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A Note on the Time-Fractional Navier–Stokes Equation and the Double Sumudu-Generalized Laplace Transform Decomposition Method

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Abstract: In this work, the time-fractional Navier–Stokes equation is discussed using a calculational method, which is called the Sumudu-generalized Laplace transform decomposition method (DGLTDM). The fractional derivatives are defined in the Caputo sense. The (DGLTDM) is a hybrid of the Sumudu-generalized Laplace transform and the decomposition method. Three examples of the time-fractional Navier–Stokes equation are studied to check the validity and demonstrate the effectiveness of the current method. The results show that the suggested method succeeds remarkably well in terms of proficiency and can be utilized to study more problems in the field of nonlinear fractional differential equations (FDEs).

Keywords: double Sumudu transform; double Sumudu-generalized Laplace transform; inverse double Sumudu-generalized Laplace transform; fractional Navier–Stokes equation; decomposition methods

MSC: 35A22; 44A30

Citation: Eltayeb, H.; Bachar, I.; Mesloub, S. A Note on the Time-Fractional Navier–Stokes Equation and the Double Sumudu-Generalized Laplace Transform Decomposition Method. *Axioms* **2024**, *13*, 44. https://doi.org/10.3390/axioms13010044

Academic Editor: Jorge E. Macías Díaz

Received: 10 December 2023
Revised: 26 December 2023
Accepted: 9 January 2024
Published: 11 January 2024

1. Introduction

Fractional partial differential equations play an important role in applied mathematics, as they have been suggested for and applied in several different areas of the physical sciences and engineering such as in fluid dynamics, acoustics, electromagnetism, visco-elasticity, electro-chemistry, etc. The authors in [1] discussed the multi-scale elastic structures consisting of matrix medium and thin coatings or inclusions. There are some approaches to solving the problem of the elastic deformation of thin-walled solids with a complex shape that is analyzed based on linear and geometrically nonlinear models using new classes of surfaces [2]. The researchers in [3] applied the variational method to solve the time-fractal heat conduction problem in the playdate–block construction.

The Navier–Stokes equations are commonly utilized to explain the motion of fluids in models related to weather, ocean currents, and water flow in a pipe. Also, Navier–Stokes equations are vector equations. Newly, several researchers have generalized the classical Navier–Stokes equation into a fractional formula depending on replacing the first-time derivative with a fractional derivative of order $0 < \beta \leq 1$, as in [4–8].

Recently, several analytical and approximate techniques for solving time-fractional Navier–Stokes equations have been developed, for example, the Adomian decomposition method [9], the q-homotopy analysis transform scheme [10], the modified Laplace decomposition method [7], the Natural Homotopy Perturbation Method [11], a reliable algorithm based on the new homotopy perturbation transform method [6], and a modified reduced differential transform method [12]. In the paper [13], the authors discussed the convergence properties of double Sumudu transformation and applied it to obtain the exact solution of the Volterra integro-partial differential equation. The double Sumudu transform is connected with the Adomian decomposition method to obtain the analytical solution of nonlinear fractional partial differential equations [14].

The double Sumudu-generalized Laplace decomposition method is a strong method that has been used to develop the double Sumudu transform and generalized Laplace transform [15,16].

This work aims to study the time-fractional Navier–Stokes equation in one and two dimensions using the double Sumudu-generalized Laplace transform decomposition method and to determine the accuracy, efficiency, and simplicity of the suggested method.

lNotations:

In this paper, we employ the following symbols:

(1) (SGLT) instead of "Sumudu-generalized Laplace transform";

(2) (DST) instead of "double Sumudu transform";

(3) (DSGLT) instead of "double Sumudu-generalized Laplace transform";

(4) (DM) instead of "decomposition method";

(5) (DSGLTDM) instead of "double Sumudu-generalized Laplace transform decomposition method".

This article is organized as follows. In Section 2, some definitions regarding fractional calculus and (SGLT) are given. In Section 3, the two main theorems are proved, which are useful to study the time-fractional Navier-Stokes equation constructed using the (SGLT). In Section 3.1, the (SGLTDM) is used to solve the one-dimensional time-fractional Navier–Stokes model. In Section 3.2, the (SGLTDM) is applied to solve the two-dimensional coupled time-fractional Navier–Stokes model. In Section 4 some numerical example are given. In Section 5, conclusions are given.

2. Basic Definitions of Fractional Derivatives and Sumudu-Generalized Laplace Transforms

In this part, some basic definitions of fractional calculus and (SGLT) are given, which are helpful for this paper.

Definition 1 ([10])**.** *A real function $f(t)$, $t > 0$ is is called in the space C_μ, $\mu \in \mathbb{R}$ if $\exists p$ is a real number $p(> \mu)$, so that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0,\infty)$, and it is reportedly in the space C_μ^m if and only if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.*

Definition 2 ([17–19])**.** *The Caputo time-fractional derivative operator of order $\tau > 0$ is given by*

$$D_t^\tau u(x,t) = \{ \begin{matrix} \frac{1}{\Gamma(m-\tau)} \int_0^t (t-\sigma)^{m-\tau-1} \frac{\partial^m u(x,\sigma)}{\partial \sigma^m} d\sigma, \\ \frac{\partial^m u(x,t)}{\partial t^m},\ for\ m=\tau\in\mathbb{N} \end{matrix} \quad m-1 < \tau < m. \tag{1}$$

Definition 3 ([20])**.** *Let f be a function of two variables x and t , where $x,t > 0$. The Sumudu-generalized Laplace transform of f is defined by*

$$S_x G_t(f(x,t)) = F(u_1,s) = \frac{s^\alpha}{u_1} \int_0^\infty \int_0^\infty e^{-\left(\frac{x}{u_1}+\frac{t}{s}\right)} f(x,t)dtdx, \tag{2}$$

where the symbol $S_x G_t$ denoted the (SGLT), and the symbols u_1 and s denoted transforms of the variables x and t in (SGLT), respectively. Double Sumudu-generalized Laplace transform, which is defined by

$$S_x S_y G_t(f(x,y,t)) = \frac{s^\alpha}{u_1 u_2} \int_0^\infty \int_0^\infty \int_0^\infty e^{-\left(\frac{x}{u_1}+\frac{y}{u_2}+\frac{t}{s}\right)} f(x,y,t)dtdydx. \tag{3}$$

Similarly, the (SGLT) for the second partial derivative with respect to x and t is defined as follows

$$L_x G_t[\psi_{xx}] = \frac{\Psi_\alpha(u_1,s)}{u_1^2} - \frac{\Psi_\alpha(0,s)}{u_1^2} - \frac{\partial \Psi_\alpha(0,s)}{\partial x}.$$

$$\begin{aligned} S_x G_t[\psi_t] &= \frac{\Psi_\alpha(u_1, s)}{s} - s^\alpha \Psi(u_1, 0), \\ S_x G_t[\psi_{tt}] &= \frac{\Psi_\alpha(u_1, s)}{s^2} - s^{\alpha-1}\Psi(u_1, 0) - s^\alpha \Psi_t(u_1, 0). \end{aligned}$$

In general,

$$\begin{aligned} S_x G_\alpha\left[\frac{\partial^m f(x,t)}{\partial t^m}\right] &= \frac{F_\alpha(u_1, s)}{s^m} \\ &\quad - s^\alpha \sum_{k=1}^{n} \frac{1}{s^{m-k}} S_x\left[\frac{\partial^{k-1} f(x,0)}{\partial t^{k-1}}\right]. \end{aligned} \tag{4}$$

where u_1, s are complex values. The inverse (SGLT) $S_{u_1}^{-1} G_s^{-1}[S_x G_t(f(x,t))] = f(x,t)$ is defined as in [20] by the complex double integral formula

$$f(x,t) = \frac{1}{(2\pi i)^2} \int_{\tau-i\infty}^{\tau-i\infty} \int_{y-i\infty}^{y-i} e^{\frac{1}{u_1}x + \frac{1}{s}t} S_x G_t[f(x,t)] ds\, du_1.$$

3. Main Results

In the following theorem, we present the (SGLT) of the partial fractional Caputo derivatives

Theorem 1. *The (SGLT) of the fractional partial derivatives $D_t^\beta \psi$ is denoted by*

$$S_x G_t\left[D_t^\beta \psi\right] = \frac{\Psi_\alpha(u_1, s)}{s^\beta} - s^\alpha \sum_{k=1}^{\infty} \frac{1}{s^{\beta-k}} S_x\left[\frac{\partial^{k-1}}{\partial t^{k-1}} \psi(x, 0)\right]$$

Proof. By utilizing the definition of (SGLT), we have

$$S_x G_t\left[D_t^\beta \psi\right] = \frac{s^\alpha}{u_1} \int_0^\infty \int_0^\infty e^{-\left(\frac{x}{u_1} + \frac{t}{s}\right)} D_t^\beta \psi dt dx,$$

and with the help of Equation (1), we obtain

$$\begin{aligned} & S_x G_t\left[D_t^\beta \psi\right] \\ =\ & \frac{s^\alpha}{u_1} \int_0^\infty \int_0^\infty e^{-\left(\frac{x}{u_1} + \frac{t}{s}\right)} \frac{1}{\Gamma(m-\beta)} \int_0^t (t-\zeta)^{m-\beta-1} \frac{\partial^m \psi(x,\zeta)}{\partial \zeta^m} d\zeta\, dt dx \\ =\ & \frac{1}{u_1} \int_0^\infty e^{-\frac{x}{u_1}} \left(\frac{1}{\Gamma(m-\beta)} s^\alpha \int_0^\infty \int_\zeta^\infty \frac{e^{-\frac{t}{s}}}{(t-\zeta)^{\beta-m+1}} \frac{\partial^m \psi(x,\zeta)}{\partial \zeta^m} dt d\zeta \right) dx. \end{aligned}$$

