} + \\ \qquad \frac{\partial^3\varphi_{B_1}}{\partial r^3}] - \frac{1}{4r}\frac{\partial\varphi_{T_0}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{T_2}}{\partial r}) + \frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{T_2}}{\partial r^2}) - 2F\frac{\partial\varphi_{T_2}}{\partial\xi}] - \\ \qquad \frac{1}{4r}\frac{\partial\varphi_{T_1}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{T_1}}{\partial r}) + \frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{T_1}}{\partial r^2}) - 2F\frac{\partial\varphi_{T_1}}{\partial\xi}] + \frac{1}{4r}\frac{\partial\varphi_{T_1}}{\partial\xi}[\\ \qquad -\frac{1}{r^2}\frac{\partial\varphi_{T_1}}{\partial r} + \frac{1}{r}\frac{\partial^2\varphi_{T_1}}{\partial r^2} + \frac{\partial^3\varphi_{T_1}}{\partial r^3} - 2F\frac{\partial\varphi_{T_1}}{\partial r}] + \frac{1}{4r}\frac{\partial\varphi_{T_2}}{\partial\xi}[\\ \qquad -\frac{1}{r^2}\frac{\partial\varphi_{T_0}}{\partial r} + \frac{1}{r}\frac{\partial^2\varphi_{T_0}}{\partial r^2} + \frac{\partial^3\varphi_{T_0}}{\partial r^3} - 2F\frac{\partial\varphi_{T_0}}{\partial r}], \\ L_2\varphi_{T_3} = \frac{\partial}{\partial\tau_1}(-\frac{1}{r}\frac{\partial\varphi_{T_1}}{\partial r} - \frac{\partial^2\varphi_{T_1}}{\partial r^2} + 2F\partial\varphi_{T_1}) + (\frac{c}{r^2} - \frac{1}{r^3}\frac{\partial\varphi_{B_0}}{\partial r})\frac{\partial^3\varphi_{T_1}}{\partial\xi^3} - \\ \qquad \frac{1}{r}\frac{\partial\varphi_{B_1}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{T_1}}{\partial r}) + \frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{T_1}}{\partial r^2})] + \frac{1}{r}\frac{\partial\varphi_{B_1}}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{T_1}}{\partial r} + \\ \qquad \frac{1}{r}\frac{\partial^2\varphi_{T_1}}{\partial r^2} + \frac{\partial^3\varphi_{T_1}}{\partial r^3} - 2F\frac{\partial\varphi_{T_1}}{\partial r}] + \frac{1}{r}\frac{\partial\varphi_{B_2}}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{T_0}}{\partial r} + \frac{1}{r}\frac{\partial^2\varphi_{T_0}}{\partial r^2} + \\ \qquad \frac{\partial^3\varphi_{T_0}}{\partial r^3}] - \frac{1}{r}\frac{\partial\varphi_{T_0}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{B_2}}{\partial r}) + \frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{B_2}}{\partial r^2})] + \frac{1}{r}\frac{\partial\varphi_{T_1}}{\partial\xi}[\\ \qquad -\frac{1}{r^2}\frac{\partial\varphi_{B_1}}{\partial r} + \frac{1}{r}\frac{\partial^2\varphi_{B_1}}{\partial r^2} + \frac{\partial^3\varphi_{B_1}}{\partial r^3}], \end{cases} \tag{15}$$

with the boundary conditions

$$\varphi_{B_1} = \varphi_{T_1} = \varphi_{B_2} = \varphi_{T_2} = \cdots = 0, r = r_1, r_2. \tag{16}$$

When considering the lowest order, we find that Equation (12) is governed by $\varphi_{B_1}, \varphi_{T_1}$. Thus, it is feasible to introduce the following separable solutions:

$$\begin{cases} \varphi_{B_1} = A_1(\xi,\tau)M_1(r), \\ \varphi_{T_1} = B_1(\xi,\tau)N_1(r), \end{cases} \tag{17}$$

where $M_1(r_1) = M_1(r_2) = N_1(r_1) = N_1(r_2) = 0$.

Substituting Equation (17) into Equation (12) gives

$$\begin{cases} \frac{\partial^2 M_1}{\partial r^2} + \frac{1}{r}\frac{\partial M_1}{\partial r} + [\beta - q(r)](\frac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}M_1 = 0, \\ \frac{\partial^2 N_1}{\partial r^2} + \frac{1}{r}\frac{\partial N_1}{\partial r} + [2cFr - 2F\frac{\partial \varphi_{B_0}}{\partial r} + \beta - q(r)](\frac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}N_1 = 0. \end{cases} \tag{18}$$

The solutions to Equation (14) are assumed to be

$$\begin{cases} \varphi_{B_2} = A_2(\xi,\tau)M_2(r), \\ \varphi_{T_2} = B_2(\xi,\tau)N_2(r), \end{cases} \tag{19}$$

where $M_2(r_1) = M_2(r_2) = N_2(r_1) = N_2(r_2) = 0$.

Upon substituting Equations (17) and (19) into Equation (14), we obtain

$$\begin{cases} A_2 = B_1, \\ B_2 = A_1, \end{cases} \tag{20}$$

and

$$\begin{cases} \frac{\partial^2 M_2}{\partial r^2} + \frac{1}{r}\frac{\partial M_2}{\partial r} + [\beta - q(r)](\frac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}M_2 + \frac{1}{4}\frac{\partial \varphi_{T_0}}{\partial r} \\ (\frac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}(\frac{1}{r}\frac{\partial N_1}{\partial r} + \frac{\partial^2 N_1}{\partial r^2} - F) - \frac{1}{4}N_1 p(r)(\frac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1} = 0, \\ \frac{\partial^2 N_2}{\partial r^2} + \frac{1}{r}\frac{\partial N_2}{\partial r} + [2cFr - 2F\frac{\partial \varphi_{B_0}}{\partial r} + \beta - q(r)](\frac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}N_1 + \\ \frac{\partial \varphi_{T_0}}{\partial r}(\frac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}(\frac{1}{r}\frac{\partial M_1}{\partial r} + \frac{\partial^2 M_1}{\partial r^2}) - M_1 p(r)(\frac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1} = 0, \end{cases} \tag{21}$$

where $p(r) = -\frac{1}{r^2}\frac{\partial \varphi_{T_0}}{\partial r} + \frac{1}{r}\frac{\partial^2 \varphi_{T_0}}{\partial r^2} + \frac{\partial^3 \varphi_{T_0}}{\partial r^3} - 2F\frac{\partial \varphi_{T_0}}{\partial r}$,$q(r) = -\frac{1}{r^2}\frac{\partial \varphi_{B_0}}{\partial r} + \frac{1}{r}\frac{\partial^2 \varphi_{B_0}}{\partial r^2} + \frac{\partial^3 \varphi_{B_0}}{\partial r^3}$.

Equation (8) combined with Equation (10) can be represented by the following form:

$$\begin{cases} L_1\frac{\partial \varphi_B}{\partial \xi} = F, \\ L_2\frac{\partial \varphi_T}{\partial \xi} = G, \end{cases} \tag{22}$$

(see Appendix A for details of F, G).

The compatibility conditions for problem (22) with the boundary conditions (16) are

$$\begin{cases} \int_{r_1}^{r_2} FM_1 dr = \int_{r_1}^{r_2} L_1 M_1 \frac{\partial \varphi_B}{\partial \xi} dr = 0, \\ \int_{r_1}^{r_2} GN_1 dr = \int_{r_1}^{r_2} L_2 N_1 \frac{\partial \varphi_T}{\partial \xi} dr = 0. \end{cases} \tag{23}$$

For a combination of the compatibility conditions (23) at $O(\epsilon^4)$, Equations (17) and (19) generate the following equation:

$$\begin{cases} \dfrac{\partial A_1}{\partial \tau_1} + \alpha_1 \dfrac{\partial A_1}{\partial \xi} + \alpha_2 A_1 \dfrac{\partial A_1}{\partial \xi} + \alpha_3 B_1 \dfrac{\partial B_1}{\partial \xi} + \alpha_4 \dfrac{\partial^3 A_1}{\partial \xi^3} = 0, \\ \dfrac{\partial B_1}{\partial \tau_1} + \beta_1 \dfrac{\partial B_1}{\partial \xi} + \beta_2 A_1 \dfrac{\partial B_1}{\partial \xi} + \beta_3 B_1 \dfrac{\partial A_1}{\partial \xi} + \beta_4 \dfrac{\partial^3 B_1}{\partial \xi^3} = 0. \end{cases} \quad (24)$$

Equation (24) is a KdV-like equation describing the amplitude evolution of Rossby waves with the barotropic-baroclinic interaction of a two-layer rotating fluid with equal depths. This is the first time that Rossby wave propagation under a baroclinic coherent structure has been considered in the polar co-ordinate system. When we consider the plane co-ordinate system, the equation can be found in Zhang (2020) [20]. It is worth noting that Equation (24) is in the moving co-ordinate system. By using Equation (10), we can get the following equations in the fixed co-ordinate system.

$$\begin{cases} \dfrac{\partial A_1}{\partial t} + c\dfrac{\partial A_1}{\partial \theta} + \epsilon^2(\alpha_1 \dfrac{\partial A_1}{\partial \theta} + \alpha_2 A_1 \dfrac{\partial A_1}{\partial \theta} + \alpha_3 B_1 \dfrac{\partial B_1}{\partial \theta} + \alpha_4 \dfrac{\partial^3 A_1}{\partial \theta^3}) = 0, \\ \dfrac{\partial B_1}{\partial t} + c\dfrac{\partial B_1}{\partial \theta} + \epsilon^2(\beta_1 \dfrac{\partial B_1}{\partial \theta} + \beta_2 A_1 \dfrac{\partial B_1}{\partial \theta} + \beta_3 B_1 \dfrac{\partial A_1}{\partial \theta} + \beta_4 \dfrac{\partial^3 B_1}{\partial \theta^3}) = 0. \end{cases} \quad (25)$$

Many scholars have found that all kinds of nonlinear physical phenomena can not be described by a single or a set of low-order equations in nature. Therefore, the study of higher-order equations and higher-order terms is necessary. We will derive higher-order equations in order to describe the propagation of barotropic and baroclinic Rossby waves more truthfully and more accurately in a two-layer rotating fluid.

2.3. Derivation of the Fifth-Order Coupled KdV-mKdV Equations

We further set the solutions of Equation (15) as

$$\begin{cases} \varphi_{B_3} = I_1(r)A_1 + I_2(r)A_1^2 + I_3(r)B_1^2 + I_4(r)\dfrac{\partial^2 A_1}{\partial \xi^2}, \\ \varphi_{T_3} = J_1(r)B_1 + J_2(r)A_1B_1 + J_3(r)\dfrac{\partial^2 B_1}{\partial \xi^2}. \end{cases} \quad (26)$$

where $I_1, I_2, I_3, I_4, J_1, J_2$ and J_3 should satisfy

$$\begin{cases} \dfrac{\partial^2 I_1}{\partial r^2} = -\dfrac{1}{r}\dfrac{\partial I_1}{\partial r} - [\beta - q(r)](\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1} I_1 + r(\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}[\alpha_1(\dfrac{1}{r}\dfrac{\partial M_1}{\partial r} + \\ \quad \dfrac{\partial^2 M_1}{\partial r^2}) - \dfrac{1}{4r}\dfrac{\partial \varphi_{T_0}}{\partial r}(\dfrac{1}{r}\dfrac{\partial N_2}{\partial r} + \dfrac{\partial^2 N_2}{\partial r^2} - 2FN_2) + \dfrac{1}{4r}N_2 p(r)], \\ \dfrac{\partial^2 I_2}{\partial r^2} = -\dfrac{1}{r}\dfrac{\partial I_2}{\partial r} - [\beta - q(r)](\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1} I_2 + \dfrac{1}{2}r(\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}[(\alpha_2 - \\ \quad \dfrac{1}{r}\dfrac{\partial M_1}{\partial r})(\dfrac{1}{r}\dfrac{\partial M_1}{\partial r} + \dfrac{\partial^2 M_1}{\partial r^2}) + \dfrac{1}{r}M_1(-\dfrac{1}{r^2}\dfrac{\partial M_1}{\partial r} + \dfrac{1}{r}\dfrac{\partial^2 M_1}{\partial r^2} + \dfrac{\partial^3 M_1}{\partial r^3})], \\ \dfrac{\partial^2 I_3}{\partial r^2} = -\dfrac{1}{r}\dfrac{\partial I_3}{\partial r} - [\beta - q(r)](\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1} I_3 + \dfrac{1}{2}r(\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}[\\ \quad \alpha_3(\dfrac{1}{r}\dfrac{\partial M_1}{\partial r} + \dfrac{\partial^2 M_1}{\partial r^2}) - \dfrac{1}{4r}\dfrac{\partial N_2}{\partial r}(\dfrac{1}{r}\dfrac{\partial N_1}{\partial r} + \dfrac{\partial^2 N_1}{\partial r^2} - 2FN_1) + \dfrac{1}{4r}N_1(- \\ \quad \dfrac{1}{r^2}\dfrac{\partial N_1}{\partial r} + \dfrac{1}{r}\dfrac{\partial^2 N_1}{\partial r^2} + \dfrac{\partial^3 N_1}{\partial r^3} - 2F\dfrac{\partial N_1}{\partial r})], \\ \dfrac{\partial^2 I_4}{\partial r^2} = -\dfrac{1}{r}\dfrac{\partial I_4}{\partial r} - [\beta - q(r)](\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1} I_4 + r(\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}[\alpha_4(\dfrac{1}{r}\dfrac{\partial M_1}{\partial r} + \\ \quad \dfrac{cM1}{r^2} - \dfrac{M_1}{r^3}\dfrac{\partial \varphi_{B_0}}{\partial r}], \end{cases} \quad (27)$$

and

$$\begin{cases} \frac{\partial^2 J_1}{\partial r^2} = -\frac{1}{r}\frac{\partial J_1}{\partial r} - [2cFr - 2F\frac{\partial \varphi_{B_0}}{\partial r} + \beta - q(r)](\frac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}J_1 + \\ \quad r(\frac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}[\beta_1(\frac{1}{r}\frac{\partial N_1}{\partial r} + \frac{\partial^2 N_1}{\partial r^2} - 2FN_1) - \frac{1}{r}\frac{\partial \varphi_{T_0}}{\partial r}(\frac{1}{r}\frac{\partial M_2}{\partial r} + \\ \quad \frac{\partial^2 M_2}{\partial r^2}) + \frac{1}{r}M_2 p(r)], \\ \frac{\partial^2 J_2}{\partial r^2} = -\frac{1}{r}\frac{\partial J_2}{\partial r} - [2cFr - 2F\frac{\partial \varphi_{B_0}}{\partial r} + \beta - q(r)](\frac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}J_2 + \\ \quad r(\frac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}[\beta_2(\frac{1}{r}\frac{\partial N_1}{\partial r} + \frac{\partial^2 N_1}{\partial r^2} - 2FN_1) - \frac{1}{r}\frac{\partial M_1}{\partial r}(\frac{1}{r}\frac{\partial N_1}{\partial r} + \\ \quad \frac{\partial^2 N_1}{\partial r^2} - 2FN_1) + \frac{1}{r}N_1(-\frac{1}{r^2}\frac{\partial M_1}{\partial r} + \frac{1}{r}\frac{\partial^2 M_1}{\partial r^2} + \frac{\partial^3 M_1}{\partial r^3}], \\ \frac{\partial^2 J_3}{\partial r^2} = -\frac{1}{r}\frac{\partial J_3}{\partial r} - [2cFr - 2F\frac{\partial \varphi_{B_0}}{\partial r} + \beta - q(r)](\frac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}J_3 + \\ \quad r(\frac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}[\beta_3(\frac{1}{r}\frac{\partial N_1}{\partial r} + \frac{\partial^2 N_1}{\partial r^2} - 2FN_1) - \frac{c}{r^2}N_1 - \frac{1}{r^3}\frac{\partial \varphi_{B_0}}{\partial r}N_1]. \end{cases} \tag{28}$$

In order to describe the propagation of barotropic and baroclinic Rossby waves more truthfully and more accurately, the next order of ϵ needs to be considered. Equation (8) at $O(\epsilon^5)$ can be expressed as follows:

$$\epsilon^5 : \begin{cases} L_1\varphi_{B_4} = F_{11}, \\ L_2\varphi_{T_4} = F_{21}, \end{cases} \tag{29}$$

(see Appendix A for details of F_{11}, F_{21}).

