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An Analysis of Type-I Generalized Progressive Hybrid Censoring for the One Parameter Logistic-Geometry Lifetime Distribution with Applications

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Abstract: Based on Type-I generalized progressive hybrid censored samples (GPHCSs), the parameter estimate for the unit-half logistic-geometry (UHLG) distribution is investigated in this work. Using maximum likelihood estimation (MLE) and Bayesian estimation, the parameters, reliability, and hazard functions of the UHLG distribution under GPHCSs have been assessed. Likewise, the computation is carried out for the asymptotic confidence intervals (ACIs). Furthermore, two bootstrap CIs, bootstrap-p and bootstrap-t, are mentioned. For symmetric loss functions, like squared error loss (SEL), and asymmetric loss functions, such as linear exponential loss (LL) and general entropy loss (GEL), there are specific Bayesian approximations. The Metropolis–Hastings samplers methodology were used to construct the credible intervals (CRIs). In conclusion, a genuine data set measuring the mortality statistics of a group of male mice with reticulum cell sarcoma is regarded as an application of the methods given.

Keywords: type-I generalized progressive hybrid censoring; unit-half logistic-geometry distribution; maximum likelihood estimation; Bayesian estimation; Markov chain Monte Carlo; simulation study

MSC: Primary 62G30; Secondary 62F15

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1. Introduction

Due to budgetary or scheduling restrictions, participants in life and reliability tests have the option to end the experiment before all units fail. The term "censored samples" applies in this instance to these units. Censored samples come in several forms. These kinds are mostly determined by cost and timing factors. They are known as time-factor-dependent Type-I censored samples. Type-II censored samples are those that are restricted based on quantity or cost. The duration and cost of the experiment play a significant role in hybrid censored samples. The duration and cost of the experiment play a significant role in hybrid censored samples.

A hybrid system was created out of the censoring systems Type-I and Type-II. Moreover, the Type-I and Type-II hybrid censoring algorithms were provided by Epstein et al. and Childs et al. in [1,2], appropriately. These censored systems, albeit widely utilized, only allow the units to be removed from the test at the predetermined endpoint. Balakrishnan and Aggarwala [3] proposed a Type-II progressive censoring to address the issues, but it still needed a lengthy test time. Recently, Cho et al. [4] suggested a generalized Type-I progressive hybrid censored scheme (GPHCS), and Lee et al. [5] introduced Type-II GPHCS. Bayesian estimation has been examined by Nagy et al. in [6,7], and Nagy and Alrasheedi in [8,9], based on GPHCS from different distributions.

The squared error loss is without a doubt the most-often-used loss function in Bayesian inference. Symmetric loss functions may not be appropriate in many situations, especially when the implications of positive and negative errors differ. One of the most widely utilized asymmetric loss functions is the linear exponential loss function (LL). It was initially observed by Varian [10]. With Type-I hybrid censored data, Nagy et al. [11] computed the Bayesian and E-Bayesian estimates of parameters of an exponential model based on simple step stress. The majority of these loss functions were used in this process.

In statistical analysis, data sets are represented by various distributions. In several well-known distribution families, new distributions have emerged recently, such as the new distribution, to be more flexible when modeling real data. In the first step of data modeling, in [12], Ramadan and colleagues created a new continuous distribution known as the unit half logistic geometry distribution, which included mixing several distributions in some way. Model versatility is offered by the UHLG distribution. The probability density function (PDF), cumulative distribution function (CDF), reliability function (RF), and hazard function (HF) are given in that succession for the random variable X with a UHLG distribution.

$$f(z;\beta) = \frac{2\beta}{[\beta + (2-\beta)z]^2}; \qquad 0 < z < 1, \quad 0 < \beta; \tag{1}$$

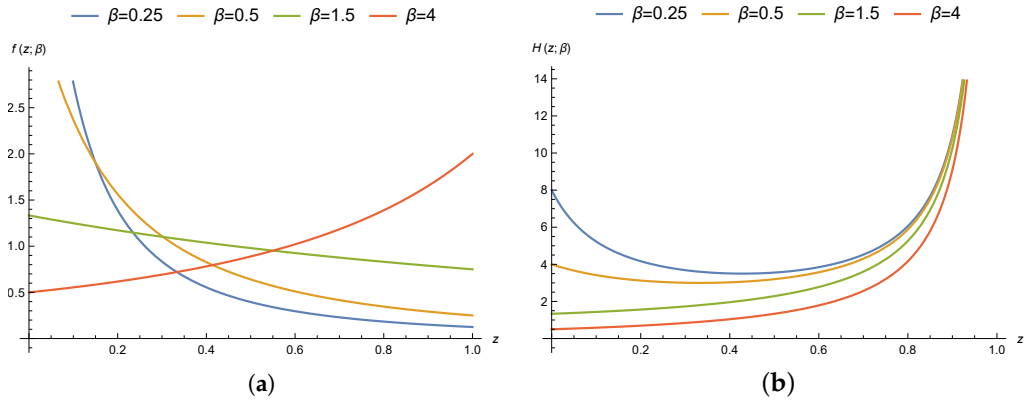
$$F(z;\beta) = 1 - \frac{\beta(1-z)}{\beta + (2-\beta)z}; \tag{2}$$

$$R(z;\beta) = \frac{\beta(1-z)}{\beta + (2-\beta)z}; \tag{3}$$

and

$$H(z;\beta) = \frac{2}{(1-z)[\beta + (2-\beta)z]}. \tag{4}$$

The UHLG distribution is a new distribution that comes with several merits since it has only one parameter, as it can model bounded data between $(0,1)$. It also has a hazard function which has decreasing, increasing, and bathtub shapes. Figure 1 shows the different shapes of the UHLG's PDF and HF.



(**a**) (**b**)

Figure 1. (**a**) The PDF, and (**b**) HF for the UHLGD with different values of β.

The work in this research is motivated by the importance of UHLG, since it can be used to represent bounded data in the interval (0, 1), and it has only one parameter, resulting in statistical inference in a closed form. It is also prompted by the GPHCS's extensive scope, which includes a variety of censoring situations. The novelty of this study is that it is the first time that a comparison has been made between the classical and the Bayesian methods for the estimation and statistical inference of UHLG distribution parameters, using statistical data resulting from GPHCS.

The remainders of this research are arranged as follows: In Section 2, the model of study under the GPHCS is clarified. While in Section 3, we offer the maximum-likelihood estimators with the asymptotic confidence intervals (ACI) for SF, HF, and the unknown UHLG distribution parameter β. The bootstrap confidence intervals for the unknown parameters, SF and HF, are computed in Section 4, utilizing two parametric bootstrapping strategies, p and t. In addition, Section 5 derives Bayes estimates for the previously described parameters and functions for a range of loss functions, including the squared error loss function (SELF), the LINEX (linear exponential) loss function (LLF), and the general entropy loss function (GELF). The actual data set has been examined in Section 6. In Section 7, a simulation study is conducted to assess the performance of the different estimators developed in this paper. Lastly, in Section 8, we wrap up the article.

2. The Model Clarification

Think about testing n equivalent units over the course of a lifetime. With the predefined censoring scheme

$$R = (R_1, R_2, ..., R_m) \quad \text{satisfying} \quad n = m + R_1 + \ldots + R_m.$$

Let $T > 0$ and the prefixed integers $0 < k < m < n$ be. The ending time is

$$T^* = \max\{Z_{k:m:n}, \min\{Z_{m:m:n}, T\}\}$$

The Figure 2 shows GPHCS, where we have three different cases.