Let $v = t - \zeta$.

$$\begin{aligned} & S_x G_t\left[D_t^\beta \psi\right] \\ =\ & \frac{1}{u_1} \int_0^\infty e^{-\frac{x}{u_1}} \left(\frac{s^\alpha}{\Gamma(m-\beta)} \int_0^\infty \frac{\partial^m \psi(x,\zeta)}{\partial \zeta^m} d\zeta \int_0^\infty v^{m-\beta-1} e^{-\frac{(v+\zeta)}{s}} dv \right) dx \\ =\ & \frac{1}{u_1} \int_0^\infty e^{-\frac{x}{u_1}} \left(\frac{s^\alpha}{\Gamma(m-\beta)} \int_0^\infty e^{-\frac{\zeta}{s}} \frac{\partial^m \psi(x,\zeta)}{\partial \zeta^m} d\zeta \int_0^\infty v^{m-\beta-1} e^{-\frac{v}{s}} dv \right) dx \\ =\ & \frac{1}{u_1} \int_0^\infty e^{-\frac{x}{u_1}} \left(\frac{s^\alpha}{\Gamma(m-\beta)} \int_0^\infty e^{-\frac{\zeta}{s}} \frac{\partial^m \psi(x,\zeta)}{\partial \zeta^m} d\zeta \frac{\Gamma(m-\beta)}{s^{\beta-m}} \right) dx \\ =\ & \frac{1}{u_1} \int_0^\infty e^{-\frac{x}{u_1}} \left(s^\alpha \int_0^\infty e^{-\frac{\zeta}{s}} \frac{\partial^m \psi(x,\zeta)}{\partial \zeta^m} d\zeta \right) \frac{1}{s^{\beta-m}} dx, \end{aligned}$$

where

$$\frac{\Gamma(m-\beta)}{s^{\beta-m}} = \int_0^\infty v^{m-\beta-1} e^{-\frac{v}{s}} dv.$$

$$\begin{aligned} & S_x G_t\left[D_t^\beta \psi\right] \\ = & \frac{1}{u_1} \int_0^\infty e^{-\frac{x}{u_1}} \left(G_t\left[\frac{\partial^m \psi(x,\zeta)}{\partial \zeta^m}\right]\right) \frac{1}{s^{\beta-m}} dx; \end{aligned}$$

by implementing Equation (4), we can obtain

$$S_x G_t\left[D_t^\beta \psi\right] = \frac{1}{s^{\beta-m}}\left[\frac{\Psi_\alpha(u_1,s)}{s^m} - s^\alpha \sum_{k=1}^{n} \frac{1}{s^{m-k}} S_x\left[\frac{\partial^{k-1}\psi(x,0)}{\partial t^{k-1}}\right]\right];$$

by rewriting the equation above, we obtain

$$S_x G_t\left[D_t^\beta \psi\right] = \frac{\Psi_\alpha(u_1,s)}{s^\beta} - s^\alpha \sum_{k=1}^{n} \frac{1}{s^{\beta-k}} S_x\left[\frac{\partial^{k-1}\psi(x,0)}{\partial t^{k-1}}\right].$$

□

In the next theorem, we utilize the (SGLT) for fractional partial derivatives $xD_t^\beta \psi$.

Theorem 2. *The (SGLT) of the fractional partial derivatives $xD_t^\beta \psi$ is achieved by*

$$\begin{aligned} S_x G_t\left[x D_t^\beta \psi\right] &= \frac{u_1}{s^\beta} \frac{d}{du_1}(u_1 \Psi_\alpha(u_1,s)) \\ & -u_1 s^{\alpha-\beta+1} \frac{d}{du_1}(u_1 \Psi(u_1,0)). \end{aligned} \quad (5)$$

Proof. By utilizing the derivatives with respect to u_1 for Equation (2), one can obtain

$$\begin{aligned} \frac{d}{du_1}\left(S_x G_t\left[D_t^\beta \psi\right]\right) &= \frac{d}{du_1} \int_0^\infty \int_0^\infty \frac{s^\alpha}{u_1} e^{-\left(\frac{1}{u_1}x + \frac{1}{s}t\right)} D_t^\beta \psi dx dt, \\ &= \int_0^\infty s^\alpha e^{-\frac{1}{s}t} \left(\int_0^\infty \frac{d}{du_1} \frac{1}{u_1} e^{-\frac{1}{u_1}x} D_t^\beta \psi dx\right) dt; \end{aligned} \quad (6)$$

the derivative between the brackets can be calculated as follows:

$$\begin{aligned} \int_0^\infty \frac{d}{du_1} \frac{1}{u_1} e^{-\frac{1}{u_1}x} D_t^\beta \psi dx &= \int_0^\infty \left(\frac{1}{u_1^3} x - \frac{1}{u_1^2}\right) e^{-\frac{1}{u_1}x} D_t^\beta \psi dx \\ &= \int_0^\infty \frac{1}{u_1^3} x e^{-\frac{1}{u_1}x} D_t^\beta \psi dx \\ & \quad - \int_0^\infty \frac{1}{u_1^2} e^{-\frac{1}{u_1}x} D_t^\beta \psi dx; \end{aligned} \quad (7)$$

by putting Equation (7) into Equation (6), we obtain

$$\begin{aligned} \frac{d}{d\mu_1}\left(S_x G_t\left[D_t^\beta \psi\right]\right) &= \int_0^\infty s^\alpha e^{-\frac{1}{s}t} \int_0^\infty \frac{1}{u_1^3} x e^{-\frac{1}{u_1}x} D_t^\beta \psi dx dt \\ & - \int_0^\infty s^\alpha e^{-\frac{1}{s}t} \int_0^\infty \frac{1}{u_1^2} e^{-\frac{1}{u_1}x} D_t^\beta \psi dx dt l \end{aligned} \quad (8)$$

consequently Equation (8) becomes

$$\frac{d}{d\mu_1}\left(S_x G_t\left[D_t^{\beta}\psi\right]\right) = \frac{1}{u_1^2}\left(\frac{s^{\alpha}}{u_1}\int_0^{\infty}\int_0^{\infty} e^{-\left(\frac{1}{u_1}x+\frac{1}{s}t\right)} x D_t^{\beta}\psi dx dt\right) - \frac{1}{u_1}\left(\frac{s^{\alpha}}{u_1}\int_0^{\infty}\int_0^{\infty} e^{-\left(\frac{1}{u_1}x+\frac{1}{s}t\right)} D_t^{\beta}\psi dx dt\right). \quad (9)$$

Hence,

$$\frac{d}{d\mu_1}\left(S_x G_t\left[D_t^{\beta}\psi\right]\right) = \frac{1}{u_1^2} S_x G_t\left[x D_t^{\beta}\psi\right] - \frac{1}{u_1} S_x G_t\left[D_t^{\beta}\psi\right]; \quad (10)$$

by managing the above equation, we will obtain the proof of Equation (5) as follows

$$S_x G_t\left[x D_t^{\beta}\psi\right] = \frac{u_1}{s^{\beta}}\frac{d}{du_1}(u_1 \Psi_{\alpha}(u_1, s)) - u_1 s^{\alpha-\beta+1}\frac{d}{du_1}(u_1 \Psi_{\alpha}(u_1, 0)).$$

The proof is complete. □

The double Sumudu-generalized Laplace transform of the partial derivatives $D_t^{\beta}\psi(x, y, t)$ is given by

$$S_x G_t\left[\frac{\partial^{\beta}\psi(x, y, t)}{\partial t^{\beta}}\right] = \frac{\Psi_{\alpha}(u_1, u_2, s)}{s^{\beta}} - s^{\alpha-\beta+1} S_x S_y[\Psi(x, y, 0)], \quad (11)$$

where β represents the order of the derivative.

3.1. Analysis of the Sumudu-Generalized Laplace Decomposition Method

This subsection gives the main concept of the (SGLTDM) for the fractional partial differential equation, to demonstrate the essential strategy of the Sumudu-generalized Laplace Adomian decomposition method. The Navier–Stokes equation with time-fractional is denoted by

$$D_t^{\beta}\psi(x, t) = D_x^2 u(x, t) + \frac{1}{x} D_x u(x, t) + f(x, t), \quad x, t > 0, \qquad m-1 < \alpha < m, \quad (12)$$

with the initial condition

$$\psi(x, 0) = f_1(x),$$

where $D_t^{\beta} = \frac{\partial^{\beta}}{\partial t^{\beta}}$ is the fractional Caputo derivative, $D_x^2 = \frac{\partial^2}{\partial x^2}$, $D_x = \frac{\partial}{\partial x}$, and the right-hand-side function $f(x, t)$ is the source term. With a view to applying the (SGLTDM), the following steps are needed.

