axioms

Review

A Comprehensive Review of Golden Riemannian Manifolds

Bang-Yen Chen [1,*,†], Majid Ali Choudhary [2,†] and Afshan Perween [2,†]

[1] Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027, USA
[2] Department of Mathematics, School of Sciences, Maulana Azad National Urdu University, Hyderabad 500032, India; majid_alichoudhary@manuu.edu.in (M.A.C.); 23phmm002hy@manuu.edu.in (A.P.)
* Correspondence: chenb@msu.edu
† These authors contributed equally to this work.

Abstract: In differential geometry, the concept of golden structure represents a compelling area with wide-ranging applications. The exploration of golden Riemannian manifolds was initiated by C. E. Hretcanu and M. Crasmareanu in 2008, following the principles of the golden structure. Subsequently, numerous researchers have contributed significant insights with respect to golden Riemannian manifolds. The purpose of this paper is to provide a comprehensive survey of research on golden Riemannian manifolds conducted over the past decade.

Keywords: golden manifold; golden structure; lightlike submanifold; warped product; Chen invariant; Chen inequality; golden semi-Riemannian manifold

MSC: 53C15; 53C25; 53C40; 53C42; 11B39; 53B20; 53B25; 53C20

1. Introduction

For nearly two thousand years, the concept of the golden ratio has fascinated scholars from diverse disciplines. The allure of the golden ratio extends beyond mathematics; it captivates biologists, artists, musicians, historians, architects, and psychologists alike. The golden ratio, known for its aesthetic harmony and proportionality, is extensively employed in iconic architectural structures and artworks, musical composition frameworks, harmonious frequency ratios, and human body measurements. It is likely fair to say that the golden ratio has inspired more scholars in various fields than any other number throughout the history of mathematics [1].

Polynomial structures on a manifold, as discussed in [2], were the foundation for the concept of golden structure. C. E. Hretcanu and M. Crasmareanu investigated some characteristics of the induced structure on an invariant submanifold within a golden Riemannian manifold in [3]. In [4], Crasmareanu and Hretcanu used a corresponding almost product structure to study the geometry of the golden structure on a manifold. In [5], Hretcanu and Crasmareanu demonstrated that a golden structure also induces a golden structure on each invariant submanifold. The issue of integrability for golden Riemannian structures was examined by Gezer et al. in [6]. Ozkan studied a golden semi-Riemannian manifold in [7], where he defined the golden structure's horizontal lift in the tangent bundle. Other structures on golden Riemannian manifolds have been studied by many authors (see, e.g., [8–10]).

However, in the study of differential geometry, the theory of submanifolds is an intriguing subject. Its roots are in Fermat's work on the geometry of surfaces and plane curves. Since then, it has been developed in different directions of differential geometry and mechanics. It is still a vibrant area of study that has contributed significantly to the advancement of differential geometry in the present era. Among all the submanifolds of an ambient manifold, there are two well known types: invariant submanifolds and anti-invariant submanifolds. The differential geometry of submanifolds in golden Riemannian

Citation: Chen, B.-Y.; Choudhary, M.A.; Perween, A. A Comprehensive Review of Golden Riemannian Manifolds. *Axioms* **2024**, *13*, 724. https://doi.org/10.3390/axioms13100724

Academic Editor: Juan De Dios Pérez

Received: 29 August 2024
Revised: 13 October 2024
Accepted: 15 October 2024
Published: 18 October 2024

manifolds was initially studied by Crasmareanu and Hretcanu. Certain characteristics of invariant submanifolds in a Riemannian manifold with a golden structure were investigated by Hretcanu and Crasmareanu in [3], which have been advanced considerably since then. Various classifications of submanifolds within golden Riemannian manifolds have been established based on how their tangent bundles react to the golden structure of the ambient manifold and been explored by numerous geometers. Erdoğan and Yıldırım presented the idea of semi-invariant submanifolds within golden Riemannian manifolds as a generalization of both invariant and anti-invariant types, followed by an analysis of the geometry of their defining distributions [11]. The properties and distributions associated with semi-invariant submanifolds in golden Riemannian manifolds were further explored by Gök, Keleş, and Kılıç [12]. The notion of pointwise slant submanifolds and pointwise bi-slant submanifolds in golden Riemannian manifolds was introduced by Hretcanu and Blaga in [13].

R. L. Bishop and B. O'Neill [14] proposed the idea of a warped product. Warped-product CR submanifolds in a Kähler manifold, which consist of warped products of holomorphic and totally real submanifolds, were first studied by B.-Y. Chen in [15–19]. Later, Riemannian manifolds with golden warped products were studied by Blaga and Hretcanu [20], who also investigated submanifolds with pointwise semi-slant and hemi-slant warped products within locally golden Riemannian manifolds [13].

The similarities between the geometries of semi-Riemannian submanifolds and their Riemannian counterparts are well established, yet the study of lightlike submanifolds presents unique challenges due to the intersection of their normal vector bundles with the tangent bundle. This complexity adds to the intrigue of researching lightlike geometry, which finds practical applications in mathematical physics, notably in general relativity and electromagnetism [21]. Duggal and Bejancu were pioneers in the study of lightlike submanifolds within semi-Riemannian manifolds [21]. In 2017, Poyraz and Yasar explored lightlike hypersurfaces in golden semi-Riemannian manifolds [22] and further extended their research to define lightlike submanifolds of golden semi-Riemannian manifolds in 2019 [23]. Subsequently, numerous researchers have investigated various types of lightlike submanifolds in golden semi-Riemannian manifolds, as evidenced by references [24–29], among others. Additionally, the concept of a lightlike hypersurface in meta-golden Riemannian manifolds was introduced by Erdoğan et al. [30].

Finding the optimal inequality between the intrinsic and extrinsic invariants of a Riemannian submanifold is a key problem in submanifold geometry. Chen [31,32] developed the δ invariants in this context, which are known today as Chen invariants. Utilizing these invariants, along with the mean curvature, which is the key extrinsic invariant of Riemannian submanifolds, he formulated sharp inequality relationships, which are well known as Chen inequalities. The study of Chen invariants and Chen inequalities across various submanifolds in diverse ambient spaces has been thoroughly pursued (see, e.g., [32–34]). Research on Chen inequalities within golden Riemannian manifolds and golden-like statistical manifolds was conducted by Choudhary and Uddin [35] and Bahadir et al. [36], respectively.

Thanks to F. Casorati [37], the widely accepted Gaussian curvature can be substituted with the Casorati curvature. Within the framework of Casorati curvatures for submanifolds across diverse ambient spaces, geometric inequalities have been formulated. The rationale behind the introduction of this curvature by F. Casorati is that it disappears precisely when both principal curvatures of a surface in $\mathbb{E}^3$ are zero, aligning more closely with the typical understanding of curvature. Numerous scholars have explored Casorati curvatures to derive sharp inequalities for specific submanifolds in varied golden ambient spaces (refer to [38–41], etc.). Moreover, in 1979, P. Wintgen [42] introduced a significant geometric inequality involving Gauss curvature, normal curvature, and square mean curvature known as Wintgen's inequality [43]. For slant, invariant, C-totally real, and Lagrangian submanifolds in golden Riemannian spaces, generalized Wintgen-type inequalities were introduced by Choudhary et al. [44].

Motivated by the previously mentioned advancements in the subject, the purpose of this paper is to provide a comprehensive survey of the latest progress on golden Riemannian manifolds achieved over the past decade.

2. Preliminaries

Throughout this paper, let $\Gamma(TM)$ denote the set consisting of all smooth vector fields on a (smooth) manifold (M).

2.1. Golden Structure and Golden Riemannian Manifolds

Take a semi-Riemannian manifold $(\bar{M}, \bar{g})$ and a $(1,1)$ tensor field (F) on $\bar{M}$. Assume that F satisfies the

$$L(X) = X^n + a_n X^{n-1} + \ldots + a_2 X + a_1 I = 0,$$

where I is the identity transformation and $F^{n-1}(p), F^{n-2}(p), \ldots, F(p), I$ (for $X = F$) are linearly independent at each point (p) on $\bar{M}$. Consequently, the polynomial $L(X)$ is referred to as the structure polynomial [2,4,45]. In particular, if we choose the structure polynomial as

$$L(X) = X^2 + I, \text{ then the structure is almost complex;}$$

$$L(X) = X^2 - I, \text{ then the structure is an almost product;}$$

$$L(X) = X^2, \text{ then the structure is an almost tangent.}$$

Definition 1. *Suppose $(\bar{M}, \bar{g})$ is a semi-Riemannian manifold equipped with a $(1,1)$ tensor field (ϕ) on $\bar{M}$. If*

$$\phi^2 - \phi - I = 0,$$

holds, then the tensor field (ϕ) is referred to as a golden structure. If the Riemannian metric ($\bar{g}$) is compatible with ϕ, i.e., $\bar{g}(\phi X, Y) = \bar{g}(X, \phi Y)$ for all $X, Y \in \Gamma(T\bar{M})$, then $(\bar{M}, \bar{g}, \phi)$ is called a golden Riemannian manifold *[2,45,46].*

If we replace X in the foregoing equation with ϕX, we obtain

$$\bar{g}(\phi X, \phi Y) = \bar{g}(\phi^2 X, Y) = \bar{g}(\phi X, Y) + \bar{g}(X, Y).$$

Consider a manifold ($\bar{M}$) endowed with a type-$(1,1)$ tensor field (F), where $F^2 = I$ but F is not equal to $\pm I$. This tensor field (F) is known as an almost product structure. If this structure (F) admits a Riemannian metric ($\bar{g}$) satisfying

$$\bar{g}(FX, Y) = \bar{g}(X, FY), \quad \forall X, Y \in \Gamma(T\bar{M}),$$

hence, $(\bar{M}, \bar{g})$ is called an almost-product Riemannian manifold. Moreover, a golden structure can be produced by an almost product structure (F) in the manner described below.

$$\phi = \frac{1}{2}(I + \sqrt{5}F).$$

On the contrary, if ϕ is a golden structure, then

$$F = \frac{1}{\sqrt{5}}(2\phi - I)$$

is an almost product structure [4,45].

Example 1 ([5])**.** *Take an Euclidean 6-space $\mathbb{R}^6$ with natural coordinates $(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6)$ and suppose $\phi : \mathbb{R}^6 \to \mathbb{R}^6$ symbolizes a $(1,1)$ tensor field defined by*

$$\phi(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6) = (\psi\zeta_1, \psi\zeta_2, \psi\zeta_3, (1-\psi)\zeta_4, (1-\psi)\zeta_5, (1-\psi)\zeta_6)$$

for any vector field $(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6) \in \mathbb{R}^6$, *where* $\psi = \frac{1+\sqrt{5}}{2}$ *and* $1 - \psi = \frac{1-\sqrt{5}}{2}$ *are the roots of the equation* $\zeta^2 = \zeta + 1$. *Then, one can obtain*

$$\begin{aligned}\phi^2(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6) =& \Big(\psi^2\zeta_1, \psi^2\zeta_2, \psi^2\zeta_3, (1-\psi)^2\zeta_4, (1-\psi)^2\zeta_5, (1-\psi)^2\zeta_6\Big) \\ =& (\psi\zeta_1, \psi\zeta_2, \psi\zeta_3, (1-\psi)\zeta_4, (1-\psi)\zeta_5, (1-\psi)\zeta_6) \\ &+ (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6).\end{aligned}$$

Thus, $\phi^2 - \phi - I = 0$. *Moreover,*

$$\langle \phi(\zeta_1, \ldots, \zeta_6), (\Omega_1, \ldots, \Omega_6) \rangle = \langle (\zeta_1, \ldots, \zeta_6), \phi(\Omega_1, \ldots, \Omega_6) \rangle,$$

for every vector field $(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6), (\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6) \in \mathbb{R}^6$, *where the standard metric on* $\mathbb{R}^6$ *is denoted by* $\langle \, , \rangle$. *As a result, the manifold* $(\mathbb{R}^6, \langle \, , \rangle, \phi)$ *is a golden riemannian manifold.*

2.2. Golden-like Statistical Manifolds

An interesting characteristic of golden structures is that they always occur in pairs; for instance, if ϕ represents a golden structure, then $\phi^* = I - \phi$ also qualifies as one. This phenomenon is similarly observed in almost tangent structures (J and $-J$) and almost complex structures (F and $-F$). Consequently, exploring the relationship between golden and product structures becomes a pertinent question. Consider $\bar{M}$ as a Riemannian manifold and let ∇ represent a torsion-free affine connection. The configuration $(\bar{M}, \bar{g}, \nabla)$ is termed a statistical manifold when $\nabla\bar{g}$ is symmetric. Another affine connection (∇^*) is defined by

$$X\bar{g}(Y, Z) = \bar{g}(\nabla_X Y, Z) + \bar{g}(\nabla^*_X Z, Y)$$

for vector fields $X, Y,$, and Z on $\bar{M}$. The affine connection (∇^*) is referred to as the conjugate (or dual) to ∇ relative to $\bar{g}$. This connection (∇^*) is torsion-free, ensures that $\nabla^*\bar{g}$ is symmetric, and complies with the equation $\nabla^* = \frac{\nabla + \nabla^*}{2}$. It is evident that the set $(\bar{M}, \nabla^*, \bar{g},)$ forms a statistical manifold. Curvature tensors R and R^* on $\bar{M}$ correspond to affine connections ∇ and ∇^*, respectively. Additionally, the curvature tensor field (R^0) linked with ∇^0 is known as the Riemannian curvature tensor. Then [4,22,46]

$$\bar{g}(R(X, Y)Z, W) = -\bar{g}(Z, R^*(X, Y)W)$$

for vector fields $X, Y, Z,$ and W on $\bar{M}$, where

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z.$$

Generally, since the dual connections are not metric, the sectional curvature is unable to be defined in a statistical setting as it is in semi-Riemannian geometry. Hence, Opozda introduced two types of sectional curvature on statistical manifolds (See [47,48]).

Considering that π is a plane section in $T\bar{M}$ with an orthonormal basis ($\{X, Y\}$), where $\bar{M}$ is a statistical manifold, the definition of the sectional K curvature is [47]

$$K(\pi) = \frac{1}{2}[\bar{g}(R(X, Y)Y, X) + \bar{g}(R^*(X, Y)Y, X) - \bar{g}(R^0(X, Y)Y, X)].$$

Definition 2 ([36])**.** *Let* $(\bar{M}, \bar{g}, \phi)$ *be a golden semi-Riemannian manifold equipped with a tensor field* ϕ^* *of type* $(1, 1)$ *satisfying*

$$\bar{g}(\phi X, Y) = \bar{g}(X, \phi^* Y) \tag{1}$$

for vector fields $X, Y \in \Gamma(T\bar{M})$. *In view of (1), we have*

$$(\phi^*)^2 X = \phi^* X + X, \tag{2}$$

$$\bar{g}(\phi X, \phi^* Y) = \bar{g}(\phi X, Y) + \bar{g}(X, Y). \tag{3}$$

Then, $(\bar{M}, \bar{g}, \phi)$ is called a golden-like statistical manifold.

According to (2) and (3), the tensor $\phi + \phi^*$ and $\phi - \phi^*$ are symmetric and skew-symmetric with respect to $\bar{g}$, respectively.

Equations (1)–(3) imply the following proposition.

Proposition 1 ([36])**.** *$(\bar{M}, \bar{g}, \phi)$ is a golden-like statistical manifold if and only if it is $(\bar{M}, \bar{g}, \phi^*)$.*

Remark 1. *If one chooses $\phi = \phi^*$ in a golden-like statistical manifold, then we have a golden semi-Riemannian manifold.*

2.3. Golden Lorentzian Manifolds

For the locally golden space form of $\bar{M} = \bar{M}_p(c_p) \times \bar{M}_q(c_q)$, where c_p and c_q are constant sectional curvatures of Riemannian manifolds $\bar{M}_p$ and $\bar{M}_q$, respectively, the Riemannian curvature tensor (R) is expressed as follows in [49]:

$$\begin{aligned} R(X,Y)Z =& \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q}{10}[\bar{g}(Y,Z)X - \bar{g}(X,Z)Y] \\ &+ \frac{(\mp\sqrt{5}-1)c_p + (\pm\sqrt{5}-1)c_q}{10}[\bar{g}(\phi Y,Z)X - \bar{g}(\phi X,Z)Y \\ &+ \bar{g}(Y,Z)\phi X - \bar{g}(X,Z)\phi Y] + \frac{c_p + c_q}{5}[\bar{g}(\phi Y,Z)\phi X - \bar{g}(\phi X,Z)\phi Y] \end{aligned}$$

where X, Y and $Z \in \Gamma(TN)$.

The golden Lorentzian manifold is defined as follows.

Definition 3 ([50])**.** *Let us consider a semi-Riemannian manifold $(\bar{M}^n, \bar{g})$ where $\bar{g}$ has a signature of $(-,+,+,\cdots,+)$ ($+$ appears $(n-1)$ times). Then, $\bar{M}$ stands for a golden Lorentzian manifold if it is endowed with a golden structure (ϕ) and $\bar{g}$ is ϕ-compatible.*

Example 2 ([50])**.** *Let $\mathbb{R}^3_1$ represent the semi-Euclidean space and consider the signature of $\bar{g}$ as $(-,+,+)$. If ϕ stands for a $(1,1)$ tensor field, then it is easy to show that if*

$$\phi(\zeta_1,\zeta_2,\zeta_3) = \frac{1}{2}\Big(\zeta_1 + \sqrt{5}\zeta_2, \zeta_2 + \sqrt{5}\zeta_1, 2\psi\zeta_3\Big),$$

for any vector field $(\zeta_1,\zeta_2,\zeta_3) \in \mathbb{R}^3_1$, where $\psi = \frac{1+\sqrt{5}}{2}$ is the golden mean, then

$$\phi^2 = \phi + I,$$

and hence, ϕ is a golden structure on $\mathbb{R}^3_1$. Moreover, $\bar{g}$ may also be verified to be ϕ-compatible. Thus, $(\mathbb{R}^3_1, \bar{g}, \phi)$ becomes a golden Lorentzian manifold.

2.4. Meta-Golden Semi-Riemannian Manifolds

The following structure is comparable to the golden Ratio (see [51]). The authors of [30] obtained $\chi = \frac{1}{\psi} + \frac{1}{\chi'}$, where the meta-golden Chi ratio is $\chi = \frac{1+\sqrt{4\psi+5}}{2\psi}$ and $\psi = \frac{1+\sqrt{5}}{2}$, which suggests that $\chi^2 - \frac{1}{\psi}\chi - 1 = 0$. Thus, the roots are are given by $\frac{1}{2}\big(\frac{1}{\psi} \pm \sqrt{4+\frac{1}{\psi^2}}\big)$. The correlation between the meta-golden Chi ratio (χ) and continued fractions was found in [51]. If we denote the positive root by $\chi = \frac{1}{2}\big(\frac{1}{\psi} + \sqrt{4+\frac{1}{\psi^2}}\big)$ and the negative root by $\chi^{\cdot} = \frac{1}{2}\big(\frac{1}{\psi} - \sqrt{4+\frac{1}{\psi^2}}\big)$, then we have [51]

$$\chi^{\cdot} = \frac{\chi}{\psi}, \tag{4}$$

$$\psi\chi^2 = \psi + \chi, \tag{5}$$

$$\psi\chi^{\cdot 2} = \psi + \chi^{\cdot}. \tag{6}$$

It was stated in [4] that an endomorphism (ϕ) on a manifold ($\bar{M}$) is an almost golden structure if it satisfies

$$\phi^2 X = \phi X + X \tag{7}$$

for any $X \in \Gamma(T\bar{M})$. Hence, given a semi-Riemannian metric ($\bar{g}$) on $\bar{M}$, $(\bar{g}, \phi)$ is referred to as an almost golden semi-Riemannian structure if

$$\bar{g}(\phi X, Y) = \bar{g}(X, \phi Y) \tag{8}$$

for $X, Y \in \Gamma(T\bar{M})$. Therefore, $(\bar{M}, \bar{g}, \phi)$ is called an almost golden semi-Riemannian manifold. Because of (8), we obtain [4]

$$\bar{g} = (\phi X, \phi Y) = \bar{g}(X, \phi Y) + \bar{g}(X, Y). \tag{9}$$

Definition 4 ([52])**.** *Let F be a $(1,1)$ tensor field on an almost golden manifold $(\bar{M}, \phi)$ that satisfies*

$$\phi F^2 X = \phi X + FX$$

for every $X \in \Gamma(T\bar{M})$. Then, F is called an almost meta-golden structure and $(\bar{M}, \phi, F)$ is called an almost meta-golden manifold.

Theorem 1 ([52])**.** *A $(1,1)$ tensor field (F) on an almost golden manifold $(\bar{M}, \phi)$ is an almost meta-golden structure if*

$$F^2 = \phi F - F + 1.$$

Definition 5 ([30])**.** *Let F be an almost meta-golden structure on $(\bar{M}, \phi, \bar{g})$. If F is compatible with a semi-Riemannian metric ($\bar{g}$) on $\bar{M}$, namely*

$$\bar{g}(FX, Y) = \bar{g}(X, FY),$$

or

$$\bar{g}(FX, FY) = \bar{g}(\phi X, FY) - \bar{g}(X, FY) + \bar{g}(X, Y),$$

then $(\bar{M}, \phi, F, \bar{g})$ is known as an almost meta-golden semi-Riemannian manifold, where for X, $Y \in \Gamma(T\bar{M})$.