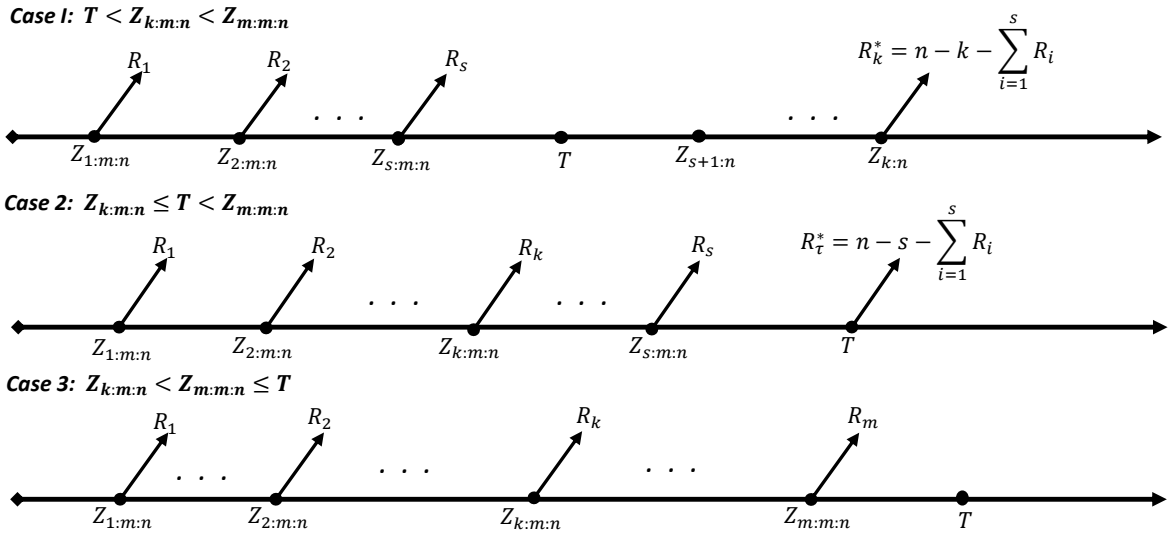


Figure 2. A schematic representation of a Type-I extended GPHCS.

Based on the GPHCS for all cases, s, n^* indicate the number of failures seen up to T and T^*, respectively. This yields the joint PDF as written by Cho et al. [4] and updated by Nagy et al. in [7] and Nagy and Alrasheedi in [9].

$$f_{\underline{\mathbf{Z}}}(\underline{\mathbf{z}}) = C\left[\bar{F}(T)\right]^{R_\tau^*} \prod_{i=1}^{n^*}\left[f(z_{i:n^*:n})[\bar{F}(z_{i:n^*:n})]^{R_i^*}\right], \tag{5}$$

where,

$$C = \prod_{i=1}^{n^*}\left(\sum_{j=i}^{m}\left(R_j^* + 1\right)\right) \quad \text{and} \quad R_j^* \quad \text{is the} \quad jth \quad \text{value of the vector} \quad R^*$$

$$[\underline{\mathbf{z}}, R^*] = \begin{cases} \left[(z_{1:m:n}, R_1), \ldots, (z_{s:m:n}, R_s), (z_{s+1:m:n}, 0), \ldots, (z_{k-1:m:n}, 0), \right. \\ \left. \left(z_{k:m:n}, R_k^* = n - k - \sum_{j=1}^{s} R_j\right)\right], & \text{Case-I,} \\ [(z_{1:m:n}, R_1), \ldots, (z_{s:m:n}, R_s)], & \text{Case-II,} \\ [(z_{1:m:n}, R_1), \ldots, (z_{m:m:n}, R_m)], & \text{Case-III,} \end{cases} \tag{6}$$

with the number of units eliminated at the time T is R_τ^*, as determined by

$$R_\tau^* = \begin{cases} 0, & \text{Case-I,} \\ n - s - \sum_{j=1}^{s} R_j, & \text{Case-II,} \\ 0, & \text{Case-III,} \end{cases} \tag{7}$$

and

$$n^* = \begin{cases} k & \text{Case-I,} \\ s & \text{Case-II,} \\ m & \text{Case-III,} \end{cases} \tag{8}$$

under the GPHCS, β's likelihood function may be obtained by applying (1) and (3) in (5), as

$$\begin{aligned} L(\beta; \underline{\mathbf{z}}) &= C \prod_{i=1}^{n^*} \frac{2\beta}{[\beta + (2-\beta)z_i]^2} \left[\frac{\beta(1-z_i)}{\beta + (2-\beta)z_i}\right]^{R_i^*} \left[\frac{\beta(1-T)}{\beta + (2-\beta)T}\right]^{R_\tau^*} \\ &= C(2\beta)^{n^*} \left[\frac{\beta(1-T)}{\beta + (2-\beta)T}\right]^{R_\tau^*} \prod_{i=1}^{n^*} \frac{1}{[\beta + (2-\beta)z_i]^2} \left[\frac{\beta(1-z_i)}{\beta + (2-\beta)z_i}\right]^{R_i^*}, \end{aligned} \tag{9}$$

where $z_i = z_{i:n^*:n}$ for simplicity of notation.

3. Maximum Likelihood Evaluation

Equation (9) provides the associated log-likelihood function, which is as follows:

$$\begin{aligned} \log L(\beta|\underline{\mathbf{z}}) \propto\ & n^* \log(2\beta) + R_\tau^* \{\log[\beta(1-T)] - \log[\beta + (2-\beta)T]\} - 2\sum_{i=1}^{n^*} \log[\beta + (2-\beta)z_i] \\ & + \sum_{i=1}^{n^*} R_i^* \{\log[\beta(1-z_i)] - \log[\beta + (2-\beta)z_i]\}. \end{aligned} \tag{10}$$

The likelihood equations can be obtained by determining the first and second partial derivatives of (10) with respect to β and equating them to zero.

$$\begin{aligned} \frac{\partial \ln L(\beta|\underline{\mathbf{z}})}{\partial \beta} &= \frac{n^*}{\beta} + R_\tau^* \left\{\frac{1}{\beta} - \frac{1-T}{\beta + (2-\beta)T}\right\} - 2\sum_{i=1}^{n^*} \frac{1-z_i}{\beta + (2-\beta)z_i} \\ &\quad + \sum_{i=1}^{n^*} R_i^* \left\{\frac{1}{\beta} - \frac{1-z_i}{\beta + (2-\beta)z_i}\right\}. \end{aligned} \tag{11}$$

It is useful to produce a range of values that, with a particular degree of confidence, could accommodate the precise parameter as well as a point estimate for the unknown parameter. Interval estimation is the term used in statistical inference to describe this process. We suggest using the asymptotic normality of the MLE to build the ACI of β. Equation (10), which provides the log-likelihood function, gives us

$$\frac{\partial^2 \ln L(\beta|\underline{\mathbf{z}})}{\partial \beta^2} = \frac{-n^*}{\beta^2} + R_\tau^* \left\{ \frac{-1}{\beta^2} - \frac{(1-T)^2}{[\beta+(2-\beta)T]^2} \right\} - 2\sum_{i=1}^{n^*} \frac{(1-z_i)^2}{[\beta+(2-\beta)z_i]^2} + \sum_{i=1}^{n^*} R_i^* \left\{ \frac{(1-z_i)^2}{[\beta+(2-\beta)z_i]^2} - \frac{1}{\beta^2} \right\}, \tag{12}$$

Since there are no closed-form solutions for Equations (11) and (12), the estimates are obtained using the Newton–Raphson iteration technique. For more details and information regarding the phases of the algorithm's operation, we can consult [13–15]. The algorithm is described as follows:

1. After setting $k = 0$, estimate the parameter β using the moments technique or another way to serve as the starting value for iterations; the estimates are then denoted as (β_0).
2. Now, we can determine $\left[\frac{\partial \ln L(\beta|\underline{\mathbf{z}})}{\partial \beta}\right]_{(\beta_k)}$, along with the Fisher information matrix that was seen $I^{-1}(\beta)$.
3. Update β as
$$(\beta_{k+1}) = (\beta_k) + \left[\frac{\partial \ln L(\beta|\underline{\mathbf{z}})}{\partial \beta}\right]_{(\beta_k)} I^{-1}(\beta). \tag{13}$$
4. After setting $k = k + 1$, return to Step 1.
5. Iterate repeatedly until $\beta_{k+1} - \beta_k$ is less than a predetermined threshold. The MLE of the parameters, represented as $\widehat{\beta}_{ML}$, represents the final estimations of β.