Step 1: We multiply first Equation (12) by x, and we obtain

$$x D_t^{\alpha}\psi = x D_x^2\psi + D_x\psi + x f(x, t), \quad x, t > 0. \quad (13)$$

Step 2: Applying the (SGLT) on both sides of Equation (13), we have

$$S_x G_t[x D_t^{\alpha}\psi] = S_x G_t\left[x D_x^2\psi + D_x\psi + x f(x, t)\right], \quad x, t > 0. \quad (14)$$

Using Theorem 2, we obtain

$$\begin{aligned} & \frac{u_1}{s^\beta}\frac{d}{du_1}(u_1\Psi_\alpha(u_1,s)) - u_1 s^{\alpha-\beta+1}\frac{d}{du_1}(u_1\Psi_\alpha(u_1,0)) \\ = \; & S_x G_t\Big[xD_x^2\psi + D_x\psi + xf(x,t)\Big]; \end{aligned}$$

after an algebraic handling, we obtain

$$\begin{aligned} & \frac{d}{du_1}(u_1\Psi_\alpha(u_1,s)) = s^{\alpha+1}\frac{d}{du_1}(u_1F_1(u_1,0)) \\ & + \frac{s^\beta}{u_1}S_xG_t\Big[xD_x^2\psi + D_x\psi + xf(x,t)\Big]. \end{aligned} \tag{15}$$

Step 3: By employing the integral for both sides of Equation (15) from 0 to u_1 with respect to u_1, one can obtain

$$\begin{aligned} \Psi_\alpha(u_1,s) = \; & s^{\alpha+1}F_1(u_1,0) + \frac{1}{u_1}\int_0^{u_1}\left(\frac{s^\beta}{u_1}S_xG_t[xf(x,t)]\right)du_1 \\ & + \frac{1}{u_1}\int_0^{u_1}\left(\frac{s^\beta}{u_1}S_xG_t\Big[xD_x^2\psi + D_x\psi\Big]\right)du_1. \end{aligned} \tag{16}$$

Step 4: By utilizing the inverse (SGLT) for Equation (16), we obtain

$$\begin{aligned} \psi(x,t) = \; & f_1(x) + S_{u_1}^{-1}G_s^{-1}\left[\frac{1}{u_1}\int_0^{u_1}\left(\frac{s^\beta}{u_1}S_xG_t\Big[xD_x^2\psi + D_x\psi\Big]\right)du_1\right] \\ & + S_{u_1}^{-1}G_s^{-1}\left[\frac{1}{u_1}\int_0^{u_1}\left(\frac{s^\beta}{u_1}S_xG_t[xf(x,t)]\right)du_1\right], \end{aligned} \tag{17}$$

where the symbol $S_{u_1}^{-1}G_s^{-1}$ indicates the inverse (SGLT). The method (SGLTD M) designates the solution as an infinite series, as

$$\psi(x,t) = \sum_{m=0}^{\infty}\psi_m(x,t); \tag{18}$$

by placing Equation (18) into Equation (16), we obtain

$$\begin{aligned} \sum_{m=0}^{\infty}\psi_m(x,t) = \; & f_1(x) + S_{u_1}^{-1}G_s^{-1}\left[\frac{1}{u_1}\int_0^{u_1}\left(\frac{s^\beta}{u_1}S_xG_t[xf(x,t)]\right)du_1\right] \\ & + S_{u_1}^{-1}G_s^{-1}\left[\frac{1}{u_1}\int_0^{u_1}\left(\frac{s^\beta}{u_1}S_xG_t\left[\sum_{m=0}^{\infty}\left(xD_x^2\psi_m + D_x\psi_m\right)\right]\right)du_1\right]. \end{aligned} \tag{19}$$

By using (SGLTDM), we present the iteration relations as:

$$\psi_0(x,t) = f_1(x) + S_{u_1}^{-1}G_s^{-1}\left[\frac{1}{u_1}\int_0^{u_1}\left(\frac{s^\beta}{u_1}S_xG_t[xf(x,t)]\right)du_1\right], \tag{20}$$

and the remaining terms can be acquired from the next formula

$$\psi_{m+1}(x,t) = S_{u_1}^{-1}G_s^{-1}\left[\frac{1}{u_1}\int_0^{u_1}\left(\frac{s^\beta}{u_1}S_xG_t\Big[\left(xD_x^2\psi_m + D_x\psi_m\right)\Big]\right)du_1\right]. \; m \geq 1 \tag{21}$$

We consider that the inverse exists for all terms on the right-hand side of Equations (20) and (21), respectively, where S_xG_t is the (SGLT) with respect to x, t, and the inverse (SGLT) is given by $S_{u_1}^{-1}G_s^{-1}$ with respect to u_1, s.

3.2. Analysis of the Double Sumudu-Generalized Laplace Transforms Decomposition Method

In this part of the paper, we present the fundamental concept of the (DSGLTDM) for the time-fractional partial differential equation. To show the elementary plan of (DSGLTDM), we consider in the following a general coupled system two-dimensional time-fractional Navier–Stokes equations.

$$\begin{aligned} D_t^{\beta}\psi + \psi\psi_x + \varphi\psi_y &= \rho_0\big(\psi_{xx} + \psi_{yy}\big) - \frac{1}{\rho}\frac{\partial p}{\partial x}, \quad x, y, t > 0, \\ D_t^{\beta}\varphi + \psi\varphi_x + \varphi\varphi_y &= \rho_0\big(\varphi_{xx} + \varphi_{yy}\big) - \frac{1}{\rho}\frac{\partial p}{\partial y}, \quad x, y, t > 0, \\ n-1 &< \beta < n, \end{aligned} \tag{22}$$

subject to the conditions

$$\psi(x, y, 0) = f_1(x, y), \quad \varphi(x, y, 0) = g_1(x, y), \tag{23}$$

where $D_t^{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is the fractional Caputo derivative, p is pressure; in addition, if p is known, then $q_1 = \frac{1}{\rho}\frac{\partial p}{\partial x}$, and $q_2 = -\frac{1}{\rho}\frac{\partial p}{\partial y}$. The approach requires applying the (DSGLT) for both sides of Equation (22), and we obtain

$$\begin{aligned} \frac{\Psi(u_1, u_2, s)}{s^{\beta}} - s^{\alpha-\beta+1}\Psi(u_1, u_2, 0) &= -S_x S_y G_t\big(\psi\psi_x + \varphi\psi_y\big) \\ &\quad + S_x S_y G_t\big(\rho_0\big(\psi_{xx} + \psi_{yy}\big)\big) - S_x S_y G_t(q_1), \\ \frac{\Phi(u_1, u_2, s)}{s^{\beta}} - s^{\alpha-\beta+1}\Phi(u_1, u_2, 0) &= -S_x S_y G_t\big(\psi\varphi_x + \varphi\varphi_y\big) \\ &\quad + S_x S_y G_t\big(\rho_0\big(\varphi_{xx} + \varphi_{yy}\big)\big) + S_x S_y G_t(q_2). \end{aligned} \tag{24}$$

Now, using the differentiation property of the (DST), we have

$$\begin{aligned} \Psi(u_1, u_2, s) &= s^{\alpha-\beta+1}F_1(u_1, u_2) - s^{\beta}S_x S_y G_t\big(\psi\psi_x + \varphi\psi_y\big) \\ &\quad + s^{\beta}S_x S_y G_t\big(\rho_0\big(\psi_{xx} + \psi_{yy}\big)\big) - s^{\beta}S_x S_y G_t(q_1), \\ \Phi(u_1, u_2, s) &= s^{\alpha-\beta+1}G_1(u_1, u_2) - s^{\beta}S_x S_y G_t\big(\psi\varphi_x + \varphi\varphi_y\big) \\ &\quad + s^{\beta}S_x S_y G_t\big(\rho_0\big(\varphi_{xx} + \varphi_{yy}\big)\big) + s^{\beta}S_x S_y G_t(q_2). \end{aligned} \tag{25}$$

By involving the inverse (DSGLT) for Equation (25), we obtain

$$\begin{aligned} \psi(x, y, t) &= S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\Big(s^{\alpha-\beta+1}F_1(u_1, u_2)\Big) - S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\Big(s^{\beta}S_x S_y G_t\big(\psi\psi_x + \varphi\psi_y\big)\Big) \\ &\quad + S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\Big(s^{\beta}S_x S_y G_t\big(\rho_0\big(\psi_{xx} + \psi_{yy}\big)\big)\Big) - S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\Big[s^{\beta}S_x S_y G_t(q_1)\Big], \\ \varphi(x, y, t) &= S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\Big(s^{\alpha-\beta+1}G_1(u_1, u_2)\Big) - S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\Big(s^{\beta}S_x S_y G_t\big(\psi\varphi_x + \varphi\varphi_y\big)\Big) \\ &\quad + S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\Big(s^{\beta}S_x S_y G_t\big(\rho_0\big(\varphi_{xx} + \varphi_{yy}\big)\big)\Big) + S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\Big[s^{\beta}S_x S_y G_t(q_2)\Big]. \end{aligned} \tag{26}$$

The DM presumes that the functional solutions to $\psi(x, y, t)$ and $\varphi(x, y, t)$ are given by the following infinite series

$$\psi(x, y, t) = \sum_{n=0}^{\infty}\psi_n(x, y, t), \quad \varphi(x, y, t) = \sum_{n=0}^{\infty}\varphi_n(x, y, t). \tag{27}$$

