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Hamiltonian Formulation for Continuous Systems with Second-Order Derivatives: A Study of Podolsky Generalized Electrodynamics

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Abstract: This paper presents an analysis of the Hamiltonian formulation for continuous systems with second-order derivatives derived from Dirac's theory. This approach offers a unique perspective on the equations of motion compared to the traditional Euler–Lagrange formulation. Focusing on Podolsky's generalized electrodynamics, the Hamiltonian and corresponding equations of motion are derived. The findings demonstrate that both Hamiltonian and Euler–Lagrange formulations yield equivalent results. This study highlights the Hamiltonian approach as a valuable alternative for understanding the dynamics of second-order systems, validated through a specific application within generalized electrodynamics. The novelty of the research lies in developing advanced theoretical models through Hamiltonian formalism for continuous systems with second-order derivatives. The research employs an alternative method to the Euler–Lagrange formulas by applying Dirac's theory to study the generalized Podolsky electrodynamics, contributing to a better understanding of complex continuous systems.

Keywords: Podolsky Lagrangian; Hamiltonian formulation; classical fields variables; Euler–Lagrange equations

MSC: 34C25; 92C60; 92D25

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1. Introduction

In the context of a deep understanding of theoretical physics, quantum theories have emerged as a crucial theoretical framework. These are theories that aim to provide an accurate mathematical description of natural phenomena and interactions between particles and fields. We can envision these theories as carefully crafted mathematical tools that enable us to better understand the surrounding world through the interpretation of multiple experiments and measurements. The Podolsky equation exemplifies a type of higher derivative gauge theory within the realm of quantum theories, where a kinetic term is integrated into the Maxwell Lagrangian. This category of theories has prompted extensive scholarly discussion [1–8]. Researchers are currently exploring the potential of achieving enhanced organization of interactions at close distances by incorporating higher-order derivatives akin to the framework employed in Podolsky's electromagnetic dynamics. This approach presents an exciting opportunity to formulate specific theories that surpass the indeterminacies and difficulties associated with traditional gauge theory formulations.

Moreover, systems governed by higher-order Lagrangians have gained increasing significance in investigating physical phenomena. The prominent examples encompass the organization of theories in the field of quantum measurement, in addition to the behaviors of solid strings. In the realm of theoretical physics, the behavior of solid strings is studied as vibrational patterns in space and time. Solid strings are considered theoretical models used to comprehend the behavior of particles at the quantum level, where this theory relies on the interaction of strings with variables like energy and intensity. As such, it can contribute to interpreting phenomena at the particle level [9,10]. These examples also include the study of solid particles [11,12] and relativistic particles in three-dimensional spacetime with curvature and twisting. This concept is part of the general theory of relativity, which posits that spacetime interacts as a single unit called spacetime. In the presence of mass or energy, this leads to the curvature of spacetime around it, affecting the trajectory of moving particles in the curved region [13]. Podolsky's proposal to extend electrodynamics with third-order derivatives [14] has further fueled interest in these system types.

Additionally, the effective Lagrangian in gauge theories was initially pioneered by Ostrogradski [2,15,16]. It provides a valuable framework for deducing Hamiltonian equations of motion and Euler–Lagrange equations. Exploration of systems characterized by fractional higher-order derivatives has been extensively researched in recent years. Building upon this context, Ref. [17] introduces a novel approach to address such systems, specifically those featuring third-order fractional derivatives. This method is then applied to Lee–Wick's generalized electrodynamics, illuminating the behaviors of these intricate systems. Ref. [18] further builds on this foundation, delving into advancements for systems featuring even higher-order derivatives. Drawing from this context, Awaideh et al. have effectively computed high-level fractional order derivatives using the Atangana–Baleanu fractional derivative, allowing them to determine conserved variables such as energy density, momentum, and Poynting's vector [19].

The electromagnetic potentials, φ and A_i, play a crucial role as mathematical constructs intricately connected to electric and magnetic fields through Maxwell's equations. These potentials simplify the understanding of fields, especially those that vary with time, and are essential in deriving the electric field. They effectively embody the electromagnetic field that originates from charged particles, providing an internal view of field distributions within specific spatial regions. In the past, the description relied solely on electric and magnetic fields to portray the electromagnetic field. However, the integration of electromagnetic potentials introduced a precise framework for portraying electromagnetic phenomena with greater comprehensiveness. Everyone contributes to providing an accurate framework for depicting electromagnetic phenomena. The electromagnetic potentials serve as mathematical tools to visualize the distributions of electric and magnetic fields in space, with complete integration into the traditional description of the electromagnetic field.