By substituting Equations (17), (19), and (26) into Equation (29), and by considering the compatibility conditions (23), we obtain

$$\begin{cases} \frac{\partial A_1}{\partial \tau_2} + \alpha_5\frac{\partial B_1}{\partial \xi} + \alpha_6\frac{\partial A_1 B_1}{\partial \xi} + \alpha_7\frac{\partial^3 B_1}{\partial \xi^3} = 0, \\ \frac{\partial B_1}{\partial \tau_2} + \beta_5\frac{\partial A_1}{\partial \xi} + \beta_6 A_1\frac{\partial A_1}{\partial \xi} + \beta_7 B_1\frac{\partial B_1}{\partial \xi} + \beta_8\frac{\partial^3 A_1}{\partial \xi^3} = 0. \end{cases} \tag{30}$$

The solutions to Equation (29) can be assumed as

$$\begin{cases} \varphi_{B_4} = I_5(r)B_1 + I_6(r)A_1B_1 + I_7(r)\frac{\partial^2 B_1}{\partial \xi^2}, \\ \varphi_{T_4} = J_4(r)A_1 + J_5(r)A_1^2 + J_6(r)B_1^2 + J_7(r)\frac{\partial^2 A_1}{\partial \xi^2}. \end{cases} \tag{31}$$

where $I_5, I_6, I_7, J_4, J_5, J_6$ and J_7 should satisfy

$$
\begin{cases}
\dfrac{\partial^2 I_5}{\partial r^2} = -\dfrac{1}{r}\dfrac{\partial I_5}{\partial r} - [\beta - q(r)](\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1} I_5 + r(\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}[\beta_1(\dfrac{1}{r}\dfrac{\partial M_2}{\partial r} + \\
\quad \dfrac{\partial^2 M_2}{\partial r^2}) + \alpha_5(\dfrac{1}{r}\dfrac{\partial M_1}{\partial r} + \dfrac{\partial^2 M_1}{\partial r^2}) - \dfrac{1}{4r}\dfrac{\partial \varphi_{T_0}}{\partial r}(\dfrac{1}{r}\dfrac{\partial J_1}{\partial r} + \dfrac{\partial^2 J_1}{\partial r^2} - 2FJ_1) + \\
\quad \dfrac{1}{4r} J_1 p(r)], \\
\dfrac{\partial^2 I_6}{\partial r^2} = -\dfrac{1}{r}\dfrac{\partial I_6}{\partial r} - [\beta - q(r)](\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1} I_6 + r(\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}[2(\beta_2 - \\
\quad \dfrac{1}{r}\dfrac{\partial M_1}{\partial r})(\dfrac{1}{r}\dfrac{\partial M_2}{\partial r} + \dfrac{\partial^2 M_2}{\partial r^2}) - \dfrac{1}{2r}\dfrac{\partial N_2}{\partial r}(\dfrac{1}{r}\dfrac{\partial N_1}{\partial r} + \dfrac{\partial^2 N_1}{\partial r^2} - 2FN_1) + \\
\quad \dfrac{2}{r} M_2(-\dfrac{1}{r^2}\dfrac{\partial M_1}{\partial r} + \dfrac{1}{r}\dfrac{\partial^2 M_1}{\partial r^2} + \dfrac{\partial^3 M_1}{\partial r^3}) + \dfrac{1}{2r} N_1(-\dfrac{1}{r^2}\dfrac{\partial N_2}{\partial r} + \dfrac{1}{r}\dfrac{\partial^2 N_2}{\partial r^2} + \\
\quad \dfrac{\partial^3 N_2 2}{\partial r^3} - F\dfrac{\partial N_2}{\partial r})] + r(\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}[\alpha_6(\dfrac{1}{r}\dfrac{\partial M_1}{\partial r} + \dfrac{\partial^2 M_1}{\partial r^2}) - \\
\quad \dfrac{1}{4r}\dfrac{\partial \varphi_{T_0}}{\partial r}(\dfrac{1}{r}\dfrac{\partial J_2}{\partial r} + \dfrac{\partial^2 J_2}{\partial r^2} - 2FJ_2) + \dfrac{1}{4r} J_2 p(r)], \\
\dfrac{\partial^2 I_7}{\partial r^2} = -\dfrac{1}{r}\dfrac{\partial I_7}{\partial r} - [\beta - q(r)](\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1} I_7 + r(\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}[\beta_4(\dfrac{1}{r}\dfrac{\partial M_2}{\partial r} + \\
\quad \dfrac{\partial^2 M_2}{\partial r^2}) + \alpha_7(\dfrac{1}{r}\dfrac{\partial M_1}{\partial r} + \dfrac{\partial^2 M_1}{\partial r^2}) - \dfrac{1}{4r}\dfrac{\partial \varphi_{T_0}}{\partial r}(\dfrac{1}{r}\dfrac{\partial J_3}{\partial r} + \dfrac{\partial^2 J_3}{\partial r^2} - 2FJ_3 + \\
\quad \dfrac{1}{r^2} N_1) + \dfrac{1}{4r} N_1(-\dfrac{1}{r^2}\dfrac{\partial N_1}{\partial r} + \dfrac{1}{r}\dfrac{\partial^2 N_1}{\partial r^2} + \dfrac{cM_2}{r^2} - \dfrac{M_2}{r^3}\dfrac{\partial \varphi_{T_0}}{\partial r} + \dfrac{1}{4r} J_3 p(r)],
\end{cases} \tag{32}
$$

and

$$
\begin{cases}
\dfrac{\partial^2 J_4}{\partial r^2} = -\dfrac{1}{r}\dfrac{\partial J_4}{\partial r} - [2cFr - 2F\dfrac{\partial \varphi_{B_0}}{\partial r} + \beta - q(r)](\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1} J_4 + \\
\quad r(\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}[\alpha_1(\dfrac{1}{r}\dfrac{\partial N_2}{\partial r} + \dfrac{\partial^2 N_2}{\partial r^2} - 2FN_2) + \beta_5(\dfrac{1}{r}\dfrac{\partial N_1}{\partial r} + \\
\quad \dfrac{\partial^2 N_1}{\partial r^2} - 2FN_1) - \dfrac{1}{r}\dfrac{\partial \varphi_{T_0}}{\partial r}(\dfrac{1}{r}\dfrac{\partial I_1}{\partial r} + \dfrac{\partial^2 I_1}{\partial r^2}) + \dfrac{1}{r} I_1 p(r)], \\
\dfrac{\partial^2 J_5}{\partial r^2} = -\dfrac{1}{r}\dfrac{\partial J_5}{\partial r} - [2cFr - 2F\dfrac{\partial \varphi_{B_0}}{\partial r} + \beta - q(r)](\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1} J_5 + \\
\quad \dfrac{1}{2} r(\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}[(\alpha_2 - \dfrac{1}{r}\dfrac{\partial M_1}{\partial r})(\dfrac{1}{r}\dfrac{\partial N_2}{\partial r} + \dfrac{\partial^2 N_2}{\partial r^2} - 2FN_2) + \\
\quad \beta_6(\dfrac{1}{r}\dfrac{\partial N_1}{\partial r} + \dfrac{\partial^2 N_1}{\partial r^2} - 2FN_1) - \dfrac{1}{r}\dfrac{\partial N_2}{\partial r}(\dfrac{1}{r}\dfrac{\partial M_1}{\partial r} + \dfrac{\partial^2 M_1}{\partial r^2}) + \dfrac{1}{r} M_1(\\
\quad -\dfrac{1}{r^2}\dfrac{\partial N_2}{\partial r} + \dfrac{1}{r}\dfrac{\partial^2 N_2}{\partial r^2} + \dfrac{\partial^3 N_2}{\partial r^3} - 2F\dfrac{\partial N_2}{\partial r}) + \dfrac{1}{r} N_2(-\dfrac{1}{r^2}\dfrac{\partial M_1}{\partial r} + \\
\quad \dfrac{1}{r}\dfrac{\partial^2 M_1}{\partial r^2} + \dfrac{\partial^3 M_1}{\partial r^3}) - \dfrac{2}{r}\dfrac{\partial \varphi_{T_0}}{\partial r}(\dfrac{1}{r}\dfrac{\partial I_2}{\partial r} + \dfrac{\partial^2 I_2}{\partial r^2}) + \dfrac{2}{r} I_2 p(r)], \\
\dfrac{\partial^2 J_6}{\partial r^2} = -\dfrac{1}{r}\dfrac{\partial J_6}{\partial r} - [2cFr - 2F\dfrac{\partial \varphi_{B_0}}{\partial r} + \beta - q(r)](\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1} J_6 + \\
\quad \dfrac{1}{2} r(\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}[\alpha_3(\dfrac{1}{r}\dfrac{\partial N_2}{\partial r} + \dfrac{\partial^2 N_2}{\partial r^2} - 2FN_2) + \beta_7(\dfrac{1}{r}\dfrac{\partial N_1}{\partial r} + \dfrac{\partial^2 N_1}{\partial r^2} \\
\quad - 2FN_1) - \dfrac{1}{r}\dfrac{\partial M_2}{\partial r}(\dfrac{1}{r}\dfrac{\partial N_1}{\partial r} + \dfrac{\partial^2 N_1}{\partial r^2} - 2FN_1) - \dfrac{1}{r}\dfrac{\partial N_1}{\partial r}(\dfrac{1}{r}\dfrac{\partial M_2}{\partial r} + \\
\quad \dfrac{\partial^2 M_2}{\partial r^2}) + \dfrac{1}{r} M_2(-\dfrac{1}{r^2}\dfrac{\partial N_1}{\partial r} + \dfrac{1}{r}\dfrac{\partial^2 N_1}{\partial r^2} + \dfrac{\partial^3 N_1}{\partial r^3} - 2F\dfrac{\partial N_1}{\partial r}) + \dfrac{1}{r} N_1(\\
\quad -\dfrac{1}{r^2}\dfrac{\partial M_2}{\partial r} + \dfrac{1}{r}\dfrac{\partial^2 M_2}{\partial r^2} + \dfrac{\partial^3 M_2}{\partial r^3}) - \dfrac{2}{r}\dfrac{\partial \varphi_{T_0}}{\partial r}(\dfrac{1}{r}\dfrac{\partial I_3}{\partial r} + \dfrac{\partial^2 I_3}{\partial r^2}) + \dfrac{2}{r} I_3 p(r)], \\
\dfrac{\partial^2 J_7}{\partial r^2} = \dfrac{1}{r}\dfrac{\partial J_7}{\partial r} + [2cFr - 2F\dfrac{\partial \varphi_{B_0}}{\partial r} + \beta - q(r)](\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1} J_7 + \\
\quad r(\dfrac{\partial \varphi_{B_0}}{\partial r} - cr)^{-1}[\alpha_4(\dfrac{1}{r}\dfrac{\partial N_2}{\partial r} + \dfrac{\partial^2 N_2}{\partial r^2} - 2FN_2) + \beta_8(\dfrac{1}{r}\dfrac{\partial N_1}{\partial r} + \\
\quad \dfrac{\partial^2 N_1}{\partial r^2} - 2FN_1) - \dfrac{1}{r}\dfrac{\partial \varphi_{T_0}}{\partial r}(\dfrac{1}{r}\dfrac{\partial I_4}{\partial r} + \dfrac{\partial^2 I_4}{\partial r^2} + \dfrac{M_1}{r^2}) + \\
\quad \dfrac{1}{r} I_4 p(r) + \dfrac{c}{r^2} N_2 - \dfrac{1}{r^3}\dfrac{\partial \varphi_{B_0}}{\partial r} N_2].
\end{cases} \tag{33}
$$

Unfortunately, Equation (29) does not describe the evolution of either the barotropic fluid or the baroclinic fluid, so we need to move onto the next order of Equation (8).

$$\epsilon^6 : \begin{cases} L_1\varphi_{B_5} = F_{12}, \\ L_2\varphi_{T_5} = F_{22}, \end{cases} \tag{34}$$

(see Appendix A for details of F_{12}, F_{22}).

By substituting Equations (17), (19), (26), and (31) into Equation (34) and by considering the compatibility conditions (23), the coupled KdV-mKdV equations describing the evolution of Rossby waves in barotropic-baroclinic coherent structures are obtained for a moving co-ordinate system.

$$\begin{cases} \frac{\partial A_1}{\partial \tau_3} + \alpha_8\frac{\partial A_1}{\partial \xi} + \alpha_9 A_1\frac{\partial A_1}{\partial \xi} + \alpha_{10}B_1\frac{\partial B_1}{\partial \xi} + \alpha_{11}A_1^2\frac{\partial A_1}{\partial \xi} + \alpha_{12}B_1^2\frac{\partial A_1}{\partial \xi} + \\ \alpha_{13}A_1B_1\frac{\partial B_1}{\partial \xi} + \alpha_{14}\frac{\partial^2 A_1}{\partial \xi^2}\frac{\partial A_1}{\partial \xi} + \alpha_{15}\frac{\partial^2 B_1}{\partial \xi^2}\frac{\partial B_1}{\partial \xi} + \alpha_{16}\frac{\partial^3 A_1}{\partial \xi^3} + \alpha_{17}A_1\frac{\partial^3 A_1}{\partial \xi^3} \\ + \alpha_{18}B_1\frac{\partial^3 B_1}{\partial \xi^3} + \alpha_{19}\frac{\partial^5 A_1}{\partial \xi^5} = 0, \\ \frac{\partial B_1}{\partial \tau_3} + \beta_9\frac{\partial B_1}{\partial \xi} + \beta_{10}A_1\frac{\partial B_1}{\partial \xi} + \beta_{11}B_1\frac{\partial A_1}{\partial \xi} + \beta_{12}A_1^2\frac{\partial B_1}{\partial \xi} + \beta_{13}B_1^2\frac{\partial B_1}{\partial \xi} + \\ \beta_{14}A_1B_1\frac{\partial A_1}{\partial \xi} + \beta_{15}\frac{\partial^2 A_1}{\partial \xi^2}\frac{\partial B_1}{\partial \xi} + \beta_{16}\frac{\partial^2 B_1}{\partial \xi^2}\frac{\partial A_1}{\partial \xi} + \beta_{17}\frac{\partial^3 B_1}{\partial \xi^3} + \beta_{18}A_1\frac{\partial^3 B_1}{\partial \xi^3} \\ + \beta_{19}B_1\frac{\partial^3 A_1}{\partial \xi^3} + \beta_{20}\frac{\partial^5 B_1}{\partial \xi^5} = 0, \end{cases} \tag{35}$$

where the coefficients $\alpha_i(i = 1,\ldots,19)$ and $\beta_i(i = 1,\ldots,20)$ of the coupled equations are shown in Appendix A.

Then, by using Equations (10), (24), and (30), the same equations are obtained for the fixed co-ordinate system.

$$\begin{cases} \frac{\partial A_1}{\partial t} + c\frac{\partial A_1}{\partial \theta} + \epsilon^2(\alpha_1\frac{\partial A_1}{\partial \theta} + \alpha_2 A_1\frac{\partial A_1}{\partial \theta} + \alpha_4\frac{\partial^3 A_1}{\partial \theta^3} + \alpha_3 B_1\frac{\partial B_1}{\partial \theta}) + \epsilon^3(\alpha_5\frac{\partial B_1}{\partial \theta} \\ + \alpha_6 A_1\frac{\partial B_1}{\partial \theta} + \alpha_6 B_1\frac{\partial A_1}{\partial \theta} + \alpha_7\frac{\partial^3 B_1}{\partial \theta^3}) + \epsilon^4(\alpha_8\frac{\partial A_1}{\partial \theta} + \alpha_9 A_1\frac{\partial A_1}{\partial \theta}\alpha_{11}A_1^2\frac{\partial A_1}{\partial \theta} \\ + \alpha_{14}\frac{\partial^2 A_1}{\partial \theta^2}\frac{\partial A_1}{\partial \theta} + \alpha_{16}\frac{\partial^3 A_1}{\partial \theta^3} + \alpha_{17}A_1\frac{\partial^3 A_1}{\partial \theta^3} + \alpha_{19}\frac{\partial^5 A_1}{\partial \theta^5} + \alpha_{10}B_1\frac{\partial B_1}{\partial \theta} + \\ \alpha_{12}B_1^2\frac{\partial A_1}{\partial \theta} + \alpha_{13}A_1B_1\frac{\partial B_1}{\partial \theta} + \alpha_{15}\frac{\partial^2 B_1}{\partial \theta^2}\frac{\partial B_1}{\partial \theta}) = 0, \\ \frac{\partial B_1}{\partial t} + c\frac{\partial B_1}{\partial \theta} + \epsilon^2(\beta_1\frac{\partial B_1}{\partial \theta} + \beta_2 A_1\frac{\partial B_1}{\partial \theta} + \beta_4\frac{\partial^3 B_1}{\partial \theta^3} + \beta_3 B_1\frac{\partial A_1}{\partial \theta}) + \epsilon^3(\beta_5\frac{\partial A_1}{\partial \theta} \\ + \beta_6 A_1\frac{\partial A_1}{\partial \theta} + \beta_7 B_1\frac{\partial B_1}{\partial \theta} + \beta_8\frac{\partial^3 A_1}{\partial \theta^3}) + \epsilon^4(\beta_9\frac{\partial B_1}{\partial \theta} + \beta_{10}A_1\frac{\partial B_1}{\partial \theta}\beta_{12}A_1^2\frac{\partial B_1}{\partial \theta} \\ + \beta_{15}\frac{\partial^2 A_1}{\partial \theta^2}\frac{\partial B_1}{\partial \theta} + \beta_{17}\frac{\partial^3 B_1}{\partial \theta^3} + \beta_{18}A_1\frac{\partial^3 B_1}{\partial \theta^3} + \beta_{20}\frac{\partial^5 B_1}{\partial \theta^5} + \beta_{11}B_1\frac{\partial A_1}{\partial \theta} + \\ \beta_{13}B_1^2\frac{\partial B_1}{\partial \theta} + \beta_{14}A_1B_1\frac{\partial A_1}{\partial \theta} + \beta_{16}\frac{\partial^2 B_1}{\partial \theta^2}\frac{\partial A_1}{\partial \theta}) = 0. \end{cases} \tag{36}$$

Equation (36) depicts the amplitude of the propagation of the barotropic and baroclinic Rossby waves in polar co-ordinates more accurately, and these are new equations that have never been derived; we call them the fifth-order coupled KdV-mKdV equations. This should be the first time that the fifth-order coupled KdV-mKdV equations are used to simulate the evolution of barotropic and baroclinic Rossby waves, and this is also the first time that this has been considered in the polar co-ordinates. In particular, for when $B_1 = 0$, the first expression in Equation (36), becomes a fifth-order KdV equation. This model can be found in the work by R Grimshaw (2001) [25].

The β effect and the basic background flow are indispensable in the generation of barotropic and baroclinic Bossby waves. In fact, the depth of the upper and lower fluids also affects Rossby wave amplitude. But in this paper, the depth of the upper and lower are assumed to be the same. In the next article, we will consider the higher-order coupled Boussinesq equations describing Rossby waves with unequal depths. From Equation (36), we can clearly observe the effect of barotropic and baroclinic interactions on Rossby wave amplitude. Specifically, whether barobaric-baroclinic interaction, baroclinic-baroclinic interaction, or baroclinic-baroclinic interaction controls the propagation of barotropic Rossby and baroclinic Rossby wave amplitudes. However, we find that the barotropic-baroclinic interaction has a stronger effect on the propagation of baroclinic Rossby wave amplitudes. The barotropic-baroclinic interaction has an effect on baroclinic flow propagation in the higher-order equations, and the barotropic-baroclinic flow propagation has an effect on baroclinic flow propagation, which is not described in the lower-order coupling equations. By using the fifth-order coupled KdV-mKdV equations in polar co-ordinates, we obtain more information about the effects of baroclinic interactions on barotropic and baroclinic Rossby wave propagation when compared to the lower-order coupled equations. Therefore, the derivation of higher-order equations has very important theoretical significance and research value.

3. Conservation Laws of the Fifth-Order Coupled KdV-mKdV Equations in Polar Co-Ordinates

In Section 2, we obtained the fifth-order coupled KdV-mKdV equations in polar co-ordinates, and the conservation law of this fifth-order coupled KdV-mKdV equations have never been studied before. Therefore, it is of great theoretical significance and potential application value to study it.

3.1. *Lie Symmetry Analysis*

Firstly, the group of one-parameter Lie transformations is considered as follows:

$$\begin{cases} \theta^* = \theta + \epsilon\rho_1(\theta, t, A_1.B_1) + O(\varepsilon^2), \\ t^* = t + \epsilon\rho_2(\theta, t, A_1.B_1) + O(\varepsilon^2), \\ A_1^* = A_1 + \epsilon\eta_1(\theta, t, A_1.B_1) + O(\varepsilon^2), \\ B_1^* = B_1 + \epsilon\eta_2(\theta, t, A_1.B_1) + O(\varepsilon^2), \end{cases} \tag{37}$$

where ϵ is a group parameter, and $\rho_1, \rho_2, \eta_1, \eta_2$ are infinitesimal functions. The corresponding infinitesimal generator V of the Lie algebra can be expressed as

$$V = \rho_1 \frac{\partial}{\partial\theta} + \rho_2 \frac{\partial}{\partial t} + \eta_1 \frac{\partial}{\partial A_1} + \eta_2 \frac{\partial}{\partial B_1}. \tag{38}$$

According to the infinitesimal transformation, the invariance of the fifth-order coupled KdV-mKdV contributes to the following invariance condition:

$$\begin{cases} Pr^{(5)}V(\Delta_1)|_{\Delta_1=0} = 0, \\ Pr^{(5)}V(\Delta_2)|_{\Delta_2=0} = 0, \end{cases} \tag{39}$$

$Pr^{(5)}V$ is a fifth-order continuation of V in the following form:

$$\begin{aligned} Pr^{(5)}V = & V + \eta_1^t \frac{\partial}{\partial A_{1t}} + \eta_2^t \frac{\partial}{\partial B_{1t}} + \eta_1^\theta \frac{\partial}{\partial A_{1\theta}} + \eta_2^\theta \frac{\partial}{\partial B_{2\theta}} + \eta_1^{\theta\theta} \frac{\partial}{\partial A_{1\theta\theta}} + \eta_2^{\theta\theta} \frac{\partial}{\partial B_{2\theta\theta}} \\ & + \eta_1^{\theta\theta\theta} \frac{\partial}{\partial A_{1\theta\theta\theta}} + \eta_2^{\theta\theta\theta} \frac{\partial}{\partial B_{2\theta\theta\theta}} + \eta_1^{\theta\theta\theta\theta} \frac{\partial}{\partial A_{1\theta\theta\theta\theta}} + \eta_2^{\theta\theta\theta\theta} \frac{\partial}{\partial B_{2\theta\theta\theta\theta}} + \\ & \eta_1^{\theta\theta\theta\theta\theta} \frac{\partial}{\partial A_{1\theta\theta\theta\theta\theta}} + \eta_2^{\theta\theta\theta\theta\theta} \frac{\partial}{\partial B_{2\theta\theta\theta\theta\theta}}, \end{aligned} \tag{40}$$

where Δ_1, Δ_2 (see Appendix A for details).