Remark 2. *An almost meta-golden semi-Riemannian manifold is called a meta-golden semi-Riemannian manifold if $\bar{\nabla} F = 0$, where $\bar{\nabla}$ is a Levi–Civita connection of $\bar{M}$. In this instance, we also have $\bar{\nabla}\phi = 0$. In the remainder of this paper,almost meta-golden semi-Riemannian manifolds and meta-golden semi-Riemannian manifolds are referred to as AMGsR manifolds and MGsR manifolds, respectively.*

2.5. Norden Golden Manifolds

The notion of an almost Norden golden manifold can be recalled from [4,53]. Let $\bar{M}$ be a manifold and ϕ be an endomorphism on $\bar{M}$ such that

$$\phi^2 = \phi - \frac{3}{2} I.$$

Then, $(\bar{M}, \phi)$ is called an almost complex golden manifold. Let $\bar{g}$ be a semi-Riemannian metric on $\bar{M}$ such that

$$\bar{g}(\phi X, Y) = \bar{g}(X, \phi Y), \tag{10}$$

Then, $(\bar{M}, \phi, \bar{g})$ is called an almost Norden golden manifold. Note that (10) is comparable to

$$\bar{g}(\phi X, \phi Y) = \bar{g}(\phi X, Y) - \frac{3}{2}\bar{g}(X, Y).$$

Moreover, if ϕ is parallel with respect to a vector field (X) on $\bar{M}$, $(\nabla_X \phi = 0)$, then $(\bar{M}, \phi, \bar{g})$ is called a locally decomposable almost Norden golden semi-Riemannian manifold (in short, Norden golden semi-Riemannian manifold).

2.6. Golden Warped-Product Riemannian Manifolds

Let n and m be the dimensions of two Riemannian manifolds ($(\bar{M}_1, \bar{g}_1)$ and $(\bar{M}_2, \bar{g}_2)$, respectively). Projection maps P and Q from product manifolds $\bar{M}_1 \times \bar{M}_2$ to $\bar{M}_1$ and $\bar{M}_2$ and the lift to $\bar{M}_1 \times \bar{M}_2$ of a smooth function φ on $\bar{M}_1$ are indicated by $\bar{\varphi}_e := \varphi \circ P$.

In this context, we call $\bar{M}_1$ the base and $\bar{M}_2$ the fiber of $\bar{M}_1 \times \bar{M}_2$. The unique element ($\bar{X}$ of $T(\bar{M}_1 \times \bar{M}_2)$) that is P-related to $X \in T(\bar{M}_1)$ and to the zero vector field on $\bar{M}_2$ is the horizontal lift of X, and the unique element $\bar{Y}$ of $T(\bar{M}_1 \times \bar{M}_2)$ that is Q-related to $Y \in T(\bar{M}_2)$ and to the zero vector field on $\bar{M}_1$ is called the vertical lift of Y. Furthermore, $\mathcal{L}(\bar{M}_1)$ is the set of all horizontal lifts of vector fields on $\bar{M}_1$, and $\mathcal{L}(\bar{M}_2)$ is the set of all vertical lifts of vector fields on $\bar{M}_2$.

Let $f > 0$ be a smooth function on $\bar{M}_1$ and

$$\bar{g} := P^*\bar{g}_1 + (f \circ P)^2 Q^* \bar{g}_2 \tag{11}$$

be the Riemannian metric on $\bar{M}_1 \times \bar{M}_2$ [20]. According to [14], the product manifold of $\bar{M}_1$ and $\bar{M}_2$, together with the Riemannian metric $(\bar{g})$ defined by (11) is called a *warped product* of $\bar{M}_1$ and $\bar{M}_2$ by the warping function (f) and is denoted by $\bar{M} := (\bar{M}_1 \times_f \bar{M}_2, \bar{g})$.

The golden warped-product Riemannian manifold was defined by Blaga and Hretcanu in [13] as follows.

Theorem 2 ([20])**.** *Let $\left(\bar{M} = \bar{M}_1 \times_f \bar{M}_2, \bar{g}, \phi\right)$ be the warped product of two locally golden Riemannian manifolds ($(\bar{M}_1, \bar{g}_1, \phi_1)$ and $(\bar{M}_2, \bar{g}_2, \phi_1)$). Then, $\bar{M}$ is locally golden if and only if*

$$\begin{cases} (df^2 \circ \phi_1) \otimes I = df^2 \otimes \phi_2 \\ \bar{g}_2(\phi_1 \cdot, \cdot) \cdot \mathrm{grad}(f^2) = \bar{g}_2(\cdot, \cdot) \cdot \phi_1(\mathrm{grad}(f^2)). \end{cases}$$

3. Submanifolds Immersed in Riemannian Manifolds with Golden Structure

3.1. Invariant Submanifolds with Golden Structure

Among all the submanifolds of an ambient manifold, invariant submanifolds are a common class. It is commonly known that practically every property of an ambient manifold is inherited by an invariant submanifold. As a result, invariant submanifolds represent a dynamic and productive area of study that has greatly influenced the advancement of modern differential geometry. Numerous articles concerning invariant submanifolds in golden Riemannian manifolds have been published.

In [3], Hretcanu and Crasmareanu studied invariant submanifolds in golden Riemannian manifolds and proved the following three propositions:

Proposition 2. *Let N be an n-dimensional submanifold of a golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$ of codimension r and let $(\phi', g, u_\alpha, \varepsilon\xi_\alpha, (a_{\alpha\beta})_r)$, $\alpha, \beta \in \{1, 2, ..., r\}$ be the structure on N induced by structure $(\bar{g}, \phi)$, where u_α is the 1 form, ξ_α represents tangent vector fields, and $\left(a_{\alpha\beta}\right)_r$ is an $r \times r$ matrix of a real function on N. Then, an essential and sufficient requirement for N to be invariant is that the induced structure (ϕ', g) on N be a golden Riemannian structure whenever ϕ' is non-trivial.*

Proposition 3. *If (N, g, ϕ') is an invariant submanifold of codimension r in a golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$ with $\bar{\nabla}\phi = 0$ and $(\phi', g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ is the structure on N induced by $(\phi, \bar{g})$*

(where $\bar{\nabla}$ is the Levi–Civita connection defined on N with respect to g), then the Nijenhuis torsion tensor field of ϕ' vanishes identically on N.

Proposition 4. *Let N be an invariant submanifold of codimension r in a golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$ with $\bar{\nabla}\phi = 0$ and let $\left(\phi', g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r\right)$ be the induced structure on N. If normal connection $\nabla^\perp$ on normal bundle $TN^\perp$ vanishes identically $(l_{\alpha\beta} = 0)$, then components $\mathcal{N}^{(1)}, \mathcal{N}^{(2)}, \mathcal{N}^{(3)}$, and $\mathcal{N}^{(4)}$ of the Nijenhuis torsion tensor field of ϕ' for the structure $\left(\phi', g, \xi_\alpha, u_\alpha, (a_{\alpha\beta})_r\right)$ induced on N have the following forms:*

$$\begin{cases} \text{(i) } \mathcal{N}^{(1)}(X,Y) = \mathcal{N}^{(4)}_{\alpha\beta}(X) = 0, \\ \text{(ii) } \mathcal{N}^{(2)}_{\alpha}(X,Y) = -\sum_\beta a_{\alpha\beta} g\big((\phi' A_\beta - A_\beta \phi')(X), Y\big) \\ \text{(iii) } \mathcal{N}^{(3)}_{\alpha}(X) = \sum_\beta a_{\alpha\beta}(\phi' A_\beta - A_\beta \phi')(X) - \phi'(\phi' A_\alpha - A_\alpha \phi')(X), \end{cases}$$

for any $X, Y \in \Gamma(TN)$.

Remark 3 ([3])**.** *Under the conditions of Proposition 4, if $\phi' A_\alpha = A_\alpha \phi'$, where A is the shape operator for $\alpha \in \{1, 2, \ldots, r\}$, then components $\mathcal{N}^{(1)}, \mathcal{N}^{(2)}, \mathcal{N}^{(3)}$, and $\mathcal{N}^{(4)}$ vanish identically on N.*

Inspired by [3], in [12], Gök et al. demonstrated the local decomposability of any invariant submanifold of a golden Riemannian manifold and came up with a definition of invariance for submanifolds in a golden Riemannian manifold. They also determined the prerequisites that must be met for any invariant submanifold to be totally geodesic.

Theorem 3 ([12])**.** *Let N be an invariant submanifold of a locally decomposable golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$. Then, N is a locally decomposable golden Riemannian manifold whenever the induced structure (ϕ') on N is non-trivial.*

Theorem 4 ([12])**.** *Let N be a submanifold of a golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$. Then, N is an invariant submanifold if and only if there exists a local orthonormal frame of the normal bundle $(TN^\perp)$ such that it consists of eigenvectors of the golden structure (ϕ).*

Remark 4. *The result of N as a totally geodesic invariant submanifold was also obtained in [12] by Gök et al.*

3.2. Anti-Invariant Submanifolds in Golden Riemannian Manifolds

Some properties of an anti-invariant submanifold of a golden Riemannian manifold were studied in [54], and some necessary prerequisites for any submanifold in a locally decomposable golden Riemannian manifold to be anti-invariant were obtained. Any anti-invariant submanifold (N) of a golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$ is a submanifold such that the golden structure (ϕ) of the ambient manifold ($\bar{M}$) carries each tangent vector of the submanifold (N) into its corresponding normal space in the ambient manifold ($\bar{M}$), that is,

$$\phi(T_x N) \subseteq T_x N^\perp$$

for each point $x \in N$ (see [54]).

Theorem 5 ([54])**.** *Let N be an n-dimensional submanifold of a 2n-dimensional locally decomposable golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$. Then, for any $\alpha, \beta \in \{1, 2, \ldots n\}$, N is an anti-invariant submanifold whenever $a_{\alpha\beta} = \delta_{\alpha\beta}$. In addition, the submanifold (N) is totally geodesic.*

Theorem 6 ([11])**.** *Let N be an n-dimensional submanifold of a 2n-dimensional locally decomposable golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$. If $\phi^{-}1(\eta_\alpha) \in \Gamma(TN)$ for any $\alpha \in \{1, 2, \ldots n\}$, then N is an anti-invariant submanifold. Furthermore, the submanifold (N) is totally geodesic.*

Remark 5. *In [54], M. Gök et al. also obtained results with respect to the existence of an orthonormal frame of an anti-invariant submanifold of a locally decomposable golden Riemannian manifold.*

Gök and Kılıç [55] studied a non-invariant submanifold of a locally decomposable golden Riemannian manifold in a case in which the rank of the set of tangent vector fields of the structure on the submanifold induced by the golden structure of the ambient manifold is less than or equal to the co-dimension of the submanifold.

Theorem 7 ([55]). *Let N be a submanifold of codimension r in a locally decomposable golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$. If the tangent vector fields $(\xi_1, \ldots, \xi_r)$ are linearly independent and $\nabla\phi' = 0$, then N is a totally geodesic submanifold.*

Theorem 8 ([55]). *Let N be a submanifold of codimension r in a locally decomposable golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$, where λ_α is an eigenvalue of the matrix $(a_{\alpha\beta})_{r\times r}$. If $a_{\alpha\beta} = \lambda_\alpha\delta_{\alpha\beta}, \lambda_\alpha \in (1-\psi, \psi)$ for any $\alpha, \beta \in \{1, \ldots, r\}$, and $\nabla\phi' = 0$, then N is totally geodesic.*

Theorem 9 ([55]). *Let N be a submanifold of codimension r in a locally decomposable golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$. If $a_{\alpha\beta} = \lambda_\alpha\delta_{\alpha\beta}, \lambda_\alpha \in (1-\psi, \psi)$ for any $\alpha, \beta \in \{1, \ldots, r\}$, $\mathrm{Trace}\,\phi'$ is constant, and N is totally umbilical, then N is totally geodesic.*

Remark 6. (i) *Gök and Kılıç [55] also obtained some results on the non-invariant submanifold if the tangent vector fields of the induced structure are linearly dependent.*

(ii) *The stability problem of certain anti-invariant submanifolds in golden Riemannian manifolds was discussed by the same authors in [56].*

(iii) *Effective relations for certain induced structures on a submanifold of codimension 2 in golden Riemannian manifolds were obtained in [57].*

3.3. Slant Submanifolds of Golden Riemannian Manifolds

Given a golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$, let (N, g) be one of its submanifolds and g be the induced metric on N. Then, we can write

$$\phi X = PX + QX$$

for any $X \in \Gamma(TN)$, where PX and QX are the tangent and transversal components of ϕX, respectively.

A submanifold (N, g) of a golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$ is referred to as a slant submanifold if, at any point (x), each nonzero vector tangent to N and the angle between TN and $\phi(X)$, as represented by $\theta(X)$, are independent of the selection of $x \in N$ and $X \in T_xN$. It can also be seen that N is a ϕ-invariant (resp. ϕ-anti-invariant) submanifold if the slant angle is $\theta = 0$ (resp. $\theta = \frac{\pi}{2}$). The term proper slant (or θ-slant proper) submanifold refers to a slant submanifold that is neither anti-invariant nor invariant.

Using a golden Riemannian manifold, Uddin and Bahadir [45] establish the following characterization of slant submanifolds.

Theorem 10 ([45]). *Assume that a golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$ has a submanifold (N, g). Consequently, N is a slant submanifold if and only if $c \in [0, 1]$ is a constant such that*

$$P^2 = c(\phi + I).$$

Additionally, if θ represents the slant angle of N, then $c = \cos^2\theta$.

Corollary 1 ([45]). *Take a golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$ and let (N, g) be a submanifold of it. After that, N is a slant submanifold if and only if $c \in [0, 1]$ exists and ensures*

$$\phi^2 = \frac{1}{c} P^2,$$

where $c = \cos^2\theta$ and θ are slant angles of N.

Remark 7. *In [45] Uddin and Bahadir also derived some results of ϕ-invariant and ϕ-anti-invariant submanifolds of a golden Riemannian manifold and provided some examples of such submanifolds.*

3.4. Semi-Invariant Submanifolds of Golden Riemannian Manifolds

Definition 6 ([58])**.** *Given a golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$, consider N to be a real submanifold of $\bar{M}$. If N is equipped with a pair of orthogonal distributions $(D, D^{\perp})$ that meet the given conditions, it can be deemed a semi-invariant submanifold of $\bar{M}$.*

(i) *$TN = D \oplus D^{\perp}$;*

(ii) *The distribution (D) is invariant, i.e., $\phi D_x = D_x$ for each $x \in N$;*

(iii) *The distribution $(D^{\perp})$ is anti-invariant, i.e., $\phi D^{\perp} \subset T_x N^{\perp}$ for each $x \in N$.*

For any $x \in N$, a semi-invariant submanifold (N) is considered invariant and anti-invariant if $D_x^{\perp} = 0$ and $D_x = 0$, respectively.

The following results were obtained by Erdogan et al. for semi-invariant submanifolds of the golden Riemannian manifold investigated in [58].

Theorem 11 ([58])**.** *Assume that N is a semi-invariant submanifold of $(\bar{M}, \bar{g}, \phi)$, the golden Riemannian manifold. Consequently, the distribution (D) is integrable if and only if*

$$h(X, \phi' Y) = h(Y, \phi' X)$$

where h is the second fundamental form and $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.

Theorem 12 ([58])**.** *Let N be a semi-invariant submanifold of the golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$. Then, distribution D is integrable if and only if*

$$\phi' A_{\phi' X} Y = A_{\phi' X} Y$$

has no components in D, where A is a shape operator and for every $X, Y \in \Gamma(D^{\perp})$ and $Z \in \Gamma(D)$.

Remark 8. *The conditions for distributions D and $D^{\perp}$ of semi-invariant submanifolds of golden Riemannian manifolds to be a totally geodesic foliation were examined in [58]. The condition for a semi-invariant submanifold (N) to be totally geodesic was also covered. Readers can also refer to [45] for conclusions of a similar kind for semi-invariant submanifolds of a golden Riemannian manifold.*

Remark 9. *M. Gök et al. [59] proposed specific characterizations for every submanifold of a golden Riemannian manifold to be semi-invariant in terms of canonical structures on the submanifold as a consequence of the ambient manifold's golden structure.*

Totally umbilical, semi-invariant submanifolds of golden Riemannian manifolds were studied in [11].

Theorem 13 ([11])**.** *Let N be a totally umbilical submanifold of a golden Riemannian manifold $(\bar{M})$. Then, distribution D is always integrable.*

Theorem 14 ([11])**.** *Let N be a totally umbilical submanifold of a golden Riemannian manifold $(\bar{M}, \phi)$. Then, $D^{\perp}$ is integrable.*

Remark 10. *Moreover, the properties of semi-invariant submanifolds and totally umbilical, semi-invariant submanifolds of golden Riemannian manifolds with constant sectional curvatures were studied by Sahin et al. in [60].*

3.5. Skew Semi-Invariant Submanifolds

In [61], Ahmad and Qayyoom studied skew semi-invariant submanifolds in a golden Riemannian manifold and in the a locally golden Riemannian manifold.

Definition 7 ([61])**.** *A submanifold (N) of a golden Riemannian manifold ($\bar{M}$) is defined as a skew semi-invariant submanifold if there exists an integer (k) and constant functions ($\alpha_i, 1 \leq i \leq k$) defined on N with values in the range of $(0, 1)$ such that*

(i) *Each $\alpha_i, 1 \leq i \leq k$ is a distinct eigenvalue of ϕ^2 with*

$$T_x N = D_x^0 \oplus D_x^1 \oplus D_x^{\alpha_1} \oplus \cdots \oplus D_x^{\alpha_k}$$

for $x \in N$, and (ii) *The dimensions of D_x^0, D_x^1 and $D_k^1, 1 \leq i \leq k$ are independent of $x \in N$.*

Remark 11. *The tangent bundle of N has the following decomposition:*

$$TN = D^0 \oplus D^1 \oplus D^{\alpha_1} \oplus \cdots \oplus D^{\alpha_k}.$$

If $k = 0$, then N is a semi-invariant submanifold. Also, if $k = 0$ and D_x^0 (D_x^1) are trivial, then N is an invariant (or anti-invariant) submanifold of $\bar{M}$.

Definition 8 ([61])**.** *A submanifold (N) of a locally golden Riemannian manifold ($\bar{M}$) is defined as a skew semi-invariant submanifold of order 1 if N is a skew semi-invariant submanifold with $k = 1$. In this case, we have*

$$TN = D^{\perp} \oplus D^T \oplus D^{\theta},$$

where $D^{\theta} = D^{\alpha_1}$ and α_1 are constant. A skew semi-invariant submanifold of order 1 is proper if $D^{\perp} \neq 0$ and $D^T \neq 0$.

Remark 12. *Some lemmas for proper skew semi-invariant submanifolds of a locally golden Riemannian manifold were also discussed in [61].*

3.6. Pointwise Slant Submanifolds in Golden Riemannian Manifolds

The notion of slant submanifolds in almost Hermitian manifolds was first introduced by the first author in [62–64]. Later, the first author and Garay [65] extended the notion of slant submanifolds to pointwise slant submanifolds in almost Hermitian manifolds. Hretcanu and Blaga [13] defined the notion of pointwise slant submanifolds of golden Riemannian manifolds as follows.

A submanifold N of a golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$ is referred to as a *pointwise slant* [13] if, at every point ($x \in N$), the angle ($\theta_x(X)$) between ϕ_x and $T_x N$ (called the Wirtinger angle) is consistent, regardless of the nonzero tangent vector ($X \in T_x N \setminus \{0\}$), but it depends on $x \in N$. The Wirtinger angle is a real-valued function (θ; called a Wirtinger function) verifying

$$\cos\theta_x = \frac{\bar{g}(\phi X, TX)}{\|\phi X\| \cdot \|TX\|} = \frac{\|TX\|}{\|\phi X\|},$$

for any $x \in N$ and $X \in T_x N \setminus \{0\}$. If the Wirtinger function (θ) of a pointwise slant submanifold of a golden Riemannian manifold is globally constant, it is referred to as a *slant submanifold*.

Proposition 5 ([13])**.** *In a golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$, if N is an isometrically immersed submanifold and T is the map, then N is a pointwise slant submanifold if and only if*

$$T^2 = \left(\cos^2\theta_x\right)(T + I),$$

for some real-valued function ($x \mapsto \theta_x$) for $x \in N$.

Proposition 6 ([13]). *Let N be a submanifold of a golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$ that is isometrically immersed. Given N as a pointwise slant submanifold and θ_x as its Wirtinger angle, then*

$$\left(\nabla_X T^2\right)Y = \left(\cos^2\theta_x\right)(\nabla_X T)Y - \sin(2\theta_x)X(\theta_x)(TY + Y),$$

for any $X, Y \in T_xN\backslash\{0\}$ and any $x \in N$.

3.7. Pointwise Bi-Slant Submanifolds in Golden Riemannian Manifolds

Consider N as an immersed submanifold within a golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$. We define N as a *pointwise bi-slant submanifold* of $\bar{M}$ if there exist two orthogonal distributions (D and $D^{\perp}$) on N such that

(i) $TN = D \oplus D^{\perp}$;

(ii) $\phi(D) \perp D^{\perp}$ and $\phi(D^{\perp}) \perp D$;

(iii) Distributions D and $D^{\perp}$ are pointwise slant with slant functions of θ_{1x} and θ_{2x}, respectively, for $x \in N$. The pair $\{\theta_1, \theta_2\}$ of slant functions is referred to as the bi-slant function.