Higher-order variational theory includes functions that depend on higher-order derivatives. There are two forms: the dependent form that involves derivatives and the independent form that covers the function and its derivatives from the first order up to the $(n-1)$ th order. The dependent form is helpful in calculating partial derivatives in mathematical physics and for analyzing functions that rely on higher-order derivatives. On the other hand, the independent form is valuable for solving boundary problems and for studying functions that require control over super functions, as demonstrated in the optimal endpoint problem in thermodynamics. Inspired by these discussions, researchers have extended these formulations to encompass broader applications to continuous systems with second-order derivatives. This methodology has also been expanded to include Podolsky's generalized electrodynamics. As a practical application of second-order derivatives in rotational dynamics and stability, several studies [20–23] have examined topics such as rigid body motion with gyrostatic moments, the dynamics of charged solid bodies, and periodic solutions in two degrees-of-freedom systems. The main contributions of this study can be summarized as follows:

i. Second-order Hamiltonian analysis: The study presents a comprehensive analysis of systems using Second-Order Lagrangian and Generalized Dirac's derivatives. Researchers gain insights into how the proposed methodology aligns with expected theories. Subsequently, these results can be compared with the findings of other researchers to verify alignment and attain a more generalized formulation.
ii. Enhanced understanding of Hamiltonian formalism: By applying Dirac's method to second-order Lagrangians, the research sheds light on a deeper understanding of Hamiltonian formalism. This facilitates a more comprehensive analysis of physical systems with intricate dynamics.
iii. Systems governed by higher-order Lagrangians:
    1- These systems have gained increasing significance in investigating physical phenomena, such as the organization of theories in the field of quantum measurement and the behaviors of solid strings.
    2- Other examples include the study of solid particles and relativistic particles in three-dimensional spacetime with curvature and twisting.
iv. Higher-order variational theory: This theory includes functions that depend on higher-order derivatives and is useful for calculating partial derivatives in mathematical physics and analyzing functions that rely on higher-order derivatives.

This research aims to achieve two main objectives. Firstly, it aims to explore the analysis of systems using generalized Dirac's derivatives and second-order Lagrangian. Secondly, it aims to illustrate how Podolsky's generalized electrodynamics can be enhanced through a more comprehensive approach, enabling its application to continuous systems that involve second-order derivatives.

This work is structured as follows: Section 2 provides a concise overview of the Euler–Lagrange equations of motion. Section 3 delves into the examination of the Hamiltonian formulation of continuous systems. The equations of motion, expressed in terms of (ϕ, A_i), are presented in Section 4 through Hamilton's equations. Moving on to Section 5, we explore a classical field example leading to the Podolsky equation in the second-order form. Section 5 will introduce various applications of fractional calculus and provide research recommendations. Lastly, Section 7 encompasses the conclusion of this study.

2. Classical Field Theories: Lagrangians of Second Order

We can discuss a continuous system described by a Lagrangian density, dependent on dynamic field variables and their second-order derivatives. A Lagrangian density is a function that describes a physical system's dynamics, defined as the difference between kinetic and potential energies. In our case, the Lagrangian density captures both the system's current state and its changing behavior over time. The term "continuous system" implies smooth changes in field variables over time and space, contrasting with discrete changes in other systems.

$$\mathcal{L} = \mathcal{L}\left(\psi_\rho, \partial_\eta \psi_\rho, \partial_\eta \partial_\sigma \psi_p\right).$$

Now L can be expressed as:

$$L = \int \mathcal{L}\left(\psi_p, \partial_\eta \psi_\rho, \partial_\eta \partial_\sigma \psi_p\right) d^3 r. \tag{1}$$

The variational principle and the classical variants of Lagrangian of the second order are closely related. The Lagrangian function represents the difference between the kinetic and potential energies of a physical system, and the classical variants of Lagrangian of the second order are a set of mathematical equations used to describe the system's motion based on this function. To derive these equations, the principle of least action is applied to the Lagrangian function, which is a specific form of the variational principle. Therefore, the classical variants of Lagrangian of the second order provide an example of how the variational principle can be used to determine the motion of physical systems.

