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Some Identities Related to Semiprime Ideal of Rings with Multiplicative Generalized Derivations

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Abstract: This paper investigates the relationship between the commutativity of rings and the properties of their multiplicative generalized derivations. Let $\mathcal{F}$ be a ring with a semiprime ideal Π. A map $\phi : \mathcal{F} \to \mathcal{F}$ is classified as a multiplicative generalized derivation if there exists a map $\sigma : \mathcal{F} \to \mathcal{F}$ such that $\phi(xy) = \phi(x)y + x\sigma(y)$ for all $x, y \in \mathcal{F}$. This study focuses on semiprime ideals Π that admit multiplicative generalized derivations ϕ and G that satisfy certain differential identities within $\mathcal{F}$. By examining these conditions, the paper aims to provide new insights into the structural aspects of rings, particularly their commutativity in relation to the behavior of such derivations.

Keywords: semiprime ring ideal; generalized derivation; multiplicative generalized derivation

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1. Introduction

Let $\mathcal{F}$ be an associative ring with center Z. A proper ideal Π of $\mathcal{F}$ is termed *prime* if for any elements $\vartheta_1, \vartheta_2 \in \mathcal{F}$, the inclusion $\vartheta_1\mathcal{F}\vartheta_2 \subseteq \Pi$ implies that either $\vartheta_1 \in \Pi$ or $\vartheta_2 \in \Pi$. Equivalently, the ring $\mathcal{F}$ is said to be prime if (0), the zero ideal, is a prime ideal. This is to say, $\mathcal{F}$ is prime if $\vartheta_1\mathcal{F}\vartheta_2 = 0$ implies $\vartheta_1 = 0$ or $\vartheta_2 = 0$.

In addition to prime ideals, the concept of semiprime ideals is also fundamental in ring theory. A proper ideal Π is *semiprime* if for any $\vartheta_1 \in \mathcal{F}$, the condition $\vartheta_1\mathcal{F}\vartheta_1 \subseteq \Pi$ implies $\vartheta_1 \in \Pi$. The ring $\mathcal{F}$ is semiprime if (0) is a semiprime ideal. While every prime ideal is semiprime, the converse is not generally true. Therefore, it is important to investigate the structure and properties of semiprime ideals, particularly when considering multiplicative generalized semiderivations. For any $\vartheta_1, \vartheta_2 \in \mathcal{F}$, the symbol $[\vartheta_1, \vartheta_2]$ stands for the commutator $\vartheta_1\vartheta_2 - \vartheta_2\vartheta_1$, and the symbol $\vartheta_1 \circ \vartheta_2$ denotes the anti-commutator $\vartheta_1\vartheta_2 + \vartheta_2\vartheta_1$. For any $\vartheta_1, \vartheta_2 \in \mathcal{F}$ it is expressed as $[\vartheta_1, \vartheta_2]_0 = \vartheta_1$, $[\vartheta_1, \vartheta_2]_1 = [\vartheta_1, \vartheta_2] = \vartheta_1\vartheta_2 - \vartheta_2\vartheta_1$, and for $k > 1$, it is expressed as $[\vartheta_1, \vartheta_2]_k = [[\vartheta_1, \vartheta_2]_{(k-1)}, \vartheta_2]$.

The study of derivations in rings has a rich history, originating with Posner's seminal work in 1957 [1]. A derivation σ on $\mathcal{F}$ is an additive map satisfying

$$\sigma(\vartheta_1\vartheta_2) = \sigma(\vartheta_1)\vartheta_2 + \vartheta_1\sigma(\vartheta_2) \quad \text{for all } \vartheta_1, \vartheta_2 \in \mathcal{F}.$$

Derivations are critical in understanding the internal structure of rings, particularly in the context of prime rings, where they can impose strong commutativity conditions.

Building on Posner's work, Brešar [2], introduced the concept of generalized derivations. A map $\phi : \mathcal{F} \to \mathcal{F}$ is called a generalized derivation if there exists a derivation $\sigma : \mathcal{F} \to \mathcal{F}$ such that

$$\phi(\vartheta_1\vartheta_2) = \phi(\vartheta_1)\vartheta_2 + \vartheta_1\sigma(\vartheta_2) \quad \text{for all } \vartheta_1, \vartheta_2 \in \mathcal{F}.$$

Familiar examples of generalized derivations are derivations and generalized inner derivations, and the latter include left multipliers and right multipliers (i.e., $\phi(\vartheta_1\vartheta_2) = \phi(\vartheta_1)\vartheta_1$ for all $\vartheta_1, \vartheta_2 \in \mathcal{F}$).

The commutativity of prime or semiprime rings with derivation was initiated by Posner in [1]. Thereafter, several authors have proved commutativity theorems of prime or semiprime rings with derivations. In [3], the notion of multiplicative derivation was introduced by Daif motivated by Martindale in [4]. Daif [3] introduced this concept and explored its implications in prime and semiprime rings. A multiplicative derivation σ satisfies the condition

$$\sigma(\vartheta_1\vartheta_2) = \sigma(\vartheta_1)\vartheta_2 + \vartheta_1\sigma(\vartheta_2) \quad \text{for all } \vartheta_1, \vartheta_2 \in \mathcal{F},$$

but unlike a traditional derivation, σ may not be additive. In [5], Goldman and Semrl gave the complete description of these maps. We have $\mathcal{F} = C[0,1]$, the ring of all continuous (real or complex valued) functions, and define a map $\sigma : \mathcal{F} \to \mathcal{F}$ such as

$$\sigma(\phi)(\vartheta_1) = \left\{ \begin{array}{cc} \phi(\vartheta_1)\log|\phi(\vartheta_1)|, & \phi(\vartheta_1) \neq 0 \\ 0, & \text{otherwise} \end{array} \right\}.$$

It is clear that σ is a multiplicative derivation, but σ is not additive. Inspired by the definition multiplicative derivation, the notion of multiplicative generalized derivation was extended by Daif and Tamman El-Sayiad in [6] as follows: $\phi : \mathcal{F} \to \mathcal{F}$ is called a multiplicative generalized derivation if there exists a derivation $\sigma : \mathcal{F} \to \mathcal{F}$ such that $\phi(\vartheta_1\vartheta_2) = \phi(\vartheta_1)\vartheta_2 + \vartheta_1\sigma(\vartheta_2)$ for all $\vartheta_1, \vartheta_2 \in \mathcal{F}$.

Dhara and Ali [7] provided a slight generalization of this definition by allowing σ to be any map, not necessarily an additive map or derivation. It is worth noting that if $\mathcal{F}$ is a semiprime ring, then in this case σ must be a multiplicative derivation, because for any $\vartheta_1, \vartheta_2, \vartheta_3 \in \mathcal{F}$,

$$\begin{aligned} \phi((\vartheta_1\vartheta_2)\vartheta_3) &= \phi(\vartheta_1(\vartheta_2\vartheta_3)) \\ \phi(\vartheta_1\vartheta_2)\vartheta_3 + \vartheta_1\vartheta_2\sigma(\vartheta_3) &= \phi(\vartheta_1)\vartheta_2\vartheta_3 + \vartheta_1\sigma(\vartheta_2\vartheta_3), \\ \phi(\vartheta_1)\vartheta_2\vartheta_3 + \vartheta_1\sigma(\vartheta_2)\vartheta_3 + \vartheta_1\vartheta_2\sigma(\vartheta_3) &= \phi(\vartheta_1)\vartheta_2\vartheta_3 + \vartheta_1\sigma(\vartheta_2\vartheta_3). \end{aligned}$$

This implies that $\mathcal{F}(\sigma(\vartheta_2\vartheta_3) - \sigma(\vartheta_2)\vartheta_3 - \vartheta_2\sigma(\vartheta_3)) = \{0\}$. This gives that σ is a multiplicative derivation. Further, every generalized derivation is a multiplicative generalized derivation. But the converse is not true in general (see example ([7], Example 1.1)). Hence, one may observe that the concept of multiplicative generalized derivations includes the concepts of derivations, multiplicative derivation, and the left multipliers. So, it should be interesting to extend some results concerning these notions to multiplicative generalized derivations.