A pointwise bi-slant submanifold (N) is called *proper* if its bi-slant functions ($\theta_1, \theta_2 \neq 0; \frac{\pi}{2}$) and neither θ_1 or θ_2 are constant on N. Specifically, if $\theta_1 = 0$ and $\theta_2 \neq 0; \frac{\pi}{2}$, then N is called a pointwise semi-slant submanifold; if $\theta_1 = \frac{\pi}{2}$ and $\theta_2 \neq 0; \frac{\pi}{2}$, then N is called a pointwise hemi-slant submanifold. $T(D) \subseteq D$ and $T\left(D^{\perp}\right) \subseteq D^{\perp}$ are verified by distributions D and $D^{\perp}$ on N if N is a pointwise bi-slant submanifold of $\bar{M}$ [13].

Remark 13. *Some examples of pointwise bi-slant submanifolds in golden Riemannian manifolds were given in [13], where Blaga and Hretcanu provided fundamental lemmas for pointwise bi-slant, pointwise semi-slant, and pointwise hemi-slant submanifolds in locally golden Riemannian manifolds.*

3.8. CR Submanifolds of a Golden Riemannian Manifold

A submanifold (N) within a golden Riemannian manifold ($\bar{M}$) is called a CR submanifold if there exists a differentiable distribution ($D : X \to D_x \subseteq T_xN$) on N that meets the following criteria:

(i) D is holomorphic, meaning $\phi D_x = D_x$ for every $x \in N$; and

(ii) The orthogonal complementary distribution ($D^{\perp} : x \to D^{\perp} \subseteq T_xN$) is completely real, i.e., $\phi D^{\perp} \subset T_xN^{\perp}$ for each $x \in N$. If $\dim D_x^{\perp} = 0$ (or $\dim D_x = 0$), then the CR submanifold (N) is a holomorphic submanifold (or a totally real submanifold). If $\dim D_x^{\perp} = \dim T_xN^{\perp}$, then the CR submanifold is an anti-holomorphic submanifold (or a generic submanifold). A submanifold is considered a proper CR submanifold if it is neither holomorphic nor totally real [66].

The authors of [66] defined and studied CR submanifolds of a golden Riemannian manifold.

Proposition 7 ([66]). *Let N be a CR submanifold of a locally golden Riemannian manifold ($\bar{M}$). Then,*

$$\bar{g}(\phi A_{\phi Y}X, Z) + \bar{g}(\nabla_X Y, \phi Z) + \bar{g}(\nabla_X Y, Z) = 0,$$

$$A_{\phi\xi'}Z = -A - \xi'\phi Z,$$

$$A_{\phi Y}W = A_{\phi W}Y$$

for $X \in TN, Z \in D, Y, W \in D^{\perp}$ and $\xi' \in \mathcal{V}$, where $\mathcal{V}$ is the complementary orthogonal sub-bundle of $\phi(D^{\perp})$ in $TN^{\perp}$; ξ' is the unit normal vector field; and X, Y, Z, and W are vector fields.

Lemma 1 ([66]). *Consider N a CR submanifold of $\bar{M}$, a locally golden Riemannian manifold. Given any $Y, W \in D^{\perp}$, then*

$$(\nabla^{\perp}_W \phi Y - \nabla^{\perp}_Y \phi W) \in \phi D^{\perp}.$$

Remark 14. *The integral condition of the D of CR submanifolds of a golden Riemannian manifold was also discussed in [66].*

The following outcomes were attained by Ahmad and Qayyoom in [66] from their study of totally umbilical CR submanifolds of golden Riemannian manifolds:

Lemma 2 ([66])**.** *Assume that N is a totally umbilical CR submanifold of $\bar{M}$, a locally golden Riemannian manifold. Then, either H, the mean curvature vector, is perpendicular to $\phi(D^{\perp})$ or the totally real distribution ($D^{\perp}$) is one-dimensional.*

Theorem 15 ([66])**.** *For a locally golden Riemannian manifold ($\bar{M}$), let N be a totally umbilical CR submanifold. Therefore, $\bar{K}(\pi) = 0$ for every CR-section π, i.e., the CR-sectional curvature of $\bar{M}$ vanishes.*

4. Warped Product Manifolds in Golden Riemannian Manifolds

A golden warped product Riemannian manifold was defined by Blaga et al. in [20] as mentioned in Theorem (2). The authors of [20] also studied its properties with a special view towards its curvature and attained the following outcomes:

Theorem 16 ([20])**.** *Let $\left(\bar{M} = \bar{M}_1 \times_f \bar{M}_2, \bar{g}, \phi\right)$ (with $\bar{g}$ given by Equation (11)) be the warped product of golden Riemannian manifolds $(\bar{M}_1, \bar{g}_1, \phi_1)$ and $(\bar{M}_2, \bar{g}_2, \phi_2)$. If $\bar{M}_1$ and $\bar{M}_2$ have ϕ_1- and ϕ_2-invariant Ricci tensors, respectively (i.e., $Q_{\bar{M}_i} \circ \phi_i = \phi_i \circ Q_{\bar{M}_i}, i \in \{1, 2\}$), then $\bar{M}$ has a ϕ-invariant Ricci tensor if and only if we have*

$$\mathrm{Hess}(f)(\phi_1 \cdot, \cdot) - \mathrm{Hess}(f)(\cdot, \phi_1 \cdot) \in \{0\} \times T(\bar{M}_2).$$

Remark 15. *The authors of [20] also provided examples of a golden warped-product Riemannian manifold.*

In [13], Blaga and Hretcanu studied warped-product pointwise bi-slant submanifolds and warped-product pointwise semi-slant or hemi-slant submanifolds in golden Riemannian manifolds and derived the following results:

Definition 9 ([13])**.** *The warped product ($N_1 \times_f N_2$) of two pointwise slant submanifolds (N_1 and N_2) within a golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$ is referred to as a warped-product pointwise bi-slant submanifold. Furthermore, the pointwise bi-slant submanifold ($N_1 \times_f N_2$) is termed proper if both submanifolds (N_1 and N_2) are proper pointwise slant in $(\bar{M}, \bar{g}, \phi)$.*

Definition 10 ([13])**.** *Consider $N := N_1 \times_f N_2$ as a warped-product bi-slant submanifold within a golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$, where one of the components ($N_i\,; i \in \{1, 2\}$) is either an invariant submanifold or an anti-invariant submanifold in $\bar{M}$ and the other component is a pointwise slant submanifold in $\bar{M}$ with a Wirtinger angle of $\theta_x \in [0, \frac{\pi}{2}]$. In this context, the submanifold (N) is referred to as a warped-product pointwise semi-slant submanifold or a warped-product pointwise hemi-slant submanifold in the golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$.*

Theorem 17 ([13])**.** *Consider $N := N_T \times N_\theta$ a warped-product pointwise semi-slant submanifold within a locally golden Riemannian manifold $(\bar{M}, \bar{g}, \phi)$ with a pointwise slant angle of $\theta_x \in \left(0, \frac{\pi}{2}\right)$ for $x \in N_\theta$; then, the warping function (f) is constant on the connected components of N_T.*

Remark 16. *Blaga and Hretcanu [13,67] gave examples of warped-product pointwise bi-slant submanifolds, warped-product semi-slant submanifolds, and warped-product hemi-slant submanifolds within a golden Riemannian manifold. Additionally, they discussed various results related to warped-product pointwise semi-slant submanifolds and warped-product pointwise hemi-slant submanifolds.*

In [61], Ahmad and Qayyoom introduced and examined warped-product skew semi-invariant submanifolds within a locally golden Riemannian manifold. They also explored the required and sufficient conditions for a skew semi-invariant submanifold in such a manifold to be classified as a locally warped product.

Proposition 8 ([61])**.** *For a locally golden Riemannian manifold ($\bar{M}$), let $N = N_1 \times_f N_T$ be a $\left(D^\theta, D^T\right)$-mixed, totally geodesic, proper skew semi-invariant submanifold with an integrable distribution ($D^\perp$). Then, N is a locally warped product submanifold if*

$$A_{\phi V}\phi X = -V(\ln f)(TX - X),$$

$$\bar{g}\left(A_{\eta TZ}X, Y\right) + \bar{g}\left(A_{\eta Z}X, \phi Z\right) = \sin^2\theta X(\ln f)[\bar{g}(Y, Z) + \bar{g}(Y, TZ)]$$

for any $\eta, V \in TN^\perp$.

Lemma 3 ([61])**.** *Assume that $N = N_1 \times_f N_T$ is a warped-product, proper skew semi-invariant submanifold of a locally golden Riemannian manifold. Then, we have*

$$\bar{g}(h(X, V), \phi W) = 0$$

$$\bar{g}(h(X, V), NZ) = 0.$$

Lemma 4 ([61])**.** *Let $N = N_1 \times_f N_T$ be a warped-product, proper skew semi-invariant submanifold (N) of a locally golden Riemannian manifold. Then,*

$$\bar{g}(h(X, \phi Y), \phi V) = -V(\ln f)[\bar{g}(\phi Y, X) + \bar{g}(Y, X)].$$

Theorem 18 ([61])**.** *Consider $N = N_1 \times_f N_T$ as a $(p + q + r)$-dimensional warped-product, proper skew semi-invariant submanifold within a $(2p + 2q + r)$-dimensional locally golden Riemannian manifold ($\bar{M}$). The following statements hold true:*

(i) *The squared norm of the second fundamental form of N meets the following condition:*

$$\|h\|^2 \geq r\left\{2\left\|\nabla^\perp(\ln f)\right\|^2 + 2\cos^2\theta\left\|\nabla^\theta(\ln f)\right\|^2\right\},$$

where $r = \dim(N_T)$, and $\nabla^\perp(\ln f)$ and $\nabla^\theta(\ln f)$ are gradients of $(\ln f)$ on $D^\perp$ and D^θ, respectively.

(ii) *Assume that the equality sign remains unchanged. It follows that N is a mixed, entirely geodesic submanifold and N_1 is a totally geodesic submanifold of $\bar{M}$. Furthermore, N_T will never be the minimal submanifold of $\bar{M}$.*

5. Lightlike Submanifolds of Golden Semi-Riemannian Manifolds

Research on the geometry of degenerate submanifolds shows significant differences compared to non-degenerate submanifolds. This variation stems from the fact that the tangent bundle of non-degenerate submanifolds intersects trivially with the normal vector bundle, whereas this intersection is non-trivial in degenerate submanifolds. The work of Duggal, Bejancu, and Kupeli on lightlike submanifolds within semi-Riemannian manifolds is documented in [21,68]. Furthermore, the study of lightlike submanifolds in semi-Riemannian and specifically golden semi-Riemannian manifolds is a critical field in differential geometry, attracting the attention of numerous researchers.

For a lightlike submanifold (N) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$, Duggal and Bejancu [21] defined the notion of *radical distribution* ($Rad(TN)$) and the notion of a *normal bundle* ($TN^\perp$) such that

$$Rad(TN) = TN \cap TN^\perp,$$

where $T^{\perp}N = \cup_{x \in N}\{X \in T_x\overline{M} \mid \bar{g}(X,Y) = 0, \forall Y \in T_x N\}$.

5.1. Lightlike Hypersurfaces

Poyraz and Yaşar introduced lightlike hypersurfaces of a golden semi-Riemannian manifold in [22]. Let N be a lightlike hypersurface of a golden semi-Riemannian manifold $(\bar{M}, \bar{g}, \phi)$. For any $X \in \Gamma(TN)$ and $\mathcal{T} \in \text{Trace}(TN)$,

$$\phi X = \phi' X + u_1(X)\mathcal{T}, \quad \phi\mathcal{T} = U_1 + u_2(\xi')\mathcal{T},$$

where $\phi' X \in \Gamma(TN)$ and u_1 and u_2 are 1-form defined by

$$u_1(X) = g(X, \phi\xi'), \quad u_2(X) = g(X, \phi\mathcal{T}).$$

Definition 11 ([22])**.** *Let N be a lightlike hypersurface of a golden semi-Riemannian manifold $(\bar{M}, \bar{g}, \phi)$. Then,*

(i) *N is called a screen semi-invariant lightlike hypersurface if $\phi(Rad(TN)) \subset S(TN)$. Here, a screen distribution on N, as denoted by $S(TN)$, is defined as a non-degenerate, complementary vector bundle of $TN^{\perp}$ in TN. Additionally, $\phi(\text{ltr}(TN)) \subset S(TN)$, where $\text{ltr}(TN)$ denotes the lightlike transversal bundle associated with hypersurface N;*

(ii) *N is known as a radical, anti-invariant lightlike hypersurface if $\phi(Rad(TN)) \subset \text{ltr}(TN)$.*

Theorem 19 ([22])**.** *Let N be a lightlike hypersurface of a golden semi-Riemannian manifold $(\bar{M}, \bar{g}, \phi)$ and consider the induced structure (g, ϕ') on TN. Then, the next three statements are equivalent.*

(i) *N is invariant;*

(ii) *u_1 vanishes on N;*

(iii) *ϕ' is a golden structure on N.*

Theorem 20 ([22])**.** *An anti-invariant lightlike hypersurface of a golden semi-Riemann manifold is not radical.*

In [22], Poyraz and Yaşar derived certain findings regarding screen semi-invariant lightlike hypersurfaces in a golden semi-Riemannian manifold.

Theorem 21 ([22])**.** *A golden semi-Riemannian manifold $(\bar{M}, \bar{g}, \phi)$ has a screen semi-invariant lightlike hypersurface denoted by $(N, g, S(TN))$. When N is totally geodesic in $\bar{M}$ and $u = 0$, where u is 1-form, then only the vector field ($U = \phi\xi'$) on N is parallel to ∇.*

Both mixed geodesic lightlike hypersurfaces and totally geodesic lightlike hypersurfaces were introduced in [22].

Theorem 22 ([22])**.** *Let $(N, g, S(TN))$ be a screen semi-invariant lightlike hypersurface of a golden semi-Riemannian manifold $(\bar{M}, \bar{g}, \phi)$. Then, the following claims are equivalent:*

(i) *N is mixed geodesic;*

(ii) *There is no D_2 component of $A_{\mathcal{T}}$;*

(iii) *There is no D_1 component of $A^*_{\xi'}$.*

Theorem 23 ([22])**.** *Let $(N, g, S(TN))$ be a screen semi-invariant lightlike hypersurface of a golden semi-Riemannian manifold $(\bar{M}, \bar{g}, \phi)$. It follows that N is totally geodesic if and only if, for any $X \in \Gamma(TN)$ and $Y \in \Gamma(D)$, we have*

$$(\nabla_X \phi')Y = 0, \quad (\nabla_X \phi')U_1 = A_{\mathcal{T}} X.$$

Theorem 24 ([22])**.** *Assume that the locally golden product space form is $\bar{M} = M_p(c_p) \times M_q(c_q)$ and that the screen semi-invariant lightlike hypersurface of $\bar{M}$ is N. If N is totally umbilical, then $c_p = -(\psi + 1)c_q$.*

Remark 17. *Poyraz and Yaşar presented additional findings on a screen semi-invariant lightlike hypersurface in the locally golden product space form as described in [22].*

In [22], Poyraz and Yaşar also studied conformal screen semi-invariant lightlike hypersurfaces and obtained the following outcomes:

Theorem 25 ([22])**.** *Assume that a golden semi-Riemannian manifold $(\bar{M}, \bar{g}, \phi)$ has a conformal screen semi-invariant lightlike hypersurface $(N, g, S(TN))$. The leaf (N^b) of $S(TN)$ is totally geodesic in both N and $\bar{M}$, and N is totally geodesic in $\bar{M}$ if N or $S(TN)$ is totally umbilical.*

Theorem 26 ([22])**.** *Consider $(N, g, S(TN))$ as a conformal screen semi-invariant lightlike hypersurface within a locally golden product space with the form of $\bar{M} = M_p(c_p) \times M_q(c_q)$. Then, we have $c_p = c_q = 0$.*

Corollary 2 ([22])**.** *There exists no conformal screen semi-invariant lightlike hypersurface within a locally golden product space in the form of $\bar{M} = M_p(c_p) \times M_q(c_q)$ with $c_p, c_q \neq 0$.*

5.2. Invariant Lightlike Submanifolds

In [69], researchers explored invariant lightlike submanifolds within golden semi-Riemannian manifolds and identified certain criteria for such submanifolds to qualify as local product manifolds in the context of golden semi-Riemannian manifolds.

Theorem 27 ([69])**.** *Assume that N is a lightlike submanifold of $(\bar{M}, \bar{g}, \phi)$, a golden semi-Riemannian manifold. For N to be invariant, the required and sufficient conditions are that the induced structure (ϕ', g) on N be a golden semi-Riemannian structure.*

Theorem 28 ([69])**.** *For a golden semi-Riemannian manifold $(\bar{M}, \bar{g}, \phi)$, let N be an invariant lightlike submanifold. The induced structure (ϕ') on the submanifold (N) is parallel to the induced connection (∇) if the golden structure (ϕ) is parallel to the Levi–Civita connection ($\bar{\nabla}$) of $\bar{M}$.*

Theorem 29 ([69])**.** *Assume that N is an invariant lightlike submanifold of $(\bar{M}, \bar{g}, \phi)$, a golden semi-Riemannian manifold. Then, for the induced golden structure (ϕ) on N, the Nijenhuis tensor expression is given by*

$$\begin{aligned}\mathcal{N}'_\phi(X, Y) = \left(\nabla_{\phi' X}\phi'\right)Y - \left(\nabla_{\phi' Y}\phi'\right)X + (\nabla_X \phi')\phi' Y - (\nabla_Y \phi')\phi' X \\ +(\nabla_Y \phi')X - (\nabla_X \phi')Y,\end{aligned}$$

for any $X, Y \in \Gamma(TN)$.

Theorem 30 ([69])**.** *For a golden semi-Riemannian manifold $(\bar{M}, \bar{g}, \phi)$, let N be an invariant lightlike submanifold. As a result, N is a totally geodesic, lightlike submanifold of $\bar{M}$.*

In [23], Poyraz and Yaşar explored various characteristics of semi-invariant lightlike submanifolds within golden semi-Riemannian manifolds, leading to the subsequent findings.

Definition 12 ([23])**.** *Consider a lightlike submanifold $(N, g, S(TN))$ of a golden semi-Riemannian manifold $(\bar{M}, \bar{g}, \phi)$.*

(i) *N is an invariant lightlike submanifold if $\phi(Rad(TN)) = Rad(TN)$ and $\phi(S(TN)) = S(TN)$;*

(ii) *N is a semi-invariant lightlike submanifold if*

$$\phi(Rad(TN)) \subset S(TN), \ \ \phi(\text{ltr}(TN)) \subset S(TN) \ \text{ and } \ S(TN^{\perp}) \subset S(TN);$$

(iii) *N is a radical anti-invariant lightlike submanifold if* $\phi(Rad(TN)) = \text{ltr}(TN)$;

Let $(N, g, S(TN))$ be a semi-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\bar{M}, \bar{g}, \phi)$. If we set $D_1 = \phi(Rad(TN)), D_2 = \phi(Itr(TN))$ and $D_3 = \phi(S(TN^{\perp}))$, then we have

$$S(TN) = D_0 \perp \{D_1 \oplus D_2\} \perp D_3.$$

Therefore,

$$TN = D_0 \perp \{D_1 \oplus D_2\} \perp D_3 \perp Rad(TN) \quad \text{and}$$
$$T\bar{M} = D_0 \perp \{D_1 \oplus D_2\} \perp D_3 \perp \{Rad(TN) \oplus \text{ltr}(TN)\} \perp S\left(TN^{\perp}\right).$$

According to this definition, one can write

$$D = D_0 \perp D_1 \perp Rad(TN) \ \text{ and } \ D^{\perp} = D_2 \perp D_3.$$

Thus, we have $TN = D \oplus D^{\perp}$.

Proposition 9 ([23]). *Regarding ϕ, we know that distributions D_0 and D are invariant distributions.*

Theorem 31 ([23]). *A golden semi-Riemannian manifold $(\bar{M}, \bar{g}, \phi)$ has a lightlike submanifold denoted by N. Hence, the ensuing assertions are equivalent.*
(i) *N is invariant;*
(ii) *The 1-forms u_i and w_α vanish on TN $\forall i$ and α;*
(iii) *ϕ' is a golden structure on N.*

Theorem 32 ([23]). *Let N be a totally umbilical, semi-invariant lightlike submanifold of a golden semi-Riemannian manifold $(\bar{M}, \bar{g}, \phi)$.*
Then, we have $c_p = c_q = 0$.

Remark 18. *In [28], Poyraz and Doğan identified several criteria for the integrability of distributions on semi-invariant lightlike submanifolds within golden semi-Riemannian manifolds and explored both totally geodesic and mixed geodesic distributions of such submanifolds.*

The authors explored the geometry of screen semi-invariant lightlike submanifolds within a golden semi-Riemannian manifold in [29].