In addition, the nonlinear terms $\psi\psi_x$, $\varphi\psi_y$, $\psi\varphi_x$, and $\varphi\varphi_y$ are specified by

$$\psi\psi_x = \sum_{n=0}^{\infty}A_n, \ \varphi\psi_y = \sum_{n=0}^{\infty}B_n, \ \psi\varphi_x = \sum_{n=0}^{\infty}C_n, \ \varphi\varphi_y = \sum_{n=0}^{\infty}D_n. \tag{28}$$

By placing Equation (27) into Equation (25), we obtain

$$\begin{aligned}\sum_{m=0}^{\infty}\psi_n(x,y,t) &= f_1(x,y) - S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^{\beta}S_xS_yG_t\left(\sum_{n=0}^{\infty}(A_n+B_n)\right)\right)\\ &+S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^{\beta}S_xS_yG_t\left(\rho_0\left(\sum_{n=0}^{\infty}\psi_{nxx}+\sum_{n=0}^{\infty}\psi_{nyy}\right)\right)\right)\\ &-S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left[s^{\beta}S_xS_yG_t(q_1)\right], \end{aligned} \tag{29}$$

and

$$\begin{aligned}\sum_{m=0}^{\infty}\varphi_n(x,y,t) &= g_1(x,y) - S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^{\beta}S_xS_yG_t\left(\sum_{n=0}^{\infty}(C_n+D_n)\right)\right)\\ &+S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^{\beta}S_xS_yG_t\left(\rho_0\left(\sum_{n=0}^{\infty}\varphi_{nxx}+\sum_{n=0}^{\infty}\varphi_{nyy}\right)\right)\right)\\ &+S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left[s^{\beta}S_xS_yG_t(q_2)\right]. \end{aligned} \tag{30}$$

Using (DSGLTDM), we present the recursive relations as:

$$\begin{aligned} u_0(x,y,t) &= f_1(x,y) - S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left[s^{\beta}S_xS_yG_t(q_1)\right]\\ v_0(x,y,t) &= g_1(x,y) + S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left[s^{\beta}S_xS_yG_t(q_2)\right], \end{aligned} \tag{31}$$

and the remaining elements ψ_{n+1} and $\varphi_{n+1}, n \geq 0$ are denoted by

$$\begin{aligned} \psi_{n+1}(x,y,t) &= -S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^{\beta}S_xS_yG_t(A_n+B_n)\right)\\ &+S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^{\beta}S_xS_yG_t\left(\rho_0\left(\psi_{nxx}+\psi_{nyy}\right)\right)\right), \end{aligned} \tag{32}$$

and

$$\begin{aligned} \varphi_{n+1}(x,y,t) &= -S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^{\beta}S_xS_yG_t(C_n+D_n)\right)\\ &+S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^{\beta}S_xS_yG_t\left(\rho_0\left(\varphi_{nxx}+\varphi_{nyy}\right)\right)\right). \end{aligned} \tag{33}$$

The inverse (DSGLT) is denoted by $S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}$ with respect to u_1, u_2, s. We presume that the inverse (DSGLT), with respect to u_1, u_2 and s exist for Equations (31)–(33).

4. Numerical Examples

In this section, two problems on fractional homogeneous and non-homogeneous time-fractional Navier–Stokes equations are solved to verify the ability and dependability of our method (SGLTDM) and (DSGLTDM).

Example 1. *Consider the following homogeneous one-dimensional motion of a dense fluid in a tube with the condition provided by*

$$D_t^{\beta}\psi = -\frac{\partial p}{\rho\partial z} + \frac{v}{x}\frac{\partial}{\partial x}(xD_x\psi), \quad x,\ t>0 \tag{34}$$

and the initial condition

$$\psi(x,0) = 1 - x^2. \tag{35}$$

The fractional derivative model is used to illustrate the time derivative term, and Equation (34) can be written in the following form

$$D_t^{\beta}\psi = K + \frac{v}{x}\frac{\partial}{\partial x}(xD_x\psi), \quad x,\ t > 0, \tag{36}$$

where $K = -\frac{\partial p}{\rho \partial z}$; multiplying the above equation by x, we have

$$xD_t^{\beta}\psi = Kx + v\frac{\partial}{\partial x}(xD_x\psi), \quad x,\ t > 0. \tag{37}$$

By taking the (SGLT) for both sides of Equation (37), we arrive at

$$S_xG_t[xD_t^{\alpha}\psi] = S_xG_t[Kx] + S_xG_t\left[v\frac{\partial}{\partial x}(xD_x\psi)\right]; \tag{38}$$

on using the differentiation property of the Sumudu transform and Theorem 2, we can obtain

$$\begin{aligned} \frac{d}{du_1}(u_1\Psi_{\alpha}(u_1,s)) &= s^{\alpha+1}\frac{d}{du_1}(u_1F_1(u_1,0)) \\ &+Ks^{\alpha+1+\beta} + \frac{s^{\beta}}{u_1}S_xG_t\left[v\frac{\partial}{\partial x}(xD_x\psi)\right]. \end{aligned} \tag{39}$$

Utilizing the Sumudu transform for the initial condition and substituting it into Equation (39), we have

$$\begin{aligned} \frac{d}{du_1}(u_1\Psi_{\alpha}(u_1,s)) &= \left(1-6u_1^2\right)s^{\alpha+1} + Ks^{\alpha+1+\beta} \\ &+\frac{s^{\beta}}{u_1}S_xG_t\left[v\frac{\partial}{\partial x}(xD_x\psi)\right]; \end{aligned} \tag{40}$$

by taking the integral for both sides of Equation (40) from 0 to u_1 with respect to u_1 and dividing the results by u_1, we obtain

$$\begin{aligned} \Psi_{\alpha}(u_1,s) &= \left(1-2u_1^2\right)s^{\alpha+1} + Ks^{\alpha+1+\beta} \\ &+\frac{1}{u_1}\int_0^{u_1}\frac{s^{\beta}}{u_1}S_xG_t\left[v\frac{\partial}{\partial x}(xD_x\psi)\right]du_1. \end{aligned} \tag{41}$$

Now, the inverse (SGLT) of Equation (41) is given by

$$\begin{aligned} \psi(x,t) &= 1 - x^2 + \frac{Kt^{\beta}}{\Gamma(\beta+1)} \\ &+S_{u_1}^{-1}G_s^{-1}\left[\frac{1}{u_1}\int_0^{u_1}\frac{s^{\beta}}{u_1}S_xG_t\left[v\frac{\partial}{\partial x}(xD_x\psi)\right]du_1\right], \end{aligned} \tag{42}$$

and we assume an infinite series solution of the unknown function $\psi(x,t)$ is denoted by Equation (18). By substituting Equation (18) into Equation (42), we obtain:

$$\begin{aligned} \sum_{m=0}^{\infty}\psi_m(x,t) &= 1 - x^2 + \frac{Kt^{\beta}}{\Gamma(\beta+1)} \\ &+S_{u_1}^{-1}G_s^{-1}\left[\frac{1}{u_1}\int_0^{u_1}\left(\frac{s^{\beta}}{u_1}S_xG_t\left[\sum_{m=0}^{\infty}v\frac{\partial}{\partial x}(xD_x\psi_m)\right]\right)du_1\right]. \end{aligned} \tag{43}$$

The zeroth component ψ_0 is recommended by the Adomian method, which always includes the initial condition and the source term, both of which are considered to be known. Therefore, we place

$$\psi_0 = 1 - x^2 + \frac{Kt^\beta}{\Gamma(\beta+1)}.$$

The remaining components ψ_{m+1}, $m \geq 0$ are given by using the relation

$$\psi_{m+1}(x,t) = S_{u_1}^{-1} G_s^{-1}\left[\frac{1}{u_1}\int_0^{u_1}\left(\frac{s^\beta}{u_1} S_x G_t\left[v\frac{\partial}{\partial x}(xD_x\psi_m)\right]\right)du_1\right]; \tag{44}$$

by substituting $m = 0$, into Equation (44), we obtain

$$\begin{aligned}\psi_1(x,t) &= S_{u_1}^{-1} G_s^{-1}\left[\frac{1}{u_1}\int_0^{u_1}\left(\frac{s^\beta}{u_1} S_x G_t\left[v\frac{\partial}{\partial x}(xD_x\psi_0)\right]\right)du_1\right] \\ &= -S_{u_1}^{-1} G_s^{-1}\left[\frac{1}{u_1}\int_0^{u_1}\left[4vs^{\alpha+\beta+1}\right]dp\right] = -S_{u_1}^{-1} G_s^{-1}\left[4vs^{\alpha+\beta+1}\right].\end{aligned}$$

$$\psi_1(x,t) = -\frac{4vt^\beta}{\Gamma(\beta+1)}.$$

In the same way, at $m = 1$,

$$\begin{aligned}\psi_2(x,t) &= S_{u_1}^{-1} G_s^{-1}\left[\frac{1}{u_1}\int_0^{u_1}\left(\frac{s^\beta}{u_1} S_x G_t\left[v\frac{\partial}{\partial x}(xD_x\psi_1)\right]\right)du_1\right] \\ &= S_{u_1}^{-1} G_s^{-1}\left[\frac{1}{u_1}\int_0^{u_1}\frac{s^\beta}{u_1} S_x G_t[0]du_1\right] = 0;\end{aligned}$$

similarly, at $m = 2$, we obtain

$$\psi_3(x,t) = 0.$$

Thus, the solution of Equation (34) can be expressed as

$$\psi(x,t) = 1 - x^2 + \frac{(K-4v)t^\beta}{\Gamma(\beta+1)}.$$

The error between the exact and approximation solution of example 1 is given in Table 1 below.