Furthermore, by substituting $\widehat{\beta}_{ML}$ with the MLEs of $R(t)$ and $H(t)$, the invariant property of MLEs may be established.

$$\widehat{S}_{ML}(t) = \frac{\widehat{\beta}_{ML}(1-z)}{\widehat{\beta}_{ML} + \left(2-\widehat{\beta}_{ML}\right)z}, \tag{14}$$

and

$$\widehat{H}_{ML}(t) = \frac{2}{(1-z)\left[\widehat{\beta}_{ML} - \left(2-\widehat{\beta}_{ML}\right)z\right]}. \tag{15}$$

Asymptotic Confidence Intervals

In this subsection, for large n^*, the asymptotic confidence intervals for β, RF and HF are obtained depending on the MLE of β. The $100(1-\gamma)\%$ two-sided approximate confidence interval for β is given by

$$\left(\widehat{\beta} - z_{\gamma/2}\sqrt{V\left(\widehat{\beta}\right)}, \widehat{\beta} + z_{\gamma/2}\sqrt{V\left(\widehat{\beta}\right)}\right),$$

where $z_{\gamma/2}$ is the $\gamma/2$th quantile value of the standard normal distribution z and $V\left(\widehat{\beta}\right)$ is the estimated variance of $\widehat{\beta}_{ML}$, which is given by $\left(-\frac{\partial^2 \ln L(\beta|\underline{\mathbf{x}})}{\partial \beta^2}\right)^{-1}_{\widehat{\beta}_{ML}}$.

The $100(1-\gamma)\%$ two-sided approximate confidence interval for $S(t)$ and $H(t)$ are given by

$$\left(\widehat{S}(t) - z_{\gamma/2}\sqrt{V\left(\widehat{S}(t)\right)}, \widehat{S}(t) + z_{\gamma/2}\sqrt{V\left(\widehat{S}(t)\right)}\right),$$

and

$$\left(\widehat{H}(t) - z_{\gamma/2}\sqrt{V\left(\widehat{H}(t)\right)}, \widehat{H}(t) + z_{\gamma/2}\sqrt{V\left(\widehat{H}(t)\right)}\right).$$

where the variances $V\left(\widehat{S}(t)\right)$ and $V\left(\widehat{H}(t)\right)$ are obtained using the delta method; the reader can refer to Greene [16] and Nagy et al. [6].

4. Bootstrap Interval Estimation

A parametric bootstrap interval provides much more information about the population value of the relevant quantity than a point estimate. Furthermore, asymptotic results show poor performance in confidence intervals with small sample sizes. The β, $S(t)$, and $H(t)$ bootstrap confidence intervals are computed using two parametric bootstrap techniques. Initially, the percentile bootstrap-p confidence interval was proposed, according to Efron [17]. The second is the bootstrap-t confidence interval, which he first described in Hall [18].

4.1. Bootstrap-p Interval Estimation

In this subsection, the necessary steps that describe the process of computing the bootstrap confidence interval of type p are mentioned depending on the MLE of β.

1. Based on the original data $\underline{\mathbf{z}} = z_{1:m:n}, \ldots, z_{m:m:n}, T$, obtain by maximizing Equation (10).
2. Based on the generalized progressive censoring scheme R^* generate a type-II progressive censoring sample $\underline{\mathbf{z}} = z_{1:m:n}, \ldots, z_{m:m:n}, T$ from the parameterized UHLG distribution $\widehat{\beta}_{ML}$, applying the method outlined in [19].
3. Obtain the MLEs from the bootstrap sample and use $\widehat{\Psi}_{ML}$ to denote the bootstrap estimate. In this case, Ψ might be β, $S(t)$, and $H(t)$.
4. To obtain the intended results, steps (2) and (3) need be carried out N boot times.

$$\widehat{\Psi}_{1ML}, \widehat{\Psi}_{2ML}, ..., \widehat{\Psi}_{NbootML}$$

5. Arrange $\widehat{\Psi}_{1ML}, \widehat{\Psi}_{2ML}, ..., \widehat{\Psi}_{NbootML}$ in ascending order.

Let $G_1(x) = P\left(\widehat{\Psi}_{ML} \leq x\right)$ be the cumulative distribution function of $\widehat{\Psi}_{ML}$. Define $\widehat{\Psi}_{boot-pML} = G_1^{-1}(x)$ for the given x. The approximate bootstrap-p $100(1-\gamma)\%$ CI of $\widehat{\Psi}_{ML}$, is given by

$$\left[\left(\frac{\gamma}{2}\right)\widehat{\Psi}_{boot-pML}, \left(1-\frac{\gamma}{2}\right)\widehat{\Psi}_{boot-pML}\right]$$

4.2. Bootstrap-t Interval Estimation

In this subsection, the required steps for computing the bootstrap confidence interval of type t is introduced depending on the MLE of β.

1. Equation (10) may be maximized to produce $\underline{\mathbf{z}} = z_{1:m:n}, \ldots, z_{m:m:n}, T$ based on the original data.
2. Based on the generalized progressive censoring scheme R^*, generate a Type-II progressive censoring sample $\underline{\mathbf{z}} = z_{1:m:n}, \ldots, z_{m:m:n}, T$ from the adjustable UHLG distribution $\widehat{\beta}_{ML}$, following the method stated in [19].
3. A bootstrap estimate is indicated by $\widehat{\Psi}_{ML}$ (in our example, Ψ might) be β, $R(t)$, and $H(t)$. Obtain the MLEs based on the bootstrap sample.
4. Equation (10) may be maximized to produce $\underline{\mathbf{z}} = z_{1:m:n}, \ldots, z_{m:m:n}, T$ based on the original data.
5. The value of the $T^{*\Psi}$ statistic can be defined as

$$\frac{\left(\widehat{\Psi}^*_{ML} - \widehat{\Psi}_{ML}\right)}{\sqrt{\widehat{var\left(\widehat{\Psi}^*_{ML}\right)}}}.$$

6. Steps 2 through 5 should be repeated N times to obtain $T_1^{*\Psi}, T_2^{*\Psi}$...., $T_{Nboot}^{*\Psi}$.
7. Sort $T_1^{*\Psi}, T_2^{*\Psi}$...., $T_{Nboot}^{*\Psi}$ in ascending order to produce the sequences that are ordered. $T_1^{*\Psi}, T_2^{*\Psi}$...., $T_{Nboot}^{*\Psi}$.

Let $G_2(x) = P(T^* \leq x)$ be the cumulative distribution function of T^*. Define $\widehat{\Psi}_{boot-tML} = \widehat{\Psi}_{ML} + G_1^{-1}(x)\sqrt{\widehat{var}\left(\widehat{\Psi}^*_{ML}\right)}$ for the given x. The approximate bootstrap-t $100(1-\gamma)\%$ CI of $\widehat{\Psi}_{ML}$, is given by

$$\left[\left(\frac{\gamma}{2}\right)\widehat{\Psi}_{boot-tML}, \left(1-\frac{\gamma}{2}\right)\widehat{\Psi}_{boot-tML}\right].$$

5. Bayes Estimation

Conventional methods might produce erroneous and misleading results when censored data is provided or tests have small sample sizes. By leveraging additional prior knowledge, such as historical data or an understanding of the statistical inferential process, the Bayesian technique could be applied in this scenario to produce estimates that are more precise. Here, we treat the parameters as random variables and obtain Bayesian estimates for the parameters that account for uncertainty. Before the failure data were gathered, the joint prior distribution, characterizing parameter uncertainty, was created. The Bayesian technique is particularly useful in reliability analysis because it allows for the incorporation of previous knowledge into the study (see [20,21]). This is important, since a big barrier to reliability studies is the lack of easily accessible data. The Bayesian estimating (BE) technique is applied, assuming that the prior distribution of the parameter is not informative, in order to estimate the unknown parameter β. Furthermore, the unknown values β are supplied in order to provide the prior distribution.