$$\delta \int L dt = \int \int \delta \mathcal{L}\, d^3 r dt = 0, \tag{2}$$

Equation (1) allows us to calculate the variation of the Lagrangian function in Lagrangian mechanics. The variation of L is denoted by δL and can be written as:

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \psi_\rho} \delta \psi_\rho + \frac{\partial \mathcal{L}}{\partial (\partial_\eta \psi_\rho)} \delta (\partial_\eta \psi_\rho) + \frac{\partial \mathcal{L}}{\partial (\partial_\eta \partial_\sigma \psi_p)} \delta (\partial_\eta \partial_\sigma \psi_p), \tag{3}$$

where $\mathcal{L}$ is the Lagrangian function, ψ_ρ represents the generalized coordinates that describe the configuration of the system, δ is a small parameter, and $\delta \psi_\rho$ is the corresponding variation in the generalized coordinate.

Substituting Equation (3) in Equation (2), and using the following commutation relationship [20],

$$\left.\begin{array}{c} \delta (\partial_\eta \psi_\rho) = \partial_\eta (\delta \psi_\rho) \\ \delta (\partial_\eta \partial_\sigma \psi_p) = \partial_\eta \partial_\sigma (\delta \psi_p) \end{array}\right|. \tag{4}$$

We obtain,

$$\delta L = \int \int \left(\frac{\partial \mathcal{L}}{\partial \psi_\rho} \delta \psi_\rho + \underbrace{\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_\rho)} \partial_\mu (\delta \psi_\rho)}_{second} + \underbrace{\frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu \psi_p)} \partial_\mu \partial_\nu (\psi_p)}_{third} \right) d^3 r dt = 0.$$

Integrating the second and third terms by parts we obtain:

$$\delta L = \int \left(\frac{\partial \mathcal{L}}{\partial \psi_\rho} \delta \psi_\rho - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_\rho)} \delta (\psi_\rho) + \partial_\eta \partial_\sigma \frac{\partial \mathcal{L}}{\partial (\partial_\eta \partial_\sigma \psi_p)} \delta (\psi_\rho) \right) d^3 r dt = 0.$$

This leads to Euler–Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \psi_\rho} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_\rho)} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu \psi_p)} = 0.$$

The equations of motion for a given physical system can be derived from the Euler–Lagrange equation, which relates to the fields involved in the system. In the case of a system with fields (ϕ and A_i), the equations of motion can be constructed using the Euler–Lagrange equation with these fields as variables. This will result in a set of equations that describe the motion of the system under consideration.

$$\frac{\partial \mathcal{L}}{\partial A_0} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu A_0)} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu A_0)} = 0,$$

$$\frac{\partial \mathcal{L}}{\partial A_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu A_i)} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu A_i)} = 0.$$

To simplify the Euler–Lagrange relations, we can use the notation $(\partial_\mu = \partial_v = \partial_0, \partial_i)$, as an example. With this notation, we can express the Euler–Lagrange relations in their the first of all forms in the following manner:

$$\left(\begin{array}{c} \frac{\partial \mathcal{L}}{\partial A_0} - \partial_0 \frac{\partial \mathcal{L}}{\partial (\partial_0 A_0)} - \partial_i \frac{\partial \mathcal{L}}{\partial (\partial_i A_0)} + \partial_0 \partial_0 \frac{\partial \mathcal{L}}{\partial (\partial_0 \partial_0 A_0)} \\ + \partial_i \partial_0 \frac{\partial \mathcal{L}}{\partial (\partial_i \partial_0 A_0)} + \partial_i \partial_j \frac{\partial \mathcal{L}}{\partial (\partial_i \partial_i A_0)} \end{array} \right) = 0, \tag{5}$$

$$\left(\begin{array}{c} \frac{\partial \mathcal{L}}{\partial A_i} - \partial_0 \frac{\partial \mathcal{L}}{\partial (\partial_0 A_i)} - \partial_i \frac{\partial \mathcal{L}}{\partial (\partial_i A_i)} + \partial_0 \partial_0 \frac{\partial \mathcal{L}}{\partial (\partial_0 \partial_0 A_i)} \\ + \partial_i \partial_0 \frac{\partial \mathcal{L}}{\partial (\partial_i \partial_0 A_i)} + \partial_i \partial_j \frac{\partial \mathcal{L}}{\partial (\partial_i \partial_i A_i)} \end{array} \right) = 0. \tag{6}$$

3. Analysis of Hamiltonian Formalism for Second-Order Lagrangians Using Dirac Method

In that section, we analyzed the Hamiltonian formalism using the Dirac method to adapt it to second-order Lagrangians. We focused on explaining the concepts of "independent variables" and "dependent variables" in detail.