Table 1. Comparison between the exact and approximation solutions.

Exact $\beta = 1$	The Method $\beta = 0.95$	Error	The Method $\beta = 0.99$	Error
−2.0000	−2.0616	0.0616	−2.0126	0.0126
−2.0100	−2.0716	0.0616	−2.0226	0.0126
−2.0400	−2.1016	0.0616	−2.0526	0.0126
−2.0900	−2.1516	0.0616	−2.1026	0.0126
−2.1600	−2.2216	0.0616	−2.1726	0.0126
−2.2500	−2.3116	0.0616	−2.2626	0.0126
−2.3600	−2.4216	0.0616	−2.3726	0.0126
−2.4900	−2.5516	0.0616	−2.5026	0.0126
−2.6400	−2.7016	0.0616	−2.6526	0.0126
−2.8100	−2.8716	0.0616	−2.8226	0.0126
−3.0000	−3.0616	0.0616	−3.0126	0.0126

Figure 1 presents a comparison between the exact solution and the obtained numerical solution of Equation (34); at $t = 1$ and $\beta = 1$, we obtain the exact solution, and by taking different values of β such as ($\beta = 0.95$, $\beta = 0.99$), we obtain the approximate solutions. Figure 2 shows the plot of function $\psi(x,t)$ in three dimensions.

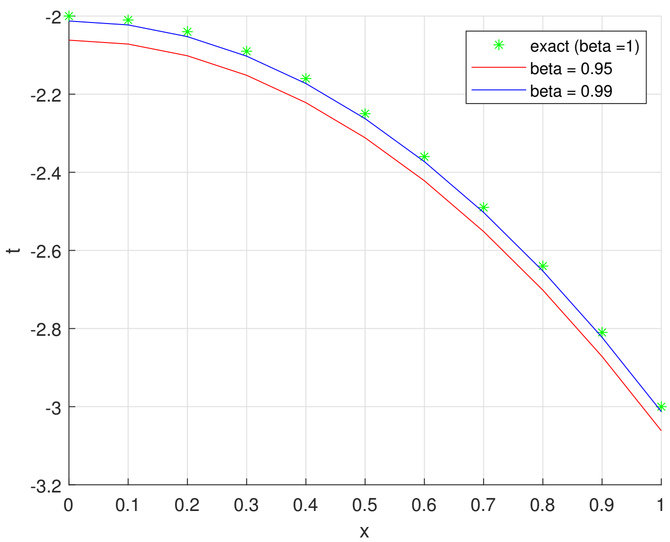

Figure 1. Comparison between the exact and numerical solutions.

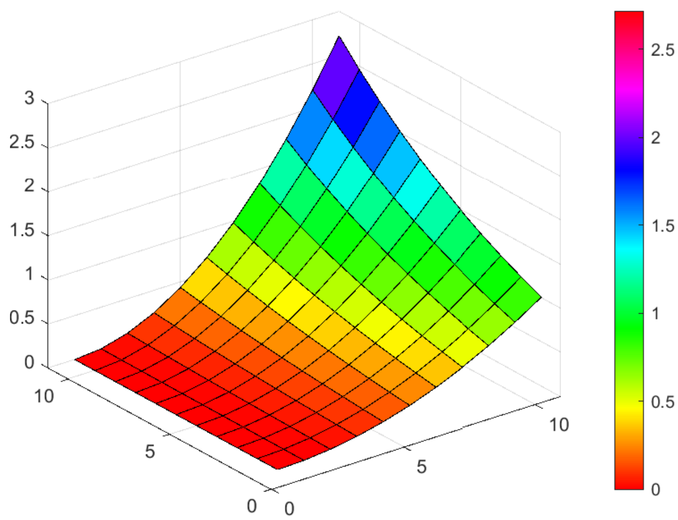

Figure 2. The surface of the function $\psi(x,t)$.

Example 2. *The non-homogenous time-fractional Navier–Stokes equation with the initial condition is*

$$D_t^{\beta}\psi = D_x^2\psi + \frac{1}{x}D_x\psi + x^2e^t - 4e^t, \quad x,t>0, \tag{45}$$

$$\psi(x,0) = x^2. \tag{46}$$

Applying the (SGLT) on both sides of Equation (45) and the Sumudu transform to the initial condition, Equation (46), we obtain

$$\begin{aligned}\Psi_\alpha(u_1,s) &= 2u_1^2 s^{\alpha+1} + 2u_1^2\frac{s^{\alpha+1+\beta}}{1-s} - \frac{4s^{\alpha+1+\beta}}{1-s} \\ &\quad + \frac{1}{u_1}\int_0^{u_1}\frac{s^\beta}{u_1}S_xG_t\left[v\frac{\partial}{\partial x}(xD_x\psi)\right]du_1.\end{aligned} \tag{47}$$

From the formula for the geometric series, the terms $\frac{s^{\alpha+1+\beta}}{1-s}$ and $\frac{4s^{\alpha+1+\beta}}{1-s}$ can be written in the form of

$$\begin{aligned} 2u_1^2 s^{\alpha+1+\beta}\frac{1}{1-s} &= 2u_1^2\left[s^{\alpha+1+\beta}+s^{\alpha+2+\beta}+s^{\alpha+3+\beta}+\dots\right] \\ s^{\alpha+1+\beta}\frac{1}{1-s} &= \left[s^{\alpha+1+\beta}+s^{\alpha+2+\beta}+s^{\alpha+3+\beta}+\dots\right]. \end{aligned}$$

Operating with the (SGLT) inverse on both sides of Equation (47) gives

$$\begin{aligned} \psi(x,t) &= x^2+x^2\left[\frac{t^\beta}{\Gamma(\beta+1)}+\frac{t^{\beta+1}}{\Gamma(\beta+2)}+\frac{t^{\beta+2}}{\Gamma(\beta+3)}+\dots\right] \\ &\quad -4\left[\frac{t^\beta}{\Gamma(\beta+1)}+\frac{t^{\beta+1}}{\Gamma(\beta+2)}+\frac{t^{\beta+2}}{\Gamma(\beta+3)}+\dots\right] \\ &\quad +S_{u_1}^{-1}G_s^{-1}\left[\frac{1}{u_1}\int_0^{u_1}\frac{s^\beta}{u_1}S_xG_t\left[\frac{\partial}{\partial x}(xD_x\psi)\right]du_1\right]. \end{aligned} \tag{48}$$

By using the above-mentioned method, if we assume an infinite series solution of the form in Equation (18), we have

$$\begin{aligned} \sum_{m=0}^{\infty}\psi_m(x,t) &= x^2+x^2\left[\frac{t^\beta}{\Gamma(\beta+1)}+\frac{t^{\beta+1}}{\Gamma(\beta+2)}+\frac{t^{\beta+2}}{\Gamma(\beta+3)}+\dots\right] \\ &\quad -4\left[\frac{t^\beta}{\Gamma(\beta+1)}+\frac{t^{\beta+1}}{\Gamma(\beta+2)}+\frac{t^{\beta+2}}{\Gamma(\beta+3)}+\dots\right] \\ &\quad +S_{u_1}^{-1}G_s^{-1}\left[\frac{1}{u_1}\int_0^{u_1}\frac{s^\beta}{u_1}S_xG_t\left[\frac{\partial}{\partial x}\left(x\sum_{m=0}^{\infty}\psi_{mx}(x,t)\right)\right]du_1\right]; \end{aligned} \tag{49}$$

the first few terms of the (SGLTDM) are given by

$$\begin{aligned} \psi_0 &= x^2+x^2\left[\frac{t^\beta}{\Gamma(\beta+1)}+\frac{t^{\beta+1}}{\Gamma(\beta+2)}+\frac{t^{\beta+2}}{\Gamma(\beta+3)}+\dots\right] \\ &\quad -4\left[\frac{t^\beta}{\Gamma(\beta+1)}+\frac{t^{\beta+1}}{\Gamma(\beta+2)}+\frac{t^{\beta+2}}{\Gamma(\beta+3)}+\dots\right], \end{aligned}$$

and

$$\psi_{m+1}(x,t)=S_{u_1}^{-1}G_s^{-1}\left[\frac{1}{u_1}\int_0^{u_1}\frac{s^\beta}{u_1}S_xG_t\left[\frac{\partial}{\partial x}(x\psi_{mx}(x,t))\right]du_1\right]$$