$$\pi(\beta) = \frac{1}{\beta}.$$

Thus, the joint posterior distribution function is given by

$$\pi^*(\beta|\underline{\mathbf{z}}) = L(\beta|\underline{\mathbf{z}})\pi(\beta) / \int L(\beta|\underline{\mathbf{z}})\pi(\beta)d\beta. \tag{16}$$

One common symmetric loss function that is used is the square error loss function. Let $g(\beta)$ be any function in terms of the parameter β under SEL; the Bayesian estimator is represented by $\widehat{g}(\beta)_{BS}$ can be obtained as

$$\widehat{g}(\beta)_{BS} = E_\beta[g(\beta)|\underline{\mathbf{z}}], \tag{17}$$

with $E_\beta[g(\beta)|\underline{\mathbf{z}}]$ is the expected value, as determined by the posterior distribution. Thus, from (17), under the SELF, the Bayesian estimates of β, $R(t)$, and $H(t)$ are, in turn, as follows:

$$\widehat{\beta}_{BS} = \int \beta\pi^*(\beta|\underline{\mathbf{z}})d\beta, \tag{18}$$

$$\widehat{R(t)}_{BS} = \int \frac{\widehat{\beta}(1-z)}{\widehat{\beta} + \left(2-\widehat{\beta}\right)z}\pi^*(\beta|\underline{\mathbf{z}})d\beta, \tag{19}$$

and

$$\widehat{H(t)}_{BS} = \int \frac{2}{(1-z)[\beta-(2-\beta)z]}\pi^*(\beta|\underline{\mathbf{z}})d\beta. \tag{20}$$

The Bayesian estimator for $g(\beta)$ under the linex loss function (LL), denoted by $\widehat{g}(\beta)_{BL}$, can be obtained as

$$\widehat{\beta}_{BL} = \frac{-1}{\upsilon}\ln\{E_\beta[\exp(-\upsilon\beta)|\underline{\mathbf{z}}]\}, \tag{21}$$

where v is constant with $E_\beta[\exp(-v\beta)|\underline{\mathbf{z}}]$ is finite. Thus, from (21), the Bayesian estimators of β, $R(t)$, and $H(t)$ are, respectively, ascertained using the LLF.

$$\widehat{\beta}_{BL} = \frac{-1}{v}\ln\left[\int \exp(-v\beta)\pi^*(\beta|\underline{\mathbf{z}})d\beta\right], \tag{22}$$

$$\widehat{R(t)}_{BL} = \frac{-1}{v}\ln\left\{\int \exp\left[\frac{-\widehat{\beta}(1-z)v}{\widehat{\beta}+\left(2-\widehat{\beta}\right)z}\right]\pi^*(\beta|\underline{\mathbf{z}})d\beta\right\}, \tag{23}$$

and

$$\widehat{H(t)}_{BL} = \frac{-1}{v}\ln\left\{\int \exp\left[\frac{-2v}{(1-z)[\beta-(2-\beta)z]}\right]\pi^*(\beta|\underline{\mathbf{z}})d\beta\right\}. \tag{24}$$

The Bayesian estimator for any parameter β GEL function denoted by $\widehat{\beta}_{BE}$ can be obtained as

$$\widehat{\beta}_{BE} = \left\{E_\beta\left[\beta^{-\kappa}|\underline{\mathbf{z}}\right]\right\}^{\frac{-1}{\kappa}}, \tag{25}$$

where κ is constant with $E_\beta[\beta^{-\kappa}|\underline{\mathbf{z}}]$ is finite. Thus, from (25), the Bayesian estimates of β, $R(t)$, and $H(t)$ pursuant to the GELF are, in turn, the following:

$$\widehat{\beta}_{BE} = \left[\int \beta^{-\kappa}\pi^*(\beta|\underline{\mathbf{z}})d\beta\right]^{\frac{-1}{\kappa}}, \tag{26}$$

$$\widehat{R(t)}_{BE} = \left\{\int \left[\frac{\widehat{\beta}(1-z)}{\widehat{\beta}+\left(2-\widehat{\beta}\right)z}\right]^{-\kappa}\pi^*(\beta|\underline{\mathbf{z}})d\beta\right\}^{\frac{-1}{\kappa}}, \tag{27}$$

and

$$\widehat{H(t)}_{BE} = \left\{\int \left[\frac{2}{(1-z)[\beta-(2-\beta)z]}\right]^{-\kappa}\pi^*(\beta|\underline{\mathbf{z}})d\beta\right\}^{\frac{-1}{\kappa}}. \tag{28}$$

where the computation of the various integrals in Equations (18)–(28) cannot be solved analytically. Consequently, the MCMC approach is used to generated samples from the joint posterior density function in Equation (16). To apply the MCMC approach, we consider the Gibbs within the Metropolis–Hastings (M-H) samplers procedures. M-H and Gibbs sampling are two efficient MCMC methods that have been widely used in statistics. These methods have been the subject of extensive discussion recently. See [22–26] for additional details. The MCMC Algorithm 1 generates β samples from conditional posterior distributions, which are then used to approximate Bayes estimates of them.

Algorithm 1: MCMC method

Step 1 start with $\beta^{(0)} = \hat{\beta}_{ML}$
Step 2 set $i = 1$
Step 3 generate a proposal $\beta^{(*)}$ from $N(\beta^{(i-1)}, V(\beta))$
Step 4 calculate the acceptance probabilities
$d_\beta = min\left[1, \frac{\pi^*(\beta^{(*)}|\underline{\mathbf{z}})}{\pi^*(\beta^{(i-1)}|\underline{\mathbf{z}})}\right]$
Step 5 generate u_1 to follow a $Uniform(0,1)$ distribution, if $u_1 \leq d_\beta$, set $\beta^{(i)} = \beta^{(*)}$, else set $\beta^{(i)} = \beta^{(i-1)}$
Step 6 set $i = i+1$, repeat steps 3 to 6, N times and obtain $\left(\beta^{(j)}\right)$, $j = 1, 2, ..., N$.
Step 7 remove the first B values for β, which is the burn-in period of $\beta^{(j)}$ where $j = 1, 2, ..., N-B$.

6. Real Data Application

In this section, a real data set has been investigated under GPHCS. The data were mentioned in Hoel [27] and they represent the mortality of 39 male mice after receiving a radiation dose of 300 r at an age of 1.25–1.5 months with a reticulum cell sarcoma as a mortality cause.

The complete data set, which is given in Table 1, are the ratios of the lifetimes with respect to the maximum lifetime, which equals 770.

Table 1. Mortality data for 39 male mice due to reticulum cell sarcoma.