The specific expression of $\eta_1^t, \eta_2^t, \eta_1^\theta, \eta_2^\theta, \eta_1^{\theta\theta}, \eta_2^{\theta\theta}, \eta_1^{\theta\theta\theta}, \eta_2^{\theta\theta\theta}, \eta_1^{\theta\theta\theta\theta}, \eta_2^{\theta\theta\theta\theta}, \ldots$ is given by

$$\begin{cases} \eta_1^t = D_t(\eta_1) - A_{1\theta} D_t(\rho_1) - A_{1t} D_t(\rho_2), \\ \eta_2^t = D_t(\eta_2) - B_{1\theta} D_t(\rho_1) - B_{1t} D_t(\rho_2), \\ \eta_1^\theta = D_\theta(\eta_1) - A_{1\theta} D_\theta(\rho_1) - A_{1t} D_\theta(\rho_2), \\ \eta_2^\theta = D_\theta(\eta_2) - B_{1\theta} D_\theta(\rho_1) - B_{1t} D_\theta(\rho_2), \\ \eta_1^{\theta\theta} = D_\theta(\eta_1^\theta) - A_{1\theta\theta} D_\theta(\rho_1) - A_{1t\theta} D_\theta(\rho_2), \\ \eta_2^{\theta\theta} = D_\theta(\eta_2^\theta) - B_{1\theta\theta} D_\theta(\rho_1) - B_{1t\theta} D_\theta(\rho_2), \\ \eta_1^{\theta\theta\theta} = D_\theta(\eta_1^{\theta\theta}) - A_{1\theta\theta\theta} D_\theta(\rho_1) - A_{1t\theta\theta} D_\theta(\rho_2), \\ \eta_2^{\theta\theta\theta} = D_\theta(\eta_2^{\theta\theta}) - B_{1\theta\theta\theta} D_\theta(\rho_1) - B_{1t\theta\theta} D_\theta(\rho_2), \\ \cdots\cdots \end{cases} \tag{41}$$

where D_t and D_θ are the total derivative operators, which are defined as

$$\begin{cases} D_t = \frac{\partial}{\partial t} + A_{1t} \frac{\partial}{\partial A_1} + B_{1t} \frac{\partial}{\partial B_1} + A_{1\theta t} \frac{\partial}{\partial A_{1\theta}} + A_{1tt} \frac{\partial}{\partial A_{1t}} + B_{1\theta t} \frac{\partial}{\partial B_{1\theta}} + \\ \quad B_{1tt} \frac{\partial}{\partial B_{1t}} + \ldots, \\ D_\theta = \frac{\partial}{\partial \theta} + A_{1\theta} \frac{\partial}{\partial A_1} + B_{1\theta} \frac{\partial}{\partial B_1} + A_{1\theta\theta} \frac{\partial}{\partial A_{1\theta}} + A_{1t\theta} \frac{\partial}{\partial A_{1t}} + B_{1\theta\theta} \frac{\partial}{\partial B_{1\theta}} + \\ \quad B_{1t\theta} \frac{\partial}{\partial B_{1t}} + A_{1\theta\theta\theta} \frac{\partial}{\partial A_{1\theta\theta}} + A_{1t\theta\theta} \frac{\partial}{\partial A_{1t\theta}} + A_{1\theta tt} \frac{\partial}{\partial A_{1tt}} + \ldots. \end{cases} \tag{42}$$

The sufficient conditions for Equation (39) to hold are

$$\begin{cases} \eta_1^t + (a_1 A_{1\theta} + a_2 A_1 A_{1\theta} + a_3 A_{1\theta\theta\theta} + a_4 B_{1\theta} + a_5 B_1 B_{1\theta})\eta_1 + (a_6 + a_7 A_1 + \\ a_8 A_1^2 + a_9 A_{1\theta\theta} + a_{10} B_1 + a_{11} B_1^2)\eta_1^\theta + a_{12} A_{1\theta} \eta_1^{\theta\theta} + (a_{13} + a_{14} A_1)\eta_1^{\theta\theta\theta} + \\ a_{15} \eta_1^{\theta\theta\theta\theta\theta} + (a_{16} A_{1\theta} + a_{17} B_{1\theta} + a_{18} B_1 A_{1\theta} + a_{19} A_1 B_{1\theta})\eta_2 + (a_{20} + \\ a_{21} A_1 + a_{22} B_1 + a_{23} A_1 B_1 + a_{24} B_{1\theta\theta})\eta_2^\theta + a_{25} B_{1\theta} \eta_2^{\theta\theta} + a_{26} \eta_2^{\theta\theta\theta} = 0, \\ \eta_2^t + (b_1 A_{1\theta} + b_2 A_{1\theta} + b_3 B_1 B_{1\theta} + b_4 A_1 A_{1\theta})\eta_2 + (b_5 + b_6 A_1 + b_7 A_1^2 + \\ b_8 A_{1\theta\theta} + b_9 B_1 + b_{10} B_1^2)\eta_2^\theta + b_{11} A_{1\theta} \eta_2^{\theta\theta} + (b_{12} + b_{13} A_1)\eta_2^{\theta\theta\theta} + b_{14} \eta_2^{\theta\theta\theta\theta\theta} + \\ (b_{15} B_{1\theta} + b_{16} A_{1\theta} + b_{17} A_1 B_{1\theta} + b_{18} B_1 A_{1\theta} + b_{19} B_{1\theta\theta\theta})\eta_1 + (b_{20} + b_{21} A_1 + \\ b_{22} B_1 + b_{23} A_1 B_1 + b_{24} B_{1\theta\theta})\eta_1^\theta + b_{25} B_{1\theta} \eta_1^{\theta\theta} + b_{26} \eta_1^{\theta\theta\theta} = 0, \end{cases} \tag{43}$$

where the coefficients $a_i (i = 1, \ldots, 26)$ and $b_j (j = 1, \ldots, 26)$ in the coupled equations are shown in Appendix A.

By substituting Equations (41) and (42) into Equation (43) when $a_1 = a_{16} = 0$ by letting the coefficient of $1, A_1, A_1B_1, A_1^2A_{1\theta}, A_1^2B_{1\theta}, A_1B_1B_{1\theta}, B_{1\theta\theta}, B_1A_{1\theta}^2, A_1^2A_{1\theta}A_{1t}, A_1^2A_{1\theta\theta\theta}^2, A_1^2A_{1\theta\theta\theta}B_{1\theta\theta\theta}, A_{1\theta\theta\theta\theta\theta}, B_{1\theta\theta\theta\theta\theta}$ be zero, we can obtain

$$\begin{cases}\rho_{1A_1} = \rho_{1B_1} = \rho_{1t} = \rho_{2A_1} = \rho_{2B_1} = \rho_{2\theta} = 0,\\ \eta_{1t} = \eta_{1\theta} = \eta_{1B_1} = \eta_{2t} = \eta_{2\theta} = \eta_{2A_1} = 0,\\ \rho_{1\theta} = \rho_{2t} = \eta_{1A_1} = \eta_{2B_1} = c_1.\end{cases} \tag{44}$$

Thus,

$$\begin{cases}\rho_1 = c_1\theta + c_2,\\ \rho_2 = c_1 t + c_3,\\ \eta_1 = c_1A_1 + c_4,\\ \eta_1 = c_1B_1 + c_5,\end{cases} \tag{45}$$

where c_1, c_2, c_3, c_4, c_5 are arbitrary constants.

Therefore, the generating elements of Equation (36) are

$$\begin{cases}V_1 = \dfrac{\partial}{\partial\theta},\\ V_2 = \dfrac{\partial}{\partial t},\\ V_3 = \theta\dfrac{\partial}{\partial\theta} + t\dfrac{\partial}{\partial t} + A_1\dfrac{\partial}{\partial A_1} + B_1\dfrac{\partial}{\partial B_1}.\end{cases} \tag{46}$$

3.2. Conservation Laws

Rewrite Equation (36) in the following form:

$$\begin{cases}A_{1t} + c_1A_{1\theta} + c_2A_1A_{1\theta} + c_3A_{1\theta\theta\theta} + c_4A_1^2A_{1\theta} + c_5A_{1\theta}A_{1\theta\theta} + c_6A_1A_{1\theta\theta\theta} +\\ c_7A_{1\theta\theta\theta\theta\theta} + c_8B_{1\theta} + c_9B_1B_{1\theta} + c_{10}A_1B_{1\theta} + c_{11}B_1A_{1\theta} + c_{12}B_{1\theta\theta\theta} +\\ c_{13}B_1^2A_{1\theta} + c_{14}A_1B_1B_{1\theta} + c_{15}B_{1\theta}B_{1\theta\theta} = 0,\\ B_{1t} + d_1B_{1\theta} + d_2A_1B_{1\theta} + d_3B_{1\theta\theta\theta} + d_4A_1^2B_{1\theta} + d_5B_{1\theta}A_{1\theta\theta} + d_6A_1B_{1\theta\theta\theta} +\\ d_7B_{1\theta\theta\theta\theta\theta} + d_8A_{1\theta} + d_9B_1A_{1\theta} + d_{10}A_1A_{1\theta} + d_{11}B_1B_{1\theta} + d_{12}A_{1\theta\theta\theta} +\\ d_{13}B_1^2B_{1\theta} + d_{14}A_1B_1A_{1\theta} + d_{15}A_{1\theta}B_{1\theta\theta} = 0,\end{cases} \tag{47}$$

where the coefficients $c_i(i = 1, \ldots, 15)$ and $d_j(j = 1, \ldots, 15)$ in the coupled equations are shown in Appendix A.

The Lagrangian form of Equation (47) is

$$\begin{aligned}L =& u(\theta,t)(A_{1t} + c_1A_{1\theta} + c_2A_1A_{1\theta} + c_3A_{1\theta\theta\theta} + c_4A_1^2A_{1\theta} + c_5A_{1\theta}A_{1\theta\theta\theta} +\\ & c_6A_1A_{1\theta\theta\theta} + c_7A_{1\theta\theta\theta\theta\theta} + c_8B_{1\theta} + c_9B_1B_{1\theta} + c_{10}A_1B_{1\theta} + c_{11}B_1A_{1\theta} +\\ & c_{12}B_{1\theta\theta\theta} + c_{13}B_1^2A_{1\theta} + c_{14}A_1B_1B_{1\theta} + c_{15}B_{1\theta}B_{1\theta\theta}) + v(\theta,t)(B_{1t} + d_1B_{1\theta} +\\ & d_2A_1B_{1\theta} + d_3B_{1\theta\theta\theta} + d_4A_1^2B_{1\theta} + d_5B_{1\theta}A_{1\theta\theta} + d_6A_1B_{1\theta\theta\theta} + d_7B_{1\theta\theta\theta\theta\theta} +\\ & d_8A_{1\theta} + d_9B_1A_{1\theta} + d_{10}A_1A_{1\theta} + d_{11}B_1B_{1\theta} + d_{12}A_{1\theta\theta\theta} + d_{13}B_1^2B_{1\theta} +\\ & d_{14}A_1B_1A_{1\theta} + d_{15}A_{1\theta}B_{1\theta\theta}),\end{aligned} \tag{48}$$

where $u(\theta,t), v(\theta,t)$ are new variables.

Consider the adjoint equation as follows:

$$\begin{cases}F_1^* = \dfrac{\delta L}{\delta A_1} = 0,\\ F_2^* = \dfrac{\delta L}{\delta B_1} = 0,\end{cases} \tag{49}$$

where $\frac{\delta}{\delta A_1}$, $\frac{\delta}{\delta B_1}$ can be represented as

$$\begin{cases} \frac{\delta}{\delta A_1} = \frac{\partial}{\partial A_1} - D_\theta \frac{\partial}{\partial A_{1\theta}} - D_t \frac{\partial}{\partial A_{1t}} + D_{\theta\theta} \frac{\partial}{\partial A_{1\theta\theta}} + D_{\theta t} \frac{\partial}{\partial A_{1\theta\theta}} - D_{\theta\theta\theta} \frac{\partial}{\partial A_{1\theta\theta\theta}} + \\ \quad D_{\theta\theta\theta\theta} \frac{\partial}{\partial A_{1\theta\theta\theta\theta}} - D_{\theta\theta\theta\theta\theta} \frac{\partial}{\partial A_{1\theta\theta\theta\theta\theta}}, \\ \frac{\delta}{\delta B_1} = \frac{\partial}{\partial B_1} - D_\theta \frac{\partial}{\partial B_{1\theta}} - D_t \frac{\partial}{\partial B_{1t}} + D_{\theta\theta} \frac{\partial}{\partial B_{1\theta\theta}} + D_{\theta t} \frac{\partial}{\partial B_{1\theta\theta}} - D_{\theta\theta\theta} \frac{\partial}{\partial B_{1\theta\theta\theta}} + \\ \quad D_{\theta\theta\theta\theta} \frac{\partial}{\partial B_{1\theta\theta\theta\theta}} - D_{\theta\theta\theta\theta\theta} \frac{\partial}{\partial B_{1\theta\theta\theta\theta\theta}}. \end{cases} \tag{50}$$

Therefore, Equation (49) can also be expressed as

$$\begin{cases} F_1^* = -u_t + (c_{10}B_{1\theta} - c_{11}B_1 - c_{13}B_1^2 + c_{14}B_1B_{1\theta})u - (c_1 + c_2 + c_4A_1^2 + \\ \quad c_5A_{1\theta\theta})u_\theta + c_5A_{1\theta}u_{\theta\theta} - (c_3 + c_6A_1)u_{\theta\theta\theta} - c_7u_{\theta\theta\theta\theta\theta} + (d_2B_{1\theta} + \\ \quad 2d_4A_1B_{1\theta} + d_5B_{1\theta\theta\theta} + d_6B_{1\theta\theta\theta} - d_9B_{1\theta} - d_{14}A_1B_{1\theta} - d_{15}B_{1\theta\theta\theta})v - \\ \quad (d_8 + d_9B_1 + d_{10}A_1 + d_{14}A_1B_1 + d_{15}B_{1\theta\theta\theta})v_\theta + d_5B_{1\theta}v_{\theta\theta} - d_{12}v_{\theta\theta\theta}, \\ F_2^* = -v_t - (d_2A_{1\theta} + 2d_4A_1A_{1\theta} + d_5A_{1\theta\theta\theta} + d_6A_{1\theta\theta\theta} - d_9A_{1\theta} - d_{14}A_1A_{1\theta} - \\ \quad d_{15}A_{1\theta\theta\theta})v - (d_1 + d_2 + d_4A_1^2 + d_5A_{1\theta\theta})v_\theta + d_{15}A_{1\theta}v_{\theta\theta} - (d_3 + \\ \quad d_6A_1)v_{\theta\theta\theta} - d_7v_{\theta\theta\theta\theta\theta} + (-c_{10}A_{1\theta} + c_{11}A_{1\theta} + 2c_{13}B_1A_{1\theta} - c_{14}B_1A_{1\theta})u - \\ \quad (c_8 + c_9B_1 + c_{10}A_1 + c_{14}A_1B_1 + c_{15}B_{1\theta\theta\theta})u_\theta + c_{15}B_{1\theta}u_{\theta\theta} - c_{12}u_{\theta\theta\theta}. \end{cases} \tag{51}$$

According to the Lie characteristic function,

$$\begin{cases} W^1 = A_1 - \theta A_{1\theta} - tA_{1t}, \\ W^2 = B_1 - \theta B_{1\theta} - tB_{1t}, \end{cases} \tag{52}$$

the conserved vectors of the fifth-order coupled KdV-mKdV equations in polar co-ordinates are expressed as

$$\begin{cases} C^1 = \rho^1 L + W^1[\frac{\partial L}{\partial A_{1t}}] + W^2[\frac{\partial L}{\partial B_{1t}}], \\ C^2 = \rho^2 L + W^1[\frac{\partial L}{\partial A_{1\theta}} - D_\theta \frac{\partial L}{\partial A_{1\theta\theta}} + D_{\theta\theta} \frac{\partial L}{\partial A_{1\theta\theta\theta}} + D_{\theta\theta\theta\theta} \frac{\partial L}{\partial A_{1\theta\theta\theta\theta\theta}}] + \\ \quad D_\theta(W^1)[\frac{\partial L}{\partial A_{1\theta\theta}} - D_\theta \frac{\partial L}{\partial A_{1\theta\theta\theta}} - D_{\theta\theta\theta} \frac{\partial L}{\partial A_{1\theta\theta\theta\theta\theta}}] + D_{\theta\theta}(W^1)[\frac{\partial L}{\partial A_{1\theta\theta\theta}} + \\ \quad D_{\theta\theta} \frac{\partial L}{\partial A_{1\theta\theta\theta\theta\theta}}] + D_{\theta\theta\theta}(W^1)[D_\theta \frac{\partial L}{\partial A_{1\theta\theta\theta\theta\theta}}] + D_{\theta\theta\theta\theta}(W^1)[\frac{\partial L}{\partial A_{1\theta\theta\theta\theta\theta}}] + \\ \quad W^2[\frac{\partial L}{\partial B_{1\theta}} - D_\theta \frac{\partial L}{\partial B_{1\theta\theta}} + D_{\theta\theta} \frac{\partial L}{\partial B_{1\theta\theta\theta}} + D_{\theta\theta\theta\theta} \frac{\partial L}{\partial B_{1\theta\theta\theta\theta\theta}}] + D_\theta(W^2) \\ \quad [\frac{\partial L}{\partial B_{1\theta\theta}} - D_\theta \frac{\partial L}{\partial B_{1\theta\theta\theta}} - D_{\theta\theta\theta} \frac{\partial L}{\partial B_{1\theta\theta\theta\theta\theta}}] + D_{\theta\theta}(W^2)[\frac{\partial L}{\partial B_{1\theta\theta\theta}} + \\ \quad D_{\theta\theta} \frac{\partial L}{\partial B_{1\theta\theta\theta\theta\theta}}] + D_{\theta\theta\theta}(W^2)[D_\theta \frac{\partial L}{\partial B_{1\theta\theta\theta\theta\theta}}] + D_{\theta\theta\theta\theta}(W^2)[\frac{\partial L}{\partial B_{1\theta\theta\theta\theta\theta}}]. \end{cases} \tag{53}$$

Equation (46) is the Lie algebra of the point symmetries of the fifth-order coupled KdV-mKdV equations, whereas Equation (53) is the conservation laws of the fifth-order coupled KdV-mKdV equations. They are all obtained here first and have potential value for physics and mathematical research.