Definition 13 ([29]). *Let N be a lightlike submanifold and $(\bar{M}, \bar{g}, \phi)$ be a golden semi-Riemannian manifold. Then, it is possible to define N as a screen semi-invariant lightlike submanifold of $\bar{M}$ if the following conditions are met:*

$$\phi(Rad(TN)) \subseteq S(TN), \ \ \phi(ltr(TN)) \subseteq S(TN).$$

Using the description above, one may also define a non-degenerate distribution (D_0) for a screen semi-invariant lightlike submanifold of a golden semi-Riemannian manifold such that $S(TN)$ is decomposed as follows:

$$S(TN) = D_0 \perp D_1 \oplus D_2,$$

where $D_1 = \phi(Rad(TN))$ and $D_2 = \phi(\text{ltr}(TN))$.

Theorem 33 ([29]). *With a screen semi-invariant lightlike submanifold, let $\bar{M}$ be a golden semi-Riemannian manifold. The invariant distribution (D) is integrable for any vector fields ($U, V \in \Gamma(D)$) if and only if we have*

$$h^l(\phi V, \phi U) = h^l(U, \phi V) + h^l(U, V),$$

where h^l is the second fundamental form on $\Gamma(ltr(TN))$.

Theorem 34 ([29]). *Let $\bar{M}$ be a golden semi-Riemannian manifold with a screen semi-invariant lightlike submanifold. The radical distribution ($Rad(TN)$) is integrable for any vector fields ($U, V \in \Gamma(Rad(TN))$) if and only if*

$$\nabla^*_U \phi V - \nabla^*_V \phi U = (A^*_U - A^*_V) \quad \text{or} \quad \phi \nabla^*_U \phi V - \phi \nabla^*_V \phi U = (\nabla^*_U \phi V - \nabla^*_V \phi U),$$

where ∇^* is a linear connection on $(S(TN))$ and A^* is a shape operator of distributions $(S(TN))$ and $Rad(TN)$.

Theorem 35 ([29]). *Let N be a screen semi-invariant lightlike submanifold of a golden semi-Riemannian manifold ($\bar{M}$). Then, for any $U, V \in \Gamma(S(TN))$, the screen distribution ($S(TN)$) is integrable if and only if*

$$\nabla^*_U \phi V - \nabla^*_V \phi U = (\nabla^*_U V - \nabla^*_V U) \ \text{ or } \ \nabla^*_U \phi V = \nabla^*_V \phi U.$$

Theorem 36 ([29]). *Let N be a screen semi-invariant lightlike submanifold of a golden semi-Riemannian manifold ($\bar{M}$). Then, for any $U \in \Gamma(TN)$ and $\xi \in \Gamma(Rad(TN))$, an induced connection (∇ on N) is a metric connection if and only if one of the following conditions is satisfied:*

$$\nabla^*_U \phi \xi' = -A^*_{\xi'} U \ \text{ or } \ A^*_{\xi'} U = 0$$

Remark 19. *The essential criteria for these distributions to form complete geodesic foliations are also established in [29].*

Remark 20. *Poyraz also researched screen semi-invariant lightlike submanifolds within golden semi-Riemannian manifolds, as documented in [70]. This study included a discussion on various characteristics of these submanifolds and the verification of certain properties specific to totally umbilical screen semi-invariant lightlike submanifolds of golden semi-Riemannian manifolds.*

5.3. Transversal Lightlike Submanifolds

Research on the geometry of screen transversal lightlike submanifolds and their anti-invariant counterparts in golden semi-Riemannian manifolds was conducted in [25]. The authors explored the geometry of distributions and established the essential and adequate conditions for the induced connections in these manifolds to qualify as metric connections. Furthermore, they provided a characterization of screen transversal, anti-invariant lightlike submanifolds within golden semi-Riemannian manifolds.

Definition 14 ([25]). *For a golden semi-Riemannian manifold ($\bar{M}$), let N be a lightlike submanifold. If*

$$\phi(Rad(TN)) \subset S(TN)^{\perp},$$

where $S(TN)^{\perp}$ refers to the screen distribution of the normal bundle $TN^{\perp}$, then N is a screen transversal lightlike submanifold of a $\bar{M}$ golden semi-Riemannian manifold.

Definition 15 ([25]). *Let N be a screen transversal lightlike submanifold of a golden semi-Riemannian manifold ($\bar{M}$). Then, if $\phi(S(TN)) \subset S(TN)^{\perp}$, N is a screen transversal anti-invariant submanifold of $\bar{M}$.*

If N is a screen transversal anti-invariant submanifold of $\bar{M}$, then

$$S(TN)^{\perp} = \phi(Rad(TN)) \oplus \phi(\text{ltr}(TN)) \oplus \phi(S(TN)) \perp D^{\perp},$$

where $D^{\perp}$ is the orthogonal non-degenerate distribution complementary to

$$\phi(Rad(TN)) \oplus \phi(\text{ltr}(TN)) \oplus \phi(S(TN)).$$

Proposition 10 ([25]). *Let N be a screen transversal, anti-invariant lightlike submanifold of a golden semi-Riemannian manifold ($\bar{M}$). In such a case, $D^{\perp}$ is an invariant distribution about ϕ.*

Theorem 37 ([25]). *Let N be a screen transversal, anti-invariant lightlike submanifold within a golden semi-Riemannian manifold ($\bar{M}$). A radical distribution is integrable if and only if*

$$\nabla^s_U \phi V = \nabla^s_V \phi U$$

for $U, V \in \Gamma(RadTN)$ and ∇^s refers to the screen connection on the screen bundle ($S(TN)$).

Theorem 38 ([25]). *Consider N a screen transversal, anti-invariant lightlike submanifold of a golden semi-Riemannian manifold ($\bar{M}$). In this instance, the screen distribution is integrable if and only if*

$$\nabla^s_U \phi V - \nabla^s_V \phi U = h^s(U, V) - h^s(V, U),$$

for $U, V \in \Gamma(S(TN))$ and h^s is the screen's second fundamental form.

Remark 21. *Erdogan also explored the geometry of screen transversal lightlike submanifolds; radical screen transversal lightlike submanifolds; and screen transversal, anti-invariant lightlike submanifolds within golden semi-Riemannian manifolds in [71].*

A study of the geometry of radical screen transversal lightlike submanifolds was carried out in [25] by Erdoğan.

Definition 16 ([25]). *Assume N is a screen transversal lightlike submanifold of a golden semi-Riemannian manifold ($\bar{M}$). If $\phi(S(TN)) = S(TN)$, then N is called a radical transversal lightlike screen submanifold of $\bar{M}$.*

Theorem 39 ([25]). *Let N be a radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold ($\bar{M}$). In this case, the screen distribution is integrable if and only if*

$$U, V \in \Gamma(S(TN)). \quad h^s(U, \phi V) = h^s(V, \phi U).$$

Theorem 40 ([25]). *Let N be a radical screen transversal lightlike submanifold of golden semi-Riemannian manifold ($\bar{M}$). The radical distribution is integrable if and only if*

$$A_{\phi V} U - A_{\phi U} V = A^*_U V - A^*_V U$$

for $U, V \in \Gamma(Rad(TN))$.

Theorem 41 ([25]). *Given a golden semi-Riemannian manifold ($\bar{M}$), let N be a radical screen transversal lightlike submanifold of it. The screen distribution defines totally geodesic foliation if and only if there is no component of $h^s(U, \phi V) - h^s(U, V)$ in $\phi ltr(TN)$ for $U, V \in \Gamma(S(TN))$.*

Theorem 42 ([25]). *Let N be a radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold ($\bar{M}$). The radical distribution defines totally geodesic foliation if and only if there is no component of $A_{\phi V} U$ in $S(TN)$ and $A^*_V U = 0$ for $U, V \in \Gamma(RadTN)$.*

Theorem 43 ([25])**.** *Let N be a radical screen transversal lightlike submanifold of a golden semi-Riemannian manifold* ($\bar{M}$)*. The connection induced on N is a metric connection if and only if there is no component of* $h^s(V,U)$ *in* $\phi(Rad(TN))$ *or of* $A_{\phi\xi'}U$ *in* $S(TN)$ *for* $U,V \in S(TN)$*.*

A study of the geometry of radical transversal lightlike submanifolds of golden semi-Riemannian manifolds was carried out in [26]. The authors also investigated the geometry of the distributions and obtained the necessary and sufficient conditions for the induced connection on the manifolds to be a metric connection in [26].

Definition 17 ([26])**.** *Consider a lightlike submanifold of a golden semi-Riemannian manifold as* $(N, g, S(TN), S(TN)^{\perp})$*. If* $\phi(Rad(TN)) = \mathrm{ltr}(TN)$ *and* $\phi(S(TN)) = S(TN)$ *are satisfied, then the lightlike submanifold is called a radical transversal lightlike submanifold.*

Proposition 11 ([26])**.** *Let* $\bar{M}$ *be a a golden semi-Riemannian manifold. In this instance, the manifold* ($\bar{M}$) *does not have a 1-lightlike radical transversal lightlike submanifold.*

Theorem 44 ([26])**.** *For a locally golden semi-Riemannian manifold* ($\bar{M}$)*, let N be a radical transversal lightlike submanifold. The induced connection* (∇) *on D is a metric connection if and only if the following conditions are met:* $U \in \Gamma(TN)$ *and* $\xi' \in \Gamma(Rad(TN))$

$$A_{\phi\xi'}U \in \Gamma(Rad(TN)).$$

Theorem 45 ([26])**.** *For a locally golden semi-Riemannian manifold* ($\bar{M}$)*, let N be a radical transversal lightlike submanifold. Here, the distribution* ($S(TN)$) *must meet the following requirements to be integrable: for all* $U,V \in \Gamma(S(TN))$*,*

$$h^l(V,SU) = h^l(U,SV).$$

Remark 22. *The necessary and sufficient condition for the radical distribution definition of totally geodesic foliation on N is also discussed in [26].*

The geometry of transversal lightlike submanifolds of golden semi-Riemannian manifolds was studied in [26]. The authors also looked into the geometry of the distributions and determined what is required for the induced connection on the manifold to be a metric connection.

Definition 18 ([26])**.** *Let* $(N, g, S(TN), S(TN)^{\perp})$ *be a lightlike submanifold of a golden semi-Riemannian manifold. If* $\phi(Rad(TN)) = \mathrm{ltr}(TN)$ *and* $\phi(S(TN)) \subseteq S(TN)$ *are satisfied, then the lightlike submanifold is referred to as a transversal lightlike submanifold.*

Proposition 12 ([26])**.** *Let N be a transversal lightlike submanifold of a golden semi-Riemannian manifold. In this case, the sub-bundle* (μ) *is an orthogonal complement to the* $\phi(S(TN))$ *in the* $S(TN)$*, and the distribution is invariant under* ϕ*.*

Proposition 13 ([26])**.** *Let* $\bar{M}$ *be a golden semi-Riemannian manifold. In this case, there exists no 1-lightlike transversal submanifold of* $\bar{M}$*.*

Remark 23. *It was also established in [26] that the screen distribution constitutes both a necessary and sufficient condition for defining totally geodesic foliation on N.*

5.4. CR Lightlike Submanifolds of Golden Semi-Riemannian Manifolds

CR lightlike submanifolds within a golden semi-Riemannian manifold were explored and characterized by Ahmad and Qayyoom in [24]. They further examined various aspects of geodesic CR submanifolds in a golden semi-Riemannian manifold.

Additionally, they documented numerous significant findings regarding totally geodesic and totally umbilical CR submanifolds in a golden Riemannian manifold.

Definition 19 ([24]). *If both of the following two requirements are met, a submanifold N of a golden semi-Riemannian manifold ($\bar{M}$) is referred to as a CR lightlike submanifold:*

(i) $\phi(Rad(TN))$ *is a distribution on N such that*

$$Rad(TN) \cap \phi(Rad(TN)) = \{0\};$$

(ii) *There exist vector bundles* $S(TN)$, $S(TN^{\perp})$, $\mathrm{ltr}(TN)$, D_0, *and* D' *over N such that*

$$S(TN) = \{\phi(Rad(TN)) \oplus D'\} \perp D_0, \quad \phi D_0 = D_0, \quad \phi D' = Z_1 \perp Z_2,$$

where D_0 *is a non-degenerate distribution on N and* Z_1 *and* Z_2 *are vector bundles of* $ltr(TN)$ *and* $S(TN^{\perp})$, *respectively.*

Lemma 5 ([24]). *Let the screen distribution be totally geodesic and N be a CR lightlike submanifold of a golden semi-Riemannian manifold. Then,* $\nabla_X Y \in \Gamma(S(TN))$, *where* $X, Y \in \Gamma(S(TN))$.

Lemma 6 ([24]). *Let N be a CR lightlike submanifold of a locally golden semi-Riemannian manifold ($\bar{M}$). Then,* $\nabla_X \phi X = \phi \nabla_X X$ *for any* $X \in \Gamma(D_0)$.

5.4.1. Geodesic CR-Lightlike Submanifolds

Definition 20 ([24]). *When the second fundamental form (h) of a CR lightlike submanifold of a golden semi-Riemannian manifold satisfies certain conditions, it is referred to as a mixed geodesic CR lightlike submanifold.*

$$h(X, U) = 0, \quad \text{where } X \in \Gamma(D) \text{ and } U \in \Gamma(D').$$

Definition 21 ([24]). *A CR lightlike submanifold in a golden semi-Riemannian manifold is referred to as a D-geodesic CR lightlike submanifold if its second fundamental form (h) satisfies* $h(X, Y) = 0$ *for* $X, Y \in \Gamma(D)$.

Definition 22 ([24]). *A CR lightlike submanifold in a golden semi-Riemannian manifold is known as a* D'*-geodesic CR lightlike submanifold if its second fundamental form (h) satisfies* $h(U, V) = 0$ *for* $U, V \in \Gamma(D')$.

Theorem 46 ([24]). *Given a golden semi-Riemannian manifold ($\bar{M}$), the submanifold (N) of $\bar{M}$ should be CR lightlike. In that case, N is totally geodesic if and only if* $(Z_{\xi'} g)(X, Y) = 0$ *and* $(Z_W g)(X, Y) = 0$ *for any* $X, Y \in \Gamma(TN), \xi' \in \Gamma(Rad(TN))$ *and* $W \in \Gamma(S(TN^{\perp}))$.

Theorem 47 ([24]). *Let $\bar{M}$ be a golden semi-Riemannian manifold and N be a CR lightlike submanifold of it. Then, N is mixed geodesic if and only if we have*

$$A^*_{\xi'} X \in \Gamma(D_0 \perp \phi Z_1)$$

and

$$A_W X \in \Gamma(D_0 \perp Rad(TN) \perp \phi Z_1),$$

for any $X \in \Gamma(D), \xi' \in \Gamma(Rad(TN))$ *and* $W \in \Gamma(S(TN^{\perp}))$.

5.4.2. Totally Umbilical CR Lightlike Submanifolds

Definition 23 ([24]). *A lightlike submanifold (N) within a semi-Riemannian manifold ($\bar{M}$) is termed totally umbilical within $\bar{M}$ if it possesses a smooth transversal vector field ($\mathcal{T}$), belonging to* $\Gamma(\mathrm{Trace}(TN))$ *on N, called the transversal curvature vector field of N, such that* $h(X, Y) = \mathcal{T} g(X, Y)$ *for* $X, Y \in \Gamma(TN)$.

Theorem 48 ([24])**.** *Consider N as a totally umbilical CR lightlike submanifold of $\bar{M}$, a golden manifold. The CR lightlike sectional curvature of N then disappears, that is, $K(\pi) = 0$ for any CR lightlike section (π).*

Remark 24. *Analogous findings for CR lightlike, totally geodesic lightlike, and totally umbilical lightlike submanifolds of a golden semi-Riemannian manifold are discussed in [72].*

5.5. Half Lightlike Submanifolds

Submanifolds of the lightlike type with a codimension 2 are termed either half lightlike or coisotropic, depending on the rank of their radical distribution. These are further divided into two subclasses [73]. A half lightlike submanifold represents a particular instance of the broader r-lightlike submanifolds where $r = 1$. Its geometric structure is more comprehensive than that of a coisotropic submanifold or a lightlike hypersurface [21].

Poyraz et al. examined half lightlike submanifolds within a golden semi-Riemannian manifold in their work [74]. They demonstrated the absence of radical anti-invariant, half lightlike submanifolds within such a context and provided findings regarding screen semi-invariant, half lightlike submanifolds. Additionally, their study encompassed screen conformal, half lightlike submanifolds within a golden semi-Riemannian manifold.

Definition 24 ([74])**.** *Suppose N is a half lightlike submanifold of a golden semi-Riemannian manifold ($\bar{M}$).*

(i) *N is an invariant half lightlike submanifold if $\phi(TN) = TN$;*

(ii) *N is a screen semi-invariant half lightlike submanifold if $\phi(Rad(TN)) \subset S(TN)$ and $\phi(\mathrm{ltr}(TN)) \subset S(TN)$.*

(iii) *N is a radical anti-invariant half lightlike submanifold if $\phi(Rad(TN)) = \mathrm{ltr}(TN)$.*

Theorem 49 ([74])**.** *If N is a half lightlike submanifold of a golden semi-Riemannian manifold ($\bar{M}$), then the following three statements are equivalent.*

(i) *N is invariant;*

(ii) *u_1 and u_2 vanish on N, where u_1 and u_2 are 1-forms on N;*

(iii) *ϕ' is a golden structure on N.*

Theorem 50 ([74])**.** *No radical, anti-invariant half lightlike submanifold exists within a golden semi-Riemannian manifold.*

Theorem 51 ([74])**.** *Suppose N is a screen conformal, totally umbilical, screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold ($\bar{M}$). Then, both N and the screen distribution ($S(TN)$) are totally geodesic.*

Corollary 3 ([74])**.** *For a screen semi-invariant half lightlike submanifold N of a golden semi-Riemannian manifold ($\bar{M}$), the condition of*

$$h_1(X, Y) = 0,$$

holds, indicating that the vector field (V) results in the degeneration of the local second fundamental form of N.

Theorem 52 ([74])**.** *If N is a screen semi-invariant half lightlike submanifold of a golden semi-Riemannian manifold ($\bar{M}$), then the distribution (D) is integrable if and only if the following condition is satisfied for any $X, Y \in \Gamma(D)$:*

$$h_1(\phi X, \phi Y) = h_1(\phi X, Y) + h_1(X, Y) \quad \text{and}$$

$$h_2(\phi X, \phi Y) = h_2(\phi X, Y) + h_2(X, Y),$$

where h_1 and h_2 are the local second fundamental forms of N.

Definition 25 ([74])**.** *If N is a screen semi-invariant half-lightlike submanifold of a golden semi-Riemannian manifold ($\bar{M}$), then N is mixed totally geodesic if and only if*

$$h_1(X,Y) = h_2(X,Y) = 0,$$

for any $X \in \Gamma(D)$ and $Y \in \Gamma(D^{\perp})$.

Remark 25. *Results regarding a totally umbilical screen semi-invariant half-lightlike submanifold of a golden semi-Riemannian manifold were also discussed in [74].*

5.6. Generic Lightlike Submanifolds

If N is a real r lightlike submanifold of a semi-Riemannian manifold ($\bar{M}$), it is called a *golden generic lightlike submanifold* if the screen distribution ($S(TN)$) of N is characterized by

$$\begin{aligned} S(TN) &= \phi(S(TN)^{\perp}) \perp D_0 \\ &= \phi(Rad(TN)) \oplus \phi(\mathrm{ltr}(TN)) \perp \phi(S(TN)^{\perp}) \perp D_0 \end{aligned} \tag{12}$$

where D_0 is a non-degenerate almost complex distribution on N with respect to ϕ [75].

Theorem 53 ([75])**.** *For a golden generic lightlike submanifold (N) of a golden semi-Riemannian manifold ($\bar{M}$), the Nijenhuis tensor field concerning the golden structure (ϕ) is null.*

Theorem 54 ([75])**.** *If N is a golden generic lightlike submanifold of a golden semi-Riemannian manifold ($\bar{M}$), then g serves as a golden structure on (D).*

Definition 26 ([75])**.** *A golden generic lightlike submanifold (N) is termed mixed geodesic if its second fundamental form (h) fulfills the following condition:*

$$h(Y,Z) = 0, \quad \textit{for } Y \in \Gamma(D) \textit{ and } Z \in \Gamma(D').$$

Theorem 55 ([75])**.** *If N is a totally umbilical golden generic lightlike submanifold of a golden semi-Riemannian manifold ($\bar{M}$), then the distribution (D) is inherently integrable.*

Remark 26. *Results of minimal golden generic lightlike submanifolds were also discussed in [75].*

Yadav and Kumar researched screen generic lightlike submanifolds, as documented in [76], yielding the following outcomes:

Definition 27 ([76])**.** *If N is a real r lightlike submanifold of a golden semi-Riemannian manifold ($\bar{M}$), it is termed a screen generic lightlike submanifold of $\bar{M}$ if the following conditions hold:*

(i) *$Rad(TN)$ is invariant with respect to ϕ, that is,*

$$\phi(Rad(TN)) = Rad(TN);$$

(ii) *There exist sub-bundles (D_0) of $S(TN)$ such that*

$$D_0 = \phi(S(TN)) \cap S(TN),$$

where D_0 is a non-degenerate distribution on N.