A functional identity is an identity relation in an algebra involving arbitrary elements, similar to a polynomial identity, but also incorporating functions that are treated as unknowns (see [8]). In [9], Ashraf and Rehman showed that a prime ring $\mathcal{F}$ with a nonzero ideal I must be commutative if it admits a derivation σ satisfying either of the properties $\sigma(\vartheta_1\vartheta_2) + \vartheta_1\vartheta_2 \in Z$ or $\sigma(\vartheta_1\vartheta_2) - \vartheta_1\vartheta_2 \in Z$ for all $\vartheta_1, \vartheta_2 \in \mathcal{F}$. In [10], the authors explored the commutativity of prime ring $\mathcal{F}$, which satisfies any one of the properties when ϕ is a generalized derivation. In [11], studied the commutativity of such a prime ring if anyone of the following is hold: $G(\vartheta_1\vartheta_2) + \phi(\vartheta_1)\phi(\vartheta_2) \pm \vartheta_1\vartheta_2 = 0$ or $G(\vartheta_1\vartheta_2) + \phi(\vartheta_1)\phi(\vartheta_2) \pm \vartheta_2\vartheta_1 = 0$ where ϕ and G are generalized derivations.

Let S be a nonempty subset of $\mathcal{F}$. A mapping ϕ from $\mathcal{F}$ to $\mathcal{F}$ is called centralizing on S if $[\phi(\vartheta_1), \vartheta_1] \in Z$ for all $\vartheta_1 \in S$ and is called commuting on S if $[\phi(\vartheta_1), \vartheta_1] = 0$ for all $\vartheta_1] \in S$. This definition has been generalized as: a map $\phi : \mathcal{F} \to \mathcal{F}$ is called a π-commuting map on S if $[\phi(\vartheta_1), \vartheta_1] \in \pi$ for all $\vartheta_1 \in S$ and some $\pi \subseteq \mathcal{F}$. In particular, if $\pi = 0$, then ϕ is called a commuting map on S. Note that every commuting map is a π-commuting map. But the converse is not true in general. Take π some a set of $\mathcal{F}$ has no zero such that $[\phi(\vartheta_1), \vartheta_1] \in \pi$; then ϕ is a π-commuting map but it is not a commuting map.

The significance of these derivations, especially in the context of commutativity, has been widely studied. A mapping ϕ from $\mathcal{F}$ to $\mathcal{F}$ is said to be commutativity-preserving on a subset $S \subseteq \mathcal{F}$ if $[\vartheta_1, \vartheta_2] = 0$ implies $[\phi(\vartheta_1), \phi(\vartheta_2)] = 0$ for all $\vartheta_1, \vartheta_2 \in S$. The concept of strong commutativity-preserving (SCP) maps, where $[\vartheta_1, \vartheta_2] = [\phi(\vartheta_1), \phi(\vartheta_2)]$ for all $\vartheta_1, \vartheta_2 \in S$, has also been extensively explored. There is a growing body of literature on strong commutativity-preserving (SCP) maps and derivations. In [12], Bell and Daif were the first to investigate the derivation of SCP maps on the ideal of a semiprime ring. Ma and Xu extended this study to generalized derivations in [13]. There are some recent articles that studied identities with multiplicative generalized derivations (see [7,14–17]). In [17], Gölbaşi Additionally, Koç and Gölbaşi generalized these results to multiplicative generalized derivations on semiprime rings in [18]. In [19], Samman demonstrated that an epimorphism of a semiprime ring is strong commutativity-preserving if and only if it is centralizing. Researchers have extensively explored derivations and SCP mappings within the framework of operator algebras, as well as in prime and semiprime rings.

This paper investigates the commutativity conditions in rings that admit multiplicative generalized derivations, particularly in the context of semiprime ideals. By extending existing results and introducing new findings, this study contributes to a deeper understanding of the interplay between derivations, semiprime ideals, and commutativity in ring theory.

2. Main Results

We will make some extensive use of the basic commutator identities:

$$[\vartheta_1, \vartheta_2\vartheta_3] = \vartheta_2[\vartheta_1, \vartheta_3] + [\vartheta_1, \vartheta_2]\vartheta_3$$
$$[\vartheta_1\vartheta_2, \vartheta_3] = [\vartheta_1, \vartheta_3]\vartheta_2 + \vartheta_1[\vartheta_2, \vartheta_3]$$
$$\vartheta_1 o(\vartheta_2\vartheta_3) = (\vartheta_1 o\vartheta_2)\vartheta_3 - \vartheta_2[\vartheta_1, \vartheta_3] = \vartheta_2(\vartheta_1 o\vartheta_3) + [\vartheta_1, \vartheta_2]\vartheta_3$$
$$(\vartheta_1\vartheta_2)o\vartheta_3 = \vartheta_1(\vartheta_2 o\vartheta_3) - [\vartheta_1, \vartheta_3]\vartheta_2 = (\vartheta_1 o\vartheta_3)\vartheta_2 + \vartheta_1[\vartheta_2, \vartheta_3].$$

Theorem 1. *Let $\mathcal{F}$ be a ring with Π as a semiprime ideal of R. Suppose that $\mathcal{F}$ admits a multiplicative generalized derivation ϕ associated with a nonzero map σ. If any of the following conditions is satisfied for all $\vartheta_1, \vartheta_2 \in \mathcal{F}$:*

(i) $\sigma(\vartheta_1) \circ \phi(\vartheta_2) \mp (\vartheta_1 \circ \vartheta_2) \in \Pi$,
(ii) $[\sigma(\vartheta_1), \phi(\vartheta_2)] \mp [\vartheta_1, \vartheta_2] \in \Pi$,
(iii) $\sigma(\vartheta_1) \circ \phi(\vartheta_2) \mp [\vartheta_1, \vartheta_2] \in \Pi$,
(iv) $[\sigma(\vartheta_1), \phi(\vartheta_2)] \mp (\vartheta_1 \circ \vartheta_2) \in \Pi$,

then $[\vartheta_1, \sigma(\vartheta_1)]_2 \in \Pi$ for all $\vartheta_1 \in \mathcal{F}$.

Proof. (i) By the hypothesis, we have

$$\sigma(\vartheta_1) \circ \phi(\vartheta_2) \mp (\vartheta_1 \circ \vartheta_2) \in \Pi \text{ for all } \vartheta_1, \vartheta_2 \in \mathcal{F}.$$

That is,

$$\sigma(\vartheta_1)\phi(\vartheta_2) + \phi(\vartheta_2)\sigma(\vartheta_1) \mp (\vartheta_1\vartheta_2 + \vartheta_2\vartheta_1) \in \Pi. \quad (1)$$

Replacing ϑ_2 by $\vartheta_2\vartheta_1$ in this expression, we have

$$\sigma(\vartheta_1)\phi(\vartheta_2\vartheta_1) + \phi(\vartheta_2\vartheta_1)\sigma(\vartheta_1) \mp (\vartheta_1\vartheta_2\vartheta_1 + \vartheta_2\vartheta_1\vartheta_1) \in \Pi$$

and so

$$\sigma(\vartheta_1)\{\phi(\vartheta_2)\vartheta_1 + \vartheta_2\sigma(\vartheta_1)\} + \{\phi(\vartheta_2)\vartheta_1 + \vartheta_2\sigma(\vartheta_1)\}\sigma(\vartheta_1) \mp (\vartheta_1\vartheta_2 + \vartheta_2\vartheta_1)\vartheta_1 \in \Pi. \quad (2)$$

Right multiplying by ϑ_1 the expression (1), we see that

$$\sigma(\vartheta_1)\phi(\vartheta_2)\vartheta_1 + \phi(\vartheta_2)\sigma(\vartheta_1)\vartheta_1 \mp (\vartheta_1\vartheta_2 + \vartheta_2\vartheta_1)\vartheta_1 \in \Pi. \quad (3)$$