4. The Solutions of the Fifth-Order Coupled KdV-mKdV Equations

It is necessary to study its solution for a new system of equations in order to have a better understanding of the nonlinear problem. In this section, The solitary wave solutions of the fifth-order coupled KdV-mKdV equation are calculated by the Jacobian elliptic function expansion method.

Firstly, the following traveling wave solutions are considered as

$$A_1 = A_1(\zeta), B_1 = B_1(\zeta), \zeta = k(\theta - ct), \tag{54}$$

where k is the wave number, and c is the wave speed. Substituting Equation (54) into Equation (47) leads to

$$\begin{cases} -cA_{1\zeta} + c_1 A_{1\zeta} + c_2 A_1 A_{1\zeta} + c_3 k^2 A_{1\zeta\zeta\zeta} + c_4 A_1^2 A_{1\zeta} + c_5 k^2 A_{1\zeta} A_{1\zeta\zeta} + \\ c_6 k^2 A_1 A_{1\zeta\zeta\zeta} + c_7 k^4 A_{1\zeta\zeta\zeta\zeta\zeta} + c_8 B_{1\zeta} + c_9 B_1 B_{1\zeta} + c_{10} A_1 B_{1\zeta} + c_{11} B_1 A_{1\zeta} + \\ c_{12} k^2 B_{1\zeta\zeta\zeta} + c_{13} B_1^2 A_{1\zeta} + c_{14} A_1 B_1 B_{1\zeta} + c_{15} k^2 B_{1\zeta} B_{1\zeta\zeta} = 0, \\ -cB_{1\zeta} + d_1 B_{1\zeta} + d_2 A_1 B_{1\zeta} + d_3 k^2 B_{1\zeta\zeta\zeta} + d_4 A_1^2 B_{1\zeta} + d_5 k^2 B_{1\zeta} A_{1\zeta} + \\ d_6 A_1 k^2 B_{1\zeta\zeta\zeta} + d_7 k^4 B_{1\zeta\zeta\zeta\zeta\zeta} + d_8 A_{1\zeta} + d_9 B_1 A_{1\zeta} + d_{10} A_1 A_{1\zeta} + d_{11} B_1 B_{1\zeta} + \\ d_{12} k^2 A_{1\zeta\zeta\zeta} + d_{13} B_1^2 B_{1\zeta} + d_{14} A_1 B_1 A_{1\zeta} + d_{15} k^2 A_{1\zeta} B_{1\zeta\zeta} = 0. \end{cases} \tag{55}$$

We suppose n = 2 in order to balance the highest order derivative terms $\frac{\partial^5 A_1}{\partial \zeta^5}, \frac{\partial^5 B_1}{\partial \zeta^5}$ and the nonlinear terms in Equation (55). Thus, $A_1(\zeta)$ and $B_1(\zeta)$ can be expressed as follows:

$$\begin{cases} A_1(\zeta) = m_0 + m_1 sn\zeta + m_2 sn^2\zeta, \\ B_1(\zeta) = n_0 + n_1 sn\zeta + n_2 sn^2\zeta. \end{cases} \tag{56}$$

Notice that

$$cn^2\zeta = 1 - sn^2\zeta, dn^2\zeta = 1 - m^2 sn^2\zeta. \tag{57}$$

By substituting Equation (56) into Equation (55) and collecting the same order of $sn\zeta$ and setting the polynomial coefficients to zero, we obtain

$$m_1 = n_1 = 0. \tag{58}$$

The solutions of the fifth-order coupled KdV-mKdV equations are obtained.

$$\begin{cases} A_1(\zeta) = m_0 + m_2 sn^2\zeta, \\ B_1(\zeta) = n_0 + n_2 sn^2\zeta, \end{cases} \tag{59}$$

where m_2 satisfies the following equation, which can be solved by Maple and is not specifically expanded in this paper.

$$\begin{aligned} &[c_4 + (c_{13} + c_{14})a_2]m_2^2 + [(c_5 + c_6)k^2 m^2 + (c_{13} + c_{14})a_1 + 6c_{15}k^2 m^2 a_2]m_2 + \\ &6c_15k^2 m^2 = 0, \end{aligned} \tag{60}$$

and n_2, m_0, n_0 can be represented by m_2, as follows:

$$\begin{cases} n_2 = \pm\sqrt{-\frac{1}{d_{13}}(6d_5 + 12d_6 + 6d_{15})k^2 m^2 m_2 - \frac{1}{d_{13}}(d_4 + d_{14})m_2^2}, \\ m_0 = (\frac{1}{2c_4 d_{14} m_2^4 + 12c_6 d_{14} k^2 m^2 m_2^3 + [c_{14}d_{14} - (2c_{13} + c_{14})(2d_4 + d_{14})]m_2^2 n_2^2} \\ \qquad - \frac{1}{12d_6 k^2 m^2 (2c_{13} + c_{14})m_2 n_2^2 + 2d_{13}(2c_{13} + c_{14})m_2^3 n_2})(bf - ce), \\ n_0 = (\frac{1}{2c_4 d_{14} m_2^4 + 12c_6 d_{14} k^2 m^2 + [c_{14}d_{14} - (2c_{13} + c_{14})(2d_4 + d_{14})]m_2^2 n_2^2} \\ \qquad - \frac{1}{12d_6 k^2 m^2 (2c_{13} + c_{14})m_2 n_2^2 + 2d_{13}(2c_{13} + c_{14})})(cd - af), \end{cases} \tag{61}$$

where the coefficients a, b, c, d, e, f are shown in Appendix A.

When $m \to 1$ and $cn\zeta \to sech\zeta$, the solitary wave solution of Equation (47) is obtained.

$$\begin{cases} A_1(\zeta) = m_0 + m_2 + m_2 sech^2\zeta, \\ B_1(\zeta) = n_0 + n_2 + n_2 sech^2\zeta. \end{cases} \tag{62}$$

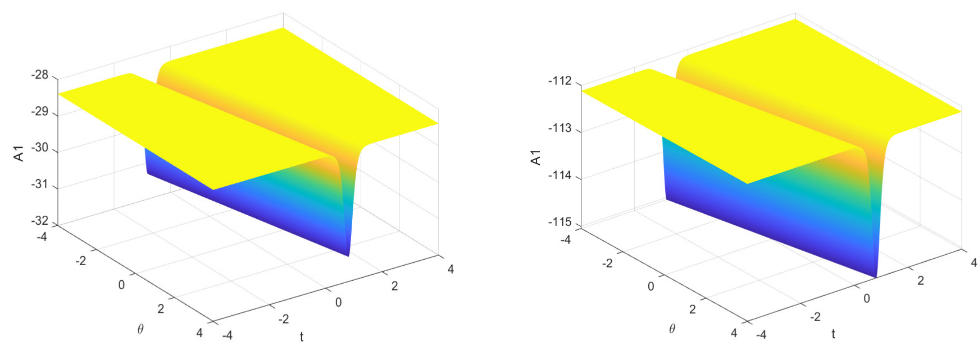
Furthermore, the approximate solutions for barotropic and baroclinic flow fields are obtained as follows:

$$\begin{cases} \overline{\psi}_B = \varphi_{B_0}(r) + \epsilon^2 M_1(r)\{m_0 + m_2 + m_2 sech^2[k(\theta - ct)]\}, \\ \overline{\psi}_T = \varphi_{T_0}(r) + \epsilon^2 N_1(r)\{n_0 + n_2 + n_2 sech^2[k(\theta - ct)]\}. \end{cases} \tag{63}$$

5. Evolution of Rossby Waves in Barotropic-Baroclinic Coherent Structures

Rosby waves are ubiquitous in the atmosphere and ocean, and large-scale weather processes are influenced by them. Thus, the simulation and evolution of Rossby wave amplitude is of great research significance for ocean and atmospheric science. In this section, we will discuss the evolution of Rossby waves with barotropic-baroclinic coherent structures. Firstly, the propagation of the barotropic Rossby wave amplitude will be discussed.

By solving the Equation (60), we can obtain two solutions, and n_2 also contains two solutions. The two solutions of m_2 are basically consistent with its corresponding Rossby wave amplitude evolution, so we select $m_2 > 0$ for the following research. Figure 1a shows the propagation of the barotropic Rossby waves A_1 where $m_2 > 0$ and $n_2 > 0$ in a three-dimensional plane, and Figure 1b shows the propagation of the barotropic Rossby waves A_1 where $m_2 > 0$ and $n_2 < 0$ in a three-dimensional plane. The barotropic Rossby waves can be excited under certain conditions.



(**a**) (**b**)

Figure 1. Evolution of the barotropic Rossby wave amplitude A_1 with $d_3 = 50$, $c_i = 1$ $(i = 1, 2, \ldots, 15)$, $d_j = 1$ $(j = 1, 2, 4, \ldots, 15)$, $k = 1$, $m = 1$, $c = 5$, and $t = 0$.

When taking $d_7 = c_7 = 0$, the higher-order terms vanish. In the following, the effect of the higher-order terms on the barotropic Rossby wave will be considered. Figure 2 shows the propagation of the wave amplitude A_1 when the higher-order effect is removed. As shown in Figure 2, the wave width and amplitude are changed due to higher-order nonlinearity and dispersion perturbation terms. Without higher-order terms, the wave moves overall downward, and the wave width becomes narrower, which results in the whole Rossby wave being steep. It can be seen that the solutions of the higher-order system of equations are closer to the evolution of the real wave.

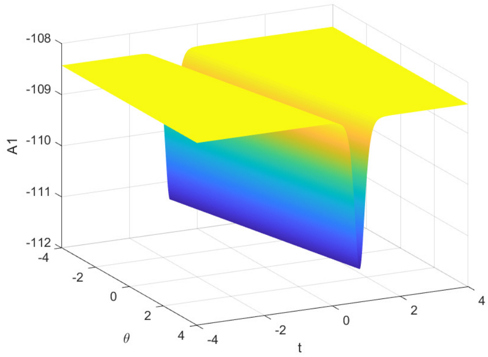
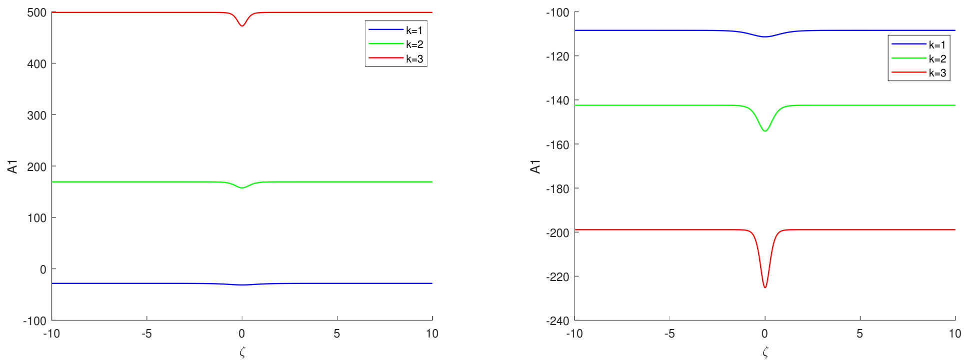


Figure 2. Evolution of the barotropic Rossby wave amplitude A_1 without high-order terms under the same parameters.

As shown in Figures 3a,b, we discuss the effect of the wave number k with and without higher-order terms. The amplitude A_1 increases with the increase in k, whereas the wave width decreases with the increase in k, which leads to the strengthening of the nonlinearity of the Rossby waves. This k-induced trend does not change when the higher-order terms vanish, but the change is more insignificant than the higher-order terms that exist.



(**a**) (**b**)

Figure 3. Influence of wave number k.

Figure 4 shows that the absolute value of amplitude A_1 of barotropic Rossby waves gradually propagates eastward with time, and the waveform and wave speed do not change during the propagation. The energy propagation process is not dissipated when friction dissipation is not considered. So, the waveform and wave velocity remain unchanged, which is consistent with the conclusion of this paper. Next, we will continue to discuss the propagation of the baroclinic Rossby wave B_1.

Figures 5a,b show the evolution of the amplitude of baroclinic Rossby waves B_1 when $m_2 > 0$ and $n_2 > 0$ and B_1 when $m_2 > 0$ and $n_2 < 0$ in a three-dimensional plane. The amplitudes of the waves are obviously different in these two cases. By taking Figure 5a as an example, the influence of the higher-order term, the wave number k, and the time t are considered below. By comparing Figures 1 and 5, we can clearly see that the amplitude of the barotropic wave is higher than that of the baroclinic wave, and the wave shape is sharper under the same conditions.

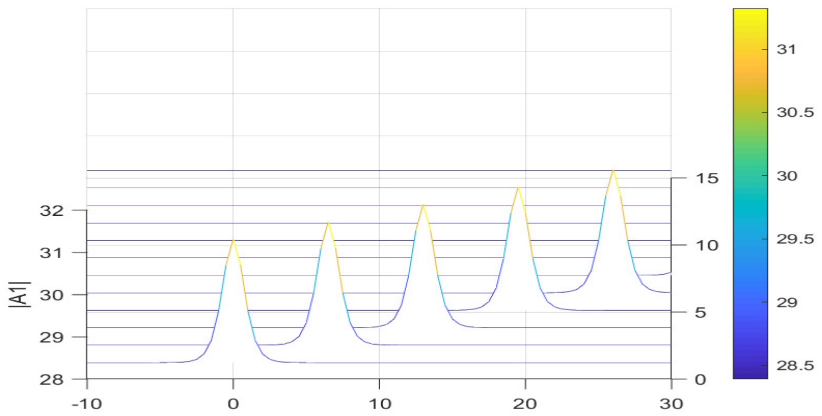
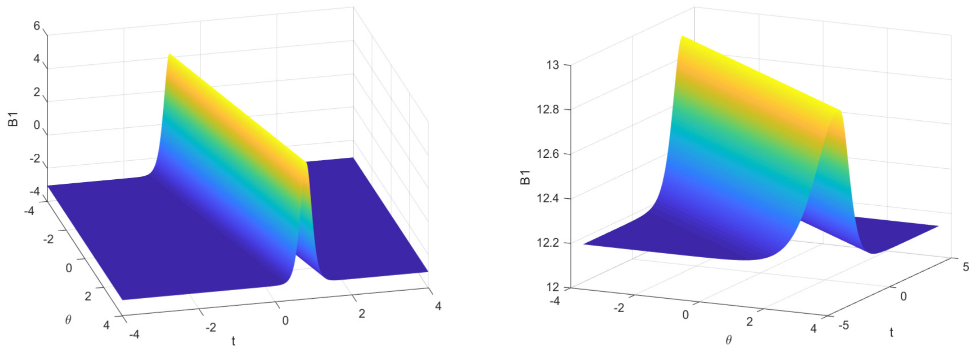


Figure 4. The propagation of A_1 with time t.



(**a**) (**b**)

Figure 5. Evolution of the barotropic Rossby wave amplitude B_1 with $d_3 = 50$, $c_i = 1$ $(i = 1, 2, \ldots, 15)$, $d_j = 1$ $(j = 1, 2, 4, \ldots, 15)$, $k = 1$, $m = 1$, $c = 5$, and $t = 0$.

Figure 6 depicts the evolution of the baroclinic wave amplitude B_1 without the influence of higher-order terms in the three-dimensional, respectively. In the absence of higher-order terms, the amplitude of the wave is increased, and the wave pattern becomes steep. This conclusion is consistent with the change in barotropic Rossby waves.

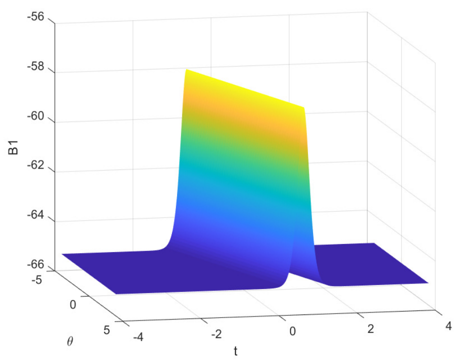
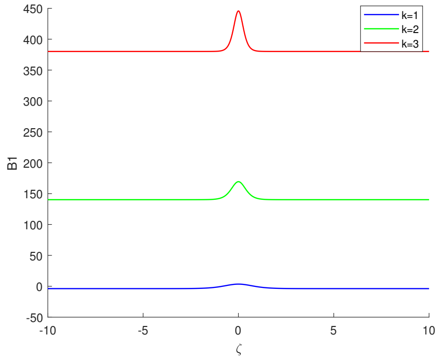
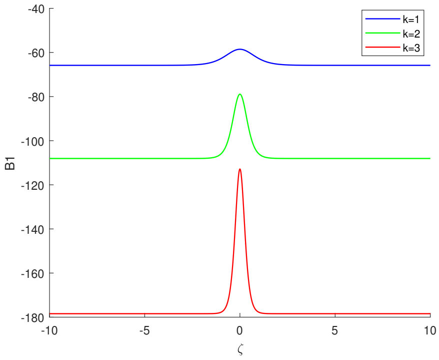


Figure 6. Evolution of the barotropic Rossby wave amplitude B_1 without high-order terms under the same parameters.

Figure 7 shows the effect of the wave number k on the baroclinic amplitude B_1. As shown in Figure 7a, the amplitude of the wave increases with the increase in k, and the width of the wave narrows when the higher-order terms are present and the other variables are the same. Figure 7b shows that the evolution trend of baroclinic wave amplitude does not change when there are no higher-order terms, meaning that the higher-order action disappears.



(a)



(b)

Figure 7. Influence of wave number k.

Figure 8 shows that the time evolution of the baroclinic Rossby wave amplitude B_1 is similar to that of the barotropic Rossby amplitude A_1, and the wave shape and velocity have not changed during the eastward movement. The number of waves generated by B_1 during the propagation process in the same time period is the same as that generated by A_1.

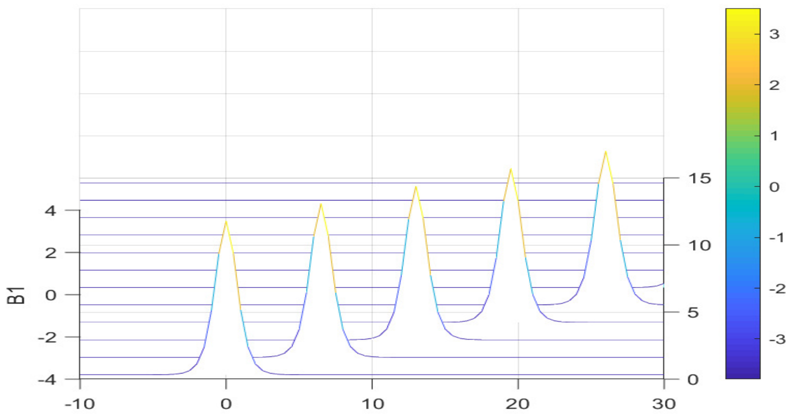


Figure 8. The propagation of B_1 with time t.