Proposition 14 ([76])**.** *A generic r lightlike submanifold is a screen generic lightlike submanifold with $\mu = 0$, where μ represents a non-degenerate invariant distribution.*

Proposition 15 ([76])**.** *No coisotropic, isotropic, or totally lightlike proper screen generic lightlike submanifold exists within a golden semi-Riemannian manifold ($\bar{M}$). Any screen generic isotropic, coisotropic, or totally lightlike submanifold in $\bar{M}$ is an invariant submanifold.*

Definition 28 ([76]). *A screen generic lightlike submanifold of a golden semi-Riemannian manifold is termed a D-geodesic screen generic lightlike submanifold if its second fundamental form (h) satisfies the following condition:* $h(X,Y) = 0$ *for any* $X, Y \in \Gamma(D)$.

Definition 29 ([76]). *A screen generic lightlike submanifold (N) of a golden semi-Riemannian manifold is called a mixed geodesic screen generic lightlike submanifold if its second fundamental form (h) satisfies the following condition:* $h(X,Y) = 0$, *for any* $X \in \Gamma(D)$ *and* $Y \in \Gamma(D')$. *Thus, N is a mixed geodesic screen generic lightlike submanifold if we have* $h^l(X,Y) = 0$ *and* $h^s(X,Y) = 0$ *for any* $X \in \Gamma(D)$ *and* $Y \in \Gamma(D')$.

Remark 27. *In [76], the conditions for the induced connection to meet the criteria of a metric connection were discussed, along with the classification of totally umbilical screen generic lightlike submanifolds of golden semi-Riemannian manifolds as totally geodesic. Moreover, the paper delved into the study of minimal screen generic lightlike submanifolds of golden semi-Riemannian manifolds.*

5.7. Golden GCR Lightlike Submanifolds

Definition 30 ([77]). *A real lightlike submanifold* $(N, g, S(TN))$ *of a golden semi-Riemannian manifold* $(\bar{M}, \bar{g}, \phi)$ *is termed a golden generalized Cauchy–Riemann (GCR) lightlike submanifold if the following conditions hold:*

(i) *There exist two sub-bundles* (D_1 *and* D_2) *of* $Rad(TN)$ *such that*

$$Rad(TN) = D_1 \oplus D_2, \;\; \phi(D_1) = D_1, \;\; \phi(D_2) \subset S(TN).$$

(ii) *There exist two sub-bundles* (D_0 *and* D' *of* $S(TN)$) *such that*

$$S(TN) = \{\phi(D_2) \oplus D'\} \perp D_0, \;\; \phi(D_0) = D_0, \;\; \phi(Z_1 \perp Z_2) = D',$$

where D_0 *is a non-degenerate distribution on N and* Z_1 *and* Z_2 *are vector sub-bundles of* $ltr(TN)$ *and* $S(TN^{\perp})$, *respectively.*

Let $\phi(Z_1) = N_1$ *and* $\phi(Z_2) = N_2$. *Then, we have*

$$D' = \phi(Z_1) \perp \phi(Z_2) = N_1 \perp N_2.$$

Thus, the following decomposition is obtained:

$$TN = D \oplus D', \;\; D = Rad(TN) \perp D_0 \perp \phi(D_2).$$

Thus, N is a proper golden GCR lightlike submanifold of a golden semi-Riemannian manifold if $D_0 \neq 0, D_1 \neq 0, D_2 \neq 0$ *and* $Z_2 \neq 0$.

Theorem 56 ([77]). *If N is a golden GCR lightlike submanifold of a golden semi-Riemannian manifold, then* ϕ' *serves as a golden structure on D.*

Theorem 57 ([77]). *If N is a golden GCR lightlike submanifold of a golden semi-Riemannian manifold, then the distribution (D) is integrable if and only if we satisfy the following conditions:*

(i) $\bar{g}(h^l(X, \phi Y), \xi') = \bar{g}(h^l(Y, \phi X), \xi')$ *and*

(ii) $\bar{g}(h^s(X, \phi Y), W) = \bar{g}(h^s(Y, \phi X), W)$

for any $X, Y \in \Gamma(D), \xi' \in \Gamma(D_2)$ *and* $W \in \Gamma(Z_2)$.

Definition 31 ([77]). *A golden GCR lightlike submanifold of a golden semi-Riemannian manifold is termed a D-geodesic golden GCR-lightlike submanifold if its second fundamental form (h) satisfies* $h(X,Y) = 0$ *for any* $X, Y \in \Gamma(D)$.

Discussions on minimal golden GCR lightlike submanifolds can also be found in [77].

Definition 32 ([77])**.** *A lightlike submanifold isometrically immersed in a semi-Riemannian manifold is considered minimal if it meets the following two conditions:*

(i) $h^s = 0$ *on* $Rad(TN)$ *and*

(ii) Trace $h = 0$, *where Trace is written with respect to* g *restricted to* $S(TN)$.

Remark 28. *In [77], Poyraz also provided examples and results of minimal golden GCR lightlike submanifolds of golden semi-Riemannian manifolds.*

Remark 29. *Poyraz discussed comparable findings on the geometry of golden GCR lightlike submanifolds in golden semi-Riemannian manifolds in [78].*

5.8. Slant Lightlike Submanifolds

Acet explored screen pseudo-slant lightlike submanifolds in golden semi-Riemannian manifolds in [79].

Definition 33 ([79])**.** *If* N *is a lightlike submanifold of a golden semi-Riemannian manifold* $(\bar{M})$*, it is termed a screen pseudo-slant submanifold of* $\bar{M}$ *if the following conditions are satisfied:*

(i) *The radical distribution* $(Rad(TN))$ *is an invariant distribution with respect to* ϕ*, i.e.,* $\phi(Rad(TN)) = Rad(TN)$*;*

(ii) *There exist non-degenerate orthogonal distributions* (D_0 *and* $D^{\perp}$) *on* N *such that* $S(TN) = D_0 \perp D^{\perp}$*;*

(iii) *Distribution* D_0 *is anti-invariant, i.e.,* $\phi(D_0) \subset S(TN^{\perp})$*;*

(iv) *Distribution* $D^{\perp}$ *is a slant with an angle of* $\theta(\neq \frac{\pi}{2})$*, i.e., for each* $x \in N$ *and each non-zero vector* ($X \in (D^{\perp})_x$)*, the angle* (θ) *between* ϕX *and the vector subspace* ($(D^{\perp})_x$) *is a constant* $(\neq \frac{\pi}{2})$ *that is independent of the choice of* $x \in N$ *and* $X \in (D^{\perp})_x$*.*

Remark 30. *Some non-trivial examples of a screen pseudo-slant lightlike submanifold in a golden semi-Riemannian manifold were studied, and the conditions for the integrability of distributions of a screen pseudo-slant lightlike submanifold of a golden semi-Riemannian manifold were also provided in [79].*

In [80], Yadav and Kumar explored characteristics of screen generic lightlike submanifolds in golden semi-Riemannian manifolds.

Definition 34 ([80])**.** *If* N *is a* $2q$*-lightlike submanifold of a golden semi-Riemannian manifold* $(\bar{M})$ *with an index of* $2q$ *(where* q *is an integer indicating the dimension of the radical bundle of the lightlike submanifold* (N) *within the golden semi-Riemannian manifold* $(\bar{M})$ *and if* $2q < dim(N)$*, then* N *is termed a screen semi-slant lightlike submanifold of* $\bar{M}$ *if the following conditions are satisfied:*

(i) $Rad(TN)$ *is invariant with respect to* ϕ*, i.e.,* $\phi(Rad(TN)) = Rad(TN)$*;*

(ii) *There exist non-degenerate orthogonal distributions* (D_0 *and* $D^{\perp}$) *on* N *such that* $S(TN) = D_0 \oplus_{orth} D^{\perp}$*;*

(iii) *Distribution* D_0 *is an invariant distribution, i.e.,* $\phi D_0 = D_0$*;*

(iv) *Distribution* $D^{\perp}$ *is a slant with an angle of* $\theta(\neq 0)$*, i.e., for each* $x \in N$ *and each non-zero vector* ($X \in (D^{\perp})_x$)*, the angle* (θ) *between* ϕX *and the vector subspace* ($(D^{\perp})_x$) *is a non-zero constant that is independent of the choice of* $x \in N$ *and* $X \in (D^{\perp})_x$*.*

Theorem 58 ([80])**.** N *is a* $2q$ *lightlike submanifold of a golden semi-Riemannian manifold* $(\bar{M})$*; it qualifies as a screen semi-slant lightlike submanifold of* $\bar{M}$ *if and only if*

(i) $ltr(TN)$ *and* D_0 *are invariant with respect to* ϕ*;*

(ii) *There exists a constant* ($c \in [0,1)$) *such that* $\phi'^2 X = c(\phi X + X)$*, for any* $X \in \Gamma(D^{\perp})$*, where* D_0 *and* $D^{\perp}$ *are non-degenerate orthogonal distributions and* ϕ' *is a* $(1,1)$*-tensor field on* N *such that* $S(TN) = D_0 \oplus_{orth} D^{\perp}$*. Moreover, in this case,* $c = \cos^2\theta$ *, and* θ *is the slant angle of* $D^{\perp}$*.*

Corollary 4 ([80]). *For N as a screen semi-slant lightlike submanifold of a golden semi-Riemannian manifold ($\bar{M}$) with a slant angle o of θ, for any $X, Y \in \Gamma(D^{\perp})$, we have the following:*

(i) $g(\phi' X, \phi' Y) = \cos^2\theta(g(X,Y) + g(X, \phi' Y))$,

(ii) $g(\mathcal{T}X, \mathcal{T}Y) = \sin^2\theta(g(X,Y) + g(\phi' X, Y))$.

Remark 31. *Yadav and Kumar investigated the essential and adequate conditions for the integrability and totally geodesic foliation of the distributions of $Rad(TN)$, D_0, and $D^{\perp}$ of screen semi-slant lightlike submanifolds of golden semi-Riemannian manifolds in [80].*

Definition 35 ([81]). *If N is a $2q$ lightlike submanifold of a golden semi-Riemannian manifold ($\bar{M}$) with an index of $2q$ and if $2q < dim(N)$, then N is termed a semi-slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:*

(i) *$\phi Rad(TN)$ is a distribution on N such that $Rad(TN) \cap \phi(Rad(TN)) = \{0\}$;*

(ii) *There exist non-degenerate orthogonal complementary distributions (D_0 and $D^{\perp}$) on N such that $S(TN) = (\phi(Rad(TN)) \oplus \phi(\mathrm{ltr}(TN)) \oplus_{orth} D_0 \oplus_{orth} D^{\perp}$;*

(iii) *Distribution D_0 is an invariant distribution, i.e., $\phi D_0 = D_0$;*

(iv) *Distribution $D^{\perp}$ is a slant with an angle of $\theta(\neq 0)$, i.e., for each $x \in N$, and for each non-zero vector ($X \in (D^{\perp})_x$), the angle (θ) between ϕX and the vector subspace ($(D^{\perp})_x$) is a non-zero constant that is independent of the choice of $x \in N$ and $X \in (D^{\perp})_x$.*

Proposition 16 ([81]). *A golden semi-Riemannian manifold $(\bar{M}, \bar{g}, \phi)$ does not have any isotropic or totally lightlike proper semi-slant lightlike submanifolds.*

Theorem 59 ([81]). *A golden semi-Riemannian manifold ($\bar{M}$) of index $2q$ has a q-lightlike submanifold, as denoted by N. Then, N is a semi-slant lightlike submanifold of $\bar{M}$ if and only if*

(i) *$\phi Rad(TN)$ is a distribution on N such that $Rad(TN) \cap \phi(Rad(TN)) = 0$;*

(ii) *The screen distribution ($S(TN)$) splits as*

$$S(TN) = \phi(Rad(TN)) \oplus \phi(\mathrm{ltr}(TN)) \oplus_{orth} D_0 \oplus_{orth} D^{\perp};$$

(iii) *There exists a constant ($c \in [0,1)$) such that $\phi'^2 X = c(\phi' X + X)$ for any $X \in \Gamma(D^{\perp})$. Moreover, in this case, we have $c = \cos^2\theta$, where θ is the slant angle of $D^{\perp}$.*

Remark 32. *Kumar and Yadav discussed the necessary and sufficient conditions for the integrability of distributions and the geometry of the leaves of the foliation determined by the distributions in [81].*

Remark 33. *Kumar and Yadav [27] explored the concept of screen slant lightlike submanifolds within golden semi-Riemannian manifolds, along with the essential conditions for the integrability of distributions and the structural details of the foliation's leaves governed by these distributions.*

Definition 36 ([82]). *A golden semi-Riemannian manifold ($\bar{M}$) of index $2q$ such that $2q < \dim(N)$ has a q lightlike submanifold called N. Then, N is a bi-slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:*

(i) *$\phi Rad(TN)$ is a distribution on N such that $Rad(TN) \cap \phi(Rad(TN)) = \{0\}$;*

(ii) *There exist non-degenerate orthogonal distributions (D, D_0, and $D^{\perp}$) on N such that*

$$S(TN) = \phi(Rad(TN)) \oplus \phi(\mathrm{ltr}(TN)) \perp D \perp D_0 \perp D^{\perp};$$

(iii) *Distribution D is an invariant distribution, i.e., $\phi D = D$;*

(iv) *Distribution D_0 is a slant with ab angle of $\theta_1(\neq 0)$, i.e., for each $x \in N$ and each non-zero vector ($X \in (D_0)_x$), the angle (θ_1) between ϕX and the vector space ($(D_0)_x$) is a non-zero constant that is independent of the choice of $x \in N$ and $X \in (D_0)_x$;*

(v) *Distribution $D^{\perp}$ is a slant with an angle of $\theta_2(\neq 0)$, i.e., for each $x \in N$ and each non-zero vector $X \in (D^{\perp})_x$, the angle (θ_2) between ϕX and the vector space ($\left(D^{\perp}\right)_x$) is a non-zero constant that is independent of the choice of $x \in N$ and $X \in (D^{\perp})_x$.*

The θ_1 and θ_2 constant angles are referred to as the slant angles of distributions D_0 and $D^\perp$, respectively. A bi-slant lightlike submanifold is said to be proper if $D_0 \neq \{0\}, D^\perp \neq \{0\}$ and $\theta_1 \neq \frac{\pi}{2}, \theta_2 \neq \frac{\pi}{2}$.

Theorem 60 ([82]). *If N is a q lightlike submanifold of a golden semi-Riemannian manifold ($\bar{M}$) with an index of 2q, then N is a bi-slant lightlike submanifold if and only if*

(i) *There exist a distribution ($\phi(Rad(TN))$) on N such that*

$$Rad(TN) \cap \phi(Rad(TN)) = \{0\};$$

(ii) *There exists a screen distribution ($S(TN)$) which can be expressed as*

$$S(TN) = \phi(Rad(TN)) \oplus \phi(\mathrm{ltr}(TN)) \perp D \perp D_0 \perp D^\perp$$

such that D is an invariant distribution on N, i.e., $\phi(D) = D$;

(iii) *There exists a constant ($c_1 \in [0,1)$) such that $\phi'^2 X = c_1(\phi' + I)X$ for any $X \in \Gamma(D_0)$;*

(iv) *There exists a constant ($c_2 \in [0,1)$) such that $\phi'^2 X = c_2(\phi' + I)X$ for any $X \in \Gamma(D^\perp)$.*

In that case, $c_1 = \cos^2\theta_1$ and $c_2 = \cos^2\theta_2$, where θ_1 and θ_2 represents the slant angles of D_0 and $D^\perp$, respectively.

Remark 34. *In [82], the integrability conditions of distributions on bi-slant lightlike submanifolds and the necessary and sufficient conditions for foliations determined by distributions on bi-slant lightlike submanifolds of golden semi-Riemannian manifolds to be geodesic were obtained.*

Definition 37 ([83]). *Given a golden semi-Riemannian manifold ($\bar{M}$) of index 2q, let N be a q lightlike submanifold such that $2q < dim(N)$. If the following criteria are met, then N is a slant lightlike submanifold of $\bar{M}$:*

(i) *$\phi Rad(TN)$ is a distribution on N such that $Rad(TN) \cap \phi(Rad(TN)) = \{0\}$;*

(ii) *There exists a non-degenerate orthogonal complementary distribution (D) on N such that*

$$S(TN) = \phi(Rad(TN)) \oplus \phi(\mathrm{ltr}(TN)) \oplus_{orth} D;$$

(iii) *Distribution D is a slant with an angle of $\theta(\neq 0)$, i.e., for each $x \in N$ and each non-zero vector ($X \in (D)_x$), the angle (θ) between ϕX and the vector subspace ($(D)_x$) is a non-zero constant that is independent of the choice of $x \in N$ and $X \in (D)_x$.*

Theorem 61 ([83]). *If N is a q lightlike submanifold of a golden semi-Riemannian manifold ($\bar{M}$) with an index of 2q, then N is a slant lightlike submanifold of $\bar{M}$ if and only if the following conditions hold:*

(i) *$\phi(Rad(TN))$ is a distribution on N such that $Rad(TN) \cap \phi(Rad(TN)) = 0$;*

(ii) *The screen distribution ($S(TN)$) is split as*

$$S(TN) = \phi(Rad(TN)) \oplus \phi(\mathrm{ltr}(TN)) \oplus_{orth} D;$$

(iii) *There exists a constant ($c \in [0,1)$) such that $\phi'^2 X = c(\phi' X + X)$ for any $X \in \Gamma(D)$. Moreover, in this case, $c = \cos^2\theta$, and θ is the slant angle of D.*

Remark 35. *The criteria for the integrability of distributions and the curvature characteristics of slant lightlike submanifolds in golden semi-Riemannian manifolds are further explored in [83].*

6. Lightlike Submanifolds of Meta-Golden Manifolds

F.E. Erdoğan et al. [30] proved the following results.

Theorem 62 ([30])**.** *Given an AMGsR manifold $(\bar{M}, \phi, F, \bar{g})$, let N be a lightlike hypersurface of $\bar{M}$. Here, the almost golden structure $(\bar{g})$ induces a structure (ϕ, g, u, X) on N that satisfies the following equalities:*

$$\phi'^2 X = \phi' X + X, \ \ u(\phi' X) = 0, \ \ \phi' U = 0,$$

$$u(W^2) - u(W) - 1 = 0, \ \ g(\phi' X, \phi' Y) = g(\phi' X, Y) + g(X, Y),$$

where for X, Y and $U \in \Gamma(TN), W \in \Gamma(ltr(TN))$.

Proposition 17 ([30])**.** *Assume that the manifold $(\bar{M}, \phi, F, \bar{g})$ is an AMGsR manifold. Then, $\nabla \phi F = 0$.*

Definition 38 ([30])**.** *Consider the AMGsR manifold $(\bar{M}, \phi, F, \bar{g},)$ and the lightlike hypersurface (N) of $\bar{M}$. Then,*

(i) *If $\phi F(TN) \subset TN$, N is called invariant;*

(ii) *If $\phi F(Rad(TN)) \subset S(TN)$ and $\phi F(ltr(TN)) \subset S(TN)$, then N is called screen semi-invariant;*

(iii) *If $\phi F(Rad(TN)) \subset (ltr(TN))$, then N is called a radical anti-invariant lightlike hypersurface.*

Theorem 63 ([30])**.** *Not every AMGsR manifold admits a radical anti-invariant lightlike hypersurface.*

The screen semi-invariant lightlike hypersurface of almost meta-golden semi-Riemannian manifolds was also studied in [30].

Corollary 5 ([30])**.** *Given an AMGsR manifold $(\bar{M}, \phi, F, \bar{g})$ and a screen semi-invariant lightlike hypersurface (N) of $\bar{M}$, we have $h(X, Z) = 0$ for any $X, Z \in \Gamma(TN)$.*

Proposition 18 ([30])**.** *Let N be a screen semi-invariant lightlike hypersurface of an AMGsR manifold $(\bar{M}, \phi, F, \bar{g})$. Then, $FD_0 \subset S(TN)$ for distribution D_0.*

Corollary 6 ([30])**.** *Let N be a screen semi-invariant lightlike hypersurface of an AMGsR manifold $(\bar{M}, \phi, F, \bar{g})$. Then, distribution D_0 is F-invariant.*

Theorem 64 ([30])**.** *Let N be a screen semi-invariant lightlike hypersurface of an AMGsR manifold $(\bar{M}, \phi, F, \bar{g})$. Then, distribution D can be integrated if and only if we have*

$$h(FY, FX) = h(X, F\phi Y) - h(X, FY) + h(X, Y)$$

for any $X, Y \in \Gamma(D)$.

Theorem 65 ([30])**.** *Consider an AMGsR manifold $(\bar{M}, \phi, F, \bar{g},)$ where N is a totally umbilicalscreen semi-invariant lightlike hypersurface of $\bar{M}$. Then, N is totally geodesic.*

Theorem 66 ([30])**.** *In an AMGsR manifold $(\bar{M}, \phi, F, \bar{g})$, suppose N is a screen semi-invariant lightlike hypersurface. If the screen distribution $(S(TN))$ is totally umbilical, then it is also totally geodesic.*

7. Lightlike Submanifolds of an Almost Norden Golden Semi-Riemannian Manifold

Investigations into certain classifications of lightlike hypersurfaces within almost Norden golden semi-Riemannian manifolds, including invariant and screen semi-invariant types, are discussed in [84].