Subtracting (2) from (3), we arrive at

$$\sigma(\vartheta_1)\vartheta_2\sigma(\vartheta_1) + \phi(\vartheta_2)[\vartheta_1, \sigma(\vartheta_1)] + \vartheta_2(\sigma(\vartheta_1))^2 \in \Pi. \tag{4}$$

Replacing ϑ_2 by $\vartheta_2[\vartheta_1, \sigma(\vartheta_1)]$ in the last expression, we have

$$\begin{aligned} \sigma(\vartheta_1)\vartheta_2[\vartheta_1, \sigma(\vartheta_1)]\sigma(\vartheta_1) + \phi(\vartheta_2)[\vartheta_1, \sigma(\vartheta_1)]^2 \\ +\vartheta_2\sigma([\vartheta_1, \sigma(\vartheta_1)])[\vartheta_1, \sigma(\vartheta_1)] + \vartheta_2[\vartheta_1, \sigma(\vartheta_1)]\sigma(\vartheta_1)^2 \in \Pi. \end{aligned} \tag{5}$$

Right multiplying by $[\vartheta_1, \sigma(\vartheta_1)]$ the expression (4), we get

$$\sigma(\vartheta_1)\vartheta_2\sigma(\vartheta_1)[\vartheta_1, \sigma(\vartheta_1)] + \phi(\vartheta_2)[\vartheta_1, \sigma(\vartheta_1)]^2 + \vartheta_2\sigma(\vartheta_1)^2[\vartheta_1, \sigma(\vartheta_1)] \in \Pi. \tag{6}$$

Subtracting (5) from (6), we arrive at

$$\sigma(\vartheta_1)\vartheta_2[[\vartheta_1, \sigma(\vartheta_1)], \sigma(\vartheta_1)] + \vartheta_2\sigma([\vartheta_1, \sigma(\vartheta_1)])[\vartheta_1, \sigma(\vartheta_1)] + \vartheta_2[[\vartheta_1, \sigma(\vartheta_1)], \sigma(\vartheta_1)^2] \in \Pi. \tag{7}$$

Writing ϑ_2 by $\vartheta_1\vartheta_2$ in (7), we obtain that

$$\begin{aligned} \sigma(\vartheta_1)\vartheta_1\vartheta_2[[\vartheta_1, \sigma(\vartheta_1)], \sigma(\vartheta_1)] + \vartheta_1\vartheta_2\sigma([\vartheta_1, \sigma(\vartheta_1)])[\vartheta_1, \sigma(\vartheta_1)] \\ +\vartheta_1\vartheta_2[[\vartheta_1, \sigma(\vartheta_1)], \sigma(\vartheta_1)^2] \in \Pi. \end{aligned} \tag{8}$$

Right multiplying by ϑ_1 the expression (7), we get

$$\begin{aligned} \vartheta_1\sigma(\vartheta_1)\vartheta_2[[\vartheta_1, \sigma(\vartheta_1)], \sigma(\vartheta_1)] + \vartheta_1\vartheta_2\sigma([\vartheta_1, \sigma(\vartheta_1)])[\vartheta_1, \sigma(\vartheta_1)] \\ +\vartheta_1\vartheta_2[[\vartheta_1, \sigma(\vartheta_1)], \sigma(\vartheta_1)^2] \in \Pi. \end{aligned} \tag{9}$$

Subtracting (8) from (9), we arrive at

$$[\vartheta_1, \sigma(\vartheta_1)]\vartheta_2[[\vartheta_1, \sigma(\vartheta_1)], \sigma(\vartheta_1)] \in \Pi. \tag{10}$$

Writing ϑ_2 by $\sigma(\vartheta_1)\vartheta_2$ in (7), we obtain that

$$[\vartheta_1, \sigma(\vartheta_1)]\sigma(\vartheta_1)\vartheta_2[[\vartheta_1, \sigma(\vartheta_1)], \sigma(\vartheta_1)] \in \Pi. \tag{11}$$

Left multiplying by $\sigma(\vartheta_1)$ the expression (10), we have

$$\sigma(\vartheta_1)[\sigma(\vartheta_1), \vartheta_1]\vartheta_2[[\vartheta_1, \sigma(\vartheta_1)], \sigma(\vartheta_1)] \in \Pi. \tag{12}$$

Subtracting (11) from (12), we arrive at

$$[[\vartheta_1, \sigma(\vartheta_1)], \sigma(\vartheta_1)]\vartheta_2[[\vartheta_1, \sigma(\vartheta_1)], \sigma(\vartheta_1)] \in \Pi \text{ for all } \vartheta_1, \vartheta_2 \in \mathcal{F}.$$

Since π is a semiprime ideal, we obtain that

$$[[\vartheta_1, \sigma(\vartheta_1)], \sigma(\vartheta_1)] \in \Pi \text{ for all } \vartheta_1 \in \mathcal{F}.$$

Thus, $[\vartheta_1, \sigma(\vartheta_1)]_2 \in \Pi$ for all $\vartheta_1 \in \mathcal{F}$.

(ii) By the hypothesis, we have

$$[\sigma(\vartheta_1), \phi(\vartheta_2)] \mp [\vartheta_1, \vartheta_2] \in \Pi \text{ for all } \vartheta_1, \vartheta_2 \in \mathcal{F}. \tag{13}$$

Replacing ϑ_2 by $\vartheta_2\vartheta_1$ in (13), we have

$$[\sigma(\vartheta_1), \phi(\vartheta_2)]\vartheta_1 + \phi(\vartheta_2)[\sigma(\vartheta_1), \vartheta_1] + [\sigma(\vartheta_1), \vartheta_2]\sigma(\vartheta_1) \mp [\vartheta_1, \vartheta_2]\vartheta_1 \in \Pi.$$

Right multiplying by ϑ_1 the expression (13), we have

$$[\sigma(\vartheta_1), \phi(\vartheta_2)]\vartheta_1 \mp [\vartheta_1, \vartheta_2]\vartheta_1 \in \Pi.$$

If the last two expressions are used, the following is found

$$\phi(\vartheta_2)[\sigma(\vartheta_1), \vartheta_1] + [\sigma(\vartheta_1), \vartheta_2]\sigma(\vartheta_1) \in \Pi. \tag{14}$$

That is,

$$\sigma(\vartheta_1)\vartheta_2\sigma(\vartheta_1) + \phi(\vartheta_2)[\sigma(\vartheta_1), \vartheta_1] - \vartheta_2\sigma(\vartheta_1)^2 \in \Pi. \tag{15}$$

Writing ϑ_2 by $\vartheta_2[\sigma(\vartheta_1), \vartheta_1]$ in the last expression, we have

$$\begin{aligned} &\sigma(\vartheta_1)\vartheta_2[\sigma(\vartheta_1), \vartheta_1]\sigma(\vartheta_1) + \phi(\vartheta_2)[\sigma(\vartheta_1), \vartheta_1]^2 \\ &+\vartheta_2\sigma([\sigma(\vartheta_1), \vartheta_1])[\sigma(\vartheta_1), \vartheta_1] - \vartheta_2[\sigma(\vartheta_1), \vartheta_1]\sigma(\vartheta_1)^2 \in \Pi. \end{aligned}$$

Right multiplying by $[\sigma(\vartheta_1), \vartheta_1]$ the expression (15), we have

$$\sigma(\vartheta_1)\vartheta_2\sigma(\vartheta_1)[\sigma(\vartheta_1), \vartheta_1] + \phi(\vartheta_2)[\sigma(\vartheta_1), \vartheta_1]^2 - \vartheta_2\sigma(\vartheta_1)^2[\sigma(\vartheta_1), \vartheta_1] \in \Pi.$$

If the last two expressions are used, the following is found

$$\sigma(\vartheta_1)\vartheta_2[[\sigma(\vartheta_1), \vartheta_1], \sigma(\vartheta_1)] + \vartheta_2\sigma([\sigma(\vartheta_1), \vartheta_1])[\sigma(\vartheta_1), \vartheta_1] + \vartheta_2[[\sigma(\vartheta_1), \vartheta_1], \sigma(\vartheta_1)^2] \in \Pi.$$