Rossby waves are closely related to various natural phenomena, such as weather changes, ocean currents, atmospheric high-pressure blocking, eddies in the Gulf of Mexico current, etc. Many scholars have used different types of KdV equations to simulate the evolution of Rossby wave amplitude, which has been widely used in oceanography, meteorology, and other fields. In this section, the amplitude evolution of Rossby waves with barotropic-baroclinic coherent structures is studied according to the solutions of the fifth-order coupled KdV-mKdV equations in polar co-ordinates. The Rossby waves can be excited and spread steadily with time under certain conditions. The above discussion focuses on the influence of higher-order terms and wave numbers on the amplitude of Rossby waves under barotropic and baroclinic interaction. It is interesting to find that higher-order equations are more practical than lower-order equations in describing the propagation of Rossby waves. These are the inheritance and development of previous theories, and they may have an important impact on many physical fields. According to the actual collected data, meteorologists can establish the fifth-order coupled KdV-mKdV equations to predict a change in Rossby waves, which has certain theoretical significance and practical value for predicting changes in the weather.

6. Conclusions

In this paper, we first obtain the fifth-order coupled KdV-mKdV equations in polar co-ordinates, according to the two-layer quasi-geostrophic vorticity equations; then, the evolution mechanism of Rossby wave amplitude under barotropic-baroclinic coherent structures was studied. The fifth-order coupled KdV-mKdV equations were analyzed through Lie symmetry and conservation laws. By utilizing the Jacobi elliptic function method, we obtained the soliton solutions for the fifth-order coupled KdV-mKdV equations. Next, we analyzed the higher-order effect, time, and wave number on Rossby wave propagation characteristics by using numerical simulations. The results show that the existence of high-order terms makes the wave deform, which shows that the amplitude decreases and the wave width widens. The amplitude of the barotropic wave is higher than that of the baroclinic wave, and the wave shape is sharper under the same conditions. The barotropic and baroclinic Rossby waves propagate steadily with time, and their amplitudes are affected by wave number.

In a word, the fifth-order coupled KdV-mKdV equations used to simulate the amplitude evolution of Rossby waves can be regarded as an extension of and supplement to the previous model. It provides ideas for scholars in the future to study nonlinear barotropic-baroclinic interaction dynamics, and it has potential application value in the study of natural phenomena, such as changes in weather.

In future research, it might be considered valuable to study the evolution of Rossby wave flow fields based on the fifth-order coupled KdV-mKdV equations, which were derived for the first time in this paper. Dipole blocking and its influence on weather changes will be further discussed. This will greatly enrich the meaning of the above-mentioned equations and the description of related physical phenomena.

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Appendix A

$$F_{11} = \frac{\partial}{\partial \tau_1}(-\frac{1}{r}\frac{\partial \varphi_{B_2}}{\partial r} - \frac{\partial^2 \varphi_{B_2}}{\partial r^2}) + \frac{\partial}{\partial \tau_2}(-\frac{1}{r}\frac{\partial \varphi_{B_1}}{\partial r} - \frac{\partial^2 \varphi_{B_1}}{\partial r^2}) + (\frac{c}{r^2} - \frac{1}{r^3}\frac{\partial \varphi_{B_0}}{\partial r})\frac{\partial^3 \varphi_{B_2}}{\partial \xi^3} -$$
$$\frac{1}{r}\frac{\partial \varphi_{B_1}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial \xi}(\frac{\partial \varphi_{B_2}}{\partial r}) + \frac{\partial}{\partial \xi}(\frac{\partial^2 \varphi_{B_2}}{\partial r^2})] - \frac{1}{r}\frac{\partial \varphi_{B_2}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial \xi}(\frac{\partial \varphi_{B_1}}{\partial r}) + \frac{\partial}{\partial \xi}(\frac{\partial^2 \varphi_{B_1}}{\partial r^2})] -$$
$$\frac{1}{4r}\frac{\partial \varphi_{T_0}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial \xi}(\frac{\partial \varphi_{T_3}}{\partial r}) + \frac{\partial}{\partial \xi}(\frac{\partial^2 \varphi_{T_3}}{\partial r^2}) + \frac{1}{r^2}\frac{\partial^3 \varphi_{T_1}}{\partial \xi^3} - 2F\frac{\partial \varphi_{T_3}}{\partial \xi}] - \frac{1}{4r}\frac{\partial \varphi_{T_1}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial \xi}(\frac{\partial \varphi_{T_2}}{\partial r}) +$$
$$\frac{\partial}{\partial \xi}(\frac{\partial^2 \varphi_{T_2}}{\partial r^2}) - 2F\frac{\partial \varphi_{T_2}}{\partial \xi}] - \frac{1}{4r}\frac{\partial \varphi_{T_2}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial \xi}(\frac{\partial \varphi_{T_1}}{\partial r}) + \frac{\partial}{\partial \xi}(\frac{\partial^2 \varphi_{T_1}}{\partial r^2}) - 2F\frac{\partial \varphi_{T_1}}{\partial \xi}] + \frac{1}{r}\frac{\partial \varphi_{B_1}}{\partial \xi}$$
$$[-\frac{1}{r^2}\frac{\partial \varphi_{B_2}}{\partial r} + \frac{1}{r}\frac{\partial^2 \varphi_{B_2}}{\partial r^2} + \frac{\partial^3 \varphi_{B_2}}{\partial r^3}] + \frac{1}{r}\frac{\partial \varphi_{B_2}}{\partial \xi}[-\frac{1}{r^2}\frac{\partial \varphi_{B_1}}{\partial r} + \frac{1}{r}\frac{\partial^2 \varphi_{B_1}}{\partial r^2} + \frac{\partial^3 \varphi_{B_1}}{\partial r^3}] +$$
$$\frac{1}{4r}\frac{\partial \varphi_{T_1}}{\partial \xi}[-\frac{1}{r^2}\frac{\partial \varphi_{T_2}}{\partial r} + \frac{1}{r}\frac{\partial^2 \varphi_{T_2}}{\partial r^2} + \frac{\partial^3 \varphi_{T_2}}{\partial r^3} - F\frac{\partial \varphi_{T_2}}{\partial r}] + \frac{1}{4r}\frac{\partial \varphi_{T_2}}{\partial \xi}[-\frac{1}{r^2}\frac{\partial \varphi_{T_1}}{\partial r} + \frac{1}{r}\frac{\partial^2 \varphi_{T_1}}{\partial r^2} +$$
$$\frac{\partial^3 \varphi_{T_1}}{\partial r^3} - F\frac{\partial \varphi_{T_1}}{\partial r}] + \frac{1}{4r}\frac{\partial \varphi_{T_3}}{\partial \xi}[-\frac{1}{r^2}\frac{\partial \varphi_{T_0}}{\partial r} + \frac{1}{r}\frac{\partial^2 \varphi_{T_0}}{\partial r^2} + \frac{\partial^3 \varphi_{T_0}}{\partial r^3} - F\frac{\partial \varphi_{T_0}}{\partial r}]$$

$$F_{21}=\frac{\partial}{\partial\tau_1}(-\frac{1}{r}\frac{\partial\varphi_{T_2}}{\partial r}-\frac{\partial^2\varphi_{T_2}}{\partial r^2}+2F\varphi_{T_2})+\frac{\partial}{\partial\tau_2}(-\frac{1}{r}\frac{\partial\varphi_{T_1}}{\partial r}-\frac{\partial^2\varphi_{T_1}}{\partial r^2}+2F\varphi_{T_1})+$$
$$(\frac{c}{r^2}-\frac{1}{r^3}\frac{\partial\varphi_{B_0}}{\partial r})\frac{\partial^3\varphi_{T_2}}{\partial\xi^3}-\frac{1}{r}\frac{\partial\varphi_{B_1}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{T_2}}{\partial r})+\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{T_2}}{\partial r^2})-2F\frac{\partial\varphi_{T_2}}{\partial\xi}]-\frac{1}{r}\frac{\partial\varphi_{B_2}}{\partial r}$$
$$[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{T_1}}{\partial r})+\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{T_1}}{\partial r^2})-2F\frac{\partial\varphi_{T_1}}{\partial\xi}]-\frac{1}{r}\frac{\partial\varphi_{T_0}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{B_3}}{\partial r})+\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{B_3}}{\partial r^2})+\frac{1}{r^2}\frac{\partial^3\varphi_{B_3}}{\partial r^3}]$$
$$-\frac{1}{r}\frac{\partial\varphi_{T_1}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{B_2}}{\partial r})+\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{B_2}}{\partial r^2})]-\frac{1}{r}\frac{\partial\varphi_{T_2}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{B_1}}{\partial r})+\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{B_1}}{\partial r^2})]+$$
$$\frac{1}{r}\frac{\partial\varphi_{B_1}}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{T_2}}{\partial r}+\frac{1}{r}\frac{\partial^2\varphi_{T_2}}{\partial r^2}+\frac{\partial^3\varphi_{T_2}}{\partial r^3}-2F\frac{\partial\varphi_{T_2}}{\partial r}]+\frac{1}{r}\frac{\partial\varphi_{B_2}}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{T_1}}{\partial r}+\frac{1}{r}\frac{\partial^2\varphi_{T_1}}{\partial r^2}+$$
$$\frac{\partial^3\varphi_{T_1}}{\partial r^3}-2F\frac{\partial\varphi_{T_1}}{\partial r}]+\frac{1}{r}\frac{\partial\varphi_{B_3}}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{T_0}}{\partial r}+\frac{1}{r}\frac{\partial^2\varphi_{T_0}}{\partial r^2}+\frac{\partial^3\varphi_{T_0}}{\partial r^3}-2F\frac{\partial\varphi_{T_0}}{\partial r}]+$$
$$\frac{1}{r}\frac{\partial\varphi_{T_1}}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{B_2}}{\partial r}+\frac{1}{r}\frac{\partial^2\varphi_{B_2}}{\partial r^2}+\frac{\partial^3\varphi_{B_2}}{\partial r^3}]+\frac{1}{r}\frac{\partial\varphi_{T_2}}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{B_1}}{\partial r}+\frac{1}{r}\frac{\partial^2\varphi_{B_1}}{\partial r^2}+\frac{\partial^3\varphi_{B_1}}{\partial r^3}]$$

$$F_{12}=\frac{\partial}{\partial\tau_1}(-\frac{1}{r}\frac{\partial\varphi_{B_3}}{\partial r}-\frac{\partial^2\varphi_{B_3}}{\partial r^2}-\frac{1}{r^2}\frac{\partial^2\varphi_{B_1}}{\partial\xi^2})+\frac{\partial}{\partial\tau_2}(-\frac{1}{r}\frac{\partial\varphi_{B_2}}{\partial r}-\frac{\partial^2\varphi_{B_2}}{\partial r^2})+$$
$$\frac{\partial}{\partial\tau_3}(-\frac{1}{r}\frac{\partial\varphi_{B_1}}{\partial r}-\frac{\partial^2\varphi_{B_1}}{\partial r^2})+(\frac{c}{r^2}-\frac{1}{r^3}\frac{\partial\varphi_{B_0}}{\partial r})\frac{\partial^3\varphi_{B_3}}{\partial\xi^3}-\frac{1}{r}\frac{\partial\varphi_{B_1}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{B_3}}{\partial r})+\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{B_3}}{\partial r^2})]$$
$$-\frac{1}{r}\frac{\partial\varphi_{B_2}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{B_2}}{\partial r})+\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{B_2}}{\partial r^2})]-\frac{1}{r}\frac{\partial\varphi_{B_3}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{B_1}}{\partial r})+\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{B_1}}{\partial r^2})]$$
$$-\frac{1}{4r}\frac{\partial\varphi_{T_0}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{T_4}}{\partial r})+\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{T_4}}{\partial r^2})+\frac{1}{r^2}\frac{\partial^3\varphi_{T_2}}{\partial\xi^3}-2F\frac{\partial\varphi_{T_4}}{\partial\xi}]-\frac{1}{4r}\frac{\partial\varphi_{T_1}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{T_3}}{\partial r})+$$
$$\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{T_3}}{\partial r^2})-2F\frac{\partial\varphi_{T_3}}{\partial\xi}+\frac{1}{r^2}\frac{\partial^3\varphi_{T_1}}{\partial\xi^3}]-\frac{1}{4r}\frac{\partial\varphi_{T_2}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{T_2}}{\partial r})+\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{T_2}}{\partial r^2})-2F\frac{\partial\varphi_{T_2}}{\partial\xi}]-$$
$$\frac{1}{4r}\frac{\partial\varphi_{T_3}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{T_1}}{\partial r})+\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{T_1}}{\partial r^2})-2F\frac{\partial\varphi_{T_1}}{\partial\xi}]+\frac{1}{r}\frac{\partial\varphi_{B_1}}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{B_3}}{\partial r}+\frac{1}{r}\frac{\partial^2\varphi_{B_3}}{\partial r^2}+$$
$$\frac{\partial^3\varphi_{B_3}}{\partial r^3}-\frac{2}{r^3}\frac{\partial^2\varphi_{B_1}}{\partial\xi^2}+\frac{1}{r^2}\frac{\partial}{\partial r}(\frac{\partial^2\varphi_{B_1}}{\partial\xi^2})]+\frac{1}{r}\frac{\partial\varphi_{B_2}}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{B_2}}{\partial r}+\frac{1}{r}\frac{\partial^2\varphi_{B_2}}{\partial r^2}+\frac{\partial^3\varphi_{B_2}}{\partial r^3}]+$$
$$\frac{1}{r}\frac{\partial\varphi_{B_3}}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{B_1}}{\partial r}+\frac{1}{r}\frac{\partial^2\varphi_{B_1}}{\partial r^2}+\frac{\partial^3\varphi_{B_1}}{\partial r^3}]+\frac{1}{4r}\frac{\partial\varphi_{T_1}}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{T_3}}{\partial r}+\frac{1}{r}\frac{\partial^2\varphi_{T_3}}{\partial r^2}+\frac{\partial^3\varphi_{T_3}}{\partial r^3}-$$
$$2F\frac{\partial\varphi_{T_3}}{\partial r}-\frac{2}{r^3}\frac{\partial^2\varphi_{T_1}}{\partial\xi^2}+\frac{1}{r^2}\frac{\partial}{\partial r}(\frac{\partial^2\varphi_{T_1}}{\partial\xi^2})]+\frac{1}{4r}\frac{\partial\varphi_{T_2}}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{T_2}}{\partial r}+\frac{1}{r}\frac{\partial^2\varphi_{T_2}}{\partial r^2}+\frac{\partial^3\varphi_{T_2}}{\partial r^3}-F\frac{\partial\varphi_{T_2}}{\partial r}]$$
$$+\frac{1}{4r}\frac{\partial\varphi_{T_3}}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{T_1}}{\partial r}+\frac{1}{r}\frac{\partial^2\varphi_{T_1}}{\partial r^2}+\frac{\partial^3\varphi_{T_1}}{\partial r^3}-F\frac{\partial\varphi_{T_1}}{\partial r}]+\frac{1}{4r}\frac{\partial\varphi_{T_4}}{\partial\xi}p(r).$$

$$F_{22}=\frac{\partial}{\partial\tau_1}(-\frac{1}{r}\frac{\partial\varphi_{T_3}}{\partial r}-\frac{\partial^2\varphi_{T_3}}{\partial r^2}+2F\varphi_{T_3}-\frac{1}{r^2}\frac{\partial^2\varphi_{T_1}}{\partial\xi^2})+\frac{\partial}{\partial\tau_2}(-\frac{1}{r}\frac{\partial\varphi_{T_2}}{\partial r}-\frac{\partial^2\varphi_{T_2}}{\partial r^2}+2F\varphi_{T_2})$$
$$+\frac{\partial}{\partial\tau_3}(-\frac{1}{r}\frac{\partial\varphi_{T_1}}{\partial r}-\frac{\partial^2\varphi_{T_1}}{\partial r^2}+2F\varphi_{T_1})+(\frac{c}{r^2}-\frac{1}{r^3}\frac{\partial\varphi_{B_0}}{\partial r})\frac{\partial^3\varphi_{T_3}}{\partial\xi^3}-\frac{1}{r}\frac{\partial\varphi_{B_1}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{T_3}}{\partial r})+$$
$$\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{T_3}}{\partial r^2})-2F\varphi_{T_3}]-\frac{1}{r}\frac{\partial\varphi_{B_2}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{T_2}}{\partial r})+\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{T_2}}{\partial r^2})-2F\varphi_{T_2}]-\frac{1}{r}\frac{\partial\varphi_{B_3}}{\partial r}[$$
$$\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{T_1}}{\partial r})+\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{T_1}}{\partial r^2})-2F\varphi_{T_1}]-\frac{1}{r}\frac{\partial\varphi_{T_0}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{B_4}}{\partial r})+\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{B_4}}{\partial r^2})+\frac{1}{r^2}\frac{\partial^3\varphi_{B_4}}{\partial r^3}]-$$
$$\frac{1}{r}\frac{\partial\varphi_{T_1}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{B_3}}{\partial r})+\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{B_3}}{\partial r^2})+\frac{1}{r^2}\frac{\partial^3\varphi_{B_3}}{\partial r^3}]-\frac{1}{r}\frac{\partial\varphi_{T_2}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{B_2}}{\partial r})+\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{B_2}}{\partial r^2})]-$$
$$\frac{1}{r}\frac{\partial\varphi_{T_3}}{\partial r}[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_{B_1}}{\partial r})+\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_{B_1}}{\partial r^2})]+\frac{1}{r}\frac{\partial\varphi_{B_1}}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{T_3}}{\partial r}+\frac{1}{r}\frac{\partial^2\varphi_{T_3}}{\partial r^2}+\frac{\partial^3\varphi_{T_3}}{\partial r^3}-2F\frac{\partial\varphi_{T_3}}{\partial r}-$$
$$\frac{2}{r^3}\frac{\partial^2\varphi_{T_1}}{\partial\xi^2}+\frac{1}{r^2}\frac{\partial}{\partial r}(\frac{\partial^2\varphi_{T_1}}{\partial\xi^2})]+\frac{1}{r}\frac{\partial\varphi_{B_2}}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{T_2}}{\partial r}+\frac{1}{r}\frac{\partial^2\varphi_{T_2}}{\partial r^2}+\frac{\partial^3\varphi_{T_2}}{\partial r^3}-2F\frac{\partial\varphi_{T_2}}{\partial r}]+$$
$$\frac{1}{r}\frac{\partial\varphi_{B_3}}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{T_1}}{\partial r}+\frac{1}{r}\frac{\partial^2\varphi_{T_1}}{\partial r^2}+\frac{\partial^3\varphi_{T_1}}{\partial r^3}-2F\frac{\partial\varphi_{T_1}}{\partial r}]+\frac{1}{r}\frac{\partial\varphi_{B_4}}{\partial\xi}p(r)+\frac{1}{r}\frac{\partial\varphi_{T_1}}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{B_3}}{\partial r}+$$
$$\frac{1}{r}\frac{\partial^2\varphi_{B_3}}{\partial r^2}+\frac{\partial^3\varphi_{B_3}}{\partial r^3}-\frac{2}{r^3}\frac{\partial^2\varphi_{B_1}}{\partial\xi^2}+\frac{1}{r^2}\frac{\partial}{\partial r}(\frac{\partial^2\varphi_{B_1}}{\partial\xi^2})]+\frac{1}{r}\frac{\partial\varphi_{T_2}}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{B_2}}{\partial r}+\frac{1}{r}\frac{\partial^2\varphi_{B_2}}{\partial r^2}+\frac{\partial^3\varphi_{B_2}}{\partial r^3}]$$
$$+\frac{1}{r}\frac{\partial\varphi_{T_3}}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_{B_1}}{\partial r}+\frac{1}{r}\frac{\partial^2\varphi_{B_1}}{\partial r^2}+\frac{\partial^3\varphi_{B_1}}{\partial r^3}].$$

