axioms

Article

Ninth-order Multistep Collocation Formulas for Solving Models of PDEs Arising in Fluid Dynamics: Design and Implementation Strategies

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Abstract: A computational approach with the aid of the Linear Multistep Method (LMM) for the numerical solution of differential equations with initial value problems or boundary conditions has appeared several times in the literature due to its good accuracy and stability properties. The major objective of this article is to extend a multistep approach for the numerical solution of the Partial Differential Equation (PDE) originating from fluid mechanics in a two-dimensional space with initial and boundary conditions, as a result of the importance and utility of the models of partial differential equations in applications, particularly in physical phenomena, such as in convection-diffusion models, and fluid flow problems. Thus, a multistep collocation formula, which is based on orthogonal polynomials is proposed. Ninth-order Multistep Collocation Formulas (NMCFs) were formulated through the principle of interpolation and collocation processes. The theoretical analysis of the NMCFs reveals that they have algebraic order nine, are zero-stable, consistent, and, thus, convergent. The implementation strategies of the NMCFs are comprehensively discussed. Some numerical test problems were presented to evaluate the efficacy and applicability of the proposed formulas. Comparisons with other methods were also presented to demonstrate the new formulas' productivity. Finally, figures were presented to illustrate the behavior of the numerical examples.

Keywords: multistep collocation formulas; convergence analysis of the formulas; orthogonal approximating function; convection diffusion reaction equations; fluid dynamics problems; partial differential equations

MSC: 65L05; 65M12; 65M20

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1. Introduction

This paper focuses on the numerical solution of a second-order partial differential equation arising from the fluid dynamics. It describes how convection effects interact with diffusion transports. Thus, we are interested in the convection diffusion reaction equation, which is given below:

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial y^2} + p\frac{\partial u}{\partial x} + q\frac{\partial u}{\partial t} + r\frac{\partial u}{\partial y} + ku = \nu \tag{1}$$

subject to the given initial conditions:

$$\left.\begin{aligned} s_1(x) &= u(x,0), \quad x \in [a,b], \\ s_2(x) &= \frac{\partial u}{\partial y}(x,0), \quad x \in [c,d] \end{aligned}\right\} \tag{2}$$

and the following boundary conditions:

$$\left.\begin{aligned} t_1(x) &= u(a,y), \quad y \geq 0 \\ t_2(x) &= u(b,t), \quad t \geq 0 \\ t_3(x) &= u(b,y), \quad y \geq 0 \\ t_4(x) &= u(b,t), \quad t \geq 0 \end{aligned}\right\} \tag{3}$$

or any of the conditions below:

$$\left.\begin{aligned} v_1(x) &= u(a,t), \quad t \geq 0 \\ v_2(x) &= u(b,y), \quad y \geq 0 \\ v_3(x) &= u(a,t), \quad t \geq 0 \\ v_4(x) &= u(b,y), \quad y \geq 0 \end{aligned}\right\}. \tag{4}$$

Symbolically, in (1), $u(x,y)$ symbolizes the dependent variable. On the other hand, x, y, and t stand for the diffusion function; likewise, v is the non-homogeneous function, also known as the prescribed source, and a_1, b_1, p_1, q_1, and r_1 are the constants. Model Equation (1) has numerous applications in physics, mathematics, and engineering (Liu et al. [1]. According to Kamran et al. [2], it can be used to describe many real-life physical systems, among which are the Langmuir wave packet estimation in plasma, electrostatics, the non-relativistic limit of the Klein–Gordon equation, fluid flow, and bimolecular dynamics (Kilic and Celik [3], Akbarov et al. [4], Chen et al. [5], Singh et al. [6], Mirzaee et al. [7], Khan et al. [8], and Aliev et al. [9]). These classes of partial differential equations lack analytic solutions and are very difficult to approximate due to the nonlinear parameters. Accordingly, the numerical method is an alternative approach for approximating the PDEs' solution, which is being proposed (Adeyefa et al. [10]). Because of the aforementioned applications, various numerical techniques for solving various application problems, such as time–frequency analysis, signal delay, convection–diffusion equations, nonlinear approximation, and Monte Carlo simulation, arising in fluid dynamics problems have been developed. For instance, predictor–corrector techniques (Su et al. [11], Awoyemi and Idowu [12], Iskandarov and Komartsova [13], Ashry et al. [14], Asif [15]); Galerkin methods (Guo et al. [16]); Haar wavelets methods (Aziz and Khan [17], Shiralashetti et al. [18], Saparova et al. [19]); Runge–Kutta methods (Takei and Iwata [20], Yakubu et al. [21], Zhao and Huang [22]); Newtral network model (Mall and Chakraverty [23]); multigrid technique (Ghaffar et al. [24], Ge [25], Gupta et al. [26]); finite difference methods (Mulla et al. [27]); and finite element methods (Harari and Hughes [28]).

In spite of the success recorded by the above-mentioned techniques, there is still a need for improvement in convergence and accuracy due to the fact that the majority of the methods discussed above are not self-starting and do not use second-derivative values in their derivations. A number of authors have proposed numerical self-starting block algorithms for solving differential equations with either boundary or initial conditions. Sunday et al. [29] constructed a class of hybrid algorithm block methods with variable steps and applied the developed method to solve application problems, namely the Kepler problem. Ramos and Vigo-Aguiar in reference [30] presented a BDF-type approach that is L-stable in nature for the numerical approximation of stiff problems through the method of line. Meanwhile, Ngwane and Jator [31] considered a numerical solution of an oscillatory second-order initial and Hamiltonian system of equations using a trigonometrically fitted block method. In reference [32], Modebei et al. proposed a class of numerical solvers for approximating a fourth-order partial differential equation via a block approach with a uniform order. In the same vein, Jator in [33] developed a block-like unification scheme for the numerical solution of a class of elliptic, telegraph, and sine-Gordon partial differential equations with an emphasis on accuracy. Likewise, Olaiya et al. in [34] presented a numerical approach for simulating the Black–Scholes partial differential equation via a two-step off-grip block of algorithms of algebraic order seven. Lastly, the work of Familua et al. in [35] considered a higher-order block technique for the numerical simulation of third-order boundary value problems with applications. The theoretical analysis of the methods was investigated and discussed comprehensively. It has been established in the literature that the block approach was first initiated by Milne [36]. Runge–Kutta's self-starting nature, high-convergence rates, and unique property of yielding approximations of the solution at various points are all still present in the block approach. The convection–diffusion equation and the Helmholtz equation, to name a couple, are examples of nonlinear partial differential equations that arise in the sciences and engineering. In light of this, ninth-order multi-step collocation formulas (NMCFs) are proposed with the potential to be a useful tool for solving these equations. This study aims to develop a method that solves nonlinear second-order PDEs more precisely and with higher convergence than other methods already described in the literature.

2. Design of NMCFS

To develop a numerical method for the solution of (1), the orthogonal polynomial of the form:

$$U(x) = \sum_{r=0}^{p+q-1} \zeta_r \Omega_r(x), \tag{5}$$

is adopted as the basis function. ζ_r are the coefficients to be determined and Ω_r is defined on the interval $(-\infty, \infty)$, with the help of the recurrence formula:

$$\Omega_{n+1}(x) = x\Omega_n(x) - \Omega_n'(x).$$

The polynomials are orthonormal with respect to the weight function e^{-x^2}.
The first five sets of orthogonal polynomials (Hermite polynomials), as contained in Adeyefa et al. [10], are as follows:

$$\left.\begin{array}{l} \Omega_0(x) = 1 \\ \Omega_1(x) = x \\ \Omega_2(x) = x^2 - 1 \\ \Omega_3(x) = x^3 - 3x \\ \Omega_4(x) = x^4 - 6x^2 + 3 \end{array}\right\}.$$

On the other hand, we differentiate (5) twice to obtain:

$$U''(x) = \sum_{r=0}^{p+q-1} \zeta_r \Omega_r''(x)\,. \tag{6}$$

Set $p = 2$, $q = 8$, with k as the step number. It follows that p is the number of distinct interpolation points, which must coincide with the order of the d.e in (1), meanwhile q denotes the number of collocation points chosen. Thus, (5) and (6) reduces to

$$U(x) = \sum_{r=0}^{10} \zeta_r \Omega_r(x), \tag{7}$$

$$U''(x) = \sum_{r=0}^{10} \zeta_r \Omega_r''(x)\,. \tag{8}$$

Now, interpolate the approximate solution to (7) at $x_{n+w}, w = 0(1)$ and also collocate (8) at $x_{n+w}, w = 0(1)8$, which produces seven non-singular equations, which can be written as a system in matrix form as

$$DJ = L \tag{9}$$

where

$$D = \begin{bmatrix} \Omega_0(x_n) & \Omega_1(x_n) & \Omega_2(x_n) & \Omega_3(x_n) & \cdots & \Omega_{k+2}(x_n) \\ \Omega_0(x_{n+1}) & \Omega_1(x_{n+1}) & \Omega_2(x_{n+1}) & \Omega_0(x_{n+1}) & \cdots & \Omega_{k+2}(x_{n+1}) \\ \Omega_0''(x_n) & \Omega_1''(x_n) & \Omega_2''(x_n) & \Omega_3''(x_n) & \cdots & \Omega_{k+2}''(x_n) \\ \Omega_0''(x_{n+1}) & \Omega_1''(x_{n+1}) & \Omega_2''(x_{n+1}) & \Omega_3''(x_{n+1}) & \cdots & \Omega_{k+2}''(x_{n+1}) \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \Omega_0''(x_{n+k}) & \Omega_1''(x_{n+k}) & \Omega_2''(x_{n+k}) & \Omega_3''(x_{n+k}) & \cdots & \Omega_{k+2}''(x_{n+k}) \end{bmatrix}$$

$$J = \left[\ \zeta_0, \zeta_1, \zeta_2, \zeta_3, \cdots, \zeta_7\ \right]^T, \quad L = \left[\ u_{m,n}, u_{m+1,n}, g_{m,n}, g_{m+1,n}, \cdots, g_{m+8,n}\ \right]^T.$$

Using computer software, like Maple, where $J = D^{-1}L$, one can solve the matrix Equation (9) for the unknown values of $\zeta_i, i = 0(1)10$. The values obtained are subsequently plugged into the Formula (7) with the setting $x = \phi h + x_n + 7$ to obtain the continuous function of the form

$$u_{m+j,n}(\phi) \quad = \quad \Psi_0 u_{m,n} + \Psi_1 u_{m+1,n} + h^2 \sum_{j=0}^{8} \Delta_j(\phi) g_{m+j,n}, \ j = 0(1)8, \tag{10}$$

where the coefficients of the continuous function (10) are given in (Appendix A).