This expression is the same as expression (7). Using the same techniques, we get the required result.
(iii) By the hypothesis, we have

$$\sigma(\vartheta_1) \circ \phi(\vartheta_2) \mp [\vartheta_1, \vartheta_2] \in \Pi \text{ for all } \vartheta_1, \vartheta_2 \in \mathcal{F}.$$

That is,

$$\sigma(\vartheta_1)\phi(\vartheta_2) + \phi(\vartheta_2)\sigma(\vartheta_1) \mp (\vartheta_1\vartheta_2 - \vartheta_2\vartheta_1) \in \Pi. \tag{16}$$

Replacing ϑ_2 by $\vartheta_2\vartheta_1$ in this expression, we have

$$\sigma(\vartheta_1)\phi(\vartheta_2\vartheta_1) + \phi(\vartheta_2\vartheta_1)\sigma(\vartheta_1) \mp (\vartheta_1\vartheta_2\vartheta_1 - \vartheta_2\vartheta_1\vartheta_1) \in \Pi,$$

and so

$$\sigma(\vartheta_1)\{\phi(\vartheta_2)\vartheta_1 + \vartheta_2\sigma(\vartheta_1)\} + \{\phi(\vartheta_2)\vartheta_1 + \vartheta_2\sigma(\vartheta_1)\}\sigma(\vartheta_1) \mp (\vartheta_1\vartheta_2 - \vartheta_2\vartheta_1)\vartheta_1 \in \Pi. \tag{17}$$

Right multiplying by ϑ_1 the expression (16), we see that

$$\sigma(\vartheta_1)\phi(\vartheta_2)\vartheta_1 + \phi(\vartheta_2)\sigma(\vartheta_1)\vartheta_1 \mp (\vartheta_1\vartheta_2 - \vartheta_2\vartheta_1)\vartheta_1 \in \Pi. \tag{18}$$

Subtracting (17) from (18), we arrive at

$$\sigma(\vartheta_1)\vartheta_2\sigma(\vartheta_1) + \phi(\vartheta_2)[\vartheta_1, \sigma(\vartheta_1)] + \vartheta_2(\sigma(\vartheta_1))^2 \in \Pi.$$

This expression is the same as expression (4), and hence applying the same lines, we complete the proof.
(iv) By the hypothesis, we have

$$[\sigma(\vartheta_1), \phi(\vartheta_2)] \mp (\vartheta_1 \circ \vartheta_2) \in \Pi \text{ for all } \vartheta_1, \vartheta_2 \in \mathcal{F}. \tag{19}$$

Replacing ϑ_2 by $\vartheta_2\vartheta_1$ in (19), we have

$$[\sigma(\vartheta_1), \phi(\vartheta_2)]\vartheta_1 + \phi(\vartheta_2)[\sigma(\vartheta_1), \vartheta_1] + [\sigma(\vartheta_1), \vartheta_2]\sigma(\vartheta_1) \mp (\vartheta_1 \circ \vartheta_2)\vartheta_1 \in \Pi.$$

Right multiplying by ϑ_1 the expression (19), we have

$$[\sigma(\vartheta_1), \phi(\vartheta_2)]\vartheta_1 \mp (\vartheta_1 \circ \vartheta_2)\vartheta_1 \in \Pi.$$

If the last two expressions are used, the following is found

$$\phi(\vartheta_2)[\sigma(\vartheta_1), \vartheta_1] + [\sigma(\vartheta_1), \vartheta_2]\sigma(\vartheta_1) \in \Pi.$$

This expression is the same as expression (14). By the same techniques, we obtain the required result. □

Theorem 2. *Let $\mathcal{F}$ be a 2-torsion-free ring with Π as a semiprime ideal of $\mathcal{F}$. Suppose that $\mathcal{F}$ admits a multiplicative generalized derivation ϕ associated with a nonzero multiplicative derivation σ. If $\phi([\vartheta_1, \vartheta_2]) - (\phi(\vartheta_1) \circ \vartheta_2) - [\sigma(\vartheta_2), \vartheta_1] \in \Pi$ for all $\vartheta_1, \vartheta_2 \in \mathcal{F}$, then σ is a Π-commuting map on $\mathcal{F}$.*

Proof. By the hypothesis, we have

$$\phi([\vartheta_1, \vartheta_2]) - (\phi(\vartheta_1) \circ \vartheta_2) - [\sigma(\vartheta_2), \vartheta_1] \in \Pi.$$

Replacing ϑ_2 by $\vartheta_2\vartheta_3$, $\vartheta_3 \in \mathcal{F}$ in the last expression, we have

$$\phi([\vartheta_1, \vartheta_2\vartheta_3]) - (\phi(\vartheta_1) \circ \vartheta_2\vartheta_3) - [\sigma(\vartheta_2\vartheta_3), \vartheta_1] \in \Pi.$$

That is,

$$\begin{array}{l}\phi([\vartheta_1, \vartheta_2])\vartheta_3 + [\vartheta_1, \vartheta_2]\sigma(\vartheta_3) + \phi(\vartheta_2)[\vartheta_1, \vartheta_3] + \vartheta_2\sigma([\vartheta_1, \vartheta_3]) \\ -(\phi(\vartheta_1) \circ \vartheta_2)\vartheta_3 + \vartheta_2[\phi(\vartheta_1), \vartheta_3] - [\sigma(\vartheta_2), \vartheta_1]\vartheta_3 - \sigma(\vartheta_2)[\vartheta_3, \vartheta_1] \\ -[\vartheta_2, \vartheta_1]\sigma(\vartheta_3) - \vartheta_2[\sigma(\vartheta_3), \vartheta_1] \in \Pi\end{array},$$

and so

$$\begin{array}{l}(\phi([\vartheta_1, \vartheta_2]) - \phi(\vartheta_1) \circ \vartheta_2 - [\sigma(\vartheta_2), \vartheta_1])\vartheta_3 \\ +[\vartheta_1, \vartheta_2]\sigma(\vartheta_3) + \phi(\vartheta_2)[\vartheta_1, \vartheta_3] + \vartheta_2\sigma([\vartheta_1, \vartheta_3]) \\ +[\vartheta_1, \vartheta_2]\sigma(\vartheta_3) + \vartheta_2[\phi(\vartheta_1), \vartheta_3] - \sigma(\vartheta_2)[\vartheta_3, \vartheta_1] - \vartheta_2[\sigma(\vartheta_3), \vartheta_1] \in \Pi.\end{array}$$

By the hypothesis, we have

$$\begin{array}{l}[\vartheta_1, \vartheta_2]\sigma(\vartheta_3) + \phi(\vartheta_2)[\vartheta_1, \vartheta_3] + \vartheta_2\sigma([\vartheta_1, \vartheta_3]) + [\vartheta_1, \vartheta_2]\sigma(\vartheta_3) \\ +\vartheta_2[\phi(\vartheta_1), \vartheta_3] - \sigma(\vartheta_2)[\vartheta_3, \vartheta_1] - \vartheta_2[\sigma(\vartheta_3), \vartheta_1] \in \Pi\end{array}.$$

Replacing ϑ_3 by ϑ_1 in this expression, we have

$$2[\vartheta_1, \vartheta_2]\sigma(\vartheta_1) + \vartheta_2[\phi(\vartheta_1), \vartheta_1] - \vartheta_2[\sigma(\vartheta_1), \vartheta_1] \in \Pi. \quad (20)$$

Writing ϑ_2 by $\vartheta_2\vartheta_3$ in (20), we have

$$2[\vartheta_1, \vartheta_2]\vartheta_3\sigma(\vartheta_1) + 2\vartheta_2[\vartheta_1, \vartheta_3]\sigma(\vartheta_1) + \vartheta_2\vartheta_3[\phi(\vartheta_1), \vartheta_1] - \vartheta_2\vartheta_3[\sigma(\vartheta_1), \vartheta_1] \in \Pi.$$