Theorem 67 ([84])**.** *In an almost Norden golden semi-Riemannian manifold $(\bar{M}, \bar{g}, \phi)$ with (N, g) as a lightlike hypersurface of $\bar{M}$, the following three statements are equivalent:*

(i) *N is ϕ-invariant;*
(ii) *The 1-form u vanishes on N;*
(iii) *ϕ is an almost Norden golden structure on N.*

Theorem 68 ([84]). *The almost Norden golden semi-Riemannian has no radical anti-invariant lightlike hypersurface.*

Proposition 19 ([84]). *Let N be a screen semi-invariant lightlike hypersurface and $(\bar{M}, \bar{g}, \phi)$ be an almost Norden golden semi-Riemannian manifold. A ϕ-invariant distribution is then D_0.*

Theorem 69 ([84]). *Let $(\bar{M}, \bar{g}, \phi)$ be an almost Norden golden semi-Riemannian manifold and N be a screen semi-invariant lightlike hypersurface. Then, the following three statements are equivalent:*
(i) *D is a parallel distribution;*
(ii) *D is totally geodesic;*
(iii) *$(\nabla_X \phi)Y = 0$, where $X, Y \in \Gamma(D)$.*

Theorem 70 ([84]). *In an almost Norden golden semi-Riemannian manifold $(\bar{M}, \bar{g}, \phi)$, if N is a screen semi-invariant lightlike hypersurface that is totally umbilical, then N is also totally geodesic in $\bar{M}$.*

Remark 36. *The authors of [84] presented illustrations of invariant and screen semi-invariant lightlike hypersurfaces in almost Norden golden semi-Riemannian manifolds.*

8. Warped Product of Screen-Real Lightlike Submanifolds of Golden Semi-Riemannian Manifolds

The study on the warped product of screen-real lightlike submanifolds in a golden semi-Riemannian manifold was discussed in [85].

Theorem 71 ([85]). *Let $N = N_1 \times_f N_T$ be a warped-product lightlike submanifold; then, for any $X \in Rad(TN)$, $Y \in \Gamma(S(T\bar{M}))$, we have $\nabla_X Y \in \Gamma(S(TN))$.*

Theorem 72 ([85]). *If $(N, g, S(TN))$ is an irrotational screen-real m lightlike submanifold of a golden semi-Riemannian manifold, then the induced connection is metric.*

Theorem 73 ([85]). *There is no concept of an irrotational screen-real m lightlike submanifold that can be expressed as warped-product lightlike submanifolds.*

9. Chen Invariants and Inequalities

Let $\bar{M}^n$ be a Riemannian n manifold. Let us choose a local field of an orthonormal frame $(e_1, \ldots, e_n)$ on $\bar{M}^n$. $K(e_i \wedge e_j)$ denotes the sectional curvatures of $\bar{M}^n$ of the plane section spanned by e_i and e_j.

The *scalar curvature* (τ) of $\bar{M}^n$ at p is defined by

$$\tau(p) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j). \tag{13}$$

Similarly, if $\mathcal{L}$ is an ℓ-dimensional linear subspace of $T_p\bar{M}^n$ with $2 < \ell < n$, then the scalar curvature $(\rho(\mathcal{L}))$ of $\mathcal{L}$ is defined as follows:

$$\tau(L) = \sum_{1 \le i < j \le \ell} K(e_i \wedge e_j), \tag{14}$$

where $e_1, \ldots, e_\ell$ is an orthonormal basis of $\mathcal{L}$.

9.1. Chen Invariants

Let n be a positive integer (≥ 3). For a positive integer ($k \leq \frac{n}{2}$), let $\mathcal{S}(n,k)$ denote the set consisting of k tuples $(n_1, \ldots, n_k)$ of integers (≥ 2) such that $n_1 < n$ *and* $n_1 + \cdots + n_k \leq n$. Furthermore, $\mathcal{S}(n) = \cup_{k\geq 1}\mathcal{S}(n,k)$.

For a given point (p) in a Riemannian n-manifold $\bar{M}^n$ and each $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, in [31,86,87], the first author introduced the following invariants:

$$\delta(n_1, \ldots, n_k)(p) = \tau(x) - \inf\{\tau(\mathcal{L}_1) + \cdots + \tau(\mathcal{L}_k)\},$$

where $\mathcal{L}_1, \ldots, \mathcal{L}_k$ run over all k mutually orthogonal subspaces of $T_p\bar{M}^n$ such that $\dim \mathcal{L}_j = n_j$ and $j = 1, \ldots, k$. In particular, we have

(a) $\delta(\emptyset) = \tau$;

(b) $\delta(2) = \tau - \inf K$, where K is the sectional curvature;

(c) $\delta(n-1)(p) = \max Ric(p)$.

Remark 37. $\delta(2)$ is known today as the first Chen invariant among all of the invariants $(\delta(n_1, \ldots, n_k))$.

9.2. Chen Inequalities

For each $(n_1, \ldots, n_k) \in \mathcal{S}(n,k)$, we set

$$a(n_1, \ldots, n_k) = \frac{1}{2}n(n-1) - \frac{1}{2}\sum_{j=1}^{k} n_j(n_j - 1),$$
$$b(n_1, \ldots, n_k) = \frac{n^2(n + k - 1 - \sum_j n_j)}{2(n + k - \sum_j n_j)}.$$

The first author proved the following optimal universal inequalities (see [32,86–88]).

Theorem 74. *Let N be an n-dimensional submanifold of a Riemannian manifold ($\bar{M}^m$). Then, for each point ($p \in N$) and each k-tuple ($(n_1, \ldots, n_k) \in \mathcal{S}(n)$), we have*

$$\delta(n_1, \ldots, n_k)(p) \leq b(n_1, \ldots, n_k)\|H\|^2(p) + a(n_1, \ldots, n_k)\max \bar{K}(p), \tag{15}$$

where $\|H\|^2$ is the squared mean curvature of N and $\max \bar{K}(p)$ *is the maximum of the sectional curvature function of $\bar{M}^m$ restricted to 2-plane sections of the tangent space (T_pN) at p.*

The equality case of inequality (15) holds at $p \in N$ if and only if the following conditions hold:

(a) *There is an orthonormal basis ($e_1, \ldots, e_n, \xi_{n+1}, \ldots, \xi_m$) at p such that the shape operators of N in $\bar{M}^m$ at p take the following form:*

$$A_{e_r} = \begin{pmatrix} \begin{matrix} A_1^r & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & A_k^r \end{matrix} & 0 \\ 0 & \mu_r I \end{pmatrix}, \quad r = n+1, \ldots, m, \tag{16}$$

where I is an identity matrix and A_j^r is a symmetric $n_j \times n_j$ submatrix such that

$$\text{trace}\,(A_1^r) = \cdots = \text{trace}\,(A_k^r) = \mu_r. \tag{17}$$

(b) *For mutual orthogonal subspaces* ($\mathcal{L}_1, \ldots, \mathcal{L}_k \subset T_pN$) *satisfying* $\delta(n_1, \ldots, n_k) = \tau - \sum_{j=1}^{k} \tau(\mathcal{L}_j)$ *at* p, *we have* $\bar{K}(e_{\alpha_i}, e_{\alpha_j}) = \max \bar{K}(p)$ *for* $\alpha_i \in \Gamma_i, \alpha_j \in \Gamma_j$ *and* $0 \leq i \neq j \leq k$, *where*

$$\Gamma_0 = \{1, \ldots, n_1\}, \ \ldots, \ \Gamma_{k-1} = \{n_1 + \cdots + n_{k-1} + 1, \ldots, n_1 + \cdots + n_k\},$$
$$\Gamma_k = \{n_1 + \cdots + n_k + 1, \ldots, n\}.$$

An important case of Theorem 74 is presented as follows.

Theorem 75 ([86,87]). *For an n-dimensional submanifold* (N) *of a real-space form* ($R^m(c)$) *of constant curvature* (c), *we have*

$$\delta(n_1, \ldots, n_k) \leq b(n_1, \ldots, n_k)\|H\|^2 + a(n_1, \ldots, n_k)c. \tag{18}$$

The equality case of inequality (18) *holds at a point* $p \in N$ *if and only if there is an orthonormal basis* ($e_1, \ldots, e_n, \xi_{n+1}, \ldots, \xi_m$) *such that the shape operators at* p *take the forms of* (16) *and* (17).

10. Inequalities in Golden Riemannian Manifolds

Following Chen's inequalities, many researchers have studied Chen-type inequalities within golden Riemannian manifolds.

10.1. Chen-Type Inequality in Golden Riemannian Manifolds

The following findings about Chen-type inequalities for slant submanifolds in golden Riemannian manifolds were discovered by Uddin and Choudhary in [35].

Theorem 76 ([35]). *The following inequality holds for any proper* θ*-slant submanifold* (N^n) *that is isometrically immersed in a locally golden product manifold* ($\bar{M}^m$).

$$\begin{aligned}\delta_N(p) \leq \frac{(n-2)}{2}&\left[\frac{n^2}{(n-1)}\|H\|^2 + \frac{1}{10}(c_p + c_q)\{3(n+1) - 2\,\mathrm{Trace}(\phi)\}\right] \\ &+ \frac{1}{10}(c_p + c_q)\left[(\mathrm{Trace}(T) + (4-n))\cos^2\theta - \mathrm{Trace}^2(\phi)\right] \\ &+ \frac{1}{4\sqrt{5}}(c_p - c_q)(n-2)[2\,\mathrm{Trace}(\phi) - (\mathrm{n}+1)], \quad p \in N^n.\end{aligned} \tag{19}$$

For the equality case, consider the following.

Theorem 77 ([35]). *When all conditions of the above Theorem 76 are met, equality in Equation* (19) *is achieved at* $p \in N$ *if and only if* $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$, *and the shape operator* (A) *has the following form:*

$$A_{n+1} = \begin{pmatrix} c & 0 & 0 & \ldots & 0 \\ 0 & d & 0 & \ldots & 0 \\ 0 & 0 & c+d & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & c+d \end{pmatrix}, \quad A_s = \begin{pmatrix} c_s & d_s & 0 & \ldots & 0 \\ d_s & -c_s & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \tag{20}$$

for $n + 2 \leq s \leq m$.

Then, an inequality involving $\delta(n_1, \ldots, n_k)$ is calculated as follows.

Theorem 78 ([35])**.** *In each proper θ-slant submanifold (N^n) immersed in $\bar{M}^m$, the following inequality is true:*

$$\delta(n_1,\ldots,n_k) \leq T_3 - \frac{1}{10}(c_p+c_q)\left\{\cos^2\theta + \mathrm{Trace}(\phi)\right\}\left(n - \sum_{j=1}^{k} n_j\right) - \frac{1}{4\sqrt{5}}(c_p-c_q)\left\{\left(n+\sum_{j=1}^{k} n_j\right) - 2\,\mathrm{Trace}(\phi) - 1\right\}\left(n - \sum_{j=1}^{k} n_j\right), \tag{21}$$

where

$$T_3 = d(n_1,\ldots,n_\mu)\|H\|^2 + \frac{3}{10}(c_p+c_q)b(n_1,\ldots,n_k).$$

Additionally, the equality sign in (21) *holds at a point $p \in N$ if and only if there exists an orthonormal basis ($\{e_1,\ldots,e_n,e_{n+1},\ldots,e_m\}$) and A such that*

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & 0 & \ldots & 0 \\ 0 & a_2 & 0 & \ldots & 0 \\ 0 & 0 & a_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & a_n \end{pmatrix}, \quad A_s = \begin{pmatrix} B_1^s & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & B_k^s & 0 & \ldots & 0 \\ 0 & \ldots & 0 & c_s & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & c_s \end{pmatrix},$$

for $s \in \{n+2,\ldots,m\}$, where $a_1,\ldots,a_n$ satisfy

$$a_1 + \cdots + a_{n_1} = \cdots = a_{n_1+\ldots n_{k-1}+1} + \cdots + a_{n_1+\ldots n_k} = a_{n_1+\ldots n_k+1} = \cdots = a_n,$$

and B_i^s is a symmetric $n_i \times n_i$ submatrix satisfying

$$\mathrm{Trace}(B_1^s) = \cdots = \mathrm{Trace}(B_k^s) = c_s.$$

Remark 38. *Additionally, in* [35]*, Uddin and Choudhary deduced a special case of Theorems 76 and 78 for ϕ-invariant submanifolds (N^n) immersed in a locally golden product manifold ($\bar{M}^m$) and inequalities for a Ricci curvature tensor.*

10.2. δ Casorati Curvature in Golden Riemannian Manifolds

In 1890, Casorati [37] introduced what is now termed Casorati curvature for surfaces in a Euclidean 3-space E^3. Casorati favored this curvature over Gaussian curvature because the latter may vanish for surfaces that intuitively seem curved, whereas the former only vanishes at planar points. The Casorati curvature (C) of a submanifold in a Riemannian manifold is generally defined as the normalized squared norm of the second fundamental form. Decu et al. introduced normalized Casorati curvatures $\delta_C(n-1)$ and $\hat{\delta}_C(n-1)$ in 2007 (refer to [89]), aligning with the essence of δ invariants. In 2008, they extended normalized Casorati curvatures to generalized normalized δ Casorati curvatures ($\delta_C(r;n-1)$ and $\hat{\delta}_C(r;n-1)$) in [90]. Concurrently, they were able to ascertain the optimal inequality concerning the (intrinsic) scalar curvature and the (extrinsic) δ Casorati curvature.

Let us recall the Weingarten and Gauss formulas in this context. For a Riemannian manifold ($\bar{M}, \bar{g}$) and a Riemannian submanifold (N) isometrically immersed in $\bar{M}$, where $\bar{\nabla}$ and ∇ are the Levi–Civita connections on $\bar{M}$ and N respectively, and h represents the second fundamental form of N, the Weingarten and Gauss formulas are expressed as follows:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y),$$
$$\bar{\nabla}_X \xi' = -A_{\xi'} X + \nabla_X^{\perp} \xi',$$

$\forall\ X, Y \in \Gamma(TN)$ and $\xi' \in \Gamma\left(T^{\perp}N\right)$, where $A_{\xi'}$ denotes the shape operator of N associated with ξ' and $\nabla^{\perp}$ represents the connection in the normal bundle. The relationship between $A_{\xi'}$ and h can then be recalled.

$$\bar{g}\left(A_{\xi'}X, Y\right) = \bar{g}(h(X,Y), \xi').$$

The Gaussian formula is written as

$$\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) - \bar{g}(h(X,W), h(Y,Z)) + \bar{g}(h(X,Z), h(Y,W))$$

for any vector fields tangent to N, such as X, Y, Z, and W. Assume that the local orthonormal tangent frame is $\{e_1, \ldots, e_n\}$ and the local orthonormal normal frame is $\{e_{n+1}, \ldots, e_m\}$. The definition of the scalar curvature is

$$\tau = \sum_{1 \le i < j \le n} R\left(e_i, e_j, e_j, e_i\right),$$

and the normalized scalar curvature (ρ) is defined as

$$\rho = \frac{2\tau}{n(n-1)}.$$

For N, the mean curvature vector (H) is

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i).$$

The components of h are

$$h_{ij}^{r} = \bar{g}\left(h\left(e_i, e_j\right), e_r\right), \ \ \forall i, j \in \{1, \ldots, n\},\ \forall r \in \{n+1, \ldots, m\}.$$

Then,

$$\|H\|^2 = \frac{1}{n^2}\sum_{r=n+1}^{m}\left(\sum_{i=1}^{n} h_{ii}^{r}\right)^2$$

and

$$\|h\|^2 = \sum_{r=n+1}^{m}\sum_{i,j=1}^{n}\left(h_{ij}^{r}\right)^2.$$

The Casorati curvature (C) of N is defined by

$$C = \frac{1}{n}\|h\|^2.$$

Let $\mathcal{L}$ be an l-dimensional subspace of $T_pN, l \ge 2$, and assume that $p \in N$. The scalar curvature of $\mathcal{L}$ for an orthonormal basis $\{e_1, \ldots, e_l\}$ can be expressed as

$$\tau(\mathcal{L}) = \sum_{1 \le i < j \le t} R\left(e_i, e_j, e_j, e_i\right).$$

One defines

$$C(\mathcal{L}) = \frac{1}{t}\sum_{r=n+1}^{m}\sum_{i,j=1}^{t}\left(h_{ij}^{r}\right)^2.$$

Assume that a hyperplane of T_pN is $\mathcal{L}$. Then, the normalized δ Casorati curvatures $\delta_c(n-1)$ and $\widehat{\delta}_c(n-1)$ are expressed by

$$[\delta_C(n-1)]_p = \frac{1}{2}C_p + \frac{n+1}{2n}\inf\{C(\mathcal{L})\},$$
$$\left[\widehat{\delta}_C(n-1)\right]_p = 2C_p - \frac{2n-1}{2n}\sup\{C(\mathcal{L})\}.$$

The generalized, normalized δ Casorati curvatures of N contain the following expression for any real number $(r > 0)$.

If $0 < r < n(n-1)$,

$$[\delta_C(r;n-1)]_p = rC_p + \frac{1}{rn}\cdot\mathcal{A}_1\inf\{C(\mathcal{L})\},$$

and if $r > n(n-1)$,

$$\left[\widehat{\delta}_C(r;n-1)\right]_p = rC_p + \frac{1}{rn}\cdot\mathcal{A}_1\sup\{C(\mathcal{L})\},$$

with $\mathcal{A}_1 = (n-1)(n+r)\left(n^2-n-r\right)$ [40].

In modern differential geometry, the study of δ Casorati curvatures is a highly active research subject. Many researchers have obtained intriguing findings on δ Casorati curvatures in golden Riemannian manifolds.

Choudhary and Park obtained the following results in [41] regarding δ Casorati curvatures of slant submanifolds of locally golden space forms.

Theorem 79 ([41])**.** *Given an $(n+m)$-dimensional locally golden product space of the form $(\bar{M}^{n+m} = M_p(c_p)\times M_q(c_q), g, \phi)$, let N be a n-dimensional θ-slant proper submanifold. Then, we have the following:*

(i) *The curvature expressed by $\delta_C(r;n-1)$, which is the generalized, normalized δ Casorati curvature, satisfies*

$$\begin{aligned}\rho \le{}& \frac{\delta_C(r;n-1)}{n(n-1)} - \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\times \\ &\left\{1 + \frac{1}{n(n-1)}\operatorname{Trace}^2\phi - \cos^2\theta\left\{\frac{1}{n-1} + \frac{1}{n(n-1)}\operatorname{Trace}\phi\right\}\right\} \\ &- \left(\frac{(1-\psi)c_p + \psi c_q}{4}\right)\frac{2}{n}\operatorname{Trace}\phi\end{aligned} \tag{22}$$

for any real number (r) such that $0 < r < n(n-1)$.

(ii) *The generalized, normalized δ Casorati curvature $(\widehat{\delta}_C(r;n-1))$ satisfies*

$$\begin{aligned}\rho \le{}& \frac{\widehat{\delta}_C(r;n-1)}{n(n-1)} - \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\times \\ &\left\{1 + \frac{1}{n(n-1)}\operatorname{Trace}^2\phi - \cos^2\theta\left\{\frac{1}{n-1} + \frac{1}{n(n-1)}\operatorname{Trace}\phi\right\}\right\} \\ &- \left(\frac{(1-\psi)c_p + \psi c_q}{4}\right)\frac{2}{n}\operatorname{Trace}\phi\end{aligned} \tag{23}$$

for any real number $(r > n(n-1))$.

Furthermore, if and only if N is an invariantly quasi-umbilical submanifold with a trivial normal connection in $(\bar{M})$, then the equalities in relations (22) and (23) hold such that the shape oper-

ators ($A_r, r \in \{n+1,\ldots,n+m\}$), with respect to some orthonormal tangent frame ($\{e_1,\ldots,e_n\}$) and orthonormal normal frame ($\{e_{n+1},\ldots,e_{n+m}\}$), have the following forms:

$$A_{n+1} = \begin{pmatrix} d & 0 & 0 & \ldots & 0 & 0 \\ 0 & d & 0 & \ldots & 0 & 0 \\ 0 & 0 & d & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & d & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{n(n-1)}{r}d \end{pmatrix}, \quad A_{n+2} = \cdots = A_{n+m} = 0. \tag{24}$$

In golden Riemannian space forms, Choudhary and Park [48] also obtained sharp inequalities for ϕ-invariant and ϕ-anti-invariant submanifolds as a consequence of Theorem 79.

Theorem 80 ([41]). *Consider N as an n-dimensional invariant submanifold of a locally golden product space of the form $(\bar{M}^{n+m} = M_p(c_p) \times M_q(c_q), g, \phi)$. Then, we have the following:*
(i) *The generalized, normalized δ Casorati curvature ($\delta_C(r; n-1)$) satisfies*

$$\begin{aligned} \rho \leq\; & \frac{\delta_C(r;n-1)}{n(n-1)} - \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left\{1 + \frac{1}{n(n-1)}\operatorname{Trace}^2\phi\right\} \\ & + \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left\{\frac{1}{n-1} + \frac{1}{n(n-1)}\operatorname{Trace}\phi\right\} \\ & - \left(\frac{(1-\psi)c_p + \psi c_q}{4}\right)\frac{2}{n}\operatorname{Trace}\phi \end{aligned} \tag{25}$$

for any real number (r) such that $0 < r < n(n-1)$.
(ii) *The generalized normalized δ Casorati curvature $\widehat{\delta}_C(r; n-1)$ satisfies*

$$\begin{aligned} \rho \leq\; & \frac{\widehat{\delta}_C(r;n-1)}{n(n-1)} - \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left\{1 + \frac{1}{n(n-1)}\operatorname{Trace}^2\phi\right\} \\ & + \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left\{\frac{1}{n-1} + \frac{1}{n(n-1)}\operatorname{Trace}\phi\right\} \\ & - \left(\frac{(1-\psi)c_p + \psi c_q}{4}\right)\frac{2}{n}\operatorname{Trace}\phi \end{aligned} \tag{26}$$

for any real number ($r > n(n-1)$).