The main formulas are produced by evaluating (10) at ϕ = −5, −4, −3, −2, −1, 0, and 1, which gives the following discrete formulas.

$$\begin{aligned} u_{m+2,n} \quad = \quad & -u_{m,n} + 2\,u_{m+1,n} + \frac{33953}{518400}\,h^2 g_{m,n} + \frac{424759}{453600}\,h^2 g_{m+1,n} - \frac{81629}{453600}\,h^2 g_{m+2,n} + \\ & \frac{11143}{28350}\,h^2 g_{m+3,n} - \frac{27533}{72576}\,h^2 g_{m+4,n} + \frac{110563}{453600}\,h^2 g_{m+5,n} - \frac{23017}{226800}\,h^2 g_{m+6,n} + \\ & \frac{5627}{226800}\,h^2 g_{m+7,n} - \frac{9829}{3628800}\,h^2 g_{m+8,n}, \end{aligned} \tag{11}$$

$$\begin{aligned} u_{m+3,n} \quad = \quad & -2u_{m,n} + 3\,u_{m+1,n} + \frac{155171}{1209600}\,h^2 g_{m,n} + \frac{9421}{4800}\,h^2 g_{m+1,n} + \frac{144847}{302400}\,h^2 g_{m+2,n} + \\ & \frac{252101}{302400}\,h^2 g_{m+3,n} - \frac{5701}{8064}\,h^2 g_{m+4,n} + \frac{135901}{302400}\,h^2 g_{m+5,n} - \frac{56473}{302400}\,h^2 g_{m+6,n} + \\ & \frac{4601}{100800}\,h^2 g_{m+7,n} - \frac{6029}{1209600}\,h^2 g_{m+8,n}, \end{aligned} \tag{12}$$

$$\begin{aligned} u_{m+4,n} \quad = \quad & -3u_{m,n} + 4\,u_{m+1,n} + \frac{12869}{67200}\,h^2 g_{m,n} + \frac{225469}{75600}\,h^2 g_{m+1,n} + \frac{94001}{75600}\,h^2 g_{m+2,n} + \\ & \frac{3271}{1575}\,h^2 g_{m+3,n} - \frac{11279}{12096}\,h^2 g_{m+4,n} + \frac{49313}{75600}\,h^2 g_{m+5,n} - \frac{3449}{12600}\,h^2 g_{m+6,n} + \\ & \frac{2537}{37800}\,h^2 f_{m+7,n} - \frac{4439}{604800}\,h^2 g_{m+8,n}, \end{aligned} \tag{13}$$

$$
\begin{aligned}
u_{m+5,n} \quad = \quad & -4u_{m,n} + 5\,u_{m+1,n} + \frac{18481}{72576}\,h^2 g_{m,n} + \frac{363173}{90720}\,h^2 g_{m+1,n} + \frac{181289}{90720}\,h^2 g_{m+2,n} + \\
& \frac{311363}{90720}\,h^2 g_{m+3,n} - \frac{13261}{36288}\,h^2 g_{m+4,n} + \frac{17551}{18144}\,h^2 g_{m+5,n} - \frac{33583}{90720}\,h^2 g_{m+6,n} + \\
& \frac{1163}{12960}\,h^2 g_{m+7,n} - \frac{3547}{362880}\,h^2 g_{m+8,n},
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
u_{m+6,n} \quad = \quad & -5u_{m,n} + 6\,u_{m+1,n} + \frac{76859}{241920}\,h^2 g_{m,n} + \frac{16883}{3360}\,h^2 g_{m+1,n} + \frac{83207}{30240}\,h^2 g_{m+2,n} + \\
& \frac{9043}{1890}\,h^2 g_{m+3,n} + \frac{2449}{8064}\,h^2 g_{m+4,n} + \frac{63047}{30240}\,h^2 g_{m+5,n} - \frac{5461}{15120}\,h^2 g_{m+6,n} + \\
& \frac{533}{5040}\,h^2 g_{m+7,n} - \frac{407}{34560}\,h^2 g_{m+8,n},
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
u_{m+7,n} \quad = \quad & -6u_{m,n} + 7\,u_{m+1,n} + \frac{7319}{19200}\,h^2 g_{m,n} + \frac{261023}{43200}\,h^2 g_{m+1,n} + \frac{152107}{43200}\,h^2 g_{m+2,n} + \\
& \frac{87827}{14400}\,h^2 g_{m+3,n} + \frac{3541}{3456}\,h^2 g_{m+4,n} + \frac{140401}{43200}\,h^2 g_{m+5,n} + \frac{7009}{14400}\,h^2 g_{m+6,n} + \\
& \frac{9143}{43200}\,h^2 f_{m+7,n} - \frac{2849}{172800}\,h^2 g_{m+8,n},
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
u_{m+8,n} \quad = \quad & -7u_{m,n} + 8\,u_{m+1,n} + \frac{57281}{129600}\,h^2 g_{m,n} + \frac{114769}{16200}\,h^2 g_{m+1,n} + \frac{67861}{16200}\,h^2 g_{m+2,n} + \\
& \frac{15506}{2025}\,h^2 g_{m+3,n} + \frac{3541}{2592}\,h^2 g_{m+4,n} + \frac{77893}{16200}\,h^2 g_{m+5,n} + \frac{9353}{8100}\,h^2 g_{m+6,n} + \\
& \frac{10157}{8100}\,h^2 f_{m+7,n} + \frac{5741}{129600}\,h^2 g_{m+8,n},
\end{aligned} \tag{17}
$$

The first derivative of (10) is given below:

$$
u'_{m+j,n}(t) \quad = \quad \Psi'_0 u_{m,n} + \Psi'_1 u_{m+1,n} + h^2 \sum_{j=0}^{8} \Delta'_j(\phi) f_{m+j,n}, \;\; j = 0(1)8. \tag{18}
$$

The coefficients of the continuous function (18) and the first derivative of (10) are given in (Appendix B).

The additional discrete formulas are generated by evaluating (18) at the points $\phi = -7, -6, -5, -4, -3, -2, -1, 0,$ and 1, which gives the following first derivative discrete formulas:

$$
\begin{aligned}
u'_{m,n} \quad = \quad & -\frac{1}{7257600h}\,(1624505\,h^2 g_{m,n} + 4124232\,h^2 g_{m+1,n} - 5225624\,h^2 g_{m+2,n} + \\
& 6488192\,h^2 g_{m+3,n} - 5888310\,g_{m+4,n}h^2 + 3698920\,g_{m+5,n}h^2 - 1522672\,h^2 g_{m+6,n} + \\
& 369744\,h^2 g_{m+7,n} - 40187\,h^2 g_{m+8,n} + 7257600\,u_{m,n} - 7257600\,u_{m+1,n}),
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
u'_{m+1,n} \quad = \quad & \frac{1}{7257600h}\,(515529\,h^2 g_{m,n} + 4809956\,h^2 g_{m+1,n} - 3983564\,h^2 g_{m+2,n} + \\
& 4702524\,h^2 g_{m+3,n} - 4177930\,g_{m+4,n}h^2 + 2593756\,g_{m+5,n}h^2 - 1059756\,h^2 g_{m+6,n} + \\
& 256004\,h^2 g_{m+7,n} - 27719\,h^2 g_{m+8,n} - 7257600\,u_{m,n} + 7257600\,u_{m+1,n}),
\end{aligned} \tag{20}
$$

$$u'_{m+2,n} = \frac{1}{7257600h}(447623\,h^2 g_{m,n} + 7561144\,h^2 g_{m+1,n} + 2506008\,h^2 g_{m+2,n} + 1197440\,h^2 g_{m+3,n} - 1543370\,g_{m+4,n}h^2 + 1083672\,g_{m+5,n}h^2 - 471184\,h^2 g_{m+6,n} + 118192\,h^2 g_{m+7,n} - 13125\,h^2 g_{m+8,n} - 7257600\,u_{m,n} + 7257600\,u_{m+1,n}), \tag{21}$$

$$u'_{m+3,n} = \frac{1}{7257600h}(462217\,h^2 g_{m,n} + 7361892\,h^2 g_{m+1,n} + 5782580\,h^2 g_{m+2,n} + 6461116\,h^2 g_{m+3,n} - 3209610\,g_{m+4,n}h^2 + 1879388\,g_{m+5,n}h^2 - 755372\,h^2 g_{m+6,n} + 181380\,h^2 g_{m+7,n} - 19591\,h^2 g_{m+8,n} - 7257600\,u_{m,n} + 7257600\,u_{m+1,n}), \tag{22}$$

$$u'_{m+4,n} = \frac{1}{7257600h}(455751\,h^2 g_{m,n} + 7434680\,h^2 g_{m+1,n} + 5350552\,h^2 g_{m+2,n} + 10280832\,h^2 g_{m+3,n} + 1239350\,g_{m+4,n}h^2 + 1027864\,g_{m+5,n}h^2 - 502800\,h^2 g_{m+6,n} + 129968\,h^2 g_{m+7,n} - 14597\,h^2 g_{m+8,n} - 7257600\,u_{m,n} + 7257600\,u_{m+1,n}), \tag{23}$$

$$u'_{m+5,n} = \frac{1}{7257600h}(460745\,h^2 g_{m,n} + 7383268\,h^2 g_{m+1,n} + 5603124\,h^2 g_{m+2,n} + 9429308\,h^2 g_{m+3,n} + 5688310\,g_{m+4,n}h^2 + 4847580\,g_{m+5,n}h^2 - 934828\,h^2 g_{m+6,n} + 202756\,h^2 g_{m+7,n} - 21063\,h^2 g_{m+8,n} - 7257600\,u_{m,n} + 7257600\,u_{m+1,n}), \tag{24}$$

$$u'_{m+6,n} = \frac{1}{7257600h}(454279\,h^2 g_{m,n} + 7446456\,h^2 g_{m+1,n} + 5318936\,h^2 g_{m+2,n} + 10225024\,h^2 g_{m+3,n} + 4022070\,g_{m+4,n}h^2 + 10111256\,g_{m+5,n}h^2 + 2341744\,h^2 g_{m+6,n} + 3504\,h^2 g_{m+7,n} - 6469\,h^2 g_{m+8,n} - 7257600\,u_{m,n} + 7257600\,u_{m+1,n}), \tag{25}$$

$$u'_{m+7,n} = \frac{1}{7257600h}(468873\,h^2 g_{m,n} + 7308644\,h^2 g_{m+1,n} + 5907508\,h^2 g_{m+2,n} + 8714940\,h^2 g_{m+3,n} + 6656630\,g_{m+4,n}h^2 + 6606172\,g_{m+5,n}h^2 + 8831316\,h^2 g_{m+6,n} + 2754692\,h^2 g_{m+7,n} - 74375\,h^2 g_{m+8,n} - 7257600\,u_{m,n} + 7257600\,u_{m+1,n}), \tag{26}$$

$$u'_{m+8,n} = \frac{1}{7257600h}(400967\,h^2 g_{m,n} + 7934392\,h^2 g_{m+1,n} + 3325080\,h^2 g_{m+2,n} + 15007616\,h^2 g_{m+3,n} - 3409610\,g_{m+4,n}h^2 + 17796888\,g_{m+5,n}h^2 - 377872\,h^2 g_{m+6,n} + 11688880\,h^2 g_{m+7,n} + 2065659\,h^2 g_{m+8,n} - 7257600\,u_{m,n} + 7257600\,u_{m+1,n}). \tag{27}$$