Using expression (20), we obtain that

$$[\vartheta_1, \vartheta_2]\vartheta_3\sigma(\vartheta_1) \in \Pi. \quad (21)$$

Replacing ϑ_2 by $\sigma(\vartheta_1)$ in (21) this expression, we have

$$[\vartheta_1, \sigma(\vartheta_1)]\vartheta_3\sigma(\vartheta_1) \in \Pi. \quad (22)$$

Writing ϑ_2 by $\vartheta_1\sigma(\vartheta_1)$ in , we get

$$[\vartheta_1, \sigma(\vartheta_1)]\vartheta_3\vartheta_1\sigma(\vartheta_1) \in \Pi. \quad (23)$$

Left multiplying (22) by ϑ_1, we get

$$[\vartheta_1, \sigma(\vartheta_1)]\vartheta_3\sigma(\vartheta_1)\vartheta_1 \in \Pi. \quad (24)$$

Subtracting (23) from (24), we get

$$[\vartheta_1, \sigma(\vartheta_1)]\vartheta_3[\vartheta_1, \sigma(\vartheta_1)] \in \Pi.$$

Since Π is a semiprime ideal of $\mathcal{F}$, we conclude that

$$[\vartheta_1, \sigma(\vartheta_1)] \in \Pi \text{ for all } \vartheta_1 \in \mathcal{F}$$

and so σ is Π-commuting map on $\mathcal{F}$. □

Theorem 3. *Let $\mathcal{F}$ be a ring with Π a semiprime ideal of R. Suppose that $\mathcal{F}$ admits multiplicative generalized derivations ϕ, G associated with the multiplicative derivation σ, and any nonzero map h, respectively. If any of the following conditions is satisfied for all $\vartheta_1, \vartheta_2 \in \mathcal{F}$*

(i) $G(\vartheta_1\vartheta_2) + \sigma(\vartheta_1)\phi(\vartheta_2) \pm \vartheta_1\vartheta_2 \in \Pi$,
(ii) $G(\vartheta_1\vartheta_2) + \sigma(\vartheta_1)\phi(\vartheta_2) \pm \vartheta_2\vartheta_1 \in \Pi$,
(iii) $G(\vartheta_1\vartheta_2) + \sigma(\vartheta_1)\phi(\vartheta_2) \pm \vartheta_1 \circ \vartheta_2 \in \Pi$,
(iv) $G(\vartheta_1\vartheta_2) + \sigma(\vartheta_1)\phi(\vartheta_2) \pm [\vartheta_1, \vartheta_2] \in \Pi$,

then σ is Π-commuting map on $\mathcal{F}$.

Proof. (i) By the hypothesis, we have

$$G(\vartheta_1\vartheta_2) + \sigma(\vartheta_1)\phi(\vartheta_2) \pm \vartheta_1\vartheta_2 \in \Pi.$$

Replacing ϑ_2 by $\vartheta_2\vartheta_3$, $\vartheta_3 \in \mathcal{F}$ in the above expression, we have

$$G(\vartheta_1\vartheta_2)\vartheta_3 + \vartheta_1\vartheta_2 h(\vartheta_3) + \sigma(\vartheta_1)\phi(\vartheta_2)\vartheta_3 + \sigma(\vartheta_1)\vartheta_2\sigma(\vartheta_3) \pm \vartheta_1\vartheta_2\vartheta_3 \in \Pi.$$

Using the hypothesis, we find that

$$\vartheta_1\vartheta_2 h(\vartheta_3) + \sigma(\vartheta_1)\vartheta_2\sigma(\vartheta_3) \in \Pi. \quad (25)$$

Taking ϑ_1 by $\vartheta_3 t$, $t \in \mathcal{F}$ in (25), we get

$$\vartheta_3 t\vartheta_2 h(\vartheta_3) + \sigma(\vartheta_3)t\vartheta_2\sigma(\vartheta_3) + \vartheta_3\sigma(t)\vartheta_2\sigma(\vartheta_3) \in \Pi.$$

Using (25), we have

$$\sigma(\vartheta_3)t\vartheta_2\sigma(\vartheta_3) \in \Pi.$$

Multiplying the last expression on the right by t, we have

$$\sigma(\vartheta_3)t\vartheta_2\sigma(\vartheta_3)t \in \Pi.$$

That is,

$$\sigma(\vartheta_3)t\mathcal{F}\sigma(\vartheta_3)t \subseteq \Pi.$$

Since Π is semiprime ideal, we get

$$\sigma(\vartheta_3)t \in \Pi \text{ for all } \vartheta_3, t \in \mathcal{F}.$$

Multiplying the last expression on the right by $\sigma(\vartheta_3)$, we have

$$\sigma(\vartheta_3)t\sigma(\vartheta_3) \in \Pi \text{ for all } \vartheta_3, t \in \mathcal{F}.$$

Since Π is semiprime ideal, we get $\sigma(\vartheta_3) \in \Pi$ for all $\vartheta_3 \in \mathcal{F}$. That is, $[\vartheta_3, \sigma(\vartheta_3)] \in \Pi$ for all $\vartheta_3 \in \mathcal{F}$. Hence, σ is Π-commuting on $\mathcal{F}$.

(ii) By the hypothesis, we have

$$G(\vartheta_1\vartheta_2) + \sigma(\vartheta_1)\phi(\vartheta_2) + \vartheta_2\vartheta_1 \in \Pi.$$

Replacing ϑ_2 by $\vartheta_2\vartheta_3$ in the hypothesis, we obtain

$$G(\vartheta_1\vartheta_2)\vartheta_3 + \vartheta_1\vartheta_2 h(\vartheta_3) + \sigma(\vartheta_1)\phi(\vartheta_2)\vartheta_3 + \sigma(\vartheta_1)\vartheta_2\sigma(\vartheta_3) + \vartheta_2\vartheta_3\vartheta_1 \in \Pi.$$

Using hypothesis, we have

$$\vartheta_1\vartheta_2 h(\vartheta_3) + \sigma(\vartheta_1)\vartheta_2\sigma(\vartheta_3) + \vartheta_2\vartheta_3\vartheta_1 - \vartheta_2\vartheta_1\vartheta_3 \in \Pi,$$

and so

$$\vartheta_1\vartheta_2 h(\vartheta_3) + \sigma(\vartheta_1)\vartheta_2\sigma(\vartheta_3) + \vartheta_2[\vartheta_3, \vartheta_1] \in \Pi. \quad (26)$$

Taking ϑ_1 by $\vartheta_1\vartheta_3$ in (26), we have

$$\vartheta_1\vartheta_3\vartheta_2 h(\vartheta_3) + \sigma(\vartheta_1)\vartheta_3\vartheta_2\sigma(\vartheta_3) + \vartheta_1\sigma(\vartheta_3)\vartheta_2\sigma(\vartheta_3) + \vartheta_2[\vartheta_3, \vartheta_1]\vartheta_3 \in \Pi. \quad (27)$$

Replacing ϑ_2 by $\vartheta_3\vartheta_2$ in (26), we get

$$\vartheta_1\vartheta_3\vartheta_2 h(\vartheta_3) + \sigma(\vartheta_1)\vartheta_3\vartheta_2\sigma(\vartheta_3) + \vartheta_3\vartheta_2[\vartheta_3, \vartheta_1] \in \Pi.$$

Subtracting the above expression from (27), we find

$$\vartheta_1\sigma(\vartheta_3)\vartheta_2\sigma(\vartheta_3) + \vartheta_2[\vartheta_3, \vartheta_1]\vartheta_3 - \vartheta_3\vartheta_2[\vartheta_3, \vartheta_1] \in \Pi.$$

That is

$$\vartheta_1\sigma(\vartheta_3)\vartheta_2\sigma(\vartheta_3) + [\vartheta_2[\vartheta_3, \vartheta_1], \vartheta_3] \in \Pi. \quad (28)$$