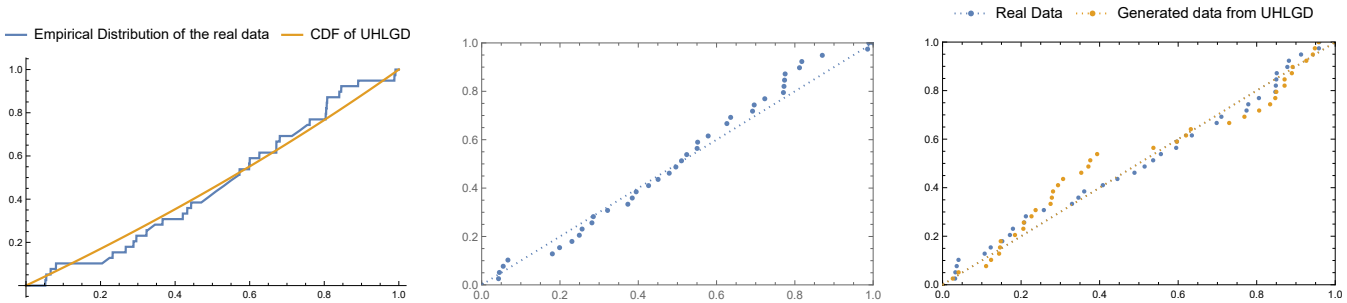
0.0519	0.0545	0.0662	0.0805	0.2117	0.2325	0.2675	0.2883	0.2961
0.3234	0.3273	0.3662	0.4207	0.4325	0.4428	0.4753	0.5000	0.5286
0.5454	0.5597	0.5727	0.5987	0.6000	0.6259	0.6714	0.6714	0.6805
0.7325	0.7364	0.7610	0.8038	0.8052	0.8065	0.8078	0.8403	0.8454
0.8909	0.9883	0.9909						

For the purpose of testing the good fitting of the UHLGD for the data set. A set of testing procedures has been executed based on the complete data; the resulting test statistic and p-value are obtained in Table 2 with an MLE of $\beta = 2.4383$.

Table 2. Goodness-of-fit test for the real data with UHLG distribution.

	Statistic	***p*-Value**
Kolmogorov–Smirnov	0.1053	0.7406
Anderson–Darling	0.5028	0.7426
Cramér-von Mises	0.0675	0.7668
Pearson χ^2	6.9231	0.5449

The difference between the empirical distribution of the data and the UHLGD distribution function is shown in Figure 3a, and the probability plot (P-P plot) for the data is displayed in Figure 3b. The P-P plot of the data in comparison to a created set of data appears in Figure 3c.



(**a**) (**b**) (**c**)

Figure 3. (**a**) The CDF, (**b**) P-P plot, and (**c**) P-P plot for the real data set compared with another set of UHLGD generated data.

To illustrate the methods of estimation that were applied in this paper, three randomly samples of GPHCS have been obtained from the complete set of data with $m = 25$ observations and the minimum number of observed lifetimes is $k = 18$. Table 3 shows the three schemes with three different values of T. The MLEs for both the complete and GPHCS samples are obtained; also, the BEs based on the non-informative priors for different loss functions of the parameter β are executed. Some reliability functions like the survival function $S(t = 0.6)$ and the hazard rate function $H(t = 0.6)$ are estimated. All of these results can be investigated in Table 4. Table 5 presents the lower and upper bounds

with average length (AL) of the asymptotic (ACI), credible (CCI), bootstrap-p (BCI-p) and bootstrap-t (BCI-t) confidence intervals of β and $S(t = 0.6)$ and $H(t = 0.6)$. Figure 4 shows the bootstrap iteration and the distributions of β for schemes 1, 2 and 3. Its obvious that the distributions are semi normal.

Table 3. GPHC schemes and samples obtained from the complete set of lifetimes.

Scheme	T	R and Sample
Sch. 1 $(T < X_k < X_m)$	0.3	$R = (1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 17)$, $R_T = 0$
		x = (0.0519, 0.0545, 0.0662, 0.0805, 0.2324, 0.2675, 0.2883 0.2961, 0.3234, 0.3273, 0.3662, 0.4207, 0.4325, 0.4428 0.4753, 0.5000, 0.5286, 0.54545)
Sch. 2 $(X_k < T < X_m)$	0.7	R = (1, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1), $R_T = 6$
		x = (0.05194, 0.0545, 0.0662, 0.0805, 0.2324, 0.2675, 0.2883 0.2961, 0.3234, 0.3273, 0.3662, 0.4208, 0.4325, 0.4428 0.4753, 0.5000, 0.5286, 0.5454, 0.5597, 0.6000, 0.6714)
Sch. 3 $(X_k < X_m < T)$	0.9	R = (1, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 2), $R_T = 0$
		x = (0.05194, 0.05454, 0.06623, 0.08052, 0.23247, 0.26753, 0.28831 0.2961, 0.3234, 0.3273, 0.3662, 0.4207, 0.4325, 0.4428 0.4753, 0.5000, 0.5286, 0.5454, 0.5597, 0.5727, 0.6000, 0.6714, 0.7325, 0.7364, 0.7610)

Table 4. The ML and Bayesian estimates of $\beta, S(t = 0.6)$ and $H(t = 0.6)$ based on the different GPHCSs for the real data.

Parameter	Scheme	MLE (Complete)	MLE	BNIFS	BNIFL	BNIFG
β	Sch. 1	2.4383	2.6948	2.8581	2.6806	2.6640
	Sch. 2		2.4775	2.6045	2.4678	2.4424
	Sch. 3		2.3639	2.4737	2.3559	2.3273
$S(t = 0.6)$	Sch. 1	0.44836	0.4732	0.4766	0.4752	0.4677
	Sch. 2		0.4523	0.4550	0.4537	0.4464
	Sch. 3		0.4407	0.4430	0.4418	0.4347
$H(t = 0.6)$	Sch. 1	2.2985	2.1950	2.1807	2.1564	2.1459
	Sch. 2		2.2821	2.2710	2.2490	2.2408
	Sch. 3		2.3304	2.3210	2.3002	2.2932

Table 5. The values of (L,U) and AL of ACI, CCI, BCI-p and BCI-t for β, $S(t = 0.6)$ and $H(t = 0.6)$ based on the different GPHCSs of the real data.

		ACI		CCI		$BCI-p$		$BCI-t$	
Parameter	**Scheme**	**(L, U)**	**AL**	**(L, U)**	**AL**	**(L, U)**	**AL**	**(L, U)**	**AL**
β	Sch. 1	(1.098, 4.292)	3.194	(1.713, 4.422)	2.710	(0.988, 3.585)	2.597	(0.000, 3.367)	5.316
	Sch. 2	(1.069, 3.886)	2.816	(1.385, 4.662)	3.277	(0.715, 2.661)	1.946	(0.000, 2.653)	6.510
	Sch. 3	(1.052, 3.676)	2.623	(1.269, 4.448)	3.179	(1.294, 4.255)	2.960	(0.535, 3.361)	2.826
$S(t = 0.6)$	Sch. 1	(0.325, 0.621)	0.295	(0.363, 0.596)	0.232	(0.248, 0.544)	0.297	(0.173, 0.545)	0.373
	Sch. 2	(0.311, 0.593)	0.282	(0.316, 0.608)	0.293	(0.192, 0.470)	0.278	(0.022, 0.470)	0.449
	Sch. 3	(0.304, 0.577)	0.274	(0.297, 0.597)	0.300	(0.301, 0.587)	0.285	(0.288, 0.581)	0.293
$H(t = 0.6)$	Sch. 1	(1.580, 2.810)	1.231	(1.675, 2.646)	0.971	(1.897, 3.133)	1.235	(1.892, 3.447)	1.555
	Sch. 2	(1.695, 2.869)	1.173	(1.631, 2.843)	1.212	(2.198, 3.362)	1.164	(2.200, 4.075)	1.875
	Sch. 3	(1.761, 2.900)	1.140	(1.676, 2.928)	1.252	(1.722, 2.906)	1.184	(1.738, 2.966)	1.228