Furthermore, the equalities in relations (25) and (26) hold if and only if N is an invariantly quasi-umbilical submanifold with a trivial normal connection in $\bar{M}$ such that the shape operators ($A_r, r \in \{n+1,\ldots,n+m\}$), take the following forms for some orthonormal tangent frame ($\{e_1,\ldots,e_n\}$) and orthonormal normal frame ($\{e_{n+1},\ldots,e_{n+m}\}$):

$$A_{n+1} = \begin{pmatrix} d & 0 & 0 & \ldots & 0 & 0 \\ 0 & d & 0 & \ldots & 0 & 0 \\ 0 & 0 & d & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & d & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{n(n-1)}{r}d \end{pmatrix}, \quad A_{n+2} = \cdots = A_{n+m} = 0. \tag{27}$$

Theorem 81 ([41]). *Let N be an n-dimensional anti-invariant submanifold within $(n+m)$-dimensional locally golden product space of the form $(\bar{M} = M_p(c_p) \times M_q(c_q), g, \phi)$. Thus,*
(i) *The generalized normalized δ Casorati curvature ($\delta_C(r; n-1)$) satisfies*

$$\rho \leq \frac{\delta_C(r;n-1)}{n(n-1)} + \left(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right) \tag{28}$$

for any real number (r) such that $0 < r < n(n-1)$*. Then, the following conditions hold:*

(ii) *The generalized normalized* δ *Casorati curvature (*$\widehat{\delta}_C(r; n-1)$*) satisfies*

$$\rho \leq \frac{\widehat{\delta}_C(r; n-1)}{n(n-1)} + \left(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right) \tag{29}$$

for any real number ($r > n(n-1)$*).*

Additionally, the equalities hold in relations (28) and (29) if and only if N is an invariantly quasi-umbilical submanifold with a trivial normal connection in $\bar{M}$ *such that the shape operators (*$A_r, r \in \{n+1, \ldots, n+m\}$*) take the following forms for some orthonormal tangent frame (*$\{e_1, \ldots, e_n\}$*) and orthonormal normal frame (*$\{e_{n+1}, \ldots, e_{n+m}\}$*):*

$$A_{n+1} = \begin{pmatrix} d & 0 & 0 & \ldots & 0 & 0 \\ 0 & d & 0 & \ldots & 0 & 0 \\ 0 & 0 & d & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & d & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{n(n-1)}{r}d \end{pmatrix}, \quad A_{n+2} = \cdots = A_{n+m} = 0. \tag{30}$$

Theorem 82 ([41])**.** *Assume that the locally golden product space form of dimension* $(n+m)$ *is* $(\bar{M} = M_p(c_p) \times M_q(c_q), g, \phi)$*. For any n-dimensional* θ*-slant proper submanifold (N) of* $\bar{M}$*,*

(i) *The normalized* δ*-Casorati curvature (*$\delta_C(n-1)$*) satisfies*

$$\begin{aligned} \rho \leq \delta_C(n-1) &- \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left\{1 + \frac{1}{n(n-1)}\operatorname{Trace}^2\phi\right\} \\ &+ \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\cos^2\theta\left\{\frac{1}{n-1} + \frac{1}{n(n-1)}\operatorname{Trace}\phi\right\} \\ &- \left(\frac{(1-\psi)c_p + \psi c_q}{4}\right)\frac{2}{n}\operatorname{Trace}\phi \end{aligned} \tag{31}$$

(ii) *The normalized* δ *Casorati curvature* $\widehat{\delta}_C(n-1)$ *satisfies*

$$\begin{aligned} \rho \leq \widehat{\delta}_C(n-1) &- \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left\{1 + \frac{1}{n(n-1)}\operatorname{Trace}^2\phi\right\} \\ &+ \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\cos^2\theta\left\{\frac{1}{n-1} + \frac{1}{n(n-1)}\operatorname{Trace}\phi\right\} \\ &- \left(\frac{(1-\psi)c_p + \psi c_q}{4}\right)\frac{2}{n}\operatorname{Trace}\phi \end{aligned} \tag{32}$$

Regarding any invariant submanifold (N) of $\bar{M}$*, we have*

(i) *The normalized* δ *Casorati curvature (*$\delta_C(n-1)$*) satisfies*

$$\begin{aligned} \rho \leq \delta_C(n-1) &- \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left\{1 + \frac{1}{n(n-1)}\operatorname{Trace}^2\phi\right\} \\ &+ \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left\{\frac{1}{n-1} + \frac{1}{n(n-1)}\operatorname{Trace}\phi\right\} \\ &- \left(\frac{(1-\psi)c_p + \psi c_q}{4}\right)\frac{2}{n}\operatorname{Trace}\phi \end{aligned} \tag{33}$$

(ii) *The normalized δ Casorati curvature* $\widehat{\delta}_C(n-1)$ *satisfies*

$$\begin{aligned}\rho \leq &\widehat{\delta}_C(n-1) - \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left\{1 + \frac{1}{n(n-1)}t^2\phi\right\} \\ &+ \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left\{\frac{1}{n-1} + \frac{1}{n(n-1)}\,\mathrm{Trace}\,\phi\right\} \\ &- \left(\frac{(1-\psi)c_p + \psi c_q}{4}\right)\frac{2}{n}\,\mathrm{Trace}\,\phi.\end{aligned} \tag{34}$$

For any n-dimensional anti-invariant submanifold (N) of $\bar{M}$*,*
(i) *The normalized δ Casorati curvature* $(\delta_C(n-1))$ *satisfies*

$$\rho \leq \delta_C(n-1) - \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right) \tag{35}$$

(ii) *The normalized δ Casorati curvature* $\widehat{\delta}_C(n-1)$ *satisfies*

$$\rho \leq \widehat{\delta}_C(n-1) - \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right). \tag{36}$$

Additionally, (31), (33), *and* (35) *hold as equalities if and only if the submanifold* (N^n) *is invariantly quasi-umbilical with a trivial normal connection in* $\bar{M}$*. In this case, the shape operators* $(A_r, r \in \{n+1, \ldots, n+m\})$ *with respect to some orthonormal tangent frame* $(\{e_1, \ldots, e_n\})$ *and orthonormal normal frame* $(\{e_{n+1}, \ldots, e_{n+m}\})$ *fulfill the following requirements:*

$$A_{n+1} = \begin{pmatrix} d & 0 & 0 & \ldots & 0 & 0 \\ 0 & d & 0 & \ldots & 0 & 0 \\ 0 & 0 & d & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & d & 0 \\ 0 & 0 & 0 & \ldots & 0 & 2d \end{pmatrix}, \quad A_{n+2} = \cdots = A_{n+m} = 0, \tag{37}$$

Furthermore, (32), (34), *and* (36) *hold as equalities if and only if the submanifold* (N^n) *is invariantly quasi-umbilical with a trivial normal connection in* $\bar{M}$ *such that the shape operators* $(A_r, r \in \{n+1, \ldots, n+m\})$ *with respect to some orthonormal tangent frame* $(\{e_1, \ldots, e_n\})$ *and orthonormal normal frame* $(\{e_{n+1}, \ldots, e_{n+m}\})$ *satisfy the following:*

$$A_{n+1} = \begin{pmatrix} 2d & 0 & 0 & \ldots & 0 & 0 \\ 0 & 2d & 0 & \ldots & 0 & 0 \\ 0 & 0 & 2d & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2d & 0 \\ 0 & 0 & 0 & \ldots & 0 & d \end{pmatrix}, \quad A_{n+2} = \cdots = A_{n+m} = 0. \tag{38}$$

Remark 39. *Some sharp inequalities for slant submanifolds immersed in golden Riemannian space forms with a semi-symmetric metric connection were deduced by Lee et al. in [91]. Furthermore, they characterized submanifolds in the equality case. They concluded by talking about these inequalities for a few special submanifolds.*

Geometric inequalities for the Casorati curvatures on submanifolds on golden Riemannian manifolds with constant golden sectional curvature were established by Choudhary and Mihai in [40]. Let $(\bar{M}, \phi, \bar{g})$ be a locally decomposable golden Riemannian manifold with a constant golden sectional curvature. Next, on a submanifold (N), the following are the optimal inequalities for $\delta_C(r; n-1)$ and $\hat{\delta}_C(r; n-1)$:

Theorem 83 ([40])**.** *Considering M as an n-dimensional Riemannian manifold isometrically immersed in* $(\bar{M}, \phi, \bar{g})$ *under the condition of* $\mathcal{A}_2 = n(n-1)$*, we have:*

(i) $\delta_C(r; n-1)$ *satisfies*

$$\rho \leq \frac{\delta_C(r; n-1)}{\mathcal{A}_2} + \frac{c}{3\mathcal{A}_2}\left\{n^2 - 3n + 2\|\phi\|^2 - 2n\|\phi\|\right\} \tag{39}$$

if $0 < r < \mathcal{A}_2$*;*

(ii) $\hat{\delta}_C(r; n-1)$ *satisfies*

$$\rho \leq \frac{\widehat{\delta}_C(r; n-1)}{\mathcal{A}_2} + \frac{c}{3\mathcal{A}_2}\left\{n^2 - 3n + 2\|\phi\|^2 - 2n\|\phi\|\right\} \tag{40}$$

if $r > \mathcal{A}_2$*.*

Furthermore, if and only if N is invariantly quasi-umbilical does the equality holds in (39) *or* (40)*. There exists an orthonormal tangent frame* $(\{e_1, \dots, e_n\})$ *and an orthonormal normal frame* $(\{e_{n+1}, \dots, e_m\})$ *such that* $A_r, r \in \{n+1, \dots, m\}$ *have the following forms and the normal connection of N in* $\bar{M}$ *is trivial.*

$$A_{n+1} = \begin{pmatrix} d & 0 & 0 & \dots & 0 & 0 \\ 0 & d & 0 & \dots & 0 & 0 \\ 0 & 0 & d & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & d & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{r}\cdot\mathcal{A}_2\cdot d \end{pmatrix}, \quad A_{n+2} = \dots = A_m = 0. \tag{41}$$

Theorem 84 ([40])**.** *Let N be an isometrically immersed n-dimensional submanifold in* $(\bar{M}, \phi, \bar{g})$*. Then,*

(i) $\delta_C(r; n-1)$ *satisfies*

$$\rho^G \leq \frac{\delta_C(r; n-1)}{\mathcal{A}_2} + \frac{c}{3\mathcal{A}_2}\{\|\phi\|(3\|\phi\| - 3 - n)\}, \tag{42}$$

for $0 < r < \mathcal{A}_2$*.*

(ii) $\hat{\delta}_C(r; n-1)$ *satisfies*

$$\rho^G \leq \frac{\widehat{\delta}_C(r; n-1)}{\mathcal{A}_2} + \frac{c}{3\mathcal{A}_2}\{\|\phi\|(3\|\phi\| - 3 - n)\}, \tag{43}$$

for $r > \mathcal{A}_2$*.*

Furthermore, A_r *achieves the following forms, and the equality holds in* (42) *or* (43) *if and only if N meets the equality's criteria as stated in Theorem* 83*.*

$$A_{n+1} = \begin{pmatrix} d & 0 & 0 & \dots & 0 & 0 \\ 0 & d & 0 & \dots & 0 & 0 \\ 0 & 0 & d & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & d & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{r}\mathcal{A}_2 d \end{pmatrix}, \quad A_{n+2} = \dots = A_m = 0. \tag{44}$$

Remark 40. *In [40], Choudhary and Mihai established the consequences of Theorems* 83 *and* 84 *and obtained inequality cases for Casorati curvature on an anti-invariant submanifold in* $(\bar{M}, \phi, \bar{g})$*.*

Remark 41. *Regarding sharp inequalities concerning δ Casorati curvatures for slant submanifolds of golden Riemannian space forms, we refer to [38].*

10.3. Wintgen-Type Inequality in Golden Riemannian Manifolds

In the context of four-dimensional Euclidean space, Wintgen inequality represents a critical geometric inequality that incorporates Gaussian curvature, normal curvature, and squared mean curvature, all of which are intrinsic invariants. In 1979, P. Wintgen [42] formulated this inequality to show that for any surface (M^2) in E^4, the Gaussian curvature (K), the normal curvature ($K^{\perp}$), and the squared mean curvature ($\|H\|^2$) meet the following condition:

$$\|H\|^2 \geq K + |K^{\perp}|$$

The equality is valid if and only if the ellipse of curvature of M^2 in E^4 is a circle. This result was further generalized by I. V. Guadalupe et al. in [92] for an arbitrary codimension of m in real-space forms ($\bar{M}^{m+2}(c)$) as follows:

$$\|H\|^2 + c \geq K + |K^{\perp}|.$$

They also discussed the conditions under which equality is achieved.

De Smet, Dillen, Verstraelen, and Vrancken [93] proposed an inequality for submanifolds in real-space forms, referred to as the generalized Wintgen inequality or DDVV conjecture, which extends the Wintgen inequality. This conjecture was independently proven by Ge and Tang in [94]. Different researchers have obtained DDVV inequality for various classes of submanifolds in various ambient manifolds in recent years. For slant, invariant, C-totally real, and Lagrangian submanifolds in golden Riemannian space forms, researchers obtained generalized Wintgen-type inequalities in [44] and discussed the equality cases.

For slant submanifolds, the generalized Wintgen inequality is outlined as follows:

Theorem 85 ([44])**.** *Let N be an n-dimensional θ-slant proper submanifold of a locally golden product space of the form $(\bar{M} = M_p(c_p) \times M_q(c_q), g, \phi)$. Then, we have the following:*

$$\begin{aligned} \rho_\eta \leq \|\mathcal{H}\|^2 - 2\rho - 2\left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left\{1 + \frac{1}{n(n-1)}\operatorname{Trace}^2\phi\right\} \\ + 2\left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left\{\frac{1}{n-1} + \frac{1}{n(n-1)}\operatorname{Trace} P\right\} \\ - \left(\frac{(1-\psi)c_p + \psi c_q}{4}\right)\frac{4}{n}\operatorname{Trace}\phi. \end{aligned} \tag{45}$$

For scalar normal curvature, ρ_η is used.

Choudhary et al. [44] established the generalized Wintgen inequality for an invariant submanifold of golden Riemannian space forms with the aid of Theorem 85.

Theorem 86 ([44])**.** *Consider N an n-dimensional invariant submanifold within a locally golden product space of the form $(\bar{M} = M_p(c_p) \times M_q(c_q), g, \phi)$. Then,*

$$\begin{aligned} \rho_\eta \leq \|\mathcal{H}\|^2 - 2\rho - 2\left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left\{1 + \frac{1}{n(n-1)}\operatorname{Trace}^2\phi\right\} \\ + 2\left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left\{\frac{1}{n-1} + \frac{1}{n(n-1)}\operatorname{Trace} P\right\} \\ - \left(\frac{(1-\psi)c_p + \psi c_q}{4}\right)\frac{4}{n}\operatorname{Trace}\phi. \end{aligned} \tag{46}$$

The generalized Wintgen inequality of a locally golden product space form for a Lagrangian submanifold is outlined as follows.

Theorem 87 ([44])**.** *Suppose N is a Lagrangian submanifold in a locally golden product space of the form* $\left(\bar{M} = M_p(c_p) \times M_q(c_q), g, \phi\right)$*. Then,*

$$\begin{aligned}\left(\rho^{\perp}\right)^2 \geq \rho_N^2 - \frac{2}{n(n-1)}\left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)^2 \\ - \frac{4}{n(n-1)}\left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left\{\left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right) + \rho\right\}\end{aligned} \tag{47}$$

11. Inequalities in Golden-like Statistical Manifolds

11.1. Chen-Type Inequality in Golden-like Statistical Manifolds

Some fundamental inequalities for the curvature invariants of statistical submanifolds in golden-like statistical manifolds were obtained by Bahadir et al. in [36].

Theorem 88 ([36])**.** *Let a golden-like statistical manifold of dimension m be expressed by* $(\bar{M}, \bar{g}, \phi)$ *and N be its statistical submanifold of dimension n. Then,*

$$\begin{aligned}&(\tau - K(\pi)) - (\tau_0 - K_0(\pi)) \geq \\ &\quad - \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left[n(n-2) + \mathrm{Trace}^2(\phi) - \mathrm{Trace}(\phi^*)\right] \\ &\quad - \left(\frac{(1-\psi)c_p + \psi c_q}{4}\right) 2(n-1)\,\mathrm{Trace}(\phi) \\ &\quad + \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)[1 + \Psi(\pi) + \Theta(\pi)] \\ &\quad - \frac{n^2(n-2)}{4(n-1)}\left[\|H\|^2 + \|H^*\|^2\right] + 2\hat{K}_0(\pi) - 2\hat{\tau}_0.\end{aligned} \tag{48}$$

Corollary 7 ([36])**.** *Let N be the totally real statistical submanifold of dimension n of a golden-like statistical manifold* $(\bar{M}, \bar{g}, \phi)$ *of dimension m. Then,*

$$\begin{aligned}(\tau - K(\pi)) - (\tau_0 - K_0(\pi)) \geq - \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)[n(n-2) - 1] \\ - \frac{n^2(n-2)}{4(n-1)}\left[\|H\|^2 + \|H^*\|^2\right] + 2\hat{K}_0(\pi) - 2\hat{\tau}_0.\end{aligned} \tag{49}$$

11.2. δ Casorati Curvature in Golden-like Statistical Manifolds

In [36], Bahadir et al. deduced optimal relationships for the generalized, normalized δ Casorati curvature of a statistical submanifold in a golden-like statistical manifold.

Theorem 89 ([36])**.** *Let* N^n *be a statistical submanifold in a golden-like statistical manifold* $(\bar{M}^m)$*. Then, we have the following optimal relationships for the generalized normalized* δ *Casorati curvature:*

(i) *For any real number* (r)*, such that* $0 < r < n(n-1)$*,*

$$\begin{aligned}\rho \leq \frac{\delta_C^0(r; n-1)}{n(n-1)} + \frac{1}{(n-1)}C^0 - \frac{n}{(n-1)}g(H, H^*) - \frac{2n}{n(n-1)}\left\|H^0\right\|^2 \\ - \frac{1}{n(n-1)}\left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left[n(n-2) + \mathrm{Trace}^2(\phi) - \mathrm{Trace}(\phi^*)\right] \\ - \frac{2}{n}\left(\frac{(1-\psi)c_p + \psi c_q}{4}\right)\mathrm{Trace}(\phi),\end{aligned} \tag{50}$$

where $\delta_C^0(r; n-1) = \frac{1}{2}\left[\delta_C(r; n-1) + \delta_C^*(r; n-1)\right]$.

(ii) *For any real number* $(r > n(n-1))$,

$$\rho \leq \frac{\widehat{\delta}_C^0(r;n-1)}{n(n-1)} + \frac{1}{(n-1)}C^0 - \frac{n}{(n-1)}g(H,H^*) - \frac{2n}{n(n-1)}\left\|H^0\right\|^2 - \frac{1}{n(n-1)}\left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left[n(n-2) + \mathrm{Trace}^2(\phi) - \mathrm{Trace}(\phi^*)\right] - \frac{2}{n}\left(\frac{(1-\psi)c_p + \psi c_q}{4}\right)\mathrm{Trace}(\phi), \tag{51}$$

where $\widehat{\delta}_C^0(r;n-1) = \frac{1}{2}\left[\widehat{\delta}_C(r;n-1) + \widehat{\delta}_C^*(r;n-1)\right]$.

Corollary 8 ([36])**.** *Given a golden-like statistical manifold* $(\bar{M}^m)$, *let* N^n *be a totally real statistical submanifold of it. Then, for the generalized normalized* δ *Casorati curvature, we obtain the following optimal relationships:*

(i) *For any real number* (r), *such that* $0 < r < n(n-1)$,

$$\rho \leq \frac{\delta_C^0(r;n-1)}{n(n-1)} + C^0 - \frac{n}{(n-1)}g(H,H^*) - \frac{2n}{n(n-1)}\left\|H^0\right\|^2 - \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left(\frac{n-2}{n-1}\right), \tag{52}$$

where $\delta_C^0(r;n-1) = \frac{1}{2}\left[\delta_C(r;n-1) + \delta_C^*(r;n-1)\right]$.

(ii) *For any real number* $(r > n(n-1))$,

$$\rho \leq \frac{\hat{\delta}_C^0(r;n-1)}{n(n-1)} + \frac{1}{(n-1)}C^0 - \frac{n}{(n-1)}g(H,H^*) - \frac{2n}{n(n-1)}\left\|H^0\right\|^2 - \left(\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\right)\left(\frac{n-2}{n-1}\right), \tag{53}$$

where $\hat{\delta}_C^0(r;n-1) = \frac{1}{2}\left[\hat{\delta}_C(r;n-1) + \widehat{\delta}_C^*(r;n-1)\right]$.