Replacing ϑ_3 by ϑ_1 in this expression, we get

$$\vartheta_1\sigma(\vartheta_1)\vartheta_2\sigma(\vartheta_1) \in \Pi.$$

Taking ϑ_2 by $\vartheta_2\vartheta_1$ in the last expression, we have

$$\vartheta_1\sigma(\vartheta_1)\vartheta_2\vartheta_1\sigma(\vartheta_1) \in \Pi.$$

Since Π is semiprime ideal, we get

$$\vartheta_1\sigma(\vartheta_1) \in \Pi \text{ for all } \vartheta_1 \in \mathcal{F}. \quad (29)$$

On the other hand, replacing ϑ_1 by $\vartheta_1\vartheta_3$ in (28), we get

$$\vartheta_1\vartheta_3\sigma(\vartheta_3)\vartheta_2\sigma(\vartheta_3) + [\vartheta_2[\vartheta_3, \vartheta_1], \vartheta_3]\vartheta_3 \in \Pi. \quad (30)$$

Right multiplying by ϑ_3 in (28), we have

$$\vartheta_1\sigma(\vartheta_3)\vartheta_2\sigma(\vartheta_3)\vartheta_3 + [\vartheta_2[\vartheta_3, \vartheta_1], \vartheta_3]\vartheta_3 \in \Pi.$$

Subtracting the above expression from (30), we find

$$\vartheta_1\vartheta_3\sigma(\vartheta_3)\vartheta_2\sigma(\vartheta_3) - \vartheta_1\sigma(\vartheta_3)\vartheta_2\sigma(\vartheta_3)\vartheta_3 \in \Pi.$$

That is,

$$\vartheta_1[\sigma(\vartheta_3)\vartheta_2\sigma(\vartheta_3), \vartheta_3] \in \Pi.$$

Replacing ϑ_1 by $[\sigma(\vartheta_3)\vartheta_2\sigma(\vartheta_3), \vartheta_3]$ in this expression, we get

$$[\sigma(\vartheta_3)\vartheta_2\sigma(\vartheta_3), \vartheta_3]\vartheta_1[\sigma(\vartheta_3)\vartheta_2\sigma(\vartheta_3), \vartheta_3] \in \Pi \text{ for all } \vartheta_1, \vartheta_2, \vartheta_3 \in \mathcal{F}.$$

Since Π is semiprime ideal, we have

$$[\sigma(\vartheta_3)\vartheta_2\sigma(\vartheta_3), \vartheta_3] \in \Pi \text{ for all } \vartheta_3, \vartheta_2 \in \mathcal{F}.$$

and so

$$\sigma(\vartheta_3)\vartheta_2\sigma(\vartheta_3)\vartheta_3 - \vartheta_3\sigma(\vartheta_3)\vartheta_2\sigma(\vartheta_3) \in \Pi.$$

Using $\vartheta_1\sigma(\vartheta_1) \in \Pi$ for all $\vartheta_1 \in \mathcal{F}$, we get

$$\sigma(\vartheta_3)\vartheta_2\sigma(\vartheta_3)\vartheta_3 \in \Pi.$$

Replacing ϑ_2 by $\vartheta_3\sigma(\vartheta_3)$ in the last expression, we obtain

$$\sigma(\vartheta_3)\vartheta_3\vartheta_2\sigma(\vartheta_3)\vartheta_3 \in \Pi.$$

Since Π is semiprime ideal, we have

$$\sigma(\vartheta_3)\vartheta_3 \in \Pi \text{ for all } \vartheta_3 \in \mathcal{F}. \quad (31)$$

Subtracting (29) from (31), we arrive at $[\vartheta_3, \sigma(\vartheta_3)] \in \Pi$ for all $\vartheta_3 \in \mathcal{F}$. Hence, σ is Π-commuting. This completes the proof.

It is proved analogously using $G(\vartheta_1\vartheta_2) + \sigma(\vartheta_1)\phi(\vartheta_2) - \vartheta_2\vartheta_1 \in \Pi$ for all $\vartheta_1, \vartheta_2 \in \mathcal{F}$.

(iii) By the hypothesis, we have

$$G(\vartheta_1\vartheta_2) + \sigma(\vartheta_1)\phi(\vartheta_2) \pm \vartheta_1 \circ \vartheta_2 \in \Pi.$$

Replacing ϑ_2 by $\vartheta_2\vartheta_3, \vartheta_3 \in \mathcal{F}$ in the above expression, we have

$$G(\vartheta_1\vartheta_2)\vartheta_3 + \vartheta_1\vartheta_2 h(\vartheta_3) + \sigma(\vartheta_1)\phi(\vartheta_2)\vartheta_3 + \sigma(\vartheta_1)\vartheta_2\sigma(\vartheta_3) \pm (\vartheta_1 o \vartheta_2)\vartheta_3 \mp \vartheta_2[\vartheta_1, \vartheta_3] \in \Pi.$$

Using the hypothesis, we have

$$\vartheta_1\vartheta_2 h(\vartheta_3) + \sigma(\vartheta_1)\vartheta_2\sigma(\vartheta_3) \mp \vartheta_2[\vartheta_1, \vartheta_3] \in \Pi. \quad (32)$$

Taking ϑ_1 by $\vartheta_3 t$ in (32), we get

$$\vartheta_3 t\vartheta_2 h(\vartheta_3) + \sigma(\vartheta_3)t\vartheta_2\sigma(\vartheta_3) + \vartheta_3\sigma(t)\vartheta_2\sigma(\vartheta_3) \mp \vartheta_2\vartheta_3[t, \vartheta_3] \pm \vartheta_3\vartheta_2[t, \vartheta_3] \mp \vartheta_3\vartheta_2[t, \vartheta_3] \in \Pi.$$

Using (32), we have

$$\sigma(\vartheta_3)t\vartheta_2\sigma(\vartheta_3) \mp \vartheta_2\vartheta_3[t, \vartheta_3] \pm \vartheta_3\vartheta_2[t, \vartheta_3] \in \Pi.$$

Replacing t by ϑ_3 in this expression, we get

$$\sigma(\vartheta_3)\vartheta_3\vartheta_2\sigma(\vartheta_3) \in \Pi.$$

Right multiplying by ϑ_3 in this expression, we have

$$\sigma(\vartheta_3)\vartheta_3\vartheta_2\sigma(\vartheta_3)\vartheta_3 \in \Pi \text{ for all } \vartheta_3 \in \mathcal{F}.$$

Since Π is semiprime ideal, we have $\sigma(\vartheta_3)\vartheta_3 \in \Pi$ for all $\vartheta_3 \in \mathcal{F}$.

On the other hand, taking ϑ_1 by $\vartheta_1\vartheta_3$ in (32), we have

$$\vartheta_1\vartheta_3\vartheta_2 h(\vartheta_3) + \sigma(\vartheta_1)\vartheta_3\vartheta_2\sigma(\vartheta_3) + \vartheta_1\sigma(\vartheta_3)\vartheta_2\sigma(\vartheta_3) \mp \vartheta_2[\vartheta_1, \vartheta_3]\vartheta_3 \in \Pi. \quad (33)$$

Replacing ϑ_3 by $\vartheta_3\vartheta_2$ in (32), we have

$$\vartheta_1\vartheta_3\vartheta_2 h(\vartheta_3) + \sigma(\vartheta_1)\vartheta_3\vartheta_2\sigma(\vartheta_3) \mp \vartheta_3\vartheta_2[\vartheta_1, \vartheta_3] \in \Pi. \quad (34)$$

Subtracting (33) from (34), we arrive at

$$\vartheta_1\sigma(\vartheta_3)\vartheta_2\sigma(\vartheta_3) \mp \vartheta_2[\vartheta_1, \vartheta_3]\vartheta_3 \mp \vartheta_3\vartheta_2[\vartheta_1, \vartheta_3] \in \Pi.$$

Writing ϑ_1 by ϑ_3 in the last expression, we have

$$\vartheta_3\sigma(\vartheta_3)\vartheta_2\sigma(\vartheta_3) \in \Pi.$$

Replacing ϑ_2 by $\vartheta_2\vartheta_3$ in the above expression, we have

$$\vartheta_3\sigma(\vartheta_3)\vartheta_2\vartheta_3\sigma(\vartheta_3) \in \Pi \text{ for all } \vartheta_3 \in \mathcal{F}.$$