Implementation Strategies of the NMCFs

Here, we combined the discrete Formulas (11)–(17) and its derivatives (19) in matrix form below:

$$PU_m = Q\eta_0 + R\eta_1 + h^2[S\eta_2 + T\eta_3], \tag{28}$$

where

$$P = \begin{bmatrix} -7257600 & 3628800 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3628800 & 0 & 1209600 & 0 & 0 & 0 & 0 & 0 \\ -2419200 & 0 & 0 & 604800 & 0 & 0 & 0 & 0 \\ -1814400 & 0 & 0 & 0 & 362880 & 0 & 0 & 0 \\ -1451520 & 0 & 0 & 0 & 0 & 241920 & 0 & 0 \\ -1209600 & 0 & 0 & 0 & 0 & 0 & 172800 & 0 \\ -1036800 & 0 & 0 & 0 & 0 & 0 & 0 & 129600 \\ -7257600 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, U_m = \begin{bmatrix} u_{m+1,n} \\ u_{m+2,n} \\ u_{m+3,n} \\ u_{m+4,n} \\ u_{m+5,n} \\ u_{m+6\,n} \\ u_{m+7,n} \\ u_{m+8,n} \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3628800 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2419200 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1814400 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1451520 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1209600 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1036800 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 907200 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7257600 \end{bmatrix}, \eta_0 = \begin{bmatrix} u_{m-1,n} \\ u_{m-2,n} \\ u_{m-3,n} \\ u_{m-4,n} \\ u_{m-5,n} \\ u_{m-6\,n} \\ u_{m-7,n} \\ u_{m,n} \end{bmatrix}$$

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7257600 \end{bmatrix}, \eta_1 = \begin{bmatrix} u'_{m-1,n} \\ u'_{m-2,n} \\ u'_{m-3,n} \\ u'_{m-4,n} \\ u'_{m-5,n} \\ u'_{m-6\,n} \\ u'_{m-7,n} \\ u'_{m,n} \end{bmatrix}$$

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 237671 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 155171 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 115821 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 92405 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 76859 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 65871 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 57281 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1624505 \end{bmatrix}, \eta_2 = \begin{bmatrix} g_{m-1,n} \\ g_{m-2,n} \\ g_{m-3,n} \\ g_{m-4,n} \\ g_{m-5,n} \\ g_{m-6\,n} \\ g_{m-7,n} \\ g_{m,n} \end{bmatrix}$$

$$T = \begin{bmatrix} 3398072 & -653032 & 1426304 & -1376650 & 884504 & -368272 & 90032 & -9829 \\ 2374092 & 579388 & 1008404 & -855150 & 543604 & -225892 & 55212 & -6029 \\ 1803752 & 752008 & 1256064 & -563950 & 394504 & -165552 & 40592 & -4439 \\ 1452692 & 725156 & 1245452 & -132610 & 351020 & -134332 & 32564 & -3547 \\ 1215576 & 665656 & 1157504 & 73470 & 504376 & -87376 & 25584 & -2849 \\ 1044092 & 608428 & 1053924 & 177050 & 561604 & 84108 & 36572 & -2849 \\ 918152 & 542888 & 992384 & 177050 & 623144 & 149648 & 162512 & 5741 \\ -4124232 & 5225624 & -6488192 & 5888310 & -3698920 & 1522672 & -369744 & 40187 \end{bmatrix}$$

$$\eta_3 = \begin{bmatrix} g_{m+1,n} \\ g_{m+2,n} \\ g_{m+3,n} \\ g_{m+4,n} \\ g_{m+5,n} \\ g_{m+6\,n} \\ g_{m+7,n} \\ g_{m+8,n} \end{bmatrix}.$$

The NMCFs are obtained by multiplying the matrix Equation (28) by the inverse of P:

$$U_m = \bar{Q}\eta_0 + \bar{R}\eta_1 + h^2[\bar{S}\eta_2 + \bar{T}\eta_3] \tag{29}$$

$$U_m = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \bar{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\bar{R} = h\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}, \bar{S} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{324901}{1451520} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{58193}{113400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{71661}{89600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{15406}{14175} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{56975}{41472} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{93}{56} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2019731}{1036800} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{31648}{14175} \end{bmatrix}$$

$$\bar{T} = \begin{bmatrix} \frac{8183}{14400} & -\frac{653203}{907200} & \frac{50689}{56700} & -\frac{196277}{241920} & \frac{92473}{181440} & -\frac{95167}{453600} & \frac{7703}{151200} & -\frac{5741}{1036800} \\ \frac{29384}{14175} & -\frac{81}{50} & \frac{30916}{14175} & -\frac{22703}{11340} & \frac{5968}{4725} & -\frac{14773}{28350} & \frac{1796}{14175} & -\frac{521}{37800} \\ \frac{1467}{400} & -\frac{4707}{2800} & \frac{225}{64} & -\frac{28143}{8960} & \frac{11079}{5600} & -\frac{9141}{11200} & \frac{2223}{11200} & -\frac{387}{17920} \\ \frac{24832}{4725} & -\frac{928}{567} & \frac{80128}{14175} & -\frac{188}{45} & \frac{38144}{14175} & -\frac{15776}{14175} & \frac{256}{945} & -\frac{418}{14175} \\ \frac{248375}{36288} & -\frac{19375}{12096} & \frac{143375}{18144} & -\frac{641875}{145152} & \frac{225}{64} & -\frac{12875}{9072} & \frac{3125}{9072} & -\frac{3625}{96768} \\ \frac{1476}{175} & -\frac{549}{350} & \frac{1776}{175} & -\frac{639}{140} & \frac{36}{7} & -\frac{81}{50} & \frac{72}{175} & -\frac{9}{200} \\ \frac{216433}{21600} & -\frac{98441}{64800} & \frac{1601467}{129600} & -\frac{160867}{34560} & \frac{55223}{8100} & -\frac{127253}{129600} & \frac{8183}{14400} & -\frac{57281}{1036800} \\ \frac{23552}{2025} & -\frac{7424}{4725} & \frac{41984}{2835} & -\frac{14528}{2835} & \frac{41984}{4725} & -\frac{7424}{14175} & \frac{23552}{14175} & 0 \end{bmatrix}.$$

The matrix equation in (29) could be written explicitly as follows:

$$\begin{aligned} u_{m+1,n} \quad = \quad & u_{m,n} + u'_{m,n}h + \frac{324901}{1451520}\,h^2 g_{m,n} + \frac{8183}{14400}\,h^2 g_{m+1,n} - \frac{653203}{907200}\,h^2 g_{m+2,n} \\ & + \frac{50689}{56700}\,h^2 g_{m+3,n} - \frac{196277}{241920}\,h^2 g_{m+4,n} + \frac{92473}{181440}\,h^2 g_{m+5,n} - \frac{95167}{453600}\,h^2 g_{m+6,n} \\ & + \frac{7703}{151200}\,h^2 g_{m+7,n} - \frac{5741}{1036800}\,h^2 g_{m+8,n}, \end{aligned} \tag{30}$$

$$\begin{aligned} u_{m+2,n} \quad = \quad & u_{m,n} + 2hu'_{m,n} + \frac{58193}{113400}\,h^2 g_{m,n} + \frac{29384}{14175}\,h^2 g_{m+1,n} - \frac{81}{50}\,h^2 g_{m+2,n} \\ & + \frac{30916}{14175}\,h^2 g_{m+3,n} - \frac{22703}{11340}\,h^2 g_{m+4,n} + \frac{5968}{4725}\,h^2 g_{m+5,n} - \frac{14773}{28350}\,h^2 g_{m+6,n} + \\ & \frac{1796}{14175}\,h^2 g_{m+7,n} - \frac{521}{37800}\,h^2 g_{m+8,n}, \end{aligned} \tag{31}$$

$$\begin{aligned} u_{m+3,n} \quad = \quad & u_{m,n} + 3hu'_{m,n} + \frac{71661}{89600}\,h^2 g_{m,n} + \frac{1467}{400}\,h^2 g_{m+1,n} - \frac{4707}{2800}\,h^2 g_{m+2,n} \\ & + \frac{225}{64}\,h^2 g_{m+3,n} - \frac{28143}{8960}\,h^2 f_{m+4,n} + \frac{11079}{5600}\,h^2 g_{m+5,n} - \frac{9141}{11200}\,h^2 g_{m+6,n} + \\ & \frac{2223}{11200}\,h^2 g_{m+7,n} - \frac{387}{17920}\,h^2 g_{m+8,n}, \end{aligned} \tag{32}$$

$$u_{m+4,n} = u_{m,n} + 4hu'_{m,n} + \frac{15406}{14175}h^2 g_{m,n} + \frac{24832}{4725}h^2 g_{m+1,n} - \frac{928}{567}h^2 g_{m+2,n} + \frac{80128}{14175}h^2 g_{m+3,n} - \frac{188}{45}h^2 f_{m+4,n} + \frac{38144}{14175}h^2 g_{m+5,n} - \frac{15776}{14175}h^2 g_{m+6,n} + \frac{256}{945}h^2 g_{m+7,n} - \frac{418}{14175}h^2 g_{m+8,n}, \quad (33)$$

$$u_{m+5,n} = u_{m,n} + 5hu'_{m,n} + \frac{56975}{41472}h^2 g_{m,n} + \frac{248375}{36288}h^2 g_{m+1,n} - \frac{19375}{12096}h^2 g_{m+2,n} + \frac{143375}{18144}h^2 g_{m+3,n} - \frac{641875}{145152}h^2 g_{m+4,n} + \frac{225}{64}h^2 g_{m+5,n} - \frac{12875}{9072}h^2 g_{m+6,n} + \frac{3125}{9072}h^2 f_{m+7,n} - \frac{3625}{96768}h^2 g_{m+8,n}, \quad (34)$$