Independent variables are those that are determined independently and are not connected to any other variables in the system. For example, when studying the motion of an object on a flat surface, time can be an independent variable as it is determined separately from the position or velocity. On the other hand, dependent variables are those that depend on independent variables and change based on their values. For instance, if we have a rectangle's length and width, the rectangle's area depends on these variables and represents a dependent variable. Let us clarify this with a simple mathematical example: Let us assume we have an equation for the motion of an object influenced by force and acceleration. In this context, the independent variables are time and distance. The dependent variables are velocity and acceleration, where velocity depends on distance and time, and acceleration depends on the change in velocity over time. Using this example, we can clearly understand the distinction between independent and dependent variables in the context of Hamiltonian analysis. It is important to note that this formalism can be universally applied to any second-order Lagrangian.

We analyze the Lagrangian by utilizing time derivatives of coordinates of the specified type, $\mathcal{L} = \mathcal{L}(\psi_\rho, \partial_\eta\psi_\rho, \partial_\eta\partial_\sigma\psi_p)$. The generalized momenta π_1 and π_2 corresponding to $\dot{\psi}_p = \partial_0\psi_p$ and $\ddot{\psi}_p = \partial_0^2\psi_p$ can be defined as follows [21]:

$$\pi_1 = \frac{\partial L}{\partial \dot{\psi}_\rho} - \partial_0\left(\frac{\partial L}{\partial \ddot{\psi}_\rho}\right),$$

$$\pi_2 = \frac{\partial L}{\partial \ddot{\psi}_\rho}.$$

The Hamiltonian, depending on the time derivatives, reads as:

$$\mathcal{H} = \pi_1\dot{\psi}_\rho + \pi_2\ddot{\psi}_\rho - \mathcal{L}(\psi_\rho, \partial_\eta\psi_\rho, \partial_\eta\partial_\sigma\psi_p). \tag{7}$$

Based on the equation provided earlier, the total differential corresponding to this specific function can be articulated in the subsequent manner:

$$d\mathcal{H} = \left\{ \begin{array}{c} \dot{\psi}_\rho d\pi_1 + \pi_1 d\dot{\psi}_\rho + \ddot{\psi}_\rho d\pi_2 + \pi_2 d\ddot{\psi}_\rho \\ -\frac{d\mathcal{L}}{d\psi_\rho}d\psi_\rho - \frac{d\mathcal{L}}{d(\partial_0\psi_p)}d(\partial_0\psi_p) - \frac{d\mathcal{L}}{d(\partial_i\psi_p)}d(\partial_i\psi_p) \\ -\frac{d\mathcal{L}}{d(\partial_0^2\psi_p)}d(\partial_0^2\psi_p) - \frac{d\mathcal{L}}{d(\partial_i\partial_i\psi_p)}d(\partial_i\partial_i\psi_p) \end{array} \right\}.$$

Now, we can rewrite equations of Hamiltonian's derivative of the above equation in terms of Lagrangian density using the defined values of the momenta given as follows:

$$d\mathcal{H} = \left\{ \begin{array}{c} \dot{\psi}_\rho d\pi_1 + \pi_1 d\dot{\psi}_\rho + \ddot{\psi}_\rho d\pi_2 + \pi_2 d\ddot{\psi}_\rho \\ -\frac{d\mathcal{L}}{d\psi_\rho}d\psi_\rho - \pi_1 d\dot{\psi}_\rho - \pi_2 d\ddot{\psi}_\rho \\ -\frac{d\mathcal{L}}{d(\partial_i\psi_p)}d(\partial_i\psi_p) - \frac{d\mathcal{L}}{d(\partial_i\partial_i\psi_p)}d(\partial_i\partial_i\psi_p) \\ -\frac{d\mathcal{L}}{d(\partial_i\partial_0\psi_p)}d(\partial_i\partial_0\psi_p) \end{array} \right\}. \tag{8}$$