Since Π is semiprime ideal, we have $\vartheta_3\sigma(\vartheta_3) \in \Pi$ for all $\vartheta_3 \in \mathcal{F}$. Hence, we conclude that $[\vartheta_3, \sigma(\vartheta_3)] \in \Pi$ for all $\vartheta_3 \in \mathcal{F}$, and so σ is Π-commuting.
(iv) By the hypothesis, we have

$$G(\vartheta_1\vartheta_2) + \sigma(\vartheta_1)\phi(\vartheta_2) \pm [\vartheta_1, \vartheta_2] \in \Pi.$$

Replacing ϑ_2 by $\vartheta_2\vartheta_3, \vartheta_3 \in \mathcal{F}$ in the above expression, we have

$$G(\vartheta_1\vartheta_2)\vartheta_3 + \vartheta_1\vartheta_2 h(\vartheta_3) + \sigma(\vartheta_1)\phi(\vartheta_2)\vartheta_3 + \sigma(\vartheta_1)\vartheta_2\sigma(\vartheta_3) \pm [\vartheta_1, \vartheta_2]\vartheta_3 \pm \vartheta_2[\vartheta_1, \vartheta_3] \in \Pi.$$

Using the hypothesis, we have

$$\vartheta_1\vartheta_2 h(\vartheta_3) + \sigma(\vartheta_1)\vartheta_2\sigma(\vartheta_3) \pm \vartheta_2[\vartheta_1, \vartheta_3] \in \Pi.$$

This expression is the same as (32) in (iii). Applying the same lines, we find that σ is Π-commuting. This completes the proof. □

Definition 1. *An additive mapping $\phi : \mathcal{F} \to \mathcal{F}$ is called a multiplicative right generalized derivation if there exists a map $\sigma : \mathcal{F} \to \mathcal{F}$ such that*

$$\phi(\vartheta_1\vartheta_2) = \phi(\vartheta_1)\vartheta_2 + \vartheta_1\sigma(\vartheta_2) \textit{ for all } \vartheta_1, \vartheta_2 \in \mathcal{F}$$

and ϕ is called a multiplicative left generalized derivation if there exists a map $\sigma : \mathcal{F} \to \mathcal{F}$ such that

$$\phi(\vartheta_1\vartheta_2) = \sigma(\vartheta_1)\vartheta_2 + \vartheta_1\phi(\vartheta_2) \textit{ for all } \vartheta_1, \vartheta_2 \in \mathcal{F}.$$

ϕ is said to be a multiplicative generalized derivation with associated map σ if it is both a multiplicative left and right generalized derivation with associated derivation σ.

Theorem 4. *Let $\mathcal{F}$ be a ring with Π a prime ideal of R. Suppose that $\mathcal{F}$ admits a multiplicative left generalized derivation ϕ associated with a nonzero map σ. If any of the following conditions is satisfied for all $\vartheta_1, \vartheta_2 \in \mathcal{F}$*

(i) $[\vartheta_1, \vartheta_2] - [\phi(\vartheta_1), \phi(\vartheta_2)] \in \Pi$,
(ii) $\vartheta_1 \circ \vartheta_2 - \phi(\vartheta_1) \circ \phi(\vartheta_2) \in \Pi$,
(iii) $[\vartheta_1, \vartheta_2] - \phi(\vartheta_1) \circ \phi(\vartheta_2) \in \Pi$,
(iv) $\vartheta_1 \circ \vartheta_2 - [\phi(\vartheta_1), \phi(\vartheta_2)] \in \Pi$,

then ϕ is Π-commuting map on $\mathcal{F}$.

Proof. (i) By the hypothesis, we get

$$[\vartheta_1, \vartheta_2] - [\phi(\vartheta_1), \phi(\vartheta_2)] \in \Pi \text{ for all } \vartheta_1, \vartheta_2 \in \mathcal{F}.$$

Replacing ϑ_2 by $\vartheta_1\vartheta_2$ in this expression, we obtain

$$\vartheta_1[\vartheta_1,\vartheta_2]-[\phi(\vartheta_1),\sigma(\vartheta_1)\vartheta_2+\vartheta_1\phi(\vartheta_2)]\in\Pi.$$

and so,

$$\vartheta_1[\vartheta_1,\vartheta_2]-[\phi(\vartheta_1),\sigma(\vartheta_1)]\vartheta_2-\sigma(\vartheta_1)[\phi(\vartheta_1),\vartheta_2]-[\phi(\vartheta_1),\vartheta_1]\phi(\vartheta_2)-\vartheta_1[\phi(\vartheta_1),\phi(\vartheta_2)]\in\Pi.$$

Using the hypothesis, we get

$$[\phi(\vartheta_1),\sigma(\vartheta_1)]\vartheta_2+\sigma(\vartheta_1)[\phi(\vartheta_1),\vartheta_2]+[\phi(\vartheta_1),\vartheta_1]\phi(\vartheta_2)\in\Pi.$$

Taking ϑ_2 by $\vartheta_2\vartheta_3$, $\vartheta_3\in\mathcal{F}$ in the above expression and using this expression, we have

$$\sigma(\vartheta_1)\vartheta_2[\phi(\vartheta_1),\vartheta_3]+[\phi(\vartheta_1),\vartheta_1]\vartheta_2\sigma(\vartheta_3)\in\Pi. \tag{35}$$

Replacing ϑ_3 by $\phi(\vartheta_1)$ in (35), we have

$$[\phi(\vartheta_1),\vartheta_1]\vartheta_2\sigma(\phi(\vartheta_1))\in\Pi \text{ for all } \vartheta_1,\vartheta_2\in\mathcal{F}.$$

Since Π is prime ideal, we get

$$[\phi(\vartheta_1),\vartheta_1]\in\Pi \text{ or } \sigma(\phi(\vartheta_1))\in\Pi.$$

Assume that there exists $\vartheta_1\in\mathcal{F}$ such that $[\phi(\vartheta_1),\vartheta_1]\notin\Pi$. Then $\sigma(\phi(\vartheta_1))\in\Pi$. By the hypothesis, we have for all $\vartheta_2\in\mathcal{F}$,

$$\begin{aligned}&[\vartheta_1,\vartheta_2\phi(\vartheta_1)]-[\phi(\vartheta_1),\phi(\vartheta_2\phi(\vartheta_1))]\\&=[\vartheta_1,\vartheta_2]\phi(\vartheta_1)+\vartheta_2[\vartheta_1,\phi(\vartheta_1)]-[\phi(\vartheta_1),\phi(\vartheta_2)\phi(\vartheta_1)+\vartheta_2\sigma(\phi(\vartheta_1))]\\&=[\vartheta_1,\vartheta_2]\phi(\vartheta_1)+\vartheta_2[\vartheta_1,\phi(\vartheta_1)]-[\phi(\vartheta_1),\phi(\vartheta_2)]\phi(\vartheta_1)+[\phi(\vartheta_1),\vartheta_2\sigma(\phi(\vartheta_1))]\\&=[\vartheta_1,\vartheta_2]\phi(\vartheta_1)+\vartheta_2[\vartheta_1,\phi(\vartheta_1)]-([\vartheta_1,\vartheta_2]+\Pi)\phi(\vartheta_1)+[\phi(\vartheta_1),\vartheta_2\sigma(\phi(\vartheta_1))]\in\Pi.\end{aligned}$$

That is

$$\vartheta_2[\vartheta_1,\phi(\vartheta_1)]-[\phi(\vartheta_1),\vartheta_2\sigma(\phi(\vartheta_1))]\in\Pi.$$

Using $\sigma(\phi(\vartheta_1))\in\Pi$, we have $\vartheta_2[\vartheta_1,\phi(\vartheta_1)]\in\Pi$. Since $\mathcal{F}$ is prime, we obtain that $[\vartheta_1,\phi(\vartheta_1)]\in\Pi$, which is a contradiction. In both cases, $[\vartheta_1,\phi(\vartheta_1)]\in\Pi$ for all $\vartheta_1\in\mathcal{F}$ is obtained.