$$u_{m+6,n} = u_{m,n} + 6hu'_{m,n} + \frac{93}{56}h^2 g_{m,n} + \frac{1476}{175}h^2 g_{m+1,n} - \frac{549}{350}h^2 g_{m+2,n} + \frac{1776}{175}h^2 g_{m+3,n} - \frac{639}{140}h^2 g_{m+4,n} + \frac{36}{7}h^2 g_{m+5,n} - \frac{81}{50}h^2 f_{m+6,n} + \frac{72}{175}h^2 g_{m+7,n} - \frac{9}{200}h^2 g_{m+8,n}, \quad (35)$$

$$u_{m+7,n} = u_{m,n} + 7hu'_{m,n} + \frac{2019731}{1036800}h^2 g_{m,n} + \frac{216433}{21600}h^2 g_{m+1,n} - \frac{98441}{64800}h^2 g_{m+2,n} + \frac{1601467}{129600}h^2 g_{m+3,n} - \frac{160867}{34560}h^2 g_{m+4,n} + \frac{55223}{8100}h^2 g_{m+5,n} - \frac{127253}{129600}h^2 g_{m+6,n} + \frac{8183}{14400}h^2 g_{m+7,n} - \frac{57281}{1036800}h^2 g_{m+8,n}, \quad (36)$$

$$u_{m+8,n} = u_{m,n} + 8hu'_{m,n} + \frac{31648}{14175}h^2 g_{m,n} + \frac{23552}{2025}h^2 g_{m+1,n} - \frac{7424}{4725}h^2 g_{m+2,n} + \frac{41984}{2835}h^2 g_{m+3,n} - \frac{14528}{2835}h^2 g_{m+4,n} + \frac{41984}{4725}h^2 g_{m+5,n} - \frac{7424}{14175}h^2 g_{m+6,n} + \frac{23552}{14175}h^2 g_{m+7,n}. \quad (37)$$

In order to obtain the first derivatives of the NMCFs, substitute (30)–(37) into (20)–(27), which gives

$$u'_{m+1,n} = u'_{m,n} + \frac{1070017}{3628800}hg_{m,n} + \frac{2233547}{1814400}hg_{m+1,n} - \frac{2302297}{1814400}hg_{m+2,n} + \frac{2797679}{1814400}hg_{m+3,n} - \frac{31457}{22680}hg_{m+4,n} + \frac{1573169}{1814400}hg_{m+5,n} - \frac{645607}{1814400}hg_{m+6,n} + \frac{156437}{1814400}hg_{m+7,n} - \frac{33953}{3628800}hg_{m+8,n}, \quad (38)$$

$$u'_{m+2,n} = u'_{m,n} + \frac{32377}{113400}hg_{m,n} + \frac{22823}{14175}hg_{m+1,n} - \frac{21247}{56700}hg_{m+2,n} + \frac{15011}{14175}hg_{m+3,n} - \frac{2903}{2835}hg_{m+4,n} + \frac{9341}{14175}hg_{m+5,n} - \frac{15577}{56700}hf_{m+6,n} + \frac{953}{14175}hg_{m+7,n} - \frac{119}{16200}hg_{m+8,n}, \quad (39)$$

$$\begin{aligned} u'_{m+3,n} \;=\;& u'_{m,n} + \frac{12881}{44800} hg_{m,n} + \frac{35451}{22400} hg_{m+1,n} + \frac{1719}{22400} hg_{m+2,n} + \frac{39967}{22400} hg_{m+3,n} \\ & - \frac{351}{280} hg_{m+4,n} + \frac{17217}{22400} hg_{m+5,n} - \frac{7031}{22400} hg_{m+6,n} + \\ & \frac{243}{3200} hg_{m+7,n} - \frac{369}{44800} hg_{m+8,n}, \end{aligned} \tag{40}$$

$$\begin{aligned} u'_{m+4,n} \;=\;& u'_{m,n} + \frac{4063}{14175} hg_{m,n} + \frac{22576}{14175} hg_{m+1,n} + \frac{244}{14175} hg_{m+2,n} + \frac{32752}{14175} hg_{m+3,n} \\ & - \frac{1816}{2835} hg_{m+4,n} + \frac{9232}{14175} hg_{m+5,n} - \frac{3956}{14175} hg_{m+6,n} + \\ & \frac{976}{14175} hg_{m+7,n} - \frac{107}{14175} hg_{m+8,n}, \end{aligned} \tag{41}$$

$$\begin{aligned} u'_{m+5,n} \;=\;& u'_{m,n} + \frac{41705}{145152} hg_{m,n} + \frac{115075}{72576} hg_{m+1,n} + \frac{3775}{72576} hg_{m+2,n} + \frac{159175}{72576} hg_{m+3,n} - \\ & \frac{125}{4536} hg_{m+4,n} + \frac{85465}{72576} hg_{m+5,n} - \frac{24575}{72576} hf_{m+6,n} + \\ & \frac{5725}{72576} hg_{m+7,n} - \frac{175}{20736} hf_{m+8,n}, \end{aligned} \tag{42}$$

$$\begin{aligned} u'_{m+6,n} \;=\;& u'_{m,n} + \frac{401}{1400} hg_{m,n} + \frac{279}{175} hg_{m+1,n} + \frac{9}{700} hf_{m+2,n} + \frac{403}{175} hg_{m+3,n} \\ & - \frac{9}{35} hg_{m+4,n} + \frac{333}{175} hf_{m+5,n} + \frac{79}{700} hg_{m+6,n} + \\ & \frac{9}{175} hg_{m+7,n} - \frac{9}{1400} hg_{m+8,n}, \end{aligned} \tag{43}$$

$$\begin{aligned} u'_{m+7,n} \;=\;& u'_{m,n} + \frac{149527}{518400} hg_{m,n} + \frac{408317}{259200} hg_{m+1,n} + \frac{24353}{259200} hg_{m+2,n} + \frac{542969}{259200} hg_{m+3,n} \\ & + \frac{343}{3240} hg_{m+4,n} + \frac{368039}{259200} hg_{m+5,n} + \frac{261023}{259200} hf_{m+6,n} \\ & + \frac{111587}{259200} hg_{m+7,n} - \frac{8183}{518400} hg_{m+8,n}, \end{aligned} \tag{44}$$

$$\begin{aligned} u'_{m+8,n} \;=\;& u'_{m,n} + \frac{3956}{14175} hg_{m,n} + \frac{23552}{14175} hg_{m+1,n} - \frac{3712}{14175} hg_{m+2,n} + \frac{41984}{14175} hg_{m+3,n} \\ & - \frac{3632}{2835} hg_{m+4,n} + \frac{41984}{14175} hg_{m+5,n} - \frac{3712}{14175} hf_{m+6,n} + \\ & \frac{23552}{14175} hg_{m+7,n} + \frac{3956}{14175} hf_{m+8,n}. \end{aligned} \tag{45}$$

Remark 1. *The matrix D in Equation* (9) *must be a square matrix. Otherwise, the computation will be indeterminable.*

Remark 2. *The matrix D in Equation* (9) *must be a nonsingular matrix. Otherwise, the determinant will be equal to zero, which implies that the solution will not exist.*

Remark 3. *The Equations* (30)–(37) *and* (38)–(44) *formed the NMCFs required to simultaneously solve the second-order PDEs.*

Remark 4. *The Equations* (30)–(37) *and* (38)–(44) *possessed a uniform order of accuracy as shown below.*

3. Theoretical Analysis of the NMCFs

3.1. Introduction of the Analysis of the NMCFs

The order, error constants, zero-stability, and consistency of the NMCFs, will be examined in this section in accordance with (Familua et al. [35], Jain et al. [36]).

The Formula (18) together with their associated derivatives (29) can be expressed by the linear operator below:

$$L[u(x);h] = U_m - \bar{Q}\eta_0 - \bar{R}\eta_1 - h^2[\bar{S}\eta_2 + \bar{T}\eta_3], \tag{46}$$

where $u(m)$ is continuously differentiable, and $U_m, \bar{Q}\eta_0, \bar{R}\eta_1, \bar{S}\eta_2$, and $\bar{T}\eta_3$ have their usual meaning as stated above. Expanding U_m, η_2, and η_3 in (46), respectively, in Taylor series about x_n, and collecting their like terms in powers of h and u yields

$$L[u(x);h] = C_0 u(x) + C_1 h u'(x) + C_2 h^2 u''(x) + \cdots + C_q h^q u^{(q)}(x), \tag{47}$$

where $C_q, q = 1, 2, \ldots$

Definition 1 (Yakubu et al. [21]). *The NMCFs* (18) *and their linear operators are said to have order p if $C_0 = C_1 = \cdots = C_p = 0, C_{p+2} \neq 0$.*

Definition 2 (Sunday et al. [29]). *The term C_{p+2} in Definition 1 is referred to as the error constants, which indicate the local truncation error* (18) *given as*

$$LTE = C_{p+2} h^{p+2} u^{(p+2)}(x_n) + Oh^{(p+3)}. \tag{48}$$

Definition 3 (Modebei et al. [32]). *Any LMM class with an order greater than or equal to one is said to be consistent.*

Definition 4 (Olaiya et al. [34]). *If the roots of any class of LMM do not exceed the order of the differential equations considered, the class is said to be zero-stable.*

Definition 5 (Lambert [37]). *If an LMM class is zero-stable and consistent, it is said to be convergent.*

Definition 6 (Lambert [37]). *A matrix whose determinant is zero is called a singular matrix.*

Definition 7 (Jain et al. [38]). *If the Linear Multistep Method has order* $2k$, *where k is even, and order* $2k - 1$, *where k is odd, it is said to be of maximal order.*

Definition 8 (Jain et al. [38]). *A matrix whose determinant is not equal to zero is called a nonsingular matrix.*

Definition 9 (Awoyemi and Idowu [12]). *An LMM is said to be P-stable if its periodicity interval is* $(0, \infty)$.

Definition 10 (Jain et al. [38]). *A matrix that has one in the leading diagonal and zero elsewhere.*

Definition 11 (Henricin [39]). *An LMM is said to be A-stable if its periodicity interval is* $(-\infty, 0)$.

3.1.1. Order and Error Constant of the NMCFs

The order and error constant of the NMCFs are analysed following the approach and procedure discussed in Definition 1. Each of (30)–(37), which make up the NMCFs, is analysed.

Hence, the NMCFs is of order $p = [9,9,9,9,9,9,9,9,9,9,9,9,9,9,9,9]^T$, with error constants $C_{p+2} =$

$$C_{11} = \left(4.718068 \times 10^{-3}, 1.179146 \times 10^{-2}, 1.851157 \times 10^{-2}, 2.531131 \times 10^{-2},\right)^T$$

$$\left(3.211106 \times 10^{-2}, 3.883117 \times 10^{-2}, 4.590456 \times 10^{-2}, 5.062263 \times 10^{-2}\right)^T,$$

together with its derivatives (38)–(44):

$$\left(7.892554 \times 10^{-3}, 6.428571 \times 10^{-3}, 6.975446 \times 10^{-3}, 6.631393 \times 10^{-3},\right)^T$$

$$\left(6.975446 \times 10^{-3}, 6.428571 \times 10^{-3}, 7.892554 \times 10^{-3}, 0\right)^T.$$

3.1.2. Consistency of the NMCFs (Omole et al. [40,41])

Applying the definition 3 of consistency to the NMCFs (30)–(37), it is said to be consistent if it has an order of more than or equal to one. Therefore, the NMCFs are consistent according to Lambert [37].