This means that the Hamiltonian can be expressed as a mathematical function in the following manner:

$$\mathcal{H} = \mathcal{H}\left[(\psi_\rho,\ \pi_1,\ \partial_i\psi_p, \pi_2, \partial_0\partial_i\psi_p)\right].$$

Given the equation above, the complete differential for this specific function can be formulated as follows:

$$d\mathcal{H} = \left\{ \begin{array}{c} \frac{\partial \mathcal{H}}{\partial(\psi_p)} d(\psi_p) + \frac{\partial \mathcal{H}}{\partial(\partial_g \psi_p)} d(\partial_g \psi_p) + \frac{\partial \mathcal{H}}{\partial \pi_1} d\pi_1 + \frac{\partial \mathcal{H}}{\partial(\pi_2)} d(\pi_2) \\ + \frac{\partial \mathcal{H}}{\partial(\partial_0 \partial_j \psi_p)} d(\partial_0 \partial_j \psi_p) + \frac{\partial \mathcal{H}}{\partial(\partial_i \partial_0 \psi_p)} d(\partial_i \partial_0 \psi_p) \\ + \frac{\partial \mathcal{H}}{\partial(\partial_i \partial_j \psi_p)} d(\partial_i \partial_j \psi_p) \end{array} \right\}. \tag{9}$$

In cases 1 and 2, we can express the variation of the Hamiltonian resulting from changes in the independent variables as a second-order derivative, similar to how we define the variation in the Lagrangian.

Case 1: All variables are independent $(\psi_\rho, \pi_1, \pi_2)$:

$$dH = \left(\frac{\partial H}{\partial \psi_\rho} d\psi_\rho + \frac{\partial H}{\partial \pi_1} d\pi_1 + \frac{\partial H}{\partial \pi_2} d\pi_2 \right). \tag{10}$$

We can rewrite Hamilton's equations of motion by comparing Equations (8) and (9) as outlined below.

$$\left\{ \frac{\partial \mathcal{H}}{\partial \psi_\rho} = -\frac{\partial \mathcal{L}}{\partial_\rho \psi_p}, \right. \tag{11}$$

$$\left\{ \frac{\partial \mathcal{H}}{\partial(\partial_i \psi_p)} = -\frac{d\mathcal{L}}{\partial(\partial_i \psi_p)}, \right.$$

$$\left\{ \begin{array}{l} \frac{\partial \mathcal{H}}{\partial \pi_1} = \dot{\psi}_\rho, \\ \frac{\partial \mathcal{H}}{\partial \pi_2} = \ddot{\psi}_\rho, \end{array} \right.$$

$$\left\{ \frac{\partial \mathcal{H}}{\partial(\partial_i \partial_i \psi_p)} = -\frac{\partial \mathcal{L}}{\partial(\partial_i \partial_i \psi_p)}. \right.$$

The Euler–Lagrange equation may be used to rewrite Equation (10). This equation has the following form:

$$\frac{\partial \mathcal{H}}{\partial \psi_\rho} = -\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_\rho)} - \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \psi_\rho)} \right). \tag{12}$$

Extending $\partial_\mu \partial_\nu$ in terms of $(0, i)$, we obtain:

$$\frac{\partial \mathcal{H}}{\partial \psi_\rho} = \left(\begin{array}{c} -\partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_\rho)} - \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i \psi_\rho)} + \partial_i \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_i \psi_\rho)} \\ + \partial_0 \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_0 \psi_\rho)} + \partial_i \partial_i \psi_p \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_i \psi_p)} \end{array} \right). \tag{13}$$

Case 2: The value of π_1 depends on the variable (ψ_ρ), while the value of π_2 depends on the variable $(\partial_0 \psi_\rho)$. Therefore, when taking a variation, we only need to consider variations in the independent variables ψ_ρ and $\partial_0 \psi_\rho$. This approach allows us to simplify the calculation and focus only on the variables that have a direct impact on the values of π_1 and π_2.

$$dH = \frac{\partial H}{\partial \psi_\rho} d\psi_\rho + \frac{\partial H}{\partial(\partial_0 \psi_\rho)} d(\partial_0 \psi_\rho). \tag{14}$$