Hence, at $m=0$, we obtain

$$\begin{aligned} \psi_1 &= S_{u_1}^{-1}G_s^{-1}\left[\frac{1}{u_1}\int_0^{u_1}\frac{s^\beta}{u_1}S_xG_t\left[\frac{\partial}{\partial x}(x\psi_{0x}(x,t))\right]du_1\right] \\ &= S_{u_1}^{-1}G_s^{-1}\left[\frac{1}{u_1}\int_0^{u_1}\frac{s^\beta}{u_1}S_xG_t\left[4x+4x\left[\begin{array}{c}\frac{t^\beta}{\Gamma(\beta+1)}+\frac{t^{\beta+1}}{\Gamma(\beta+2)}\\ +\frac{t^{\beta+2}}{\Gamma(\beta+3)}+\dots\end{array}\right]\right]du_1\right] \\ &= S_{u_1}^{-1}G_s^{-1}\left[4s^{\alpha+\beta+1}+4\left[s^{\alpha+1+2\beta}+s^{\alpha+2+2\beta}+s^{\alpha+3+2\beta}+\dots\right]\right] \\ \psi_1 &= \frac{4t^\beta}{\Gamma(\alpha+1)}+4\left[\frac{t^{2\beta}}{\Gamma(2\beta+1)}+\frac{t^{2\beta+1}}{\Gamma(2\beta+2)}+\frac{t^{2\beta+2}}{\Gamma(2\beta+3)}+\dots\right]. \end{aligned}$$

In the same manner,

$$\begin{aligned}\psi_2 &= S_{u_1}^{-1}G_s^{-1}\left[\frac{1}{u_1}\int_0^{u_1}\frac{s^\beta}{u_1}S_xG_t\left[\frac{\partial}{\partial x}(x\psi_{1x}(x,t))\right]du_1\right]\\ &= S_{u_1}^{-1}G_s^{-1}\left[\frac{1}{u_1}\int_0^{u_1}\frac{s^\beta}{u_1}S_xG_t[0]du_1\right]\\ \psi_2 &= 0,\end{aligned}$$

and

$$\psi_3 = 0, \quad \psi_4 = 0, \ldots.$$

So, our required solutions are given below

$$\psi(x,t) = \psi_0 + \psi_1 + \psi_2 + \ldots.$$

$$\begin{aligned}\psi(x,t) &= x^2 + x^2\left[\frac{t^\beta}{\Gamma(\beta+1)} + \frac{t^{\beta+1}}{\Gamma(\beta+2)} + \frac{t^{\beta+2}}{\Gamma(\beta+3)} + \ldots\right]\\ &\quad -4\left[\frac{t^\beta}{\Gamma(\beta+1)} + \frac{t^{\beta+1}}{\Gamma(\beta+2)} + \frac{t^{\beta+2}}{\Gamma(\beta+3)} + \ldots\right]\\ &\quad +\frac{4t^\beta}{\Gamma(\beta+1)} + 4\left[\frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{2\beta+1}}{\Gamma(2\beta+2)} + \frac{t^{2\beta+2}}{\Gamma(2\beta+3)} + \ldots\right].\end{aligned}$$

When we set $\beta = 1$ in Equation (45), we obtain the exact solution of the non-time-fractional Navier–Stokes equation as follows

$$\psi(x,t) = x^2e^t.$$

The error between the exact and approximation solutions to example two is given in Table 2 below.

Table 2. Comparison between the exact and approximation solutions.

Exact $\beta = 1$	The Method $\beta = 0.95$	Error	The Method $\beta = 0.99$	Error
2.3333	2.5599	0.2266	2.4077	0.0744
2.3600	2.5869	0.2269	2.4345	0.0745
2.4400	2.6679	0.2279	2.5148	0.0748
2.5733	2.8029	0.2295	2.6487	0.0754
2.7600	2.9918	0.2318	2.8361	0.0761
3.0000	3.2348	0.2348	3.0771	0.0771
3.2933	3.5317	0.2384	3.3716	0.0783
3.6400	3.8827	0.2427	3.7197	0.0797
4.0400	4.2876	0.2476	4.1214	0.0814
4.4933	4.7465	0.2532	4.5766	0.0832
5.0000	5.2594	0.2594	5.0853	0.0853

Figure 3 presents a comparison between the exact and numerical solutions of Equation (45). The exact solution is obtained when $t = 1$ and $\beta = 1$, and we obtain the numerical solutions by taking different values of β such as ($\beta = 0.95$, $\beta = 0.99$). Figure 4 shows the surface of function $\psi(x,t)$ in three dimensions.

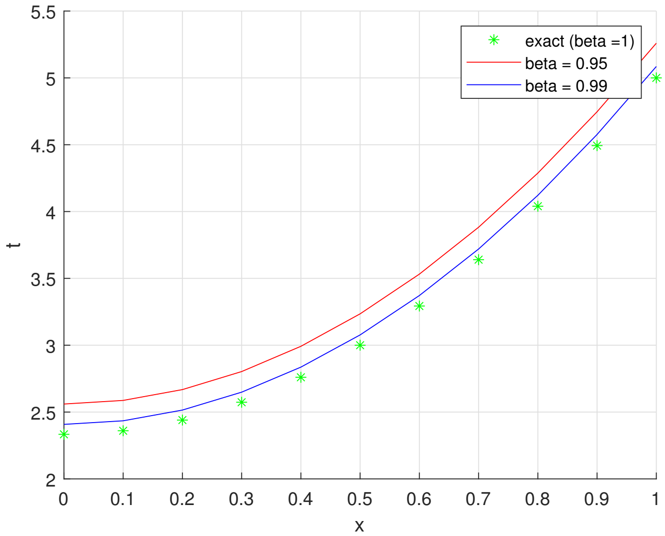

Figure 3. Comparison between the exact and numerical solutions.

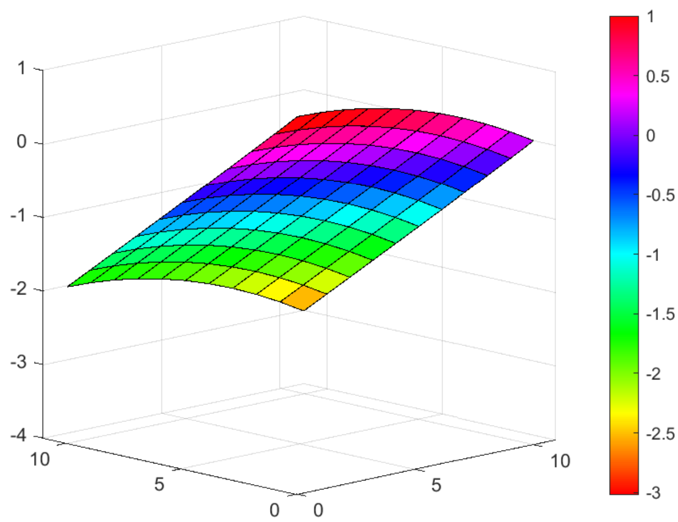

Figure 4. The surface of the function $\psi(x,t)$.

Example 3. *Consider a time-fractional order two-dimensional Navier–Stokes equation with [21,22]*

$$\begin{aligned} D_t^{\alpha}\psi + \psi\psi_x + \varphi\psi_y &= \rho_0\big(\psi_{xx} + \psi_{yy}\big) + q, \quad x,y,t > 0, \\ D_t^{\alpha}\varphi + \psi\varphi_x + \varphi\varphi_y &= \rho_0\big(\varphi_{xx} + \varphi_{yy}\big) - q, \quad x,y,t > 0, \\ n-1 &< \alpha < n, \end{aligned} \tag{50}$$

subject to the condition

$$\psi(x,y,0) = -\sin(x+y), \quad \varphi(x,y,0) = \sin(x+y);$$

by using the (DSGLT) on both sides of Equation (50), we obtain

$$\begin{aligned} &S_xS_yG_t\big[D_t^{\alpha}\psi + \psi\psi_x + \varphi\psi_y = \rho_0\big(\psi_{xx} + \psi_{yy}\big) + q\big] \\ &S_xS_yG_t\big[D_t^{\alpha}\varphi + \psi\varphi_x + \varphi\varphi_y = \rho_0\big(\varphi_{xx} + \varphi_{yy}\big) - q\big], \end{aligned}$$

and using the differentiation property of the double Sumudu transform, we have

$$\begin{aligned}\frac{\Psi(u_1,u_2,s)}{s^\beta}-s^{\alpha-\beta+1}\Psi(u_1,u_2,0) &= -S_xS_yG_t(\psi\psi_x+\varphi\psi_y)\\ &\quad +S_xS_yG_t(\rho_0(\psi_{xx}+\psi_{yy}))+S_xS_yG_t(q),\\ \frac{\Phi(u_1,u_2,s)}{s^\beta}-s^{\alpha-\beta+1}\Phi(u_1,u_2,0) &= -S_xS_yG_t(\psi\varphi_x+\varphi\varphi_y)\\ &\quad +S_xS_yG_t(\rho_0(\varphi_{xx}+\varphi_{yy}))+S_xS_yG_t(q).\end{aligned} \tag{51}$$

Replacing the initial condition and arranging Equation (51), we have

$$\begin{aligned}\Psi(u_1,u_2,s) &= -\frac{(u_1+u_2)s^{\alpha+1}}{(u_1^2+1)(u_2^2+1)}-s^\beta S_xS_yG_t(\psi\psi_x+\varphi\psi_y)\\ &\quad +s^\beta S_xS_yG_t(\rho_0(\psi_{xx}+\psi_{yy}))-s^\beta S_xS_yG_t(q)\\ \Phi(u_1,u_2,s) &= \frac{(u_1+u_2)s^{\alpha+1}}{(u_1^2+1)(u_2^2+1)}-s^\beta S_xS_yG_t(\psi\varphi_x+\varphi\varphi_y)\\ &\quad +s^\beta S_xS_yG_t(\rho_0(\varphi_{xx}+\varphi_{yy}))-s^\beta S_xS_yG_t(q).\end{aligned} \tag{52}$$