3.1.3. Zero-Stability of the NMCFs (Fatunla [42])

Similarly, the zero-stability of the NMCFs can be obtained using the first characteristics polynomial of the NMCFs given by

$$\Pi(z) = \det(zU_m - \bar{Q}) = 0. \tag{49}$$

Thus,

$$\Pi(z) = \left[z \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right] = 0$$

$$\Pi(z) = z^7(z-1) = 0, \tag{50}$$

Solving (50) for the values of z, $z = 0,0,0,0,0,0,0,1$.
Hence, the NMCFs are zero-stable.

3.1.4. Convergence of the NMCFs (Henrici [39], Rufai et al. [43])

According to Definition 5, consistency and zero-stability are all that are required for the Linear Multistep Method to be convergent, so the NMCFs' consistency and zero-stability imply that they are convergent at all points, concluding the prove.

3.1.5. Region of Absolute Stability of the NMCFs (Yakubu et al. [21], Lambert [37])

Finally, the stability of the NMCFs is examined and discussed using the procedure described in Yakubu et al. [21] and Lambert [37].

$$M(z) = V + zB(I - zA)^{-1}U, \tag{51}$$

In addition to the stability function,

$$p(n,z) = det(nI - M(z)), \tag{52}$$

For the stability properties, the Formulas (30)–(37) were formulated as

$$\begin{bmatrix} Y \\ --- \\ Y_{i+1} \end{bmatrix} = \begin{bmatrix} A & & U \\ --- & --- & --- \\ B & & V \end{bmatrix} \begin{bmatrix} h^2 f(u) \\ --- \\ Y_{i-1} \end{bmatrix}, \tag{53}$$

where

$$Y_{i-1} = \begin{bmatrix} u_{m+1,n} \\ u_{m,n} \end{bmatrix}, Y_{i+1} = \begin{bmatrix} u_{m+1,n} \\ u_{m+8,n} \end{bmatrix}, V = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

The stability polynomial (54) and its first derivative (55) are then obtained by changing the values of A, B, U, V, M, and I (see Appendix C) in Equations (51) and (52). This is then coded in the MATLAB (R2012a) environment. Figure 1 in the text below depicts the NMCFs' stability nature.

$$f(z) = \left(\eta + \frac{963008}{42525} z^2 + \frac{390304}{14175} z - 1\right)\eta^7 \tag{54}$$

$$f'(z) = \frac{32}{42525}(60188\, z + 36591)\eta^7 \tag{55}$$

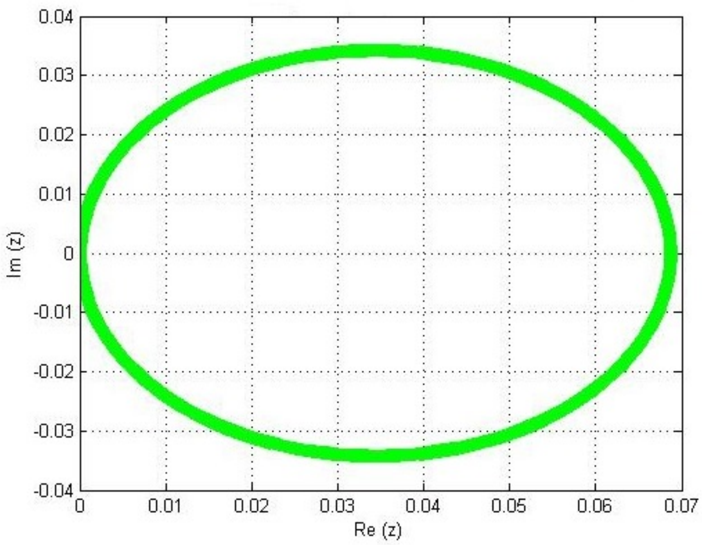

Figure 1. Region of absolute stability of the NMCFs

The NMCFs' absolute stability region is P-stable because it consists of the complex plane outside the enclosed figure and its periodicity interval lies between $(0.069, 0)$, which falls within the periodicity interval for P-stability $(\infty, 0)$.

4. Implementation Strategy

In this section, the implementation strategy is discussed in detail. It follows that variable y is discretized as follows:

$$h = \frac{b-a}{M}, \ \ x_i = a + ih, \ \ i = 0, 1, \cdots, M., \quad h = \frac{d-c}{N}, \ \ t_i = c + ih, \ \ i = 0, 1, \cdots, N. \tag{56}$$

For a fixed x in the interval $[a, b]$, $i = 0, \cdots, M$, and for a fixed t in the interval $[c, d]$, $i = 0, \cdots, N$.

N is the number of subintervals or iterations. The spatial derivative is approximated by the difference operator and replaced accordingly:

$$\frac{\partial u}{\partial t} \approx \frac{u(x_{m+1}, t) - u(x_{m-1}, t)}{2h}, \tag{57}$$

$$\frac{\partial u}{\partial y} \approx \frac{u(x, y_{m+1}) - u(x, y_{m-1})}{2h}, \tag{58}$$

$$\frac{\partial^2 u}{\partial y^2} \approx \left[\frac{u(x, y_{m+1}) - 2u(x, y_m) + u(x, y_{m-1}, y)}{(h)^2}\right], \tag{59}$$

$u(x, y_{m+1})$ is termed the numerical approximation to $u(x, y_{m+1})$ and, consequently, (1) has been converted to a system of ordinary differential equations and (1) has the semi-discretized form given below:

$$\begin{aligned} \frac{d^2 u_{m,n}}{dx^2} \;=\; & \frac{1}{a_{m,n}}[-b_{m,n}\left[\frac{u(x, y_{m+1}) - 2u(x, y_m) + u(x, y_{m-1}, y)}{(h)^2}\right] - p_{m,n}\frac{du_{m,n}}{dx} - \\ & q_{m,n}\left[\frac{u(x_{m+1}, t) - u(x_{m-1}, t)}{2h}\right] - r_{m,n}\left[\frac{u(x, y_{m+1}) - u(x, y_{m-1})}{2h}\right] - k_{m,n}u_{m,n} + g_{m,n}], \end{aligned} \tag{60}$$

where it follows that g are the non-homogeneous terms, h is the step size, and W is the tridiagonal matrix generated from (60) (shown in Appendix D).

The proposed formulas, namely the NMCFs, are then applied to solve the resulting equations of the ODEs with initial or boundary conditions (60), with the aid of Mathematica 11.0, with the features Nsolve for linear and Findroot for nonlinear, with an HP Laptop G250, 8GB RAM, ITERABYTE.

5. Numerical Examples

The accuracy and convergence of the NMCFs are presented in this section. Five numerical examples including nonlinear partial differentials originating from fluid dynamics were resolved from the literature. A comparison of the numerical solution produced by the NMCFs, and the exact results were made and also compared with the errors produced by other existing methods in the literature. To highlight the accuracy of the NMCFs and their benefits over existing techniques, the results are presented in tabular form.

For instance, the absolute error of the approximate solutions is computed alongside the exact solution and compared with the results from the other existing methods, particularly those proposed by Yagider and Karabacak [44], Lima et al. [45], Biala and Jator [46], Xu and Wang [47], and Volkov et al. [48]. The results from the methods are also discussed here.

The absolute errors (AEs) are given by = Max $|\; u(x_m, y_m) - u_m(y_n) \;|$. $u(x_m, y_m)$ symbolises the exact solution, and $u_m(y_n)$ denotes the approximate solution at the mesh point (x_m, y_m).

5.1. Test Problem One

Consider the following Helmholtz equation:

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) + 8u(x, y) = 0, \quad 0 < x, y < 1, \tag{61}$$

subjected to the following conditions:

$$\begin{aligned} u(0, y) &= \sin(2y), \quad \text{for } 0 \le x \le 1, \\ \frac{\partial u}{\partial x}(0, y) &= 0, \quad \text{for } 0 \le y \le 1, \end{aligned} \tag{62}$$

the theoretical solution of (61) is given as

$$u(x, y) = \cos(2x)\sin(2y). \tag{63}$$

Source: Yagider and Karabacak [44].

The prominent Helmholtz equation, which is an example of the elliptic PDEs considered in test problem one, was solved using the newly formulated formulas named NMCFs. The numerical solution is demonstrated in Table 1. The problem had been solved earlier by Yagider and Karabacak [44]. The numerical results were generated at x-values ranging between $0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$, and 1.0. The NMCFs results obtained show some level of improvement over the method constructed by Yagider and Karabacak [44], who proposed a multivariate Padé approximation as displayed in Table 2. Seemingly, the results generated by the NMCFs compared favorably with Yagider and Karabacak [44], despite using the same mesh size. A comparison of the errors is also displayed in Figure 2 below to show the behaviour of the performance of the NMCFs versus Yagider and Karabacak [44].