To represent the equations of motion obtained from Equation (14), write π_1 and π_2 in the general case $\pi_1 = R(\psi_\rho)$ and $\pi_2 = f(\partial_0 \psi_\rho)$. As a result, their derivative can be expressed as follows:

$$d\pi_1 = \frac{\partial R}{\partial \psi_\rho} d\psi_\rho, \tag{15}$$

$$d\pi_2 = \frac{\partial f}{\partial(\partial_0 \psi_\rho)} d(\partial_0 \psi_\rho). \tag{16}$$

As a consequence of substituting Equations (15) and (16) for Equation (10) and comparing the results with Equation (14), the following Hamiltonian density formulas are obtained:

$$\frac{\partial \mathcal{H}}{\partial \psi_\rho} = -\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_\rho)} - \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \psi_\rho)}\right) + \left(\frac{\partial R}{\partial \psi_\rho} + \frac{\partial f}{\partial\left(\partial_0 \psi_\rho\right)}\right) \partial_0 \psi_\rho. \tag{17}$$

Equations (13) and (15) are powerful tools for analyzing dynamic systems that are constrained by explicit or implicit limitations. In order to incorporate these constraints, Lagrange multipliers are introduced. The Hamiltonian Dirac approach utilizes generalized variables and their conjugate momenta as independent variables while emphasizing the Hamiltonian as the dependent variable. This method is particularly useful for studying complex systems that have constraints.

4. Hamilton Formalism in Form (ϕ, A_i) for Second-Order Lagrangians

Hamiltonian mechanics and classical field theory use the Lagrangian density and field variables in the Hamiltonian formalism to derive equations of motion. This process gives rise to two distinct scenarios: When all variables are independent, the equations derive directly from the Euler–Lagrange equations. This results in a set of coupled partial differential equations that effectively depict the evolution of the field. On the other hand, when certain variables depend on others, a heightened level of complexity emerges. In such cases, the equations of motion must account for these interdependencies. This intricate consideration is achieved by implementing constrained variations, a method that expresses changes in dependent variables (ϕ, A_i) with respect to independent ones. Consequently, this approach leads to the emergence of modified Euler–Lagrange equations.

4.1. In the First Case

All variables $\left(\psi_\rho, \pi_1, \pi_2\right)$ are independent, and we can use the definition given in Equation (12) to rewrite the equations of motion (i.e., Equation (13)) in terms of Lagrangian density. Specifically, we can express the Hamiltonian H as a function of the field variables (ϕ, A_i) and their conjugate momenta (π_ϕ, π_{A_i}), and take partial derivatives, with respect to ϕ and A_i, to obtain Equations (16) and (17), respectively.

$$\frac{\partial \mathcal{H}}{\partial A_0} = \begin{pmatrix} -\partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_0 A_0)} - \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i A_0)} + \partial_i \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_i A_0)} \\ +\partial_0 \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_0 A_0)} \end{pmatrix}, \tag{18}$$

$$\frac{\partial \mathcal{H}}{\partial A_i} = \begin{pmatrix} -\partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i)} - \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i A_i)} + \partial_i \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_i A_i)} \\ +\partial_0 \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_0 A_i)} \end{pmatrix}. \tag{19}$$

These equations give us the rate of change of the Hamiltonian with respect to changes in the field variables ϕ and A_i.

4.2. In the Second Case

The variables are dependent on one other; π_1 depends on ψ_ρ, and π_2 depends on $\left(\partial_0 \psi_\rho\right)$. To generate equations of motion in terms of Lagrangian density in this scenario, we can still follow the same steps as in the previous one, but we also need to add some more terms. Specifically, we need to add terms related to the variation of the dependent variables with respect to ϕ and A_i. These terms are given by $\frac{\partial R}{\partial(A_0,\ A_i)}$ and $\frac{\partial f}{\partial(A_0,\ A_i)}$:

$$\frac{\partial \mathcal{H}}{\partial A_0} = \begin{pmatrix} -\partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_0 A_0)} - \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i A_0)} + \partial_i \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_i A_0)} \\ +\partial_0 \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_0 A_0)} \end{pmatrix} + \left(\frac{\partial R}{\partial A_0} + \frac{\partial f}{\partial(\partial_0 A_0)}\right) \partial_0 A_0, \tag{20}$$

$$\frac{\partial \mathcal{H}}{\partial A_i} = \begin{pmatrix} -\partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i)} - \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i A_i)} + \partial_i \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_i A_i)} \\ +\partial_0 \partial_0 \frac{\partial \mathcal{L}}{\partial(\partial_0 \partial_0 A_i)} \end{pmatrix} + \left(\frac{\partial R}{\partial A_i} + \frac{\partial f}{\partial(\partial_0 A_i)}\right) \partial_0 A_i. \tag{21}$$