$$F=(\epsilon^2\frac{\partial}{\partial\tau_1}+\epsilon^3\frac{\partial}{\partial\tau_2}+\epsilon^4\frac{\partial}{\partial\tau_3})(-\frac{1}{r}\frac{\partial\varphi_B}{\partial r}-\frac{\partial^2\varphi_B}{\partial r^2})-\frac{1}{r^2}(-c\frac{\partial}{\partial\xi}+\epsilon^2\frac{\partial}{\partial\tau_1}+\epsilon^3\frac{\partial}{\partial\tau_2}+$$
$$\epsilon^4\frac{\partial}{\partial\tau_3})\epsilon^2\frac{\partial^2\varphi_B}{\partial\xi^2}-\frac{1}{r^3}\frac{\partial\varphi_{B_0}}{\partial r}\epsilon^2\frac{\partial^3\varphi_B}{\partial\xi^3}-\frac{1}{r}\frac{\partial\varphi_B}{\partial r}\epsilon[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_B}{\partial r})+\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_B}{\partial r^2})+\frac{1}{r^2}\epsilon^2\frac{\partial^3\varphi_B}{\partial\xi^3}]-$$
$$\frac{1}{2r}\epsilon(\frac{\partial\varphi_{T_0}}{\partial r}+\frac{\partial\varphi_T}{\partial r})[\frac{1}{2r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_T}{\partial r})+\frac{1}{2}\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_T}{\partial r^2})+\frac{1}{2r^2}\epsilon^2\frac{\partial^3\varphi_T}{\partial\xi^3}-F\frac{\partial\varphi_T}{\partial\xi}]+\frac{1}{r}\epsilon\frac{\partial\varphi_B}{\partial\xi}$$
$$[-\frac{1}{r^2}\frac{\partial\varphi_B}{\partial r}+\frac{1}{r}\frac{\partial^2\varphi_B}{\partial r^2}+\frac{\partial^3\varphi_B}{\partial r^3}-\frac{2}{r^3}\epsilon^2\frac{\partial^2\varphi_B}{\partial\xi^2}+\frac{1}{r^2}\epsilon^2\frac{\partial}{\partial r}(\frac{\partial^2\varphi_B}{\partial\xi^2})]+\frac{1}{2r}\epsilon\frac{\partial\varphi_T}{\partial\xi}[$$
$$-\frac{1}{2r^2}(\frac{\partial\varphi_{T_0}}{\partial r}+\frac{\partial\varphi_T}{\partial r})+\frac{1}{2r}(\frac{\partial^2\varphi_{T_0}}{\partial r^2}+\frac{\partial^2\varphi_T}{\partial r^2})+\frac{1}{2}(\frac{\partial^3\varphi_{T_0}}{\partial r^3}+\frac{\partial^3\varphi_T}{\partial r^3})-\frac{1}{r^3}\epsilon^2\frac{\partial^2\varphi_T}{\partial\xi^2}+$$
$$\frac{1}{2r^2}\epsilon^2\frac{\partial}{\partial r}(\frac{\partial^2\varphi_T}{\partial\xi^2})-F(\frac{\partial\varphi_{T_0}}{\partial r}+\frac{\partial\varphi_T}{\partial r})].$$

$$G=(\epsilon^2\frac{\partial}{\partial\tau_1}+\epsilon^3\frac{\partial}{\partial\tau_2}+\epsilon^4\frac{\partial}{\partial\tau_3})(-\frac{1}{r}\frac{\partial\varphi_T}{\partial r}-\frac{\partial^2\varphi_T}{\partial r^2}+2F\varphi_T)-\frac{1}{r^2}(-c\frac{\partial}{\partial\xi}+\epsilon^2\frac{\partial}{\partial\tau_1}+$$
$$\epsilon^3\frac{\partial}{\partial\tau_2}+\epsilon^4\frac{\partial}{\partial\tau_3})\epsilon^2\frac{\partial^2\varphi_T}{\partial\xi^2}-\frac{1}{r^3}\frac{\partial\varphi_{B_0}}{\partial r}\epsilon^2\frac{\partial^3\varphi_T}{\partial\xi^3}-\frac{1}{r}\frac{\partial\varphi_B}{\partial r}\epsilon[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_T}{\partial r})+\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_T}{\partial r^2})+$$
$$\frac{1}{r^2}\epsilon^2\frac{\partial^3\varphi_T}{\partial\xi^3}-2F\frac{\partial\varphi_T}{\partial\xi}]-\frac{1}{r}\epsilon(\frac{\partial\varphi_{T_0}}{\partial r}+\frac{\partial\varphi_T}{\partial r})[\frac{1}{r}\frac{\partial}{\partial\xi}(\frac{\partial\varphi_B}{\partial r})+\frac{\partial}{\partial\xi}(\frac{\partial^2\varphi_B}{\partial r^2})+\frac{1}{r^2}\epsilon^2\frac{\partial^3\varphi_B}{\partial\xi^3}]+$$
$$\frac{1}{r}\epsilon\frac{\partial\varphi_B}{\partial\xi}[-\frac{1}{r^2}(\frac{\partial\varphi_{T_0}}{\partial r}+\frac{\partial\varphi_T}{\partial r})+\frac{1}{r}(\frac{\partial^2\varphi_{T_0}}{\partial r^2}+\frac{\partial^2\varphi_T}{\partial r^2})+(\frac{\partial^3\varphi_{T_0}}{\partial r^3}+\frac{\partial^3\varphi_T}{\partial r^3})-\frac{2}{r^3}\epsilon^2\frac{\partial^2\varphi_T}{\partial\xi^2}+$$
$$\frac{1}{r^2}\epsilon^2\frac{\partial}{\partial r}(\frac{\partial^2\varphi_T}{\partial\xi^2})-2F(\frac{\partial\varphi_{T_0}}{\partial r}+\frac{\partial\varphi_T}{\partial r})]+\frac{1}{r}\epsilon\frac{\partial\varphi_T}{\partial\xi}[-\frac{1}{r^2}\frac{\partial\varphi_T}{\partial r}+\frac{1}{r}\frac{\partial^2\varphi_T}{\partial r^2}+\frac{\partial^3\varphi_T}{\partial r^3}-$$
$$\frac{2}{r^3}\epsilon^2\frac{\partial^2\varphi_B}{\partial\xi^2}+\frac{1}{r^2}\epsilon^2\frac{\partial}{\partial r}(\frac{\partial^2\varphi_B}{\partial\xi^2})].$$

$$a = \int_{r_1}^{r_2} (-\frac{M_1}{r}\frac{\partial M_1}{\partial r} - M_1\frac{\partial^2 M_1}{\partial r^2})dr,$$

$$\alpha_1 = \frac{1}{a}\{\int_{r_1}^{r_2} -\frac{M_1}{2r}\frac{\partial \varphi_{T_0}}{\partial r}(\frac{1}{2r}\frac{\partial N_2}{\partial r} + \frac{1}{2}\frac{\partial^2 N_2}{\partial r^2} - FN_2) + \frac{M_1 N_2}{2r}(-\frac{1}{2r^2}\frac{\partial \varphi_{T_0}}{\partial r} + \frac{1}{2r}\frac{\partial^2 \varphi_{T_0}}{\partial r^2} + \frac{1}{2}\frac{\partial^3 \varphi_{T_0}}{\partial r^3} - F\frac{\partial \varphi_{T_0}}{\partial r})dr\},$$

$$\alpha_2 = \frac{1}{a}\int_{r_1}^{r_2} -\frac{M_1}{r}\frac{\partial M_1}{\partial r}(\frac{1}{r}\frac{\partial M_1}{\partial r} + \frac{\partial^2 M_1}{\partial r^2}) + \frac{M_1^2}{r}(-\frac{1}{r^2}\frac{\partial M_1}{\partial r} + \frac{1}{r}\frac{\partial^2 M_1}{\partial r^2} + \frac{\partial^3 M_1}{\partial r^3})dr,$$

$$\alpha_3 = \frac{1}{a}\{\int_{r_1}^{r_2} -\frac{M_1}{4r}\frac{\partial N_2}{\partial r}(\frac{1}{r}\frac{\partial N_1}{\partial r} + \frac{\partial^2 N_1}{\partial r^2} - 2FN_1) + \frac{M_1 N_1}{4r}(-\frac{1}{r^2}\frac{\partial N_1}{\partial r} + \frac{1}{r}\frac{\partial^2 N_1}{\partial r^2} + \frac{\partial^3 N_1}{\partial r^3} - 2F\frac{\partial N_1}{\partial r})dr\},$$

$$\alpha_4 = \frac{1}{a}\int_{r_1}^{r_2} M_1^2(\frac{c}{r^2} - \frac{1}{r^3}\frac{\partial \varphi_{B_0}}{\partial r})dr,$$

$$\alpha_5 = \frac{1}{a}\int_{r_1}^{r_2} (\frac{M_1}{r}\frac{\partial M_2}{\partial r} + M_1\frac{\partial^2 M_2}{\partial r^2})\beta_1 - \frac{M_1}{2r}\frac{\partial \varphi_{T_0}}{\partial r}(\frac{1}{2r}\frac{\partial J_1}{\partial r} + \frac{1}{2}\frac{\partial^2 J_1}{\partial r^2} - FJ_1) + \frac{M_1 P(r)}{4r}J_1 dr,$$

$$\alpha_6 = \frac{1}{a}\int_{r_1}^{r_2} (\frac{M_1}{r}\frac{\partial M_2}{\partial r} + M_1\frac{\partial^2 M_2}{\partial r^2})\beta_2 + [-\frac{M_1}{r}\frac{\partial M_2}{\partial r}(\frac{1}{r}\frac{\partial M_1}{\partial r} + \frac{\partial^2 M_1}{\partial r^2}) - \frac{M_1}{4r}\frac{\partial N_1}{\partial r}(\frac{1}{r}\frac{\partial N_2}{\partial r} + \frac{\partial^2 N_2}{\partial r^2} - FN_2) + \frac{M_1^2}{r}(-\frac{1}{r^2}\frac{\partial M_2}{\partial r} + \frac{1}{r}\frac{\partial^2 M_2}{\partial r^2} + \frac{\partial^3 M_2}{\partial r^3}) + \frac{M_1 N_2}{4r}(-\frac{1}{r^2}\frac{\partial N_1}{\partial r} + \frac{1}{r}\frac{\partial^2 N_1}{\partial r^2} + \frac{1}{2}\frac{\partial^3 N_1}{\partial r^3} - F\frac{\partial N_1}{\partial r})] - \frac{M_1}{4r}\frac{\partial \varphi_{T_0}}{\partial r}(\frac{1}{r}\frac{\partial J_2}{\partial r} + \frac{\partial^2 J_2}{\partial r^2} - 2FJ_2) + \frac{M_1 P(r)}{2r}J_2 dr,$$

$$\alpha_7 = \frac{1}{a}\int_{r_1}^{r_2} (\frac{M_1}{r}\frac{\partial M_2}{\partial r} + M_1\frac{\partial^2 M_2}{\partial r^2})\beta_4 + (\frac{M_1 M_2 c}{r^2} - \frac{M_1 M_2}{r^3}\frac{\partial \varphi_{B_0}}{\partial r} - \frac{M_1 N_1}{4r^3}\frac{\partial \varphi_{T_0}}{\partial r}) - \frac{M_1}{4r}\frac{\partial \varphi_{T_0}}{\partial r}(\frac{1}{r}\frac{\partial J_3}{\partial r} + \frac{\partial^2 J_3}{\partial r^2} - 2FJ_3) + \frac{M_1 P(r)}{2r}J_3,$$

$$\alpha_8 = \frac{1}{a}\{\int_{r_1}^{r_2} \frac{M_1}{r}(\frac{\partial I_1}{\partial r}\alpha_1 + \frac{\partial M_2}{\partial r}\beta_5) + M_1(\frac{\partial^2 I_1}{\partial r^2}\alpha_1 + \frac{\partial^2 M_2}{\partial r^2}\beta_5) - \frac{M_1}{4r}\frac{\partial \varphi_{T_0}}{\partial r}(\frac{1}{r}\frac{\partial J_5}{\partial r} + \frac{\partial^2 J_5}{\partial r^2} - 2FJ_5) + \frac{M_1}{4r}J_5 p(r)dr\},$$

$$\alpha_9 = \frac{1}{a}\{\int_{r_1}^{r_2} \frac{M_1}{r}(\frac{\partial I_1}{\partial r}\alpha_2 + 2\frac{\partial I_2}{\partial r}\alpha_1 + \frac{\partial M_2}{\partial r}\beta_6) + M_1(\frac{\partial^2 I_1}{\partial r^2}\alpha_2 + 2\frac{\partial^2 I_2}{\partial r^2}\alpha_1 + \frac{\partial^2 M_2}{\partial r^2}\beta_6) - \frac{M_1}{r}\frac{\partial M_1}{\partial r}(\frac{1}{r}\frac{\partial I_1}{\partial r} + \frac{\partial^2 I_1}{\partial r^2}) - \frac{M_1}{r}\frac{\partial I_1}{\partial r}(\frac{1}{r}\frac{\partial M_1}{\partial r} + \frac{\partial^2 M_1}{\partial r^2}) - \frac{M_1}{4r}\frac{\partial \varphi_{T_0}}{\partial r}(\frac{1}{r}\frac{\partial J_6}{\partial r} + \frac{\partial^2 J_6}{\partial r^2} - 2FJ_6) - \frac{M_1}{4r}\frac{\partial N_2}{\partial r}(\frac{1}{r}\frac{\partial N_2}{\partial r} + \frac{\partial^2 N_2}{\partial r^2} - FN_2) + \frac{M_1^2}{r}(-\frac{1}{r^2}\frac{\partial I_1}{\partial r} + \frac{1}{r}\frac{\partial^2 I_1}{\partial r^2} + \frac{\partial^3 I_1}{\partial r^3}) + \frac{M_1 I_1}{r}(-\frac{1}{r^2}\frac{\partial M_1}{\partial r} + \frac{1}{r}\frac{\partial^2 M_1}{\partial r^2} + \frac{\partial^3 M_1}{\partial r^3}) + \frac{M_1 N_2}{4r}(-\frac{1}{r^2}\frac{\partial N_2}{\partial r} + \frac{1}{r}\frac{\partial^2 N_2}{\partial r^2} + \frac{\partial^3 M_1}{\partial r^3} - F\frac{\partial N_2}{\partial r}) + \frac{M_1}{2r}p(r)J_6 dr\},$$

$$\alpha_{10} = \frac{1}{a}\{\int_{r_1}^{r_2} \frac{M_1}{r}(\frac{\partial I_1}{\partial r}\alpha_3 + 2\frac{\partial I_3}{\partial r}\beta_1 + \frac{\partial M_2}{\partial r}\beta_7) + M_1(\frac{\partial^2 I_1}{\partial r^2}\alpha_3 + 2\frac{\partial^2 I_3}{\partial r^2}\beta_1 + \frac{\partial^2 M_2}{\partial r^2}\beta_7) - \frac{M_1}{r}\frac{\partial M_2}{\partial r}(\frac{1}{r}\frac{\partial M_2}{\partial r} + \frac{\partial^2 M_2}{\partial r^2}) - \frac{M_1}{4r}\frac{\partial J_1}{\partial r}(\frac{1}{r}\frac{\partial N_1}{\partial r} + \frac{\partial^2 N_1}{\partial r^2} - FN_1) - \frac{M_1}{4r}\frac{\partial \varphi_{T_0}}{\partial r}(\frac{1}{r}\frac{\partial J_7}{\partial r} + \frac{\partial^2 J_7}{\partial r^2} - 2FJ_7) - \frac{M_1}{4r}\frac{\partial N_1}{\partial r}(\frac{1}{r}\frac{\partial J_1}{\partial r} + \frac{\partial^2 J_1}{\partial r^2} - FJ_1) + \frac{M_1 M_2}{r}(-\frac{1}{r^2}\frac{\partial M_2}{\partial r} + \frac{1}{r}\frac{\partial^2 M_2}{\partial r^2} + \frac{\partial^3 M_2}{\partial r^3}) + \frac{M_1 N_1}{4r}(-\frac{1}{r^2}\frac{\partial J_1}{\partial r} + \frac{1}{r}\frac{\partial^2 J_1}{\partial r^2} + \frac{\partial^3 J_1}{\partial r^3} - F\frac{\partial J_1}{\partial r}) + \frac{M_1 J_1}{4r}(-\frac{1}{r^2}\frac{\partial N_1}{\partial r} + \frac{1}{r}\frac{\partial^2 N_1}{\partial r^2} + \frac{\partial^3 N_1}{\partial r^3} - F\frac{\partial N_1}{\partial r}) + \frac{M_1}{2r}p(r)J_7 dr\},$$

$$\alpha_{11} = \frac{1}{a}\{\int_{r_1}^{r_2} 2\frac{M_1}{r}\frac{\partial I_2}{\partial r}\alpha_2 + 2M_1\frac{\partial^2 I_2}{\partial r^2}\alpha_2 - \frac{2M_1}{r}\frac{\partial M_1}{\partial r}(\frac{1}{r}\frac{\partial I_2}{\partial r} + \frac{\partial^2 I_2}{\partial r^2}) - \frac{M_1}{r}\frac{\partial I_2}{\partial r}(\frac{1}{r}\frac{\partial M_1}{\partial r} + \frac{\partial^2 M_1}{\partial r^2}) + \frac{M_1^2}{r}(-\frac{1}{r^2}\frac{\partial I_2}{\partial r} + \frac{1}{r}\frac{\partial^2 I_2}{\partial r^2} + \frac{\partial^3 I_2}{\partial r^3}) + 2\frac{M_1 I_2}{r}(-\frac{1}{r^2}\frac{\partial M_1}{\partial r} + \frac{1}{r}\frac{\partial^2 M_1}{\partial r^2} + \frac{\partial^3 M_1}{\partial r^3})dr\},$$