Table 1. Showing the numerical results for test problem 5.1.

x	Exact Results	NMCFs Results	AEs in NMCFs
0.1	0.1947083556680159200	0.1947091711543252300	$8.1548630 \times 10^{-07}$
0.2	0.1829813709520361500	0.1829865712999870800	$5.2003480 \times 10^{-06}$
0.3	0.1639638132942570800	0.1639688742954361300	$5.0610010 \times 10^{-06}$
0.4	0.1384113212579074500	0.1384142557064305700	$2.9344490 \times 10^{-06}$
0.5	0.1073395314304028500	0.1073414975338517800	$1.9661030 \times 10^{-06}$
0.6	0.0719877982915558400	0.0719893725902818500	$1.5742990 \times 10^{-06}$
0.7	0.0337668596353071160	0.0337672585371394250	$3.9890180 \times 10^{-07}$
0.8	-0.005800535872526165	-0.005801049555132515	$5.1368260 \times 10^{-07}$
0.9	-0.045138469521280010	-0.045138088107911740	$3.8141340 \times 10^{-07}$
1.0	-0.082675613529302500	-0.082675613529302500	0.0000000

Table 2. Comparison of errors in NMCFs and Yagider and Karabacak [44] for test problem 5.1.

x	AEs in NMCFs	AEs in Yagider and Karabacak [44]
0.1	$8.1548630 \times 10^{-07}$	$2.7000000 \times 10^{-09}$
0.2	$5.2003480 \times 10^{-06}$	$1.6800000 \times 10^{-07}$
0.3	$5.0610010 \times 10^{-06}$	$1.8551000 \times 10^{-06}$
0.4	$2.9344490 \times 10^{-06}$	$1.0017300 \times 10^{-05}$
0.5	$1.9661030 \times 10^{-06}$	$3.6361800 \times 10^{-05}$
0.6	$1.5742990 \times 10^{-06}$	$1.0226850 \times 10^{-04}$
0.7	$3.9890180 \times 10^{-07}$	$2.4043250 \times 10^{-04}$
0.8	$5.1368260 \times 10^{-07}$	$4.9437380 \times 10^{-04}$
0.9	$3.8141340 \times 10^{-07}$	$9.1526730 \times 10^{-04}$
1.0	$0.0000000 \times 10^{-00}$	$1.5558685 \times 10^{-03}$

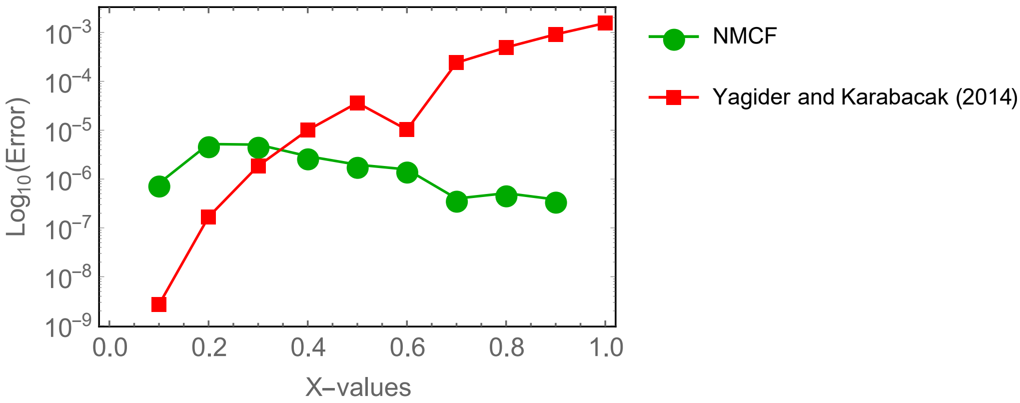

Figure 2. Comparison of errors in NMCFs versus Yagider and Karabacak [44] for test problem 5.1.

5.2. Test Problem Two

To demonstrate the accuracy of various numerical methods and gain a conceptual understanding of physical flows, the modified Burgers–Fisher equation, which combines the reaction, convection, and diffusion mechanism, has been taken into consideration by a number of authors. The Burgers–Fisher equation refers to this equation because it combines characteristics of the Fisher equation for diffusion, transport, and reactions with those of the Burgers–Fisher equation for the convective phenomenon. We are also interested in analyzing the numerical solution of the modified Burgers–Fisher equation because it exhibits relatively quick convergence and accuracy, and this will help to show the accuracy of the suggested method. If we take into account the modified Burgers–Fisher equation formed below:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \quad (x,t) \in u \equiv [0,T] \times (0,T], T > 0, \tag{64}$$

subjected to the following initial conditions:

$$u(x,0) = \frac{1}{2} - \frac{1}{2}tanh(\frac{x}{4}), \quad x \in \Omega \tag{65}$$

and the boundary conditions given by

$$\begin{aligned} u(0,t) &= \frac{1}{2} + \frac{1}{2}tanh(\frac{5t}{8}), \quad t \in (0,T], x \in \delta\Omega, \\ u(1,t) &= \frac{1}{2} + \frac{1}{2}tanh(\frac{5t}{8} - \frac{1}{4}), \quad t \in (0,T], x \in \delta\Omega \end{aligned} \tag{66}$$

the theoretical solution of (64) is given as

$$u(x,y) = \frac{1}{2} - \frac{1}{2}tanh(\frac{5t}{8} - \frac{x}{4}). \tag{67}$$

Source: Lima et al. [45].

The second test problem considered belongs to a class of convection–diffusion reaction equations known as the modified Burgers–Fisher equation. The numerical solution is taken into consideration as a means of applying the proposed formulas (NMCFs). This problem was earlier solved by Lima et al. [45], who proposed a finite element method. In Table 3, the numerical results were presented, while, in Table 4, the comparison of the absolute errors in the NMCFs against that of Lima et al. [45] is illustrated within the interval of integration. The results revealed that the performance of the NMCFs is far better than the method of Lima et al. [45] in terms of accuracy and give better convergence. The comparison of the errors is displayed in Figure 3 for better interpretation and reader understanding.

Table 3. Showing the computational results for test problem 5.2.

x	Exact Results	NMCFs Results	AEs in NMCFs
0.1	0.49106345176865770	0.49106297509207303 0	4.8×10^{-07}
0.2	0.47732805754796130	0.47732695529646950 0	1.1×10^{-06}
0.3	0.46225942047677904	0.46225766194634066 0	1.8×10^{-06}
0.4	0.45270432190392235	0.45270224249180313 0	2.1×10^{-06}
0.5	0.44047118471204494	0.44046894006482223 0	2.2×10^{-06}
0.6	0.42696371653650966	0.42696185723447140 0	1.9×10^{-06}
0.7	0.42696185723447140	0.41623374116473144 0	8.9×10^{-07}
0.8	0.40425905953324964	0.40426015992536700 0	1.1×10^{-06}
0.9	0.39108589473886196	0.39109034639973217 0	4.5×10^{-06}
1.0	0.38195699940974250	0.38196415606275036 0	7.2×10^{-06}

Table 4. Comparison of AEs in NMCFs against Lima et al. [45] for test problem 5.2.

x	AEs in NMCFs	AEs in Lima et al. [45]
0.1	4.8×10^{-07}	1.0×10^{-04}
0.2	1.1×10^{-06}	8.0×10^{-04}
0.3	1.8×10^{-06}	6.0×10^{-04}
0.4	2.1×10^{-06}	7.0×10^{-04}
0.5	2.2×10^{-06}	8.0×10^{-04}
0.6	1.9×10^{-06}	8.0×10^{-04}
0.7	8.9×10^{-07}	9.0×10^{-04}
0.8	1.1×10^{-06}	8.0×10^{-04}
0.9	4.5×10^{-06}	1.4×10^{-03}
1.0	7.2×10^{-06}	7.0×10^{-04}

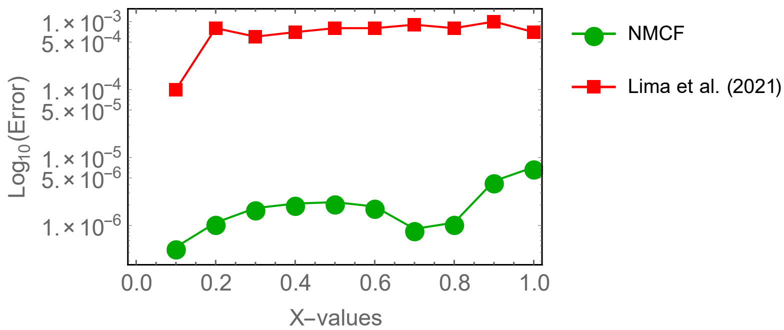

Figure 3. Comparison of errors in NMCFs versus Lima et al. [45] for test problem 5.2.

5.3. Test Problem Three

The Burgers equation is a convection–diffusion equation that can explain the evolutionary process by which a convective phenomenon can maintain balance with a diffusive behavior in a variety of areas of applied mathematics to comprehend the Navier–Stokes equations' fundamental characteristics. The pressure can be ignored in this simple equation, but the nonlinear and viscous terms still have an impact. The Reynolds number in the Navier–Stokes equations is the proportion of a flow's advective to viscous contributions. It is crucial to take into account an accurate and reliable numerical method for the simulation in order to obtain the evolution of this flow. The behavior of the modified Burgers equation in the following form was examined by the authors in Lima et al. [45].

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2}, \quad (x,t) \in u \equiv [0,T] \times (0,T], T > 0, \tag{68}$$

subjected to the following initial conditions:

$$u(x,0) = \frac{1}{(1+e^{2x})} \quad x \in \Omega, \tag{69}$$

and the boundary conditions given by

$$\begin{aligned} u(0,t) &= \frac{1}{(1+e^{-t})}, \quad t \in (0,T], x \in \delta\Omega, \\ u(1,t) &= \frac{1}{(1+e^{2-t})}, \quad t \in (0,T], x \in \delta\Omega \end{aligned} \tag{70}$$

the theoretical solution of (68) is given as

$$u(x,y) = \frac{1}{(1+e^{\frac{(2x-t)}{4\mu_0}})}. \tag{71}$$

Source: Lima et al. [45].

Next, in test problem three, the second-order nonlinear Navier–Stokes equations, which are a classical example of convection–diffusion equations, were put into consideration and solved using the NMCFs. The numerical solution, which comprises the exact results, NMCFs results, and the absolute errors, are shown in Table 5. The problem had been solved earlier by Lima et al. [45]. The numerical solution was executed at x-values ranging between $0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$, and 1.0. The NMCFs results generated show some level of outstanding performance over the method proposed by Lima et al. (2021), who constructed a finite element method with error correlation. Apparently, the results generated by the NMCFs give a minimal error against the method proposed by Lima et al. [45], as seen in Table 6. From the results, one can see that the proposed method has demonstrated excellently, with good performance with varying mesh points. The behavior of the performance of test problem three with the NMCFs and the method proposed by the authors in Lima et al. [45] is presented in plot form for better clarification in Figure 4.