5. Utilizing Euler and Hamiltonian Methods for Modeling the Podolsky Equation

In this section, we investigate the practical implementation of the aforementioned methodology, focusing on the Podolsky Equation. This equation involves a Lagrangian density featuring second-order derivatives [22]. The Lagrangian can be expressed as follows:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + a^2\partial_\lambda F^{\alpha\lambda}\partial_\rho F_{\rho\alpha},$$

where $F_{\mu\nu}$ is defined as $\partial_\mu A_v - \partial_v A_m$.

Expanding L with respect to $(0, i)$, we can rewrite the Lagrangian as:

$$\mathcal{L} = \frac{1}{2}(\partial_0 A_i - \partial_i A_0)^2 - a^2\left(\partial_i\left(\partial^0 A^i - \partial^i A^0\right)^2 - \partial_0\left(\partial^0 A^i - \partial^i A^0\right)^2\right).$$

To obtain the conjugated momenta, we use the Euler–Lagrange Equations (5) and (6) separately for the variables $(\phi,\ A_i)$. We then apply derivatives to each variable, which allows us to obtain distinct outcomes for each variable.

$$\left(1 + 2a^2\left({\partial_0}^2 - {\partial_i}^2\right)\right)(\partial_0 + \partial_i)\left(\partial^0 A^i - \partial^i A^0\right) = 0. \tag{22}$$

The result in (22) is the same as the result in [24–26].

To obtain conjugated momenta, we introduce the generalized momenta with respect to the variables (ϕ, A_i).

$$\pi_1^1 = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_0)} - \partial_0\left(\frac{\partial\mathcal{L}}{\partial(\partial_0^2 A_0)}\right) = 0, \tag{23}$$

$$\pi_1^2 = \left(\frac{\partial\mathcal{L}}{\partial(\partial_0 A_i)} - \partial_0\left(\frac{\partial\mathcal{L}}{\partial(\partial_0^2 A_i)}\right) = F_{0i} - a^2\left(2\partial_0^2 A_i - \partial_0\partial_i A_0\right)\right), \tag{24}$$

$$\pi_2^1 = \frac{\partial\mathcal{L}}{\partial(\partial_0^2 A_0)} = 0, \tag{25}$$

$$\pi_2^2 = \frac{\partial\mathcal{L}}{\partial(\partial_0^2 A_i)} = -2a^2\partial_0 F_{0i}. \tag{26}$$

By using the canonical Hamiltonian, which is defined as:

$$\mathcal{H} = \pi_1^1\partial_0 A_0 + \pi_1^2\partial_0 A_i + \pi_2^1\partial_0^2 A_0 + \pi_2^2\partial_0^2 A_i - \mathcal{L}.$$

Substituting the Lagrangian, we obtain:

$$\mathcal{H} = \begin{pmatrix} \left[F_{0i} - a^2\left(2\partial_0^2 A_i - \partial_0\partial_i A_0\right)\right]\partial_0 A_i \\ -\left[2a^2\partial_0 F_{0i}\right]\partial_0^2 A_i - \frac{1}{2}(\partial_0 A_i - \partial_i A_0)^2 \\ a^2\left(\partial_i\left(\partial^0 A^i - \partial^i A^0\right)^2 - \partial_0\left(\partial^0 A^i - \partial^i A^0\right)^2\right) \end{pmatrix}. \tag{27}$$

Let us use Equations (16) and (17) to represent the dependent variables (ϕ, A_i). In case one, we obtain:

$$\left(1 + 2a^2\left({\partial_0}^2 - {\partial_i}^2\right)\right)(\partial_0 + \partial_i)\left(\partial^0 A^i - \partial^i A^0\right) = 0. \tag{28}$$

The equation obtained above is equivalent in form to the equation obtained from Equation (20). This equivalence is achieved by taking into account the relationship: $\eta^{\mu\nu}\partial_\mu\partial_\nu = \Box = \left(\partial_0^2 - \nabla^2\right)$,

$$\left[1 - 2a^2\Box\right]\partial_\mu F_{\mu\nu} = 0. \tag{29}$$

Here, $(\Box)$ represents the Dellembrian operator. The equations presented in this study can be used to describe the motion of fourth-order systems, and the result obtained from

Equation (28) is in perfect agreement with the result presented in [25,26]. This result shows how the mathematical analysis used in this study was accurate and consistent.