$$\alpha_{12} = \frac{1}{a}\{\int_{r_1}^{r_2} 2\frac{M_1}{r}\frac{\partial I_3}{\partial r}\beta_2 + 2M_1\frac{\partial^2 I_3}{\partial r^2}\beta_2 - \frac{M_1}{r}\frac{\partial I_3}{\partial r}(\frac{1}{r}\frac{\partial M_1}{\partial r} + \frac{\partial^2 M_1}{\partial r^2}) - \frac{M_1}{4r}\frac{\partial N_1}{\partial r}(\frac{1}{r}\frac{\partial J_2}{\partial r} + \frac{\partial^2 J_2}{\partial r^2} - 2FJ_2) + \frac{M_1^2}{r}(-\frac{1}{r^2}\frac{\partial I_3}{\partial r} + \frac{1}{r}\frac{\partial^2 I_3}{\partial r^2} + \frac{\partial^3 I_3}{\partial r^3}) + 2\frac{M_1 I_3}{r}(-\frac{1}{r^2}\frac{\partial M_1}{\partial r} + \frac{1}{r}\frac{\partial^2 M_1}{\partial r^2} + \frac{\partial^3 M_1}{\partial r^3})dr\},$$

$$\alpha_{13} = \frac{1}{a}\{\int_{r_1}^{r_2} 2\frac{M_1}{r}(\frac{\partial I_2}{\partial r}\alpha_3 + \frac{\partial I_3}{\partial r}\beta_2) + 2M_1(\frac{\partial^2 I_2}{\partial r^2}\alpha_3 + \frac{\partial^2 I_3}{\partial r^2}\beta_2) - \frac{2M_1}{r}\frac{\partial M_1}{\partial r}(\frac{1}{r}\frac{\partial I_3}{\partial r} + \frac{\partial^2 I_3}{\partial r^2}) - \frac{M_1}{4r}\frac{\partial J_2}{\partial r}(\frac{1}{r}\frac{\partial M_1}{\partial r} + \frac{\partial^2 M_1}{\partial r^2} - 2FM_1) - \frac{M_1}{4r}\frac{\partial N_1}{\partial r}(\frac{1}{r}\frac{\partial J_2}{\partial r} + \frac{\partial^2 J_2}{\partial r^2} - 2FJ_2) + \frac{M_1 N_1}{4r}(-\frac{1}{r^2}\frac{\partial J_2}{\partial r} + \frac{1}{r}\frac{\partial^2 J_2}{\partial r^2} + \frac{\partial^3 J_2}{\partial r^3} - F\frac{\partial J_2}{\partial r}) + \frac{M_1 J_2}{4r}(-\frac{1}{r^2}\frac{\partial N_1}{\partial r} + \frac{1}{r}\frac{\partial^2 N_1}{\partial r^2} + \frac{\partial^3 N_1}{\partial r^3} - F\frac{\partial N_1}{\partial r})dr\},$$

$$\alpha_{14} = \frac{1}{a}\{\int_{r_1}^{r_2} -\frac{M_1}{r}\frac{\partial I_4}{\partial r}(\frac{1}{r}\frac{\partial M_1}{\partial r} + \frac{\partial^2 M_1}{\partial r^2}) + \frac{M_1^2}{r}(-\frac{1}{r^2}\frac{\partial I_4}{\partial r} + \frac{1}{r}\frac{\partial^2 I_4}{\partial r^2} + \frac{\partial^3 I_4}{\partial r^3} - \frac{2}{r^3}M_1 + \frac{1}{r^2}\frac{\partial M_1}{\partial r})dr\},$$

$$\alpha_{15} = \frac{1}{a}\{\int_{r_1}^{r_2} -\frac{M_1}{4r}\frac{\partial J_4}{\partial r}(\frac{1}{r}\frac{\partial N_1}{\partial r} + \frac{\partial^2 N_1}{\partial r^2} - FN_1) + \frac{M_1 N_1}{4r}(-\frac{1}{r^2}\frac{\partial J_4}{\partial r} + \frac{1}{r}\frac{\partial^2 J_4}{\partial r^2} + \frac{\partial^3 J_4}{\partial r^3} - \frac{2}{r^3}N_1 + \frac{1}{r^2}\frac{\partial N_1}{\partial r} - 2F\frac{\partial J_4}{\partial r})dr\},$$

$$\alpha_{16} = \frac{1}{a}\{\int_{r_1}^{r_2} \frac{M_1}{r}(\frac{\partial I_1}{\partial r}\alpha_4 + \frac{\partial I_4}{\partial r}\alpha_1 + \frac{\partial M_2}{\partial r}\beta_8) + M_1(\frac{\partial^2 I_1}{\partial r^2}\alpha_4 + 2\frac{\partial^2 I_4}{\partial r^2}\alpha_1 + \frac{\partial^2 M_2}{\partial r^2}\beta_8) + \frac{cM_1 I_1}{r^2} + \frac{M_1^2}{r^2}\alpha_1 - \frac{M_1 I_1}{r^3}\frac{\partial \varphi_{B_0}}{\partial r} - \frac{M_1}{4r}\frac{\partial \varphi_{T_0}}{\partial r}(\frac{1}{r}\frac{\partial J_8}{\partial r} + \frac{\partial^2 J_8}{\partial r^2} - 2FJ_8 + \frac{N_2}{2r^2}) + \frac{M_1}{2r}p(r)J_8 dr\},$$

$$\alpha_{17} = \frac{1}{a}\{\int_{r_1}^{r_2} \frac{M_1}{r}(2\frac{\partial I_2}{\partial r}\alpha_4 + \frac{\partial I_4}{\partial r}\alpha_2) + M_1(2\frac{\partial^2 I_2}{\partial r^2}\alpha_4 + \frac{\partial^2 I_4}{\partial r^2}\alpha_2) + 2\frac{cM_1 I_2}{r^2} + \frac{M_1^2}{r^2}\alpha_2 - 2\frac{M_1 I_2}{r^3}\frac{\partial \varphi_{B_0}}{\partial r} - \frac{M_1}{r}\frac{\partial M_1}{\partial r}(\frac{1}{r}\frac{\partial I_4}{\partial r} + \frac{\partial^2 I_4}{\partial r^2} + \frac{M_1}{r^2}) + \frac{M_1 I_4}{r}(-\frac{1}{r^2}\frac{\partial M_1}{\partial r} + \frac{1}{r}\frac{\partial^2 M_1}{\partial r^2} + \frac{\partial^3 M_1}{\partial r^3})dr\},$$

$$\alpha_{18} = \frac{1}{a}\{\int_{r_1}^{r_2} \frac{M_1}{r}(2\frac{\partial I_3}{\partial r}\beta_4 + \frac{\partial I_4}{\partial r}\alpha_2) + M_1(2\frac{\partial^2 I_3}{\partial r^2}\beta_4 + \frac{\partial^2 I_4}{\partial r^2}\alpha_4) + 2\frac{cM_1 I_3}{r^2} + \frac{M_1^2}{r^2}\alpha_3 - 2\frac{M_1 I_3}{r^3}\frac{\partial \varphi_{B_0}}{\partial r} - \frac{M_1}{4r}\frac{\partial N_1}{\partial r}(\frac{1}{r}\frac{\partial J_4}{\partial r} + \frac{\partial^2 J_4}{\partial r^2} + \frac{M_1}{r^2} - 2FJ_4)dr\},$$

$$\alpha_{19} = \frac{1}{a}\{\int_{r_1}^{r_2} \frac{M_1}{r}\frac{\partial I_4}{\partial r}\alpha_4 + M_1\frac{\partial^2 I_4}{\partial r^2}\alpha_4 + \frac{cM_1 I_4}{r^2} + \frac{M_1^2}{r^2}\alpha_4 - \frac{M_1 I_4}{r^3}\frac{\partial \varphi_{B_0}}{\partial r}dr\}.$$

$$b = \int_{r_1}^{r_2}(-\frac{N_1}{r}\frac{\partial N_1}{\partial r} - N_1\frac{\partial^2 N_1}{\partial r^2} + 2FN_1)dr$$

$$\beta_1 = \frac{1}{b}\{\int_{r_1}^{r_2} -\frac{N_1}{r}\frac{\partial \varphi_{T_0}}{\partial r}(\frac{1}{r}\frac{\partial M_2}{\partial r} + \frac{\partial^2 M_2}{\partial r^2}) + \frac{N_1 M_2}{r}(-\frac{1}{r^2}\frac{\partial \varphi_{T_0}}{\partial r} + \frac{1}{r}\frac{\partial^2 \varphi_{T_0}}{\partial r^2} + \frac{\partial^3 \varphi_{T_0}}{\partial r^3} - 2F\frac{\partial \varphi_{T_0}}{\partial r})dr\},$$

$$\beta_2 = \beta_3 = \frac{1}{b}\{\int_{r_1}^{r_2} -\frac{N_1}{r}\frac{\partial M_1}{\partial r}(\frac{1}{r}\frac{\partial N_1}{\partial r} + \frac{\partial^2 N_1}{\partial r^2} - 2FN_1) + \frac{N_1^2}{r}(-\frac{1}{r^2}\frac{\partial M_1}{\partial r} + \frac{1}{r}\frac{\partial^2 M_1}{\partial r^2} + \frac{\partial^3 M_1}{\partial r^3})dr\},$$

$$\beta_4 = \frac{1}{b}\{\int_{r_1}^{r_2} N_1^2(\frac{c}{r^2} - \frac{1}{r^3}\frac{\partial \varphi_{B_0}}{\partial r})dr\},$$

$$\beta_5 = \frac{1}{b}\{\int_{r_1}^{r_2}(\frac{N_1}{r}\frac{\partial N_2}{\partial r} + N_1\frac{\partial^2 N_2}{\partial r^2} + 2FN_1N_2)\alpha_1 - \frac{N_1}{r}\frac{\partial \varphi_{T_0}}{\partial r}(\frac{1}{r}\frac{\partial I_1}{\partial r} + \frac{\partial^2 I_1}{\partial r^2}) + \frac{N_1 p(r)}{r}I_1 dr\},$$

$$\beta_6 = \frac{1}{b}\{\int_{r_1}^{r_2}(\frac{N_1}{r}\frac{\partial N_2}{\partial r} + N_1\frac{\partial^2 N_2}{\partial r^2} + 2FN_1N_2)\alpha_2 + [-\frac{N_1}{r}\frac{\partial M_1}{\partial r}(\frac{1}{r}\frac{\partial N_2}{\partial r} + \frac{\partial^2 N_2}{\partial r^2} - 2FN_2) - \frac{N_1}{r}\frac{\partial N_2}{\partial r}(\frac{1}{r}\frac{\partial M_1}{\partial r} + \frac{\partial^2 M_1}{\partial r^2}) + \frac{M_1 N_1}{r}(-\frac{1}{r^2}\frac{\partial N_2}{\partial r} + \frac{1}{r}\frac{\partial^2 N_2}{\partial r^2} + \frac{\partial^3 N_2}{\partial r^3} - 2F\frac{\partial N_2}{\partial r}) + \frac{N_1 N_2}{r}(-\frac{1}{r^2}\frac{\partial M_1}{\partial r} + \frac{1}{r}\frac{\partial^2 M_1}{\partial r^2} + \frac{\partial^3 M_1}{\partial r^3})] - \frac{N_1}{r}\frac{\partial \varphi_{T_0}}{\partial r}(\frac{1}{r}\frac{\partial I_2}{\partial r} + \frac{\partial^2 I_2}{\partial r^2}) + \frac{N_1 P(r)}{r}I_2)dr\},$$

$$\beta_7 = \frac{1}{b}\{\int_{r_1}^{r_2}(\frac{N_1}{r}\frac{\partial N_2}{\partial r} + N_1\frac{\partial^2 N_2}{\partial r^2} + 2FN_1N_2)\alpha_3 + [-\frac{N_1}{r}\frac{\partial N_2}{\partial r}(\frac{1}{r}\frac{\partial N_1}{\partial r} + \frac{\partial^2 N_1}{\partial r^2} - 2FN_1) - \frac{N_1}{r}\frac{\partial N_1}{\partial r}(\frac{1}{r}\frac{\partial M_2}{\partial r} + \frac{\partial^2 M_2}{\partial r^2}) + \frac{M_2 N_1}{r}(-\frac{1}{r^2}\frac{\partial N_1}{\partial r} + \frac{1}{r}\frac{\partial^2 N_1}{\partial r^2} + \frac{\partial^3 N_1}{\partial r^3} - 2F\frac{\partial N_1}{\partial r}) + \frac{N_1^2}{r}(-\frac{1}{r^2}\frac{\partial M_2}{\partial r} + \frac{1}{r}\frac{\partial^2 M_2}{\partial r^2} + \frac{\partial^3 M_2}{\partial r^3})] - \frac{N_1}{r}\frac{\partial \varphi_{T_0}}{\partial r}(\frac{1}{r}\frac{\partial I_3}{\partial r} + \frac{\partial^2 I_3}{\partial r^2}) + \frac{N_1 P(r)}{r}I_3)dr\},$$

$$\beta_8 = \frac{1}{b}\{\int_{r_1}^{r_2}(\frac{N_1}{r}\frac{\partial N_2}{\partial r} + N_1\frac{\partial^2 N_2}{\partial r^2} + 2FN_1N_2)\alpha_4 + (\frac{N_1 N_2 c}{r^2} - \frac{N_1 N_2}{r^3}\frac{\partial \varphi_{B_0}}{\partial r}) - \frac{N_1}{r}\frac{\partial \varphi_{T_0}}{\partial r}(\frac{1}{r}\frac{\partial I_4}{\partial r} + \frac{\partial^2 I_4}{\partial r^2} + \frac{1}{r^2}M_1) + \frac{N_1 P(r)}{r}I_4 dr\},$$

$$\beta_9 = \frac{1}{b}\{\int_{r_1}^{r_2} \frac{N_1}{r}(\frac{\partial J_1}{\partial r}\beta_1 + \frac{\partial N_2}{\partial r}\alpha_5) + N_1(\frac{\partial^2 J_1}{\partial r^2}\beta_1 + \frac{\partial^2 N_2}{\partial r^2}\alpha_5) - 2FN_1(J_1\beta_1 + \alpha_5 N_2) - \frac{N_1}{4r}\frac{\partial \varphi_{T_0}}{\partial r}(\frac{1}{r}\frac{\partial I_5}{\partial r} + \frac{\partial^2 I_5}{\partial r^2}) + \frac{N_1}{4r}I_5 p(r)dr\},$$

$$\beta_{10} = \frac{1}{b}\{\int_{r_1}^{r_2} \frac{N_1}{r}(\frac{\partial J_1}{\partial r}\beta_2 + \frac{\partial J_2}{\partial r}\beta_1 + \frac{\partial N_2}{\partial r}\alpha_6) + N_1(\frac{\partial^2 J_1}{\partial r^2}\beta_2 + \frac{\partial^2 J_2}{\partial r^2}\beta_1 + \frac{\partial^2 N_2}{\partial r^2}\beta_6) - 2FN_1(J_1\beta_2 + J_2\beta_1 + N_2\alpha_6) - \frac{N_1}{r}\frac{\partial M_1}{\partial r}(\frac{1}{r}\frac{\partial J_1}{\partial r} + \frac{\partial^2 J_1}{\partial r^2} - 2FJ_1) - \frac{N_1}{r}\frac{\partial I_1}{\partial r}(\frac{1}{r}\frac{\partial N_1}{\partial r} + \frac{\partial^2 N_1}{\partial r^2} - 2FN_1) - \frac{N_1}{r}\frac{\partial \varphi_{T_0}}{\partial r}(\frac{1}{r}\frac{\partial I_6}{\partial r} + \frac{\partial^2 I_6}{\partial r^2}) - \frac{N_1}{r}\frac{\partial N_2}{\partial r}(\frac{1}{r}\frac{\partial M_2}{\partial r} + \frac{\partial^2 M_2}{\partial r^2}) + \frac{N_1^2}{r}(-\frac{1}{r^2}\frac{\partial I_1}{\partial r} + \frac{1}{r}\frac{\partial^2 I_1}{\partial r^2} + \frac{\partial^3 I_1}{\partial r^3}) + \frac{N_1 J_1}{r}(-\frac{1}{r^2}\frac{\partial M_1}{\partial r} + \frac{1}{r}\frac{\partial^2 M_1}{\partial r^2} + \frac{\partial^3 M_1}{\partial r^3}) + \frac{M_2 N_1}{r}(-\frac{1}{r^2}\frac{\partial N_2}{\partial r} + \frac{1}{r}\frac{\partial^2 N_2}{\partial r^2} + \frac{\partial^3 M_2}{\partial r^3} - F\frac{\partial N_2}{\partial r}) + \frac{N_1}{r}p(r)I_6 dr\},$$

$$
\begin{aligned}
\beta_{11} =&\frac{1}{b}\{\int_{r_1}^{r_2}\frac{N_1}{r}(\frac{\partial J_1}{\partial r}\beta_2+\frac{\partial J_2}{\partial r}\alpha_1+\frac{\partial N_2}{\partial r}\alpha_6)+N_1(\frac{\partial^2 J_1}{\partial r^2}\beta_2+\frac{\partial^2 J_2}{\partial r^2}\alpha_1+\frac{\partial^2 N_2}{\partial r^2}\beta_6)- \\
&2FN_1(J_1\beta_2+J_2\beta_1+N_2\alpha_6)-\frac{N_1}{r}\frac{\partial M_2}{\partial r}(\frac{1}{r}\frac{\partial N_2}{\partial r}+\frac{\partial^2 N_2}{\partial r^2}-2FN_2)- \\
&\frac{N_1}{r}\frac{\partial J_1}{\partial r}(\frac{1}{r}\frac{\partial M_1}{\partial r}+\frac{\partial^2 M_1}{\partial r^2})-\frac{N_1}{r}\frac{\partial \varphi_{T_0}}{\partial r}(\frac{1}{r}\frac{\partial I_6}{\partial r}+\frac{\partial^2 I_6}{\partial r^2})-\frac{N_1}{r}\frac{\partial N_1}{\partial r}(\frac{1}{r}\frac{\partial I_1}{\partial r}+\frac{\partial^2 I_1}{\partial r^2})+ \\
&\frac{N_1M_1}{r}(-\frac{1}{r^2}\frac{\partial J_1}{\partial r}+\frac{1}{r}\frac{\partial^2 J_1}{\partial r^2}+\frac{\partial^3 J_1}{\partial r^3}-2F\frac{\partial J_1}{\partial r})+\frac{N_1I_1}{r}(-\frac{1}{r^2}\frac{\partial M_1}{\partial r}+\frac{1}{r}\frac{\partial^2 M_1}{\partial r^2}+\frac{\partial^3 M_1}{\partial r^3})+ \\
&\frac{N_1N_2}{r}(-\frac{1}{r^2}\frac{\partial M_2}{\partial r}+\frac{1}{r}\frac{\partial^2 M_2}{\partial r^2}+\frac{\partial^3 M_2}{\partial r^3})+\frac{N_1}{r}p(r)I_6dr\}, \\
\beta_{12} =&\frac{1}{b}\{\int_{r_1}^{r_2}\frac{N_1}{r}\frac{\partial J_2}{\partial r}\beta_2+N_1\frac{\partial^2 J_2}{\partial r^2}\beta_2-2FN_1J_2\beta_2-\frac{N_1}{r}\frac{\partial M_1}{\partial r}(\frac{1}{r}\frac{\partial J_2}{\partial r}+\frac{\partial^2 J_2}{\partial r^2}-2FJ_2)- \\
&\frac{N_1}{r}\frac{\partial I_1}{\partial r}(\frac{1}{r}\frac{\partial N_1}{\partial r}+\frac{\partial^2 N_1}{\partial r^2}-2FN_1)+\frac{N_1^2}{r}(-\frac{1}{r^2}\frac{\partial I_2}{\partial r}+\frac{1}{r}\frac{\partial^2 I_2}{\partial r^2}+\frac{\partial^3 I_2}{\partial r^3})+ \\
&\frac{N_1J_2}{r}(-\frac{1}{r^2}\frac{\partial M_1}{\partial r}+\frac{1}{r}\frac{\partial^2 M_1}{\partial r^2}+\frac{\partial^3 M_1}{\partial r^3})dr\},
\end{aligned}
$$