Table 5. Showing the results for test problem 5.3.

x	Exact Results	NMCFs Results	AEs in NMCFs
0.1	0.45338575868105696	0.4533846995902536	1.1×10^{-06}
0.2	0.40492152613386595	0.40491952790249813	2.0×10^{-06}
0.3	0.3582425862347542	0.35823985347997894	2.7×10^{-06}
0.4	0.31410440319529837	0.3141011779927165	3.2×10^{-06}
0.5	0.273090250573869	0.27308678350641447	3.5×10^{-06}
0.6	0.2316531587086934	0.23164969524452944	3.4×10^{-06}
0.7	0.1982925977359844	0.19828934296755002	3.3×10^{-06}
0.8	0.1686815943978195	0.16867869569239505	2.9×10^{-06}
0.9	0.1427053084191849	0.14270286639846103	2.4×10^{-06}
1.0	0.1201511074867948	0.12014918056539994	1.9×10^{-06}

Table 6. Comparison of errors in NMCFs and Lima et al. [45] for test problem 5.3.

x	AEs in NMCFs	AEs in Lima et al. [45]
0.1	1.1×10^{-06}	1.9×10^{-03}
0.2	2.0×10^{-06}	2.4×10^{-03}
0.3	2.7×10^{-06}	2.1×10^{-03}
0.4	3.2×10^{-06}	2.0×10^{-03}
0.5	3.5×10^{-06}	1.9×10^{-03}
0.6	3.4×10^{-06}	1.7×10^{-03}
0.7	3.3×10^{-06}	1.5×10^{-03}
0.8	2.9×10^{-06}	1.3×10^{-03}
0.9	2.4×10^{-06}	1.3×10^{-03}
1.0	1.9×10^{-06}	3.0×10^{-04}

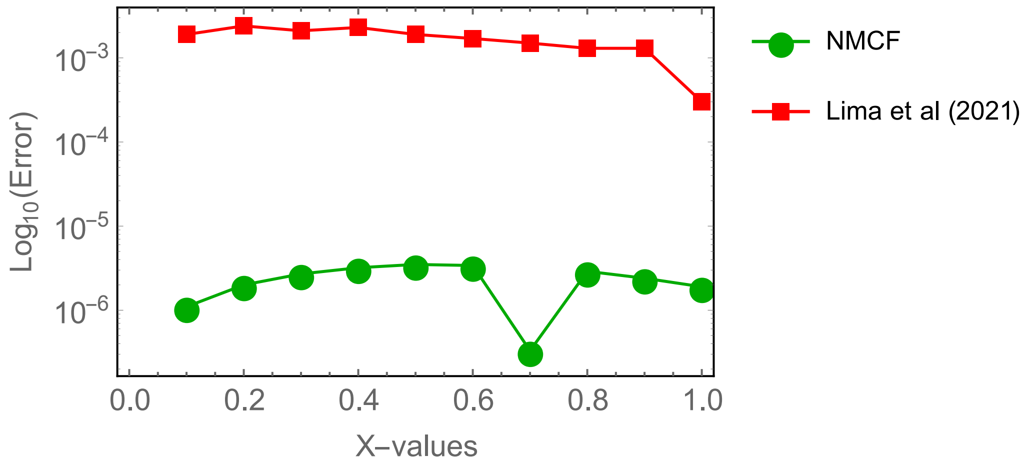

Figure 4. Comparison of errors in NMCFs versus Lima et al. [45] for test problem 5.3.

5.4. Test Problem Four

Consider the Laplace equation given below:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, x, y \in [0, 1], \tag{72}$$

subjected to the conditions on the boundary domain given below:

$$u(x, 0) = e^x, \quad u_y(0, x) = \cos(y),$$

the exact solution of (72) is given as

$$u(x, y) = e^x \cos(y). \tag{73}$$

Source: Xu and Wang [47].

Similarly, in test problem four, the Laplace equation was solved using the NMCFs. The numerical solution, which comprises the exact results, NMCFs results. and the absolute errors are shown in Tables 7 and 8. Before now, Biala et al. [46] and Xu and Wang [47] had solved the problem with their own approach. For instance, Biala et al. [46] proposed a block unification algorithm of order two via a shifted Chebychev's polynomial. Meanwhile, Xu and Wang [47] constructed a method named a parallel iterative algorithm. The numerical solution was implemented at N-values ranging between $32, 40$, and 48. The NMCFs obtained show better performances in terms of accuracy and convergence over the method proposed by the existing methods. The behavior of the performance of test problem four with the NMCFs and the method proposed by the authors in Biala et al. [46] and Xu and Wang [47] are presented in a curve for better interpretation in Figure 5.

Table 7. Showing the results for test problem 5.4.

N	Exact Results	NMCFs Results	AEs in NMCFs
16	1.0643645158455162	1.0643645183902837	2.5448×10^{-09}
24	1.0419993959674516	1.0419993959681841	7.3253×10^{-09}
32	1.0316174618605911	1.0316174648212710	2.9607×10^{-09}
40	1.0253134960263788	1.0253134960268170	4.3832×10^{-13}
48	1.0216581918640595	1.0216581918644914	4.3188×10^{-13}

Table 8. Comparison of errors in NMCFs with Xu and Wang [47] and Biala and Jator [46] for test problem 5.4.

N	AEs in NMCFs	AEs in Biala and Jator [46]	AEs in Xu and Wang [47]
16	2.5448×10^{-09}	1.2800×10^{-06}	3.9000×10^{-05}
24	7.3253×10^{-09}	2.6800×10^{-07}	1.7400×10^{-05}
32	2.9607×10^{-09}	8.7600×10^{-08}	9.7700×10^{-06}
40	4.3832×10^{-13}	3.6700×10^{-08}	6.2600×10^{-06}
48	4.3188×10^{-13}	1.8000×10^{-08}	4.3500×10^{-06}

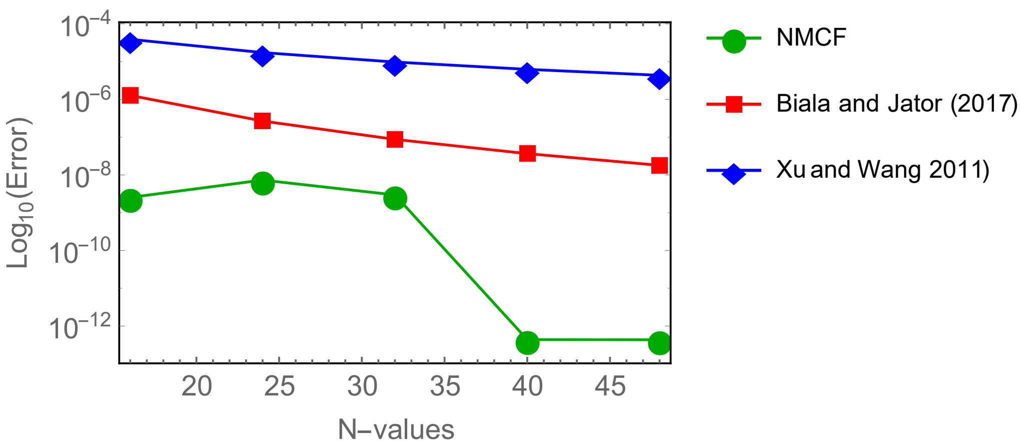

Figure 5. Comparison of errors in NMCFs versus Biala et al. [46] and Xu and Wang [47] for test problem 5.4.

5.5. Test Problem Five

Finally, the PDE with the Dirichlet boundary condition as follows is considered:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2\pi\left(2\pi y^2 - 2\pi y\right)e^{\pi(1-y)}\sin(\pi x), \quad x, y \in [0,1], \tag{74}$$

subjected to the conditions on the boundary domain given below:

$$u(x,0) = u(x,1),$$

the exact solution of (74) is given as

$$u(x,y) = e^{\pi x}\sin(\pi x) + e^{\pi(1-y)}\sin(\pi x). \tag{75}$$

Source: Volkov et al. [48].

Lastly, test problem five is solved using the NMCFs. The numerical results containing the exact results, NMCFs results, and the absolute errors are shown in Tables 9 and 10. Biala and Jator [46] and Volkov et al. [48] had solved the problem with their own techniques before now. In the work of Biala and Jator [46], a block unification algorithm of order two via a shifted Chebychev's polynomial was constructed, analyzed, and implemented. Likewise, Volkov et al. [48] presented a parallel iterative algorithm. The numerical solution was implemented at N-values ranging between $16, 32, 64$, and 128. The NMCFs obtained show superiority in performance in terms of accuracy over the method proposed by the existing methods. The behavior of the performance of test problem five with the NMCFs and the method proposed by the authors in Biala and Jator [46] and Volkov et al. [48] are presented in a figure for better usage in Figure 6.

Table 9. Showing the results for test problem 5.5.

N	Exact Results	NMCFs Results	AEs in NMCFs
16	0.21242359899585922	0.2124235991762159	1.8036×10^{-10}
32	0.11276006163890931	0.1127600617361654	9.7256×10^{-11}
64	0.06256009950548104	0.06256009955588866	5.0408×10^{-11}
128	0.24947430690102754	0.24947430707441792	1.7339×10^{-12}

Table 10. Comparison of errors in NMCFs with Volkov et al. [48] and Biala and Jator [46] for test problem 5.5.

N	AEs in NMCFs	AEs in Biala and Jator [46]	AEs in Volkov et al. [48]
16	1.8036×10^{-10}	1.2860×10^{-03}	3.2660×10^{-02}
32	9.7256×10^{-11}	3.1530×10^{-04}	8.2100×10^{-03}
64	5.0408×10^{-11}	7.9130×10^{-05}	2.053×10^{-03}
128	1.7339×10^{-12}	1.9750×10^{-05}	5.1280×10^{-04}

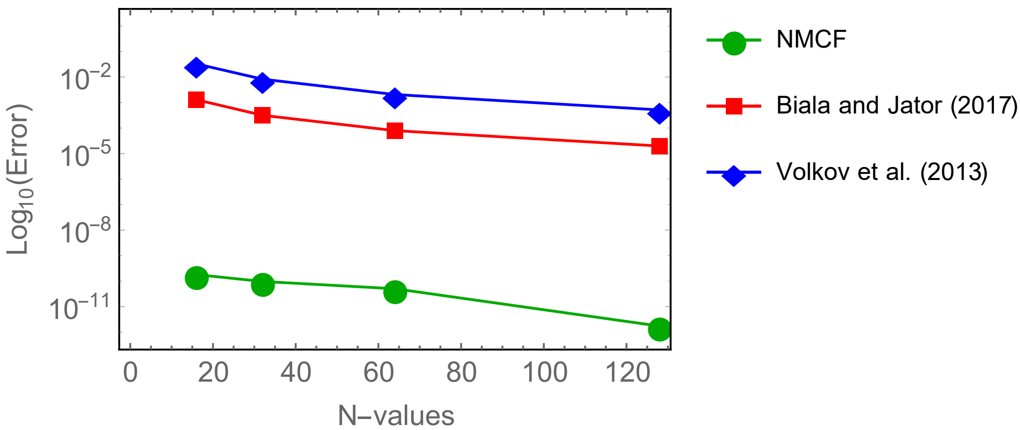

Figure 6. Comparison of errors in NMCFs versus Biala and Jator [46] and Volkov et al. [48] for test problem 5.5.

6. Conclusions

This article presents a ninth-order multistep collocation formula that was designed using orthogonal polynomial collocation. The formulas derived from the continuous function were combined in a step-by-step block approach algorithm. The proposed formulas (NMCFs) were used to solve a class of partial differential equations ranging from the Helmholtz equation to the convection diffusion reaction equations resulting from the semi-discretization of the problems studied. Because the NMCFs were implemented block by block, they do not require the starting values and predictors that are associated with the predictor-corrector method. The numerical results demonstrate a significant improvement over other methods in the literature.