To obtain the derivative with respect to the independent variables (ϕ, A_i), we use the Hamiltonian Equations (20) and (21).

$$\left(1 + 2a^2\left({\partial_0}^2 - {\partial_i}^2\right)\right)(\partial_0 + \partial_i)\left(\partial^0 A^i - \partial^i A^0\right) = 0. \tag{30}$$

The Hamilton–Dirac method and second-order differential equations show a striking similarity in the equations of motion for both independent and dependent variables, as seen in Equations (17)–(20), respectively. This similarity indicates that the equations of motion derived from both approaches are equivalent, giving researchers confidence in using either method when studying constrained systems. Understanding the mathematical foundations of physical systems, especially those with constraints, is crucial for accurately modeling and analyzing their behavior.

6. Applications of Second-Order Podolsky Lagrangians and Suggestions for Further Research

Using the innovative Dirac theory, the generalized electric dynamics fields proposed by Podolsky stands out as one of the most actively researched areas. These fields primarily focus on exploring second-order derivatives within Podolsky's equations for electrical dynamics, playing a pivotal role in developing the quantum theory for electrical dynamics. These advancements within the Hamiltonian model for continuous second-order systems have the potential to trigger a profound transformation in our understanding of physical phenomena, including Coulomb's law and the principle of superposition. Efforts by physicists have effectively demonstrated the possibility of extending the electrostatic laws derived from the principle of superposition and Coulomb's law to encompass second-order derivatives. Additionally, researchers have conducted thorough examinations of the first-order relativistic model alongside interaction fields. Their precise analyses delve into scenarios characterized by unique momentum-energy projections based on the fractional-order Podolsky equation. Currently, exploration is underway to formulate novel arrangements for individual systems, introducing new methods for investigating these complex setups.

In a broader context, this research builds on classical dynamics frameworks, including Lagrangian equations, Hamiltonians, and Euler equations. Each method has its own strengths and limitations in solving complex problems. For example, additional constraints may arise for second-order derivatives when dealing with exponential and Mittag-Leffler factors that introduce singularities. The Hamiltonian approach with second-order derivatives adds mathematical complexity compared to traditional methods like Euler–Lagrange, especially in scenarios with numerous interactions. Furthermore, applying this approach to specific systems often requires additional controls or assumptions. Challenges also include handling non-physical solutions and analyzing intricate models with implicit electronics, which complicates numerical applications. Such constraints might pave the way for the emergence of a new and distinctive realm of research.

7. Conclusions

We have devised a novel methodology for investigating continuous systems characterized by second-order derivatives. Our approach employs a Hamiltonian formulation, which has enabled us to deduce the Euler–Lagrange equations governing the dynamics of such systems. We also obtained the Hamiltonian for these systems and used it to derive the Hamiltonian equation of motion, which involves second-order derivatives. It is worth noting that our results are consistent with those obtained using the conventional Euler–Lagrange approach. To illustrate the utility of our approach, we have applied it to a specific example of a system with second-order derivatives. This exercise allowed us to validate our framework and obtain a deeper understanding of the physical behavior of such sys-

tems. Our study has presented a comprehensive Hamiltonian formulation for continuous systems with second-order derivatives. The novelty of the research lies in an alternative and complementary perspective to the traditional Euler–Lagrange framework. The results obtained through our approach have been verified through a specific example, indicating the potential utility of this framework in investigating a variety of physical systems.

The results confirm the effectiveness of the proposed approach in analyzing continuous systems with second-order derivatives. These findings enhance the understanding of complex physical systems, paving the way for practical applications in electrodynamics and developing advanced control systems. The paper presents a comprehensive mathematical framework that can be applied to a wide range of physical phenomena.

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