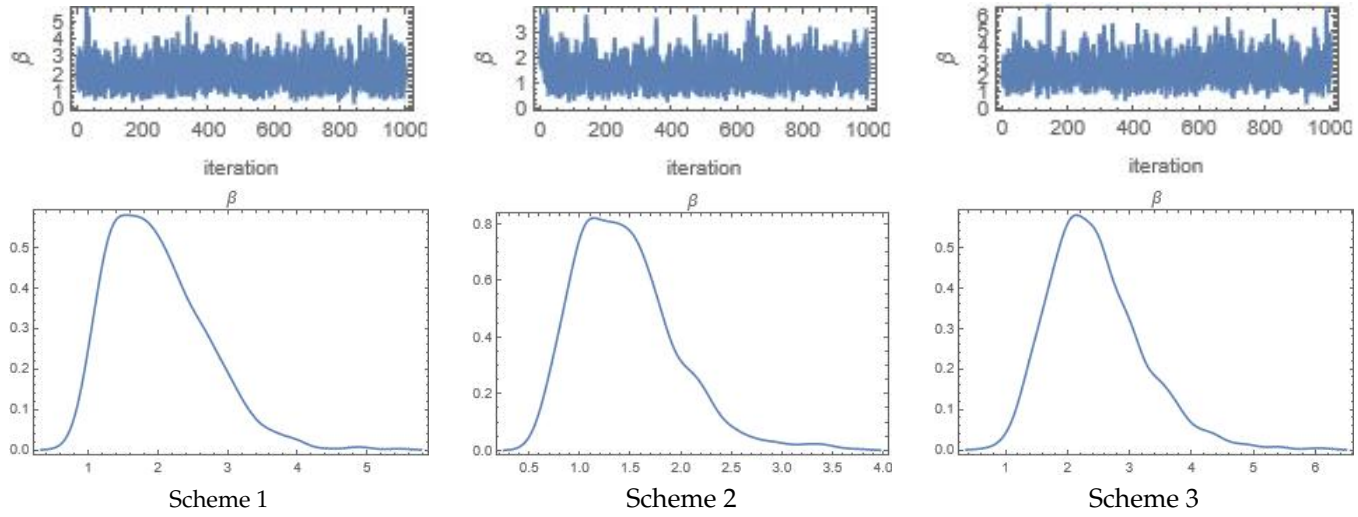


Figure 4. Bootstrap iteration and its distributions of β for schemes 1, 2 and 3.

7. Simulation Study

In this section, the proposed methods have been compared to show their efficiency and performance. The assessment and comparison of the various estimates is carried out using the Monte Carlo simulation method. For each simulation trial, 5000 generated GPHCS samples with a real value of $\beta = 0.5$ were generated from the UHLGD, and the estimates $\hat{\beta}, S(t \hat{=} 0.6)$, and $H(t \hat{=} 0.6)$ were obtained, including the average of the estimates ($AE = \frac{1}{5000}\sum_{i=1}^{5000}\hat{\beta}_i$), and the expected bias ($EB = |AE - rv|$), where rv is the real value of the parameter and the mean square error ($MSE = \frac{1}{5000}\sum_{i=1}^{5000}(\hat{\beta}_i - \beta)^2$). The average widths (AW) and coverage probabilities (CP) of the suggested 95% confidence intervals are also computed for $\beta, S(t)$, and $H(t)$, for which several confidence intervals, such as the ACI, CCI, BCI-p, and BCI-t, have been developed. The estimates BNIFS, BNIFL, and BNIFG are the Bayesian estimates that were obtained using the loss functions SEL, LINEX and GEL, respectively. Upon first generating the samples, we used the algorithm proposed by Balakrishnan and Aggarwala [3] to produce a progressive Type-II censored sample with size m from the UHLGD, then we adapted the true case of GPHCS according to the value of T, X_k, and X_m. In this simulation, we tried three different values of $T = (0.2, 0.8, 0.95)$ to increase the variability of the GPHCS cases with different values of $(n, m, k) = (50, 30, 20), (70, 50, 30)$. Three different censoring schemes have been proposed via this simulation, which are as follows:

1. Scheme 1: $R_i = 0$ if $i = 1, 2, 3, ..., m-1$, and $R_m = n - m$.
2. Scheme 2: $R_1 = n - m$ and $R_i = 0$ for $i = 2, 3, ..., m$.
3. Scheme 3: $R_i = 1$ if $i = 1, 2, 3, ..., n - m$, otherwise $R_i = 0$.

The non-informative prior of β has been assumed for two reasons: the first is that we didn't have any information about the distribution of β and the second is that the complexity of the computations is high when the informative priors are used. The hyper-parameters for the LINEX and GEL loss functions are chosen to be $(h, q) = (0.5, 0.5)$, respectively. One thousand is the number of iterations of the bootstrap algorithm which was used to obtain the bootstrap confidence intervals of $\beta, S(t = 0.6)$ & $H(t = 0.6)$.

Tables 6–8 show the AE, EB and MSE for the parameter estimate $\hat{\beta}$, and the estimates of the reliability measures $S(t \hat{=} 0.6)$, and $H(t \hat{=} 0.6)$, respectively. Tables 9–11 show the AWs and CPs for 95% ACIs, CCIs, as well as the BCI-p and BCI-t of the parameter β, and also for the reliability measures $S\&H$.

Table 6. The ML and Bayesian estimates' values of AE, EB, and MSE for β based on the various GPHCSs in order to study the simulation.

			AE				*EB*				*MSE*			
(n, m, k)	T	Schemes	MLE	BNIFS	BNIFL	BNIFG	MLE	BNIFS	BNIFL	BNIFG	MLE	BNIFS	BNIFL	BNIFG
(50, 30, 20)	0.2	sch 1	0.5152	0.5385	0.5328	0.5104	0.0152	0.0385	0.0328	0.0104	0.0150	0.0182	0.0171	0.0148
		sch 2	0.5085	0.5381	0.5299	0.4988	0.0085	0.0381	0.0299	0.0012	0.0254	0.0298	0.0274	0.0243
		sch 3	0.5095	0.5376	0.5308	0.5047	0.0095	0.0376	0.0308	0.0047	0.0193	0.0229	0.0213	0.0189
	0.8	sch 1	0.5131	0.5322	0.5272	0.5069	0.0131	0.0322	0.0272	0.0069	0.0179	0.0201	0.0191	0.0173
		sch 2	0.5294	0.5570	0.5484	0.5173	0.0294	0.0570	0.0484	0.0173	0.0313	0.0370	0.0339	0.0293
		sch 3	0.5334	0.5588	0.5520	0.5266	0.0334	0.0588	0.0520	0.0266	0.0225	0.0269	0.0250	0.0215
	0.95	sch 1	0.5174	0.5367	0.5316	0.5112	0.0174	0.0367	0.0316	0.0112	0.0166	0.0188	0.0178	0.0160
		sch 2	0.5273	0.5548	0.5463	0.5153	0.0273	0.0548	0.0463	0.0153	0.0280	0.0333	0.0305	0.0262
		sch 3	0.5275	0.5529	0.5460	0.5208	0.0275	0.0529	0.0460	0.0208	0.0249	0.0294	0.0273	0.0240
(70, 50, 30)	0.2	sch 1	0.5096	0.5255	0.5217	0.5062	0.0096	0.0255	0.0217	0.0062	0.0140	0.0157	0.0151	0.0138
		sch 2	0.5084	0.5273	0.5225	0.5027	0.0084	0.0273	0.0225	0.0027	0.0158	0.0176	0.0168	0.0154
		sch 3	0.4971	0.5153	0.5112	0.4935	0.0029	0.0153	0.0112	0.0065	0.0100	0.0109	0.0105	0.0098
	0.8	sch 1	0.5169	0.5291	0.5258	0.5118	0.0169	0.0291	0.0258	0.0118	0.0128	0.0139	0.0134	0.0124
		sch 2	0.5193	0.5352	0.5306	0.5119	0.0193	0.0352	0.0306	0.0119	0.0162	0.0180	0.0172	0.0155
		sch 3	0.5176	0.5334	0.5292	0.5121	0.0176	0.0334	0.0292	0.0121	0.0155	0.0173	0.0165	0.0150
	0.95	sch 1	0.5183	0.5305	0.5272	0.5133	0.0183	0.0305	0.0272	0.0133	0.0135	0.0147	0.0142	0.0131
		sch 2	0.5193	0.5352	0.5306	0.5119	0.0193	0.0352	0.0306	0.0119	0.0162	0.0180	0.0172	0.0155
		sch 3	0.5158	0.5314	0.5273	0.5103	0.0158	0.0314	0.0273	0.0103	0.0147	0.0163	0.0156	0.0142

Table 7. The ML and Bayesian estimates' values of AE, EB, and MSE for $S(t = 0.6)$ based on the various GPHCSs in order to study the simulation.