$$
\begin{aligned}
\beta_{13} =&\frac{1}{b}\{\int_{r_1}^{r_2}\frac{N_1}{r}\frac{\partial J_2}{\partial r}\alpha_3+N_1\frac{\partial^2 J_2}{\partial r^2}\alpha_3-2FN_1J_2\alpha_3-\frac{2N_1}{r}\frac{\partial N_1}{\partial r}(\frac{1}{r}\frac{\partial I_3}{\partial r}+\frac{\partial^2 I_3}{\partial r^2})- \\
&\frac{N_1}{r}\frac{\partial I_3}{\partial r}(\frac{1}{r}\frac{\partial N_1}{\partial r}+\frac{\partial^2 N_1}{\partial r^2}-2FN_1)+\frac{N_1^2}{r}(-\frac{1}{r^2}\frac{\partial I_3}{\partial r}+\frac{1}{r}\frac{\partial^2 I_3}{\partial r^2}+\frac{\partial^3 I_3}{\partial r^3})+ \\
&\frac{2N_1I_3}{r}(-\frac{1}{r^2}\frac{\partial N_1}{\partial r}+\frac{1}{r}\frac{\partial^2 N_1}{\partial r^2}+\frac{\partial^3 N_1}{\partial r^3}-2F\frac{\partial N_1}{\partial r})dr\}, \\
\beta_{14} =&\frac{1}{b}\{\int_{r_1}^{r_2}2\frac{N_1}{r}\frac{\partial J_2}{\partial r}(\alpha_2+\beta_2)+N_1\frac{\partial^2 J_2}{\partial r^2}(\alpha_2+\beta_2)-2FN_1J_2(\alpha_2+\beta_2)-\frac{N_1}{r}\frac{\partial M_1}{\partial r} \\
&(\frac{1}{r}\frac{\partial J_2}{\partial r}+\frac{\partial^2 J_2}{\partial r^2}-2FJ_2)-\frac{N_1}{r}\frac{\partial J_2}{\partial r}(\frac{1}{r}\frac{\partial M_1}{\partial r}+\frac{\partial^2 M_1}{\partial r^2})-\frac{2N_1}{r}\frac{\partial N_1}{\partial r}(\frac{1}{r}\frac{\partial I_2}{\partial r}+\frac{\partial^2 I_2}{\partial r^2})+ \\
&\frac{M_1N_1}{r}(-\frac{1}{r^2}\frac{\partial J_2}{\partial r}+\frac{1}{r}\frac{\partial^2 J_2}{\partial r^2}+\frac{\partial^3 J_2}{\partial r^3}-2F\frac{\partial J_2}{\partial r})+\frac{2N_1I_1}{r}(-\frac{1}{r^2}\frac{\partial N_1}{\partial r}+\frac{1}{r}\frac{\partial^2 N_1}{\partial r^2}+\frac{\partial^3 N_1}{\partial r^3}- \\
&F\frac{\partial N_1}{\partial r})+\frac{N_1J_2}{r}(-\frac{1}{r^2}\frac{\partial M_1}{\partial r}+\frac{1}{r}\frac{\partial^2 M_1}{\partial r^2}+\frac{\partial^3 M_1}{\partial r^3})dr\}, \\
\beta_{15} =&\frac{1}{b}\{\int_{r_1}^{r_2}-\frac{N_1}{r}\frac{\partial I_4}{\partial r}(\frac{1}{r}\frac{\partial N_1}{\partial r}+\frac{\partial^2 N_1}{\partial r^2}-2FN_1)+\frac{N_1J_2}{r}(-\frac{1}{r^2}\frac{\partial M_1}{\partial r}+ \\
&\frac{1}{r}\frac{\partial^2 M_1}{\partial r^2}+\frac{\partial^3 M_1}{\partial r^3}-\frac{2}{r^3}M_1+\frac{1}{r}\frac{\partial M_1}{\partial r})dr\}, \\
\beta_{16} =&\frac{1}{b}\{\int_{r_1}^{r_2}-\frac{N_1}{r}\frac{\partial J_3}{\partial r}(\frac{1}{r}\frac{\partial M_1}{\partial r}+\frac{\partial^2 M_1}{\partial r^2})+\frac{N_1M_1}{r}(-\frac{1}{r^2}\frac{\partial J_3}{\partial r}+\frac{1}{r}\frac{\partial^2 J_3}{\partial r^2}+\frac{\partial^3 J_3}{\partial r^3}-2F\frac{\partial J_3}{\partial r}- \\
&\frac{2}{r^3}N_1+\frac{1}{r^2}\frac{\partial N_1}{\partial r})dr\}, \\
\beta_{17} =&\frac{1}{b}\{\int_{r_1}^{r_2}\frac{N_1}{r}(\frac{\partial J_1}{\partial r}\beta_4+\frac{\partial J_4}{\partial r}\beta_1+\frac{\partial N_2}{\partial r}\alpha_7)+N_1(\frac{\partial^2 J_1}{\partial r^2}\beta_4+2\frac{\partial^2 J_4}{\partial r^2}\beta_1+\frac{\partial^2 N_2}{\partial r^2}\alpha_7)- \\
&2FN_1(J_1\beta_4+J_4\beta_1+N_2\alpha_7)+\frac{cN_1J_1}{r^2}+\frac{N_1^2}{r^2}\beta_1-\frac{N_1J_1}{r^3}\frac{\partial \varphi_{B_0}}{\partial r}-\frac{N_1}{r}\frac{\partial \varphi_{T_0}}{\partial r}(\frac{1}{r}\frac{\partial I_7}{\partial r}+ \\
&\frac{\partial^2 I_7}{\partial r^2}+\frac{M_2}{r^2})+\frac{N_1}{r}p(r)I_7dr\},
\end{aligned}
$$

$$
\begin{aligned}
\beta_{18} =& \frac{1}{b}\{\int_{r_1}^{r_2} \frac{N_1}{r}(\frac{\partial J_2}{\partial r}\beta_4 + \frac{\partial J_4}{\partial r}\beta_2) + N_1(\frac{\partial^2 J_2}{\partial r^2}\beta_4 + \frac{\partial^2 J_4}{\partial r^2}\beta_2) - 2FN_1(J_2\beta_4 + J_4\beta_2) + \\
& \frac{cN_1J_2}{r^2} + \frac{N_1^2}{r^2}\beta_2 - \frac{N_1J_2}{r^3}\frac{\partial \varphi_{B_0}}{\partial r} - \frac{N_1}{r}\frac{\partial M_1}{\partial r}(\frac{1}{r}\frac{\partial J_4}{\partial r} + \frac{\partial^2 J_4}{\partial r^2} - 2FJ_4) + \frac{N_1J_3}{r} \\
& (-\frac{1}{r^2}\frac{\partial M_1}{\partial r} + \frac{1}{r}\frac{\partial^2 M_1}{\partial r^2} + \frac{\partial^3 M_1}{\partial r^3})dr\}, \\
\beta_{19} =& \frac{1}{b}\{\int_{r_1}^{r_2} \frac{N_1}{r}(\frac{\partial J_2}{\partial r}\alpha_4 + \frac{\partial J_4}{\partial r}\beta_2) + N_1(\frac{\partial^2 J_2}{\partial r^2}\alpha_4 + \frac{\partial^2 J_4}{\partial r^2}\beta_2) - 2FN_1(J_2\alpha_4 + J_4\beta_2) + \\
& \frac{cN_1J_2}{r^2} + \frac{N_1^2}{r^2}\beta_2 - \frac{N_1}{r}\frac{\partial N_1}{\partial r}(\frac{1}{r}\frac{\partial I_4}{\partial r} + \frac{\partial^2 I_4}{\partial r^2} + \frac{M_1}{r^2}) + \frac{N_1M_1}{r}(-\frac{1}{r^2}\frac{\partial J_3}{\partial r} + \frac{1}{r}\frac{\partial^2 J_3}{\partial r^2} + \\
& \frac{\partial^3 J_3}{\partial r^3} - 2F\frac{\partial J_3}{\partial r} + \frac{2}{r^3}N_1 + \frac{1}{r^2}\frac{\partial N_1}{\partial r})dr\}, \\
\beta_{20} =& \frac{1}{b}\{\int_{r_1}^{r_2} \frac{N_1}{r}\frac{\partial J_4}{\partial r}\beta_4 + N_1\frac{\partial^2 J_4}{\partial r^2}\beta_4 - 2FN_1J_4\beta_4 + \frac{cN_1J_4}{r^2} + \frac{N_1^2}{r^2}\beta_4 - \frac{N_1J_4}{r^3}\frac{\partial \varphi_{B_0}}{\partial r}dr\}.
\end{aligned}
$$

$$
\begin{aligned}
\triangle_1 =& \frac{\partial A_1}{\partial \tau} + c\frac{\partial A_1}{\partial \theta} + \epsilon^2(\alpha_1\frac{\partial A_1}{\partial \theta} + \alpha_2 A_1\frac{\partial A_1}{\partial \theta} + \alpha_4\frac{\partial^3 A_1}{\partial \theta^3} + \alpha_3 B_1\frac{\partial B_1}{\partial \theta}) + \epsilon^3(\alpha_5\frac{\partial B_1}{\partial \theta} + \\
& \alpha_6 A_1\frac{\partial B_1}{\partial \theta} + \alpha_6 B_1\frac{\partial A_1}{\partial \theta} + \alpha_7\frac{\partial^3 B_1}{\partial \theta^3}) + \epsilon^4(\alpha_8\frac{\partial A_1}{\partial \theta} + \alpha_9 A_1\frac{\partial A_1}{\partial \theta}\alpha_{11}A_1^2\frac{\partial A_1}{\partial \theta} + \\
& \alpha_{14}\frac{\partial^2 A_1}{\partial \theta^2}\frac{\partial A_1}{\partial \theta} + \alpha_{16}\frac{\partial^3 A_1}{\partial \theta^3} + \alpha_{17}A_1\frac{\partial^3 A_1}{\partial \theta^3} + \alpha_{19}\frac{\partial^5 A_1}{\partial \theta^5} + \alpha_{10}B_1\frac{\partial B_1}{\partial \theta} + \alpha_{12}B_1^2\frac{\partial A_1}{\partial \theta} + \\
& \alpha_{13}A_1B_1\frac{\partial B_1}{\partial \theta} + \alpha_{15}\frac{\partial^2 B_1}{\partial \theta^2}\frac{\partial B_1}{\partial \theta}), \\
\triangle_2 =& \frac{\partial B_1}{\partial \tau} + c\frac{\partial B_1}{\partial \theta} + \epsilon^2(\beta_1\frac{\partial B_1}{\partial \theta} + \beta_2 A_1\frac{\partial B_1}{\partial \theta} + \beta_4\frac{\partial^3 B_1}{\partial \theta^3} + \beta_3 B_1\frac{\partial A_1}{\partial \theta}) + \epsilon^3(\beta_5\frac{\partial A_1}{\partial \theta} + \\
& \beta_6 A_1\frac{\partial A_1}{\partial \theta} + \beta_7 B_1\frac{\partial B_1}{\partial \theta} + \beta_8\frac{\partial^3 A_1}{\partial \theta^3}) + \epsilon^4(\beta_9\frac{\partial B_1}{\partial \theta} + \beta_{10}A_1\frac{\partial B_1}{\partial \theta}\beta_{12}A_1^2\frac{\partial B_1}{\partial \theta} + \\
& \beta_{15}\frac{\partial^2 A_1}{\partial \theta^2}\frac{\partial B_1}{\partial \theta} + \beta_{17}\frac{\partial^3 B_1}{\partial \theta^3} + \beta_{18}A_1\frac{\partial^3 B_1}{\partial \theta^3} + \beta_{20}\frac{\partial^5 B_1}{\partial \theta^5} + \beta_{11}B_1\frac{\partial A_1}{\partial \theta} + \beta_{13}B_1^2\frac{\partial B_1}{\partial \theta} + \\
& \beta_{14}A_1B_1\frac{\partial A_1}{\partial \theta} + \beta_{16}\frac{\partial^2 B_1}{\partial \theta^2}\frac{\partial A_1}{\partial \theta}).
\end{aligned}
$$

$a_1 = \epsilon^2\alpha_2 + \epsilon^4\alpha_9; a_2 = 2\epsilon^4\alpha_{11}; a_3 = \epsilon^4\alpha_{17}; a_4 = \epsilon^3\alpha_6; a_5 = \epsilon^4\alpha_{13}; a_6 = c + \epsilon^2\alpha_1 + \epsilon^4\alpha_8;$
$a_7 = \epsilon^2\alpha_2 + \epsilon^4\alpha_9; a_8 = \epsilon^4\alpha_{11}; a_9 = \epsilon^4\alpha_{14}; a_{10} = \epsilon^4\alpha_6; a_{11} = \epsilon^4\alpha_{12}; a_{12} = \epsilon^4\alpha_{14};$
$a_{13} = \epsilon^2\alpha_4 + \epsilon^4\alpha_{16}; a_{14} = \epsilon^4\alpha_{17}; a_{15} = \epsilon^4\alpha_{19}; a_{16} = \epsilon^3\alpha_6; a_{17} = \epsilon^2\alpha_3 + \epsilon^4\alpha_{10};$
$a_{18} = 2\epsilon^4\alpha_{12}; a_{19} = \epsilon^4\alpha_{13}; a_{20} = \epsilon^3\alpha_5; a_{21} = \epsilon^3\alpha_6; a_{22} = \epsilon^2\alpha_3 + \epsilon^4\alpha_{10}; a_{23} = \epsilon^4\alpha_{13};$
$a_{24} = \epsilon^4\alpha_{15}; a_{25} = \epsilon^4\alpha_{15}; a_{26} = \epsilon^3\alpha_7;$
$b_1 = \epsilon^3\beta_7; b_2 = \epsilon^2\beta_3 + \epsilon^4\beta_{11}; b - 3 = \epsilon^4\beta_{13}; b_4 = \epsilon^4\beta_{14}; b_5 = c + \epsilon^2\beta_1 + \epsilon^4\beta_9;$
$b_7 = \epsilon^4\beta_9 12; b_8 = \epsilon^4\beta_{15}; b_9 = \epsilon^3\beta_7; b_{10} = \epsilon^4\beta_{13}; b_{11} = \epsilon^4\beta_{16}; b_{12} = \epsilon^2\beta_4 + \epsilon^4\beta_{17};$
$b_{13} = \epsilon^4\beta_{18}; b_{14} = \epsilon^4\beta_{20}; b_{15} = \epsilon^2\beta_2 + \epsilon^4\beta_{10}; b_{16} = \epsilon^3\beta_6; b_{17} = 2\epsilon^4\beta_{12}; b_{18} = \epsilon^4\beta_{14};$
$b_{19} = \epsilon^4\beta_{18}; b_{20} = \epsilon^3\beta_5; b_{21} = \epsilon^3\beta_6; b_{22} = \epsilon^2\beta_3 + \epsilon^4\beta_{11}; b_{23} = \epsilon^4\beta_{14}; b_{24} = \epsilon^4\beta_{16};$
$b_{25} = \epsilon^4\beta_{15}; b_{26} = \epsilon^3\beta_8; b_6 = \epsilon^2\beta_2 + \epsilon^4\beta_{10};$

$$
\begin{aligned}
&c_1 = c + \epsilon^2\alpha_1 + \epsilon^4\alpha_8; c_2 = \epsilon^2\alpha_2 + \epsilon^4\alpha_9; c_3 = \epsilon^2\alpha_4 + \epsilon^4\alpha_{16}; c_4 = \epsilon^4\alpha_{11}; c_5 = \epsilon^4\alpha_{14};\\
&c_6 = \epsilon^4\alpha_{17}; c_7 = \epsilon^4\alpha_{19}; c_8 = \epsilon^2\alpha_3 + \epsilon^4\alpha_{10}; c_9 = \epsilon^3\alpha_5; c_{10} = \epsilon^3\alpha_6; c_{11} = c_{10}; c_{12} = \epsilon^3\alpha_7;\\
&c_{13} = \epsilon^4\alpha_{12}; c_{14} = \epsilon^4\alpha_{13}; c_{15} = \epsilon^4\alpha_{15};\\
&d_1 = c + \epsilon^2\beta_1 + \epsilon^4\beta_9; d_2 = \epsilon^2\beta_2 + \epsilon^4\beta_{10}; d_3 = \epsilon^2\beta_4 + \epsilon^4\beta_{17}; d_4 = \epsilon^4\beta_{12}; d_5 = \epsilon^4\beta_{15};\\
&d_6 = \epsilon^4\beta_{18}; d_7 = \epsilon^4\beta_{20}; d_8 = \epsilon^2\beta_3 + \epsilon^4\beta_{11}; d_9 = \epsilon^3\beta_5; d_{10} = \epsilon^3\beta_6; d_{11} = \epsilon^3\beta_7;\\
&d_{12} = \epsilon^3\beta_8; d_{13} = \epsilon^4\beta_{13}; d_{14} = \epsilon^4\beta_{14}; d_{15} = \epsilon^4\beta_{16};\\
&a = 2c_4m_2^2 + 12c_6k^2m^2m_2 + c_{14}n_2^2; b = (2c_{13} + c_{14})m_2n_2;\\
&c = [12c_3k^2m^2 - 240c_7k^4m^2(1+m^2)]m_2 + [c_2 - 4c_5k^2(1+m^2) - 4c_6k^2(1+m^2)]m_2^2+\\
&12c_{12}k^2m^2n_2 + [c_9 - 4c_{15}k^2(1+m^2)]n_2^2 + (c_{10} + c_{11})n_2m_2;\\
&d = 2d_4m_2n_2 + 12d_6k^2m^2n_2 + 2d_{13}m_2^2 + d_{14}m_2n_2; e = d_{14}m_2^2;\\
&f = 12d_{12}k^2m^2m_2 + d_{10}m_2^2 + [12d_3k^2m^2 - 240d_7k^4m^2(1+m^2]n_2 + d_{11}n_2^2+\\
&[d_2 - 4d_5k^2(1+m^2) - 4d_6k^2(1+m^2) + d_9 - 4d_{15}k^2(1+m^2)]m_2n_2.
\end{aligned}
$$

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