Now, applying the inverse (DSGLT) for both sides of Equation (52), we obtain

$$\begin{aligned}\psi(x,y,t) &= -\sin(x+y)-S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^\beta S_xS_yG_t(\psi\psi_x+\varphi\psi_y)\right)\\ &\quad +S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(\left(s^\beta S_xS_yG_t(\rho_0(\psi_{xx}+\psi_{yy}))\right)\right)\\ &\quad +\frac{qt^\beta}{\Gamma(\beta+1)},\\ \varphi(x,y,t) &= \sin(x+y)-S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^\beta S_xS_yG_t(\psi\varphi_x+\varphi\varphi_y)\right)\\ &\quad +S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^\beta S_xS_yG_t(\rho_0(\varphi_{xx}+\varphi_{yy}))\right)\\ &\quad -\frac{qt^\beta}{\Gamma(\beta+1)}.\end{aligned} \tag{53}$$

The zeroth components u_0 and v_0 are proposed by they Adomian method, and they constantly include the initial condition and the source term, both of which are supposed to be recognized. Consequently, we set

$$\psi_0=-\sin(x+y)+\frac{qt^\beta}{\Gamma(\beta+1)},\quad \psi_0=\sin(x+y)-\frac{qt^\beta}{\Gamma(\beta+1)}.$$

The remaining elements $u_{n+1}, u_{n+1}, n\geq 0$ are given as follows

$$\begin{aligned}\psi_{n+1} &= -S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^\beta S_xS_yG_t(A_n+B_n)\right)\\ &\quad +S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(\left(s^\beta S_xS_yG_t(\rho_0(\psi_{nxx}+\psi_{nyy}))\right)\right),\end{aligned} \tag{54}$$

and

$$\begin{aligned}\psi_{n+1} &= -S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^\beta S_xS_yG_t(C_n+D_n)\right)\\ &\quad +s^\beta S_xS_yG_t(\rho_0(\varphi_{nxx}+\varphi_{nyy})).\end{aligned} \tag{55}$$

The few components of the Adomian polynomials A_n, B_n, C_n, and D_n are given as follows

$$\begin{aligned} A_0 &= \psi_0\psi_{0x}, \quad A_1 = \psi_0\psi_{1x} + \psi_1\psi_{0x}, \\ A_2 &= \psi_0\psi_{2x} + \psi_1\psi_{1x} + \psi_2\psi_{0x}, \\ A_3 &= \psi_0\psi_{3x} + \psi_1\psi_{2x} + \psi_2\psi_{1x} + \psi_3\psi_{0x}, \end{aligned} \tag{56}$$

$$\begin{aligned} B_0 &= \varphi_0\psi_{0y}, \quad B_1 = \varphi_0\psi_{1y} + \varphi_1\psi_{0y}, \\ B_2 &= \varphi_0\psi_{2y} + \varphi_1\psi_{1y} + \varphi_2\psi_{0y}, \\ B_3 &= \varphi_0\psi_{3y} + \varphi_1\psi_{2y} + \varphi_2\psi_{1y} + \varphi_3\psi_{0y}, \end{aligned} \tag{57}$$

$$\begin{aligned} C_0 &= \psi_0\varphi_{0x}, \quad C_1 = \psi_0\varphi_{1x} + \psi_1\varphi_{0x}, \\ C_2 &= \psi_0\varphi_{2x} + \psi_1\varphi_{1x} + \psi_2\varphi_{0x}, \\ C_3 &= \psi_0\varphi_{3x} + \psi_1\varphi_{2x} + \psi_2\varphi_{1x} + \psi_3\varphi_{0x}. \end{aligned} \tag{58}$$

$$\begin{aligned} D_0 &= \varphi_0\varphi_{0y}, \quad D_1 = \varphi_0\varphi_{1y} + \varphi_1\varphi_{0y}, \\ D_2 &= \varphi_0\varphi_{2y} + \varphi_1\varphi_{1y} + \varphi_2\varphi_{0y}, \\ D_3 &= \varphi_0\varphi_{3y} + \varphi_1\varphi_{2y} + \varphi_2\varphi_{1y} + \varphi_3\varphi_{0y}. \end{aligned} \tag{59}$$

Setting $n = 0$ into Equations (54) and (55), we obtain

$$\begin{aligned} \psi_1 &= -\psi_{n+1} = -S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^\beta S_x S_y G_t(A_0 + B_0)\right) \\ &\quad + S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(\left(s^\beta S_x S_y G_t(\rho_0(\psi_{0xx} + \psi_{0yy}))\right)\right), \\ &= S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^\beta S_x S_y G_t(\rho_0(2\sin(x+y)))0\right) \\ &= S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(2\rho_0\frac{(u_1+u_2)s^{\alpha+\beta+1}}{(u_1^2+1)(u_2^2+1)}\right) \\ &= 2\frac{\rho_0 t^\beta}{\Gamma(\beta+1)}\sin(x+y) \end{aligned}$$

and

$$\begin{aligned} \varphi_1 &= -S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^\beta S_x S_y G_t(C_0 + D_0)\right) \\ &\quad + s^\beta S_x S_y G_t(\rho_0(\varphi_{0xx} + \varphi_{0yy})) \\ &= S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^\beta S_x S_y G_t(-\rho_0(2\sin(x+y)))\right) \\ &= -2\frac{\rho_0 t^\beta}{\Gamma(\beta+1)}\sin(x+y); \end{aligned}$$

similarly, at $n = 1$,

$$\begin{aligned} \psi_2 &= -S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^\beta S_x S_y G_t((\psi_0\psi_{1x} + \psi_1\psi_{0x} + \varphi_0\psi_{1y} + \varphi_1\psi_{0y}))\right) \\ &\quad + S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^\beta S_x S_y G_t(\rho_0(\psi_{1xx} + \psi_{1yy}))\right), \\ &= S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^\beta S_x S_y G_t\left(\frac{-4\rho_0^2\sin(x+y)t^\beta}{\Gamma(\beta+1)}\right)\right) \\ &= L_p^{-1}L_q^{-1}L_s^{-1}\left(-4\rho_0^2\frac{(u_1+u_2)s^{\alpha+2\beta+1}}{(u_1^2+1)(u_2^2+1)}\right) \\ &= -\frac{(2\rho_0)^2\sin(x+y)t^{2\beta}}{\Gamma(2\beta+1)} \end{aligned}$$

and

$$\begin{aligned}\varphi_2 &= -S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^{\beta}S_xS_yG_t\left(\psi_0\varphi_{1x}+\psi_1\varphi_{0x}+\varphi_0\varphi_{1y}+\varphi_1\varphi_{0y}\right)\right)\\ &\quad +S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^{\beta}S_xS_yG_t\left(\rho_0\left(\varphi_{1xx}+\varphi_{1yy}\right)\right)\right)\\ &= S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^{\beta}S_xS_yG_t\left(\frac{4\rho_0^2\sin(x+y)t^{\beta}}{\Gamma(\beta+1)}\right)\right)\\ &= S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(4\rho_0^2\frac{(u_1+u_2)s^{\alpha+2\beta+1}}{(u_1^2+1)(u_2^2+1)}\right)\\ &= \frac{(2\rho_0)^2\sin(x+y)t^{2\beta}}{\Gamma(2\beta+1).}\end{aligned}$$

In a similar manner, at $n=2$, we have

$$\begin{aligned}\psi_3 &= -S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^{\beta}S_xS_yG_t\left(\left(\psi_0\psi_{2x}+\psi_1\psi_{1x}+\psi_2\psi_{0x}+\varphi_0\psi_{2y}+\varphi_1\psi_{1y}+\varphi_2\psi_{0y}\right)\right)\right)\\ &\quad +S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^{\beta}S_xS_yG_t\left(\rho_0\left(\psi_{2xx}+\psi_{2yy}\right)\right)\right),\\ &= S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(s^{\beta}S_xS_yG_t\left(\frac{-8\rho_0^3\sin(x+y)t^{2\beta}}{\Gamma(2\beta+1)}\right)\right)\\ &= S_{u_1}^{-1}S_{u_2}^{-1}G_s^{-1}\left(-8\rho_0^3\frac{(u_1+u_2)s^{\alpha+2\beta+1}}{(u_1^2+1)(u_2^2+1)}\right)\\ &= \frac{-8\rho_0^3\sin(x+y)t^{3\beta}}{\Gamma(3\beta+1)}=-\frac{(2\rho_0)^3\sin(x+y)t^{3\alpha}}{\Gamma(3\alpha+1),}\end{aligned}$$

and by the same way,

$$\varphi_3=-\frac{(2\rho_0)^3\sin(x+y)t^{3\beta}}{\Gamma(3\beta+1)}.$$

In similar manner, we have

$$\psi_n=-\frac{(-2\rho_0)^n\sin(x+y)t^{n\beta}}{\Gamma(n\beta+1)},\quad \varphi_n=\frac{(-2\rho_0)^n\sin(x+y)t^{n\beta}}{\Gamma(n\beta+1)},\forall n\geq 2.$$

So, our required solutions to Equation (50) are given below

$$\begin{aligned}\psi(x,y,t) &= \psi_0+\psi_1+\psi_2+\ldots.+\psi_n\\ \varphi(x,y,t) &= \varphi_0+\varphi_1+\varphi_2+\ldots.+\varphi_n\end{aligned}$$

$$\begin{aligned}\psi(x,y,t) &= -\sin(x+y)\sum_{n=0}^{\infty}\frac{(-2\rho_0)^n t^{n\beta}}{\Gamma(n\beta+1)}+\frac{qt^{\beta}}{\Gamma(\beta+1)}\\ \varphi(x,y,t) &= \sin(x+y)\sum_{n=0}^{\infty}\frac{(-2\rho_0)^n t^{n\beta}}{\Gamma(n\beta+1)}+\frac{qt^{\beta}}{\Gamma(\beta+1);}\end{aligned}$$

substituting $\beta=1$ and $q=0$ into the above equation, we obtain the exact solution to the classical Navier–Stokes equation for the velocity as:

$$\begin{aligned}\psi(x,y,t) &= -\sin(x+y)e^{-2\rho_0 t}\\ \varphi(x,y,t) &= \sin(x+y)e^{-2\rho_0 t}.\end{aligned}$$

The error between the exact and approximation solutions to example two is given in Tables 3 and 4 below.