			AE				*EB*				*MSE*			
(n, m, k)	T	Schemes	MLE	BNIFS	BNIFL	BNIFG	MLE	BNIFS	BNIFL	BNIFG	MLE	BNIFS	BNIFL	BNIFG
(50, 30, 20)	0.2	Sch. 1	0.1456	0.1496	0.1493	0.1441	0.0027	0.0067	0.0064	0.0012	0.0009	0.0009	0.0009	0.0008
		Sch. 2	0.1433	0.1482	0.1478	0.1405	0.0004	0.0053	0.0049	0.0024	0.0014	0.0014	0.0014	0.0013
		Sch. 3	0.1439	0.1488	0.1485	0.1423	0.0010	0.0059	0.0056	0.0006	0.0011	0.0012	0.0012	0.0011
	0.8	Sch. 1	0.1449	0.1481	0.1479	0.1431	0.0020	0.0052	0.0050	0.0002	0.0010	0.0010	0.0010	0.0010
		Sch. 2	0.1480	0.1524	0.1519	0.1447	0.0051	0.0095	0.0090	0.0018	0.0017	0.0017	0.0017	0.0016
		Sch. 3	0.1495	0.1538	0.1535	0.1475	0.0066	0.0109	0.0106	0.0046	0.0012	0.0013	0.0013	0.0012
	0.95	Sch. 1	0.1460	0.1493	0.1490	0.1442	0.0031	0.0064	0.0061	0.0013	0.0009	0.0010	0.0010	0.0009
		Sch. 2	0.1477	0.1520	0.1516	0.1444	0.0048	0.0091	0.0087	0.0015	0.0015	0.0016	0.0016	0.0014
		Sch. 3	0.1480	0.1522	0.1519	0.1460	0.0051	0.0093	0.0090	0.0031	0.0013	0.0014	0.0014	0.0013
(70, 50, 30)	0.2	Sch. 1	0.1443	0.1471	0.1469	0.1432	0.0014	0.0042	0.0040	0.0003	0.0008	0.0008	0.0008	0.0008
		Sch. 2	0.1438	0.1471	0.1469	0.1422	0.0009	0.0042	0.0040	0.0007	0.0009	0.0009	0.0009	0.0009
		Sch. 3	0.1415	0.1448	0.1446	0.1403	0.0014	0.0019	0.0017	0.0026	0.0006	0.0006	0.0006	0.0006
	0.8	Sch. 1	0.1461	0.1482	0.1480	0.1447	0.0032	0.0053	0.0051	0.0018	0.0007	0.0008	0.0007	0.0007
		Sch. 2	0.1465	0.1491	0.1488	0.1444	0.0036	0.0062	0.0059	0.0015	0.0009	0.0010	0.0009	0.0009
		Sch. 3	0.1461	0.1488	0.1486	0.1445	0.0032	0.0059	0.0057	0.0016	0.0009	0.0009	0.0009	0.0008
	0.95	Sch. 1	0.1464	0.1485	0.1483	0.1450	0.0035	0.0056	0.0054	0.0021	0.0008	0.0008	0.0008	0.0007
		Sch. 2	0.1465	0.1491	0.1488	0.1444	0.0036	0.0062	0.0059	0.0015	0.0009	0.0010	0.0009	0.0009
		Sch. 3	0.1457	0.1484	0.1482	0.1442	0.0028	0.0055	0.0053	0.0013	0.0008	0.0009	0.0009	0.0008

Table 8. The ML and Bayesian estimates' values of AE, EB, and MSE for $H(t = 0.6)$ based on the various GPHCSs in order to study the simulation.

			AE				*EB*				*MSE*			
(n, m, k)	T	Schemes	MLE	BNIFS	BNIFL	BNIFG	MLE	BNIFS	BNIFL	BNIFG	MLE	BNIFS	BNIFL	BNIFG
(50, 30, 20)	0.2	Sch. 1	3.5602	3.5433	3.5381	3.5388	0.0112	0.0281	0.0333	0.0326	0.0149	0.0164	0.0172	0.0172
		Sch. 2	3.5698	3.5491	3.5418	3.5427	0.0016	0.0223	0.0296	0.0287	0.0238	0.0249	0.0262	0.0262
		Sch. 3	3.5672	3.5466	3.5405	3.5413	0.0042	0.0248	0.0309	0.0301	0.0190	0.0203	0.0212	0.0212
	0.8	Sch. 1	3.5631	3.5495	3.5449	3.5455	0.0083	0.0219	0.0265	0.0259	0.0174	0.0182	0.0188	0.0188
		Sch. 2	3.5500	3.5319	3.5245	3.5254	0.0214	0.0395	0.0469	0.0460	0.0286	0.0303	0.0320	0.0319
		Sch. 3	3.5436	3.5258	3.5198	3.5205	0.0278	0.0456	0.0516	0.0509	0.0212	0.0230	0.0242	0.0241
	0.95	Sch. 1	3.5583	3.5446	3.5400	3.5406	0.0131	0.0268	0.0314	0.0308	0.0161	0.0170	0.0177	0.0176
		Sch. 2	3.5513	3.5332	3.5258	3.5267	0.0201	0.0382	0.0456	0.0447	0.0262	0.0278	0.0294	0.0293
		Sch. 3	3.5501	3.5324	3.5264	3.5271	0.0213	0.0390	0.0450	0.0443	0.0227	0.0243	0.0255	0.0254
(70, 50, 30)	0.2	Sch. 1	3.5656	3.5538	3.5503	3.5508	0.0058	0.0176	0.0211	0.0206	0.0136	0.0144	0.0149	0.0148
		Sch. 2	3.5673	3.5537	3.5492	3.5498	0.0041	0.0177	0.0222	0.0216	0.0156	0.0163	0.0168	0.0168
		Sch. 3	3.5772	3.5634	3.5595	3.5600	0.0058	0.0080	0.0119	0.0114	0.0101	0.0104	0.0107	0.0107
	0.8	Sch. 1	3.5578	3.5492	3.5461	3.5466	0.0136	0.0222	0.0253	0.0248	0.0126	0.0131	0.0134	0.0134
		Sch. 2	3.5563	3.5455	3.5413	3.5418	0.0151	0.0259	0.0301	0.0296	0.0160	0.0166	0.0172	0.0171
		Sch. 3	3.5579	3.5467	3.5428	3.5433	0.0135	0.0247	0.0286	0.0281	0.0151	0.0158	0.0163	0.0163
	0.95	Sch. 1	3.5565	3.5480	3.5449	3.5453	0.0149	0.0234	0.0265	0.0261	0.0133	0.0137	0.0141	0.0141
		Sch. 2	3.5563	3.5455	3.5413	3.5418	0.0151	0.0259	0.0301	0.0296	0.0160	0.0166	0.0172	0.0171
		Sch. 3	3.5595	3.5483	3.5445	3.5450	0.0119	0.0231	0.0269	0.0264	0.0143	0.0149	0.0154	0.0153

Table 9. The AL and CP values of ACI, CCI, BCI-p and BCI-t for β based on the different GPHCSs in order to study the simulation.