(ii) Assume that

$$\vartheta_1\circ\vartheta_2-\phi(\vartheta_1)\circ\phi(\vartheta_2)\in\Pi \text{ for all } \vartheta_1,\vartheta_2\in\mathcal{F}.$$

Replacing ϑ_2 by $\vartheta_1\vartheta_2$ in the above expression, we get

$$\vartheta_1(\vartheta_1\circ\vartheta_2)-\phi(\vartheta_1)\circ(\sigma(\vartheta_1)\vartheta_2+\vartheta_1\phi(\vartheta_2))\in\Pi$$

and so,

$$\begin{aligned}&\vartheta_1(\vartheta_1\circ\vartheta_2)-(\phi(\vartheta_1)\circ\sigma(\vartheta_1))\vartheta_2-\sigma(\vartheta_1)[\vartheta_2,\phi(\vartheta_1)]\\&\quad-\vartheta_1(\phi(\vartheta_1)\circ\phi(\vartheta_2))-[\phi(\vartheta_1),\vartheta_1]\phi(\vartheta_2)\in\Pi.\end{aligned}$$

Using the hypothesis, we get

$$(\phi(\vartheta_1)\circ\sigma(\vartheta_1))\vartheta_2+\sigma(\vartheta_1)[\vartheta_2,\phi(\vartheta_1)]+[\phi(\vartheta_1),\vartheta_1]\phi(\vartheta_2)\in\Pi.$$

Taking ϑ_2 by $\vartheta_2\vartheta_3$, $\vartheta_3\in\mathcal{F}$ in the above expression and this expression, we have

$$\begin{aligned}&(\phi(\vartheta_1)\circ\sigma(\vartheta_1))\vartheta_2\vartheta_3+\sigma(\vartheta_1)[\vartheta_2,\phi(\vartheta_1)]\vartheta_3+\sigma(\vartheta_1)\vartheta_2[\mathcal{F},\phi(\vartheta_1)]\\&\quad+[\phi(\vartheta_1),\vartheta_1]\phi(\vartheta_2)\vartheta_3+[\phi(\vartheta_1),\vartheta_1]\vartheta_2\sigma(\vartheta_3)\in\Pi,\end{aligned}$$

and so

$$\sigma(\vartheta_1)\vartheta_2[\vartheta_3, \phi(\vartheta_1)] + [\phi(\vartheta_1), \vartheta_1]\vartheta_2\sigma(\vartheta_3) \in \Pi \text{ for all } \vartheta_1, \vartheta_2, \vartheta_3 \in \mathcal{F}.$$

This expression is the same as (35) in the proof of (i). Using the same arguments there, we get the required result.

(iii) By the hypothesis, we have

$$[\vartheta_1, \vartheta_2] - \phi(\vartheta_1) \circ \phi(\vartheta_2) \in \Pi \text{ for all } \vartheta_1, \vartheta_2 \in \mathcal{F}.$$

Replacing ϑ_2 by $\vartheta_1\vartheta_2$ in the above expression, we get

$$\vartheta_1[\vartheta_1, \vartheta_2] - \phi(\vartheta_1) \circ (\sigma(\vartheta_1)\vartheta_2 + \vartheta_1\phi(\vartheta_2)) \in \Pi$$

and so,

$$\begin{aligned} &\vartheta_1[\vartheta_1, \vartheta_2] - (\phi(\vartheta_1) \circ \sigma(\vartheta_1))\vartheta_2 - \sigma(\vartheta_1)[\vartheta_2, \phi(\vartheta_1)] \\ &\quad -\vartheta_1(\phi(\vartheta_1) \circ \phi(\vartheta_2)) - [\phi(\vartheta_1), \vartheta_1]\phi(\vartheta_2) \in \Pi. \end{aligned}$$

Using the hypothesis, we get

$$(\phi(\vartheta_1) \circ \sigma(\vartheta_1))\vartheta_2 + \sigma(\vartheta_1)[\vartheta_2, \phi(\vartheta_1)] + [\phi(\vartheta_1), \vartheta_1]\phi(\vartheta_2) \in \Pi.$$

Taking ϑ_2 by $\vartheta_2\vartheta_2$, $\vartheta_3 \in \mathcal{F}$ in the above expression and this expression, we have

$$\begin{aligned} &(\phi(\vartheta_1) \circ \sigma(\vartheta_1))\vartheta_2\vartheta_3 + \sigma(\vartheta_1)[\vartheta_2, \phi(\vartheta_1)]\vartheta_3 + \sigma(\vartheta_1)\vartheta_2[\vartheta_3, \phi(\vartheta_1)] \\ &\quad +[\phi(\vartheta_1), \vartheta_1]\phi(\vartheta_2)\vartheta_3 + [\phi(\vartheta_1), \vartheta_1]\vartheta_2\sigma(\vartheta_3) \in \Pi, \end{aligned}$$

and so

$$\sigma(\vartheta_1)\vartheta_2[\vartheta_3, \phi(\vartheta_1)] + [\phi(\vartheta_1), \vartheta_1]\vartheta_2\sigma(\vartheta_3) \in \Pi \text{ for all } \vartheta_1, \vartheta_2, \vartheta_3 \in \mathcal{F}.$$

This expression is the same as (35) in the proof of (i). By the same techniques, we get the required result.

(iv) By the hypothesis, we get

$$(\vartheta_1 \circ \vartheta_2) - [\phi(\vartheta_1), \phi(\vartheta_2)] \in \Pi \text{ for all } \vartheta_1, \vartheta_2 \in \mathcal{F}.$$

Replacing ϑ_2 by $\vartheta_1\vartheta_2$ in this expression, we obtain

$$\vartheta_1(\vartheta_1 \circ \vartheta_2) - [\phi(\vartheta_1), \sigma(\vartheta_1)\vartheta_2 + \vartheta_1\phi(\vartheta_2)] \in \Pi.$$

and so,

$$\vartheta_1(\vartheta_1 \circ \vartheta_2) - [\phi(\vartheta_1), \sigma(\vartheta_1)]\vartheta_2 - \sigma(\vartheta_1)[\phi(\vartheta_1), \vartheta_2] - [\phi(\vartheta_1), \vartheta_1]\phi(\vartheta_2) - \vartheta_1[\phi(\vartheta_1), \phi(\vartheta_2)] \in \Pi.$$

Using the hypothesis, we get

$$[\phi(\vartheta_1), \sigma(\vartheta_1)]\vartheta_2 + \sigma(\vartheta_1)[\phi(\vartheta_1), \vartheta_2] + [\phi(\vartheta_1), \vartheta_1]\phi(\vartheta_2) \in \Pi.$$

Taking ϑ_2 by $\vartheta_2\vartheta_3$, $\vartheta_3 \in \mathcal{F}$ in the above expression and using this expression, we have

$$\sigma(\vartheta_1)\vartheta_2[\phi(\vartheta_1), \vartheta_3] + [\phi(\vartheta_1), \vartheta_1]\vartheta_2\sigma(\vartheta_3) \in \Pi.$$

This expression is the same as (35) in the proof of (i). Using the same arguments in there, we obtained the required result. □

3. Conclusions

In this paper, we explored the structure and commutativity of semiprime rings under the action of multiplicative generalized derivations. Our investigation extends previous results in the literature by establishing new conditions under which a semiprime ring becomes commutative when admitting a multiplicative generalized derivation. These findings contribute to a deeper understanding of the interaction between multiplicative generalized derivations and the structural properties of rings. Moreover, our work broadens existing commutativity theorems and opens new avenues for further research in ring theory, particularly regarding the broader class of multiplicative generalized derivations and their impact on algebraic structures. These results lay the groundwork for future studies, with potential applications in operator algebras, noncommutative geometry, and other mathematical fields where ring structures play a central role.

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