When compared to existing methods in the literature, the numerical experiments presented in this paper clearly show that the NMCFs have a reasonably wide stability region displayed in Figure 1 and enjoy good accuracy and fast convergence advantages. Figures 2–6 show the comparison of errors for test problems 1–5. Because they are more accurate, provide better convergence, and are computationally faster, the NMCFs can be extended to solve problems that model real-life phenomena with numerous applications of higher-order PDEs in the future.

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Conflicts of Interest: The authors state that there are no conflicting interests

Appendix A

$$
\left.\begin{aligned}
&\Psi_0(\phi) = -(\phi+6)\\
&\Psi_1(\phi) = (\phi+7)\\
&\Delta_0(\phi) = \frac{h^2\vartheta}{7257600}\left(2\,\phi^8 + 24\,\phi^7 + 99\,\phi^6 + 105\,\phi^5 - 189\,\phi^4 - 693\,\phi^3 + 1287\,\phi^2 - 9225\,\phi + 65871\right)\\
&\Delta_1(\phi) = -\frac{h^2\vartheta}{1814400}\left(4\,\phi^8 + 53\,\phi^7 + 223\,\phi^6 + 275\,\phi^5 - 593\,\phi^4 - 439\,\phi^3 - 5327\,\phi^2 + 37289\,\phi - 261023\right)\\
&\Delta_2(\phi) = \frac{h^2\vartheta}{1814400}\left(14\,\phi^8 + 203\,\phi^7 + 913\,\phi^6 + 1145\,\phi^5 - 2033\,\phi^4 - 4777\,\phi^3 + 67\,\phi^2 - 11917\,\phi + 152107\right)\\
&\Delta_3(\phi) = -\frac{h^2\vartheta}{1814400}\left(28\,\phi^8 + 441\,\phi^7 + 2181\,\phi^6 + 2925\,\phi^5 - 4551\,\phi^4 - 12405\,\phi^3 - 2913\,\phi^2 + 29679\,\phi - 263481\right)\\
&\Delta_4(\phi) = -\frac{h^2\vartheta}{725760}\left(14\,\phi^8 + 238\,\phi^7 + 1313\,\phi^6 + 2095\,\phi^5 - 2875\,\phi^4 - 7523\,\phi^3 - 3631\,\phi^2 + 10369\,\phi + 17705\right)\\
&\Delta_5(\phi) = -\frac{h^2\vartheta}{1814400}\left(28\,\phi^8 + 511\,\phi^7 + 3161\,\phi^6 + 6445\,\phi^5 - 5791\,\phi^4 - 25937\,\phi^3 + 2063\,\phi^2 + 4135\,\phi - 140401\right)\\
&\Delta_6(\phi) = \frac{h^2\vartheta}{1814400}\left(14\,\phi^8 + 273\,\phi^7 + 1893\,\phi^6 + 5025\,\phi^5 - 393\,\phi^4 - 21729\,\phi^3 - 14757\,\phi^2 + 46059\,\phi + 21027\right)\\
&\Delta_7(\phi) = -\frac{h^2\vartheta}{1814400}\left(4\,\phi^8 + 83\,\phi^7 + 643\,\phi^6 + 2195\,\phi^5 + 2167\,\phi^4 - 5827\,\phi^3 - 17183\,\phi^2 - 13567\,\phi - 9143\right)\\
&\Delta_8(\phi) = \frac{h^2\vartheta}{7257600}\left(2\,\phi^8 + 44\,\phi^7 + 379\,\phi^6 + 1625\,\phi^5 + 3571\,\phi^4 + 3515\,\phi^3 + 343\,\phi^2 - 889\,\phi - 2849\right)\\
&\vartheta = (\phi+6)(\phi+7)
\end{aligned}\right\}
$$

Appendix B

$$
\left.\begin{aligned}
&\Psi_0'(\phi) = -1, \Psi_1'(\phi) = 1 \\
&\Delta_0'(\phi) = \frac{h^2}{7257600}\left(20\,\phi^9 + 450\,\phi^8 + 3960\,\phi^7 + 16800\,\phi^6 + 32004\,\phi^5 + 6300\,\phi^4 - 62640\,\phi^3 - 64800\,\phi^2 + 468873\right) \\
&\Delta_1'(\phi) = -\frac{h^2}{1814400}\left(40\,\phi^9 + 945\,\phi^8 + 8640\,\phi^7 + 37800\,\phi^6 + 74088\,\phi^5 + 17010\,\phi^4 - 143760\,\phi^3 - 151200\,\phi^2 - 1827161\right) \\
&\Delta_2'(\phi) = \frac{h^2}{1814400}\left(7140\,\phi^9 + 3465\,\phi^8 + 33120\,\phi^7 + 150780\,\phi^6 + 307188\,\phi^5 + 84420\,\phi^4 - 589680\,\phi^3 - 635040\,\phi^2 + 1476877\right) \\
&\Delta_3'(\phi) = -\frac{h^2}{1814400}\left(280\,\phi^9 + 7245\,\phi^8 + 72720\,\phi^7 + 348600\,\phi^6 + 750456\,\phi^5 + 256410\,\phi^4 - 1421280\,\phi^3 - 1587600\,\phi^2 - 2178735\right) \\
&\Delta_4'(\phi) = \frac{h^2}{725760}\left(140\,\phi^9 + 3780\,\phi^8 + 39960\,\phi^7 + 204120\,\phi^6 + 477036\,\phi^5 + 215460\,\phi^4 - 888720\,\phi^3 - 1058400\,\phi^2 + 665663\right) \\
&\Delta_5'(\phi) = -\frac{h^2}{1814400}\left(280\,\phi^9 + 7875\,\phi^8 + 87840\,\phi^7 + 483000\,\phi^6 + 1264536\,\phi^5 + 847350\,\phi^4 - 2313360\,\phi^3 - 3175200\,\phi^2 - 1651543\right) \\
&\Delta_6'(\phi) = \frac{h^2}{1814400}\left(140\,\phi^9 + 4095\,\phi^8 + 48240\,\phi^7 + 287700\,\phi^6 + 866628\,\phi^5 + 921060\,\phi^4 - 1254960\,\phi^3 - 3175200\,\phi^2 + 2207829\right) \\
&\Delta_7'(\phi) = -\frac{h^2}{1814400}\left(40\,\phi^9 + 1215\,\phi^8 + 15120\,\phi^7 + 98280\,\phi^6 + 346248\,\phi^5 + 572670\,\phi^4 - 7680\,\phi^3 - 1445040\,\phi^2 - 1814400\,\phi - 688673\right) \\
&\Delta_8'(\phi) = \frac{h^2}{7257600}\left(20\,\phi^9 + 630\,\phi^8 + 8280\,\phi^7 + 58800\,\phi^6 + 243684\,\phi^5 + 590940\,\phi^4 + 784080\,\phi^3 + 453600\,\phi^2 - 74375\right)
\end{aligned}\right\} \tag{A1}
$$

Appendix C

$$
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{324901}{1451520} & \frac{1169}{2} & -\frac{653203}{907200} & \frac{50689}{56700} & -\frac{196277}{241920} & \frac{92473}{181440} & -\frac{95167}{453600} & \frac{7703}{151200} & -\frac{5741}{1036800} \\
\frac{58193}{113400} & \frac{14692}{7} & -\frac{81}{50} & \frac{30916}{14175} & -\frac{22703}{11340} & \frac{5968}{4725} & -\frac{14773}{28350} & \frac{1796}{14175} & -\frac{521}{37800} \\
\frac{71661}{89600} & \frac{1467}{400} & -\frac{4707}{2800} & \frac{225}{64} & -\frac{28143}{8960} & \frac{11079}{5600} & -\frac{9141}{11200} & \frac{2223}{11200} & -\frac{387}{17920} \\
\frac{15406}{14175} & \frac{24832}{4725} & -\frac{928}{567} & \frac{80128}{14175} & -\frac{188}{45} & \frac{38144}{14175} & -\frac{15776}{14175} & \frac{256}{945} & -\frac{418}{14175} \\
\frac{56975}{41472} & \frac{248375}{36288} & -\frac{19375}{12096} & \frac{143375}{18144} & -\frac{641875}{145152} & \frac{225}{64} & -\frac{12875}{9072} & \frac{3125}{9072} & -\frac{3625}{96768} \\
\frac{93}{56} & \frac{1476}{175} & -\frac{549}{350} & \frac{1776}{175} & -\frac{639}{140} & \frac{36}{7} & -\frac{81}{50} & \frac{72}{175} & -\frac{9}{200} \\
\frac{2019731}{1036800} & \frac{216433}{21600} & -\frac{98441}{64800} & \frac{1601467}{129600} & -\frac{160867}{34560} & \frac{55223}{8100} & -\frac{127253}{129600} & \frac{8183}{14400} & -\frac{57281}{1036800} \\
\frac{31648}{14175} & \frac{23552}{2025} & -\frac{7424}{4725} & \frac{41984}{2835} & -\frac{14528}{2835} & \frac{41984}{4725} & -\frac{7424}{14175} & \frac{23552}{14175} & 0
\end{pmatrix}, U = \begin{bmatrix}
0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1
\end{bmatrix},
$$

$$
B = \begin{pmatrix}
\frac{324901}{1451520} & \frac{1169}{2} & -\frac{653203}{907200} & \frac{50689}{56700} & -\frac{196277}{241920} & \frac{92473}{181440} & -\frac{95167}{453600} & \frac{7703}{151200} & -\frac{5741}{1036800} \\
\frac{58193}{113400} & \frac{14692}{7} & -\frac{81}{50} & \frac{30916}{14175} & -\frac{22703}{11340} & \frac{5968}{4725} & -\frac{14773}{28350} & \frac{1796}{14175} & -\frac{521}{37800}
\end{pmatrix},
$$

Appendix D

$$
W = \begin{pmatrix}
\frac{-2}{(h)^2} & \frac{1}{(h)^2} & 0 & \cdots & \cdots & 0 & 0 & 0 \\
\frac{1}{(h)^2} & \frac{-2}{(h)^2} & \frac{1}{(h)^2} & \cdots & \cdots & 0 & 0 & 0 \\
0 & \frac{1}{(h)^2} & \frac{-2}{(h)^2} & \cdots & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots \\
 & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \cdots & \frac{-2}{(h)^2} & \frac{1}{(h)^2} & 0 \\
0 & 0 & 0 & \cdots & \vdots & \frac{1}{(h)^2} & \frac{-2}{(h)^2} & \frac{1}{(h)^2} \\
0 & 0 & 0 & \vdots & \cdots & 0 & \frac{1}{(h)^2} & \frac{-2}{(h)^2}
\end{pmatrix}
$$

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