			ACI		*CCI*		$BCI - p$		$BCI - t$	
(n, m, k)	T	Schemes	AL	CP	AL	CP	AL	CP	AL	CP
(50, 30, 20)	0.2	Sch. 1	0.5338	0.952	0.5471	0.950	0.3692	0.982	0.8165	0.882
		Sch. 2	0.6283	0.924	0.6481	0.942	0.3678	0.750	0.6257	0.702
		Sch. 3	0.5730	0.932	0.5897	0.960	0.3840	0.745	0.7318	0.710
	0.8	Sch. 1	0.5085	0.941	0.4927	0.923	0.5245	0.951	0.5255	0.903
		Sch. 2	0.6480	0.933	0.6381	0.905	0.6809	0.954	0.6683	0.885
		Sch. 3	0.5850	0.952	0.5758	0.932	0.6033	0.976	0.6005	0.898
	0.95	Sch. 1	0.5131	0.944	0.5059	0.921	0.5288	0.953	0.5447	0.910
		Sch. 2	0.6454	0.935	0.6352	0.917	0.6765	0.967	0.6704	0.910
		Sch. 3	0.5801	0.955	0.5747	0.930	0.5954	0.956	0.5956	0.903
(70, 50, 30)	0.2	Sch. 1	0.4435	0.933	0.4487	0.918	0.2336	0.213	0.6773	0.100
		Sch. 2	0.5002	0.943	0.5034	0.952	0.2443	0.400	0.5247	0.400
		Sch. 3	0.4653	0.960	0.4694	0.970	0.2611	0.393	0.6399	0.390
	0.8	Sch. 1	0.4245	0.938	0.4265	0.922	0.4355	0.949	0.4368	0.916
		Sch. 2	0.4942	0.940	0.4951	0.922	0.5072	0.952	0.4932	0.900
		Sch. 3	0.4711	0.938	0.4677	0.933	0.4821	0.955	0.4835	0.918
	0.95	Sch. 1	0.4254	0.936	0.4268	0.922	0.4367	0.944	0.4376	0.912
		Sch. 2	0.4942	0.940	0.4951	0.922	0.5072	0.952	0.4932	0.900
		Sch. 3	0.4685	0.946	0.4658	0.936	0.4789	0.960	0.4808	0.908

Table 10. The values of AL and CP of ACI, CCI, BCI-p and BCI-t for $S(t = 0.6)$ based on the different GPHCSs in order to study the simulation.

			ACI		CCI		$BCI-p$		$BCI-t$	
(n, m, k)	T	**Schemes**	**AL**	**CP**	**AL**	**CP**	**AL**	**CP**	**AL**	**CP**
(50, 30, 20)	0.2	Sch. 1	0.1278	0.956	0.1260	0.950	0.0951	0.982	0.1756	0.918
		Sch. 2	0.1499	0.932	0.1469	0.942	0.0905	0.750	0.1337	0.710
		Sch. 3	0.1373	0.953	0.1352	0.960	0.0963	0.745	0.1593	0.723
	0.8	Sch. 1	0.1218	0.941	0.1149	0.923	0.1216	0.951	0.1253	0.919
		Sch. 2	0.1523	0.935	0.1432	0.905	0.1518	0.954	0.1558	0.908
		Sch. 3	0.1382	0.954	0.1308	0.932	0.1369	0.976	0.1411	0.922
	0.95	Sch. 1	0.1227	0.948	0.1174	0.921	0.1226	0.953	0.1286	0.920
		Sch. 2	0.1522	0.941	0.1436	0.917	0.1511	0.967	0.1567	0.920
		Sch. 3	0.1372	0.955	0.1307	0.930	0.1354	0.956	0.1403	0.920
(70, 50, 30)	0.2	Sch. 1	0.1066	0.931	0.1049	0.918	0.0626	0.213	0.1438	0.118
		Sch. 2	0.1203	0.952	0.1174	0.952	0.0629	0.400	0.1121	0.400
		Sch. 3	0.1131	0.963	0.1106	0.970	0.0688	0.393	0.1399	0.390
	0.8	Sch. 1	0.1019	0.942	0.1000	0.922	0.1023	0.949	0.1047	0.927
		Sch. 2	0.1181	0.940	0.1151	0.922	0.1172	0.952	0.1177	0.922
		Sch. 3	0.1128	0.945	0.1092	0.933	0.1124	0.955	0.1155	0.933
	0.95	Sch. 1	0.1020	0.938	0.1000	0.922	0.1025	0.944	0.1048	0.924
		Sch. 2	0.1181	0.940	0.1151	0.922	0.1172	0.952	0.1177	0.922
		Sch. 3	0.1124	0.952	0.1090	0.936	0.1119	0.960	0.1151	0.920

Table 11. The values of AL and CP of ACI, CCI, BCI-p and BCI-t for $H(t = 0.6)$ based on the different GPHCSs in order to study the simulation.

			ACI		CCI		$BCI-p$		$BCI-t$	
(n, m, k)	T	**Schemes**	**AL**	**CP**	**AL**	**CP**	**AL**	**CP**	**AL**	**CP**
(50, 30, 20)	0.2	Sch. 1	0.5323	0.956	0.5294	0.952	0.3961	0.982	0.7316	0.918
		Sch. 2	0.6248	0.932	0.6165	0.942	0.3773	0.750	0.5572	0.710
		Sch. 3	0.5720	0.953	0.5678	0.960	0.4014	0.745	0.6637	0.723
	0.8	Sch. 1	0.5074	0.941	0.4866	0.920	0.5067	0.951	0.5222	0.919
		Sch. 2	0.6347	0.935	0.6077	0.896	0.6327	0.954	0.6492	0.908
		Sch. 3	0.5757	0.954	0.5537	0.926	0.5705	0.976	0.5881	0.922
	0.95	Sch. 1	0.5114	0.948	0.4977	0.921	0.5106	0.953	0.5359	0.920
		Sch. 2	0.6342	0.941	0.6094	0.921	0.6298	0.967	0.6428	0.920
		Sch. 3	0.5717	0.955	0.5532	0.926	0.5642	0.956	0.5843	0.920
(70, 50, 30)	0.2	Sch. 1	0.4442	0.931	0.4398	0.913	0.2608	0.213	0.5990	0.118
		Sch. 2	0.5012	0.952	0.4936	0.943	0.2620	0.400	0.4670	0.400
		Sch. 3	0.4714	0.963	0.4645	0.977	0.2867	0.393	0.5829	0.390
	0.8	Sch. 1	0.4246	0.942	0.4187	0.918	0.4263	0.949	0.4363	0.927
		Sch. 2	0.4922	0.940	0.4827	0.922	0.4883	0.952	0.4903	0.922
		Sch. 3	0.4698	0.945	0.4587	0.930	0.4685	0.955	0.4813	0.933
	0.95	Sch. 1	0.4249	0.938	0.4183	0.914	0.4270	0.944	0.4366	0.924
		Sch. 2	0.4922	0.940	0.4827	0.922	0.4883	0.952	0.4903	0.922
		Sch. 3	0.4681	0.952	0.4567	0.934	0.4664	0.960	0.4796	0.920

8. Conclusions and Discussion

In this paper, the survival and hazard rate functions of the UHLGD lifetime distribution, as well as its unique parameter, are estimated using the ML and Bayesian methods when the observed sample is a GPHCS. Together with the Bayesian estimates based on SEL, Linex, and GEL functions based on the non-informative prior distribution, the MLEs

are looked at as well. Additionally, the parameters, survival, and hazard functions are calculated, along with the 95% asymptotic (ACI), credible (CCI), bootstrap-p (BCI-p), and bootstrap-t (BCI-t) confidence intervals. The numerical analysis leads to the conclusion of the following observations:

- The MLEs and the Bayesian estimates based on the non-informative priors function semi-equally in most circumstances, as the MSE shows.
- We observed that the GEL outperforms the SEL and LINEX loss functions in the Bayesian estimation.
- Generally speaking, the MSE falls as n and m rise.
- As T increases, the average length of the confidence intervals decreases.
- The bootstrap-p confidence intervals have the highest coverage probability in a large proportion of situations.
- Compared to Schemes 2 and 3, censoring Scheme 1 performs better.
- We have observed that the projections are rather close to those of the entire sample because of the real data estimates.

Lastly, evaluating and modeling the lifespan resulting from medical conditions as well as electronic components can be carried out using the suggested methodologies that rely on the GPHCS. This will ensure that there are no time or financial losses and that the analysis is carried out as efficiently as possible based on a one-parameter lifetime distribution.

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