Table 3. Comparison between the exact and approximation solutions for $\psi(x,t)$.

Exact $\beta = 1$	The Method $\beta = 0.95$	Error	The Method $\beta = 0.99$	Error
0	0	0	0	0
0.1000	0.1145	0.0145	0.1028	0.0028
0.2000	0.2212	0.0212	0.2041	0.0041
0.3000	0.3252	0.0252	0.3049	0.0049
0.4000	0.4274	0.0274	0.4054	0.0054
0.5000	0.5283	0.0283	0.5056	0.0056
0.6000	0.6282	0.0282	0.6056	0.0056
0.7000	0.7272	0.0272	0.7055	0.0055
0.8000	0.8256	0.0256	0.8052	0.0052
0.9000	0.9233	0.0233	0.9047	0.0047
1.0000	1.0205	0.0205	1.0042	0.0042

Table 4. Comparison between the exact and approximation solutions for $\phi(x,t)$.

Exact $\beta = 1$	The Method $\beta = 0.95$	Error	The Method $\beta = 0.99$	Error
0	0	0	0	0
0.1000	0.1145	0.0145	0.1028	0.0028
0.2000	0.2212	0.0212	0.2041	0.0041
0.3000	0.3252	0.0252	0.3049	0.0049
0.4000	0.4274	0.0274	0.4054	0.0054
0.5000	0.5283	0.0283	0.5056	0.0056
0.6000	0.6282	0.0282	0.6056	0.0056
0.7000	0.7272	0.0272	0.7055	0.0055
0.8000	0.8256	0.0256	0.8052	0.0052
0.9000	0.9233	0.0233	0.9047	0.0047
1.0000	1.0205	0.0205	1.0042	0.0042

The comparison between the exact and numerical solutions for Equation (50) is shown in Figures 5 and 6. We obtain exact solution at $\beta = 1$; and the different values of β such as ($\beta = 0.95$, $\beta = 0.99$) show the approximate solution. The surfaces in Figures 7 and 8 show the exact solution of the functions $\psi(x,y,t) = -\sin(x+y)e^{-2\rho_0 t}$ and $\varphi(x,y,t) = \sin(x+y)e^{-2\rho_0 t}$ at $x = 0$, respectively.

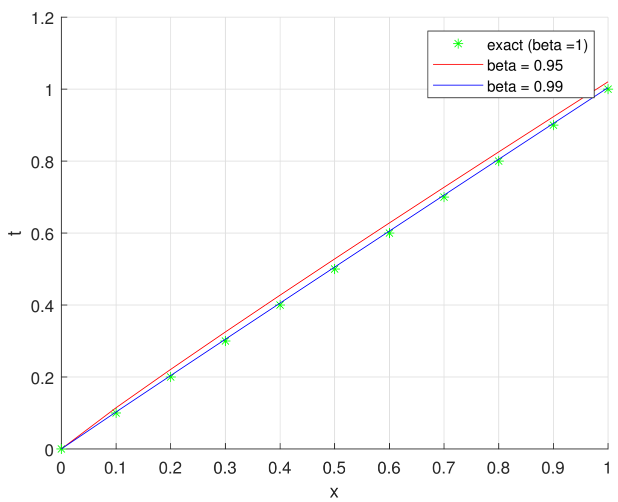

Figure 5. The comparison between the exact and numerical solutions for $\psi(x,y,t)$.

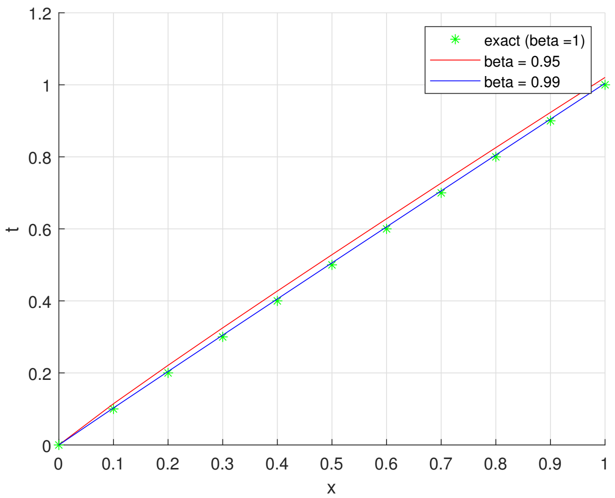

Figure 6. The comparison between the exact and numerical solutions for $\varphi(x, y, t)$.

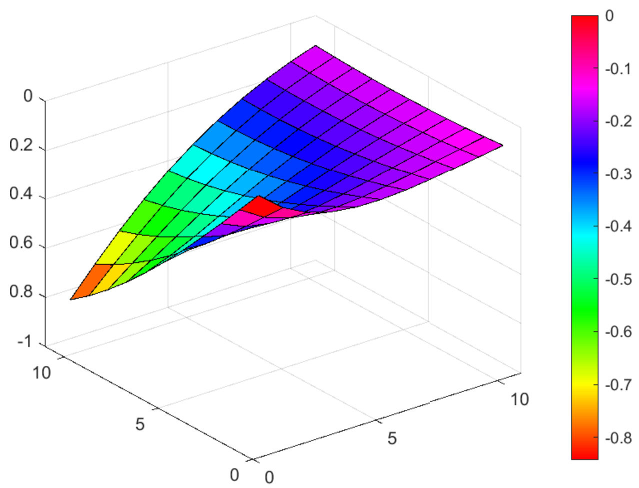

Figure 7. The surface shows the function $\psi(x, y, t) = -\sin(x + y)e^{-2\rho_0 t}$.

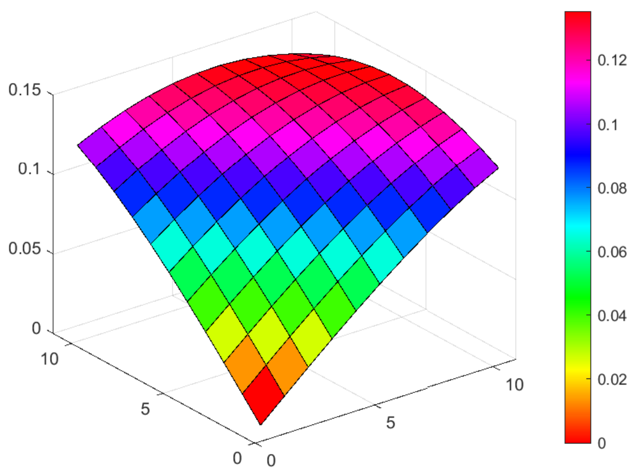

Figure 8. The surface shows the function $\varphi(x, y, t) = \sin(x + y)e^{-2\rho_0 t}$.

5. Conclusions

In this article, strong techniques, which are called (SGLTDM) and (DSGLTDM), are implemented to obtain the solution time-fractional Navier–Stokes equations. The obtained results are fascinating and agree with the exact solutions. The action and effectiveness of the introduced method are examined by utilizing some numerical examples. Thus, it can be concluded that the (SGLTDM) and (DSGLTDM) are very active in finding exact, as well as numerical, solutions for fractional partial differential equations. Moreover, the proposed method is very efficient in analyzing nonlinear systems without any categorization. The outcome shows that the present method has higher accuracy compared to the existing method in the literature. Numerical simulation was utilized to draw the exact and approximate solutions. In the future, we will use our method to develop modeling horizons in our domain.

Author Contributions: Methodology, H.E. and S.M.; Formal analysis, H.E.; Resources, I.B.; Data curation, I.B.; Writing—original draft, H.E.; Writing—review & editing, H.E. All authors have read and agreed to the published version of the manuscript.

Funding: The authors would like to extend their sincere appreciation for the Researchers Supporting Project number (RSPD 2024R948), King Saud University, Riyadh, Saudi Arabia.

Data Availability Statement: Data sharing not applicable to this article, as no datasets were generated or analyzed during the current study.

Acknowledgments: Not applicable.

Conflicts of Interest: The authors declare that they have no competing interests.

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