12. Inequalities in Golden Lorentzian Manifolds

δ Casorati Curvature in Golden Lorentzian Manifolds

In [50], Choudhary et al. deduced sharp geometric inequalities that involve generalized, normalized δ Casorati curvatures concerning submanifolds of golden Lorentzian manifolds equipped with a generalized symmetric metric U connection and obtained the following results:

Theorem 90 ([50])**.** *Below are the inequalities for the submanifold* (N^n) *of* $\bar{M}^m$.

(i) *For* $\delta_C(r;n-1)$, *we have*

$$\rho \leq \frac{1}{(n^2-n)}[\delta_C(r;n-1)] + \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q - 10\alpha^2}{10(n-1)}(n-\varepsilon) + \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q - 10\alpha\beta}{10(n^2-n)}[(2n\varepsilon-2)\mathrm{Trace}\ \phi] + \frac{c_p + c_q - 5\beta^2}{5(n^2-n)}\left[(\mathrm{Trace}\ \phi)^2 - \mathrm{Trace}\ \phi - n\varepsilon\right], \tag{54}$$

where a real number (r) *satisfies* $n^2 - n > r > 0$;

(ii) *For* $\widehat{\delta}_C(r; n-1)$, *we have*

$$\rho \leq \frac{1}{(n^2-n)}\Big[\widehat{\delta}_C(r; n-1)\Big] + \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q - 10\alpha^2}{10(n-1)}(n-\varepsilon) + \frac{(\pm\sqrt{5}-1)c_1 + (\mp\sqrt{5}-1)c_2 - 10\alpha\beta}{10(n^2-n)}[(2n\varepsilon-2)\text{Trace }\phi] + \frac{c_p+c_q-5\beta^2}{5(n^2-n)}\Big[(\text{Trace }\phi)^2 - \text{Trace }\phi - n\varepsilon\Big], \tag{55}$$

where $n^2 - n < r$.

Furthermore, if the shape operator in an orthonormal frame ($\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$) *can be expressed as follows, then the relations in Equations* (54) *and* (55) *become equalities:*

$$A_{n+1} = \begin{pmatrix} d & 0 & 0 & \ldots & 0 & 0 \\ 0 & d & 0 & \ldots & 0 & 0 \\ 0 & 0 & d & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & d & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{dn}{r}(n-1) \end{pmatrix}, \quad A_{n+2} = \cdots = A_m = 0.$$

Corollary 9 ([50]). *The following inequalities hold for any submanifold* (N^n) *immersed in a locally golden product Lorentzian manifold* ($\bar{M}^m$) *equipped with a generalized symmetric metric U connection.*

(i) *For* $\delta_C(r; n-1)$, *we have*

$$\rho \leq \delta_C(n-1) + \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q - 10\alpha^2}{10(n-1)}(n-\varepsilon) + \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q - 10\alpha\beta}{10(n^2-n)}[(2n\varepsilon-2)\text{Trace }\phi] + \frac{c_p+c_q-5\beta^2}{5(n^2-n)}\Big[(\text{Trace }\ \phi)^2 - \ \text{Trace }\ \phi - n\varepsilon\Big] \tag{56}$$

where $0 \leq r \leq n^2 - n$;

(ii) *For* $\widehat{\delta}_C(n-1)$, *we have*

$$\rho \leq \widehat{\delta}_C(n-1) + \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q - 10\alpha^2}{10(n-1)}(n-\varepsilon) + \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q - 10\alpha\beta}{10(n^2-n)}[(2n\varepsilon-2)\text{Trace }\phi] + \frac{c_p+c_q-5\beta^2}{5(n^2-n)}\Big[(\text{Trace }\phi)^2 - \text{Trace }\phi - n\varepsilon\Big], \tag{57}$$

where $n^2 - n < r$.

In addition, Equations (56) and (57) hold for equality if, for an orthonormal frame ($\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$), operator A can be expressed as follows:

$$A_{n+1} = \begin{pmatrix} d & 0 & 0 & \ldots & 0 & 0 \\ 0 & d & 0 & \ldots & 0 & 0 \\ 0 & 0 & d & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & d & 0 \\ 0 & 0 & 0 & \ldots & 0 & 2d \end{pmatrix}, \quad A_{n+2} = \cdots = A_m = 0;$$

and

$$A_{n+1} = \begin{pmatrix} 2d & 0 & 0 & \ldots & 0 & 0 \\ 0 & 2d & 0 & \ldots & 0 & 0 \\ 0 & 0 & 2d & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2d & 0 \\ 0 & 0 & 0 & \ldots & 0 & d \end{pmatrix}, \quad A_{n+2} = \cdots = A_m = 0.$$

In [50], some consequences of Theorem 90 were also derived, which are expressed as follows:

Corollary 10 ([50])**.** *We have the following for a Riemannian manifold (N^n) isometrically immersed in $\bar{M}^m$:*
(I) *For $\delta_C(r; n-1)$ with $r \in \{0, \ldots, n(n-1)\}$, the following hold:*

(a) *$\bar{M}^m$ is equipped with an α semi-symmetric metric U connection.*

$$\begin{aligned} \rho \leq & \frac{1}{(n^2-n)}[\delta_C(r; n-1)] \\ & + \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q - 10\alpha^2}{10(n-1)}(n-\varepsilon) \\ & + \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q}{10(n^2-n)}[(2m\varepsilon-2)\mathrm{Trace}\,\phi] \\ & + \frac{c_p+c_q}{5(n^2-n)}\Big[(\mathrm{Trace}\,\phi)^2 - \mathrm{Trace}\,\phi - n\varepsilon\Big] \end{aligned} \tag{58}$$

(b) *$\bar{M}^m$ is equipped with a β quarter-symmetric metric U connection.*

$$\begin{aligned} \rho \leq & \frac{1}{(n^2-n)}[\delta_C(r; n-1)] \\ & + \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q}{10(n-1)}(n-\varepsilon) \\ & + \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q}{10(n^2-n)}[(2n\varepsilon-2)\mathrm{Trace}\,\phi] \\ & + \frac{c_p+c_q-5\beta^2}{5(n^2-n)}\Big[(\mathrm{Trace}\,\phi)^2 - \mathrm{Trace}\,\phi - n\varepsilon\Big] \end{aligned} \tag{59}$$

(c) $\bar{M}^m$ *is equipped with a semi-symmetric metric U connection.*

$$\begin{aligned}\rho \leq & \frac{1}{(n^2-n)}[\delta_C(r;n-1)] \\ & + \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q - 10}{10(n-1)}(n-\varepsilon) \\ & + \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q}{10(n^2-n)}[(2n\varepsilon-2)\mathrm{Trace}\,\phi] \\ & + \frac{c_p+c_q}{5(n^2-n)}\Big[(\mathrm{Trace}\,\phi)^2 - \mathrm{Trace}\,\phi - n\varepsilon\Big]\end{aligned} \tag{60}$$

(d) $\bar{M}^m$ *is equipped with a quarter-symmetric metric U connection.*

$$\begin{aligned}\rho \leq & \frac{1}{(n^2-n)}[\delta_C(r;n-1)] \\ & + \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q}{10(n-1)}(n-\varepsilon) \\ & + \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q}{10(n^2-n)}[(2n\varepsilon-2)\mathrm{Trace}\,\phi] \\ & + \frac{c_p+c_q-5}{5(n^2-n)}\Big[(\mathrm{Trace}\,\phi)^2 - \mathrm{Trace}\,\phi - n\varepsilon\Big].\end{aligned} \tag{61}$$

(II) *For* $\widehat{\delta}_C(r;n-1)$ *with* $r > n(n-1)$*, the following hold:*

(a) $\bar{M}^m$ *is equipped with an* α *semi-symmetric metric U connection.*

$$\begin{aligned}\rho \leq & \frac{1}{(n^2-n)}\Big[\widehat{\delta}_C(r;n-1)\Big] \\ & + \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q - 10\alpha^2}{10(n-1)}(n-\varepsilon) \\ & + \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q}{10(n^2-n)}[(2n\varepsilon-2)\mathrm{Trace}\,\phi] \\ & + \frac{c_p+c_q}{5(n^2-n)}\Big[(\mathrm{Trace}\,\phi)^2 - \mathrm{Trace}\,\phi - n\varepsilon\Big]\end{aligned} \tag{62}$$

(b) $\bar{M}^m$ *is equipped with a* β *quarter-symmetric metric U connection.*

$$\begin{aligned}\rho \leq & \frac{1}{(n^2-n)}\Big[\widehat{\delta}_C(r;n-1)\Big] \\ & + \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q}{10(n-1)}(n-\varepsilon) \\ & + \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q}{10(n^2-n)}[(2n\varepsilon-2)\mathrm{Trace}\,\phi] \\ & + \frac{c_p+c_q-5\beta^2}{5(n^2-n)}\Big[(\mathrm{Trace}\,\phi)^2 - \mathrm{Trace}\,\phi - n\varepsilon\Big]\end{aligned} \tag{63}$$

(c) $\bar{M}^m$ *is equipped with a semi-symmetric metric U connection.*

$$\begin{aligned}\rho \leq &\frac{1}{(n^2-n)}\left[\widehat{\delta}_C(r;n-1)\right] \\ &+ \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q - 10}{10(n-1)}(n-\varepsilon) \\ &+ \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q}{10(n^2-n)}[(2n\varepsilon-2)\text{Trace }\phi] \\ &+ \frac{c_p+c_q}{5(n^2-n)}\left[(\text{Trace }\phi)^2 - \text{Trace }\phi - n\varepsilon\right]\end{aligned} \tag{64}$$

(d) $\bar{M}^m$ *is equipped with a quarter-symmetric metric U connection.*

$$\begin{aligned}\rho \leq &\frac{1}{(n^2-n)}\left[\widehat{\delta}_C(r;n-1)\right] \\ &+ \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q}{10(n-1)}(n-\varepsilon) \\ &+ \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q}{10(n^2-n)}[(2n\varepsilon-2)\text{Trace }\phi] \\ &+ \frac{c_p+c_q-5}{5(n^2-n)}\left[(\text{Trace }\phi)^2 - \text{Trace }\phi - n\varepsilon\right]\end{aligned} \tag{65}$$

Moreover, the relations in the above results become equalities if, in some orthonormal frame $(\{e_1,\ldots,e_n,e_{n+1},\ldots,e_m\})$ *operator* A *reduces to*

$$A_{m+1} = \begin{pmatrix} d & 0 & 0 & \cdots & 0 & 0 \\ 0 & d & 0 & \cdots & 0 & 0 \\ 0 & 0 & d & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & d & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{d(n^2-n)}{r} \end{pmatrix}, \quad A_{n+2} = \cdots = A_m = 0.$$

Corollary 11 ([50])**.** *When* N^n *represents a Riemannian manifold isometrically immersed in a golden Lorentzian manifold* $(\bar{M}^m)$ *equipped with a g.s.m. U connection, we have the following relations.*

(I) *For* $\delta_C(n-1)$ *with* $r \in \{0,\ldots(n^2-n)\}$*, the following hold:*

(a) $\bar{M}^m$ *is equipped with an* α *semi-symmetric metric U connection.*

$$\begin{aligned}\rho \leq &\frac{1}{(n^2-n)}[\delta_C(n-1)] \\ &+ \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q - 10\alpha^2}{10(n-1)}(n-\varepsilon) \\ &+ \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q}{10(n^2-n)}[(2n\varepsilon-2)\text{Trace }\phi] \\ &+ \frac{c_p+c_q}{5(n^2-n)}\left[(\text{Trace }\phi)^2 - \text{Trace }\phi - n\varepsilon\right]\end{aligned} \tag{66}$$

(b) $\bar{M}^m$ *is equipped with a β quarter-symmetric metric U connection.*

$$\begin{aligned}\rho \leq & \frac{1}{(n^2-n)}[\delta_C(n-1)] \\ & + \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q}{10(n-1)}(n-\varepsilon) \\ & + \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q}{10(n^2-n)}[(2n\varepsilon-2)\text{Trace }\phi] \\ & + \frac{c_p + c_q - 5\beta^2}{5(n^2-n)}\Big[(\text{Trace }\phi)^2 - \text{Trace }\phi - n\varepsilon\Big]\end{aligned} \tag{67}$$

(c) $\bar{M}^m$ *is equipped with a semi-symmetric metric U connection.*

$$\begin{aligned}\rho \leq & \frac{1}{(n^2-n)}[\delta_C(n-1)] \\ & + \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q - 10}{10(n-1)}(n-\varepsilon) \\ & + \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q}{10(n^2-n)}[(2n\varepsilon-2)\text{Trace }\phi] \\ & + \frac{c_p + c_q}{5(n^2-n)}\Big[(\text{Trace }\phi)^2 - \text{Trace }\phi - n\varepsilon\Big]\end{aligned} \tag{68}$$

(d) $\bar{M}^m$ *is equipped with a quarter-symmetric metric U connection.*

$$\begin{aligned}\rho \leq & \frac{1}{(n^2-n)}[\delta_C(n-1)] \\ & + \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q}{10(n-1)}(n-\varepsilon) \\ & + \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q}{10(n^2-n)}[(2n\varepsilon-2)\text{Trace }\phi] \\ & + \frac{c_p + c_q - 5}{5(n^2-n)}\Big[(\text{Trace }\phi)^2 - \text{Trace }\phi - n\varepsilon\Big]\end{aligned} \tag{69}$$

(II) *For* $\widehat{\delta}_C(n-1)$ *with* $\big(n^2-n\big) < r$*, the following hold:*
(a) $\bar{M}^m$ *is equipped with an α semi-symmetric metric U connection.*

$$\begin{aligned}\rho \leq & \frac{1}{(n^2-n)}\Big[\widehat{\delta}_C(n-1)\Big] \\ & + \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q - 10\alpha^2}{10(n-1)}(n-\varepsilon) \\ & + \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q}{10(n^2-n)}[(2n\varepsilon-2)\text{Trace }\phi] \\ & + \frac{c_p + c_q}{5(n^2-n)}\Big[(\text{Trace }\phi)^2 - \text{Trace }\phi - n\varepsilon\Big]\end{aligned} \tag{70}$$

(b) $\bar{M}^m$ *is equipped with a β quarter-symmetric metric U connection.*

$$\begin{aligned}\rho \leq \frac{1}{(n^2-n)}&\Big[\widehat{\delta}_C(n-1)\Big] \\ &+ \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q}{10(n-1)}(n-\varepsilon) \\ &+ \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q}{10(n^2-n)}[(2n\varepsilon-2)\text{Trace }\phi] \\ &+ \frac{c_p+c_q-5\beta^2}{5(n^2-n)}\Big[(\text{Trace }\phi)^2 - \text{Trace }\phi - n\varepsilon\Big]\end{aligned} \tag{71}$$

(c) $\bar{M}^m$ *is equipped with a semi-symmetric metric U-connection.*

$$\begin{aligned}\rho \leq \frac{1}{(n^2-n)}&\Big[\widehat{\delta}_C(n-1)\Big] \\ &+ \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q - 10}{10(n-1)}(n-\varepsilon) \\ &+ \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q}{10(n^2-n)}[(2n\varepsilon-2)\text{Trace }\phi] \\ &+ \frac{c_p+c_q}{5(n^2-n)}\Big[(\text{Trace }\phi)^2 - \text{Trace }\phi - n\varepsilon\Big]\end{aligned} \tag{72}$$

(d) $\bar{M}^m$ *is equipped with a quarter-symmetric metric U connection.*

$$\begin{aligned}\rho \leq \frac{1}{(n^2-n)}&\Big[\widehat{\delta}_C(n-1)\Big] \\ &+ \frac{(\mp\sqrt{5}+3)c_p + (\pm\sqrt{5}+3)c_q}{10(n-1)}(n-\varepsilon) \\ &+ \frac{(\pm\sqrt{5}-1)c_p + (\mp\sqrt{5}-1)c_q}{10(n^2-n)}[(2n\varepsilon-2)\text{Trace }\phi] \\ &+ \frac{c_p+c_q-5}{5(n^2-n)}\Big[(\text{Trace }\phi)^2 - \text{Trace }\phi - n\varepsilon\Big]\end{aligned} \tag{73}$$

Furthermore, the relations in the above results become equalities if, in some orthonormal frame ($\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$), operator A reduces to

$$A_{m+1} = \begin{pmatrix} d & 0 & 0 & \cdots & 0 & 0 \\ 0 & d & 0 & \cdots & 0 & 0 \\ 0 & 0 & d & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & d & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{d(n^2-n)}{r} \end{pmatrix}, \quad A_{n+2} = \cdots = A_m = 0.$$

13. Further Structures on Golden Riemannian Manifolds

13.1. Integrability of Golden Riemannian Structure

The significance of the golden structure on a Riemannian manifold stems from its association with pure Riemannian metrics. Given the connection between Riemannian golden structures and almost product structures, the $\varphi-$ operator technique from the theory of almost product structures applies to golden structures. Consequently, the authors of [6] established a new sufficient condition for the integrability of golden Riemannian structures while also detailing certain characteristics of twin golden Riemannian metrics and the curvature features of locally decomposable golden Riemannian manifolds.

Consider a golden manifold denoted by $\bar{M}$ equipped with a golden structure (ϕ). For ϕ to be integrable, it is both necessary and sufficient to establish a torsion-free affine connection (∇) such that the structure tensor (ϕ) remains covariantly constant under this connection. Furthermore, the integrability of ϕ correlates directly with the absence of the Nijenhuis tensor ($\mathcal{N}_{\prec}$) [4]. Recent studies [6] have investigated an additional potentially sufficient condition for the integrability of golden structures within the framework of Riemannian manifolds.

Theorem 91 ([6])**.** *A golden Riemannian manifold is denoted by* $(\bar{M}, \phi, \bar{g})$*. If* $\varphi_{\phi}\bar{g} = 0.$*, then* ϕ *is integrable.*

Corollary 12 ([6])**.** *Consider a golden Riemannian manifold* $(\bar{M}, \phi, \bar{g})$*.* $\varphi_{\phi}\bar{g} = 0$ *is the same as* $\nabla\phi = 0$*, where* ∇ *is* $\bar{g}$*'s Levi–Civita connection.*

Proposition 20 ([6])**.** *Given a golden Riemannian manifold* $(\bar{M}, \phi, \bar{g})$*, let F be the corresponding almost product structure. If* $\varphi_{F}\bar{g} = 0$*, then the golden structure (*ϕ*) is integrable.*

Proposition 21 ([6])**.** *Let* $\bar{M}, \phi, \bar{g}$ *be a golden Riemannian manifold. If and only if* $\varphi_{F}\bar{g} = 0$*, where F is the corresponding almost product structure, is the manifold (*$\bar{M}$*) a locally decomposable golden Riemannian manifold.*

Twin Golden Riemannian Metrics

A golden Riemannian manifold is denoted by $(\bar{M}, \phi, \bar{g})$. The definition of the twin golden Riemannian metric is

$$G(X, Y) = (\bar{g} \circ \phi)(X, Y) = \bar{g}(\phi X, Y) = \bar{g}(X, \phi Y)$$

for any vector field (X and Y in $\bar{M}$). It is simple to demonstrate that G is a novel pure Riemannian metric as follows:

$$\begin{aligned} G(\phi X, Y) &= (\bar{g} \circ \phi)(\phi X, Y) = \bar{g}(\phi(\phi X), Y) = \bar{g}(\phi^2 X, Y) \\ &= \bar{g}(\phi X, Y) + \bar{g}(X, Y) = \bar{g}(X, \phi Y) + \bar{g}(X, Y) \\ &= \bar{g}(X, (\phi + I)Y) = \bar{g}(X, \phi^2 Y) \\ &= (\bar{g} \circ \phi)(X, \phi Y) = G(X, \phi Y) \end{aligned}$$

which is called the twin metric of $\bar{g}$.

Theorem 92 ([6])**.** *In a golden Riemannian manifold* $(\bar{M}, \phi, \bar{g})$*, the following holds:*

$$\varphi_{\phi} G = (\varphi_{\phi}\bar{g}) \circ \phi + \bar{g} \circ (\mathcal{N}_{\prec}).$$

Corollary 13 ([6])**.** *The criteria listed below are equivalent in a locally golden Riemannian manifold* $(\bar{M}, \phi, \bar{g})$*:*

(i) $\varphi_{\phi}\bar{g} = 0$*;*
(ii) $\varphi_{\phi} G = 0$*.*

Theorem 93 ([6])**.** *Assume that a golden Riemannian manifold* $(\bar{M}, \phi, \bar{g})$ *is locally decomposable. Thus, the Levi–Civita connection of the twin golden Riemannian metric (G) and the Levi–Civita connection of the golden Riemannian metric (*$\bar{g}$*) coincide.*

Theorem 94 ([6])**.** *The Riemannian curvature tensor field in a locally decomposable golden Riemannian manifold is a* φ *tensor field.*

13.2. s-Golden Manifolds

s-golden manifolds represent a fascinating category within almost golden Riemannian manifolds, as explored by Gherici in [9].

Let $(\bar{M}, \phi, \bar{g})$ be an almost golden Riemannian manifold of dimension n. The tangent vector space $(T_p\bar{M})$ splits as follows for each $p \in \bar{M}$: $T_p\bar{M} = (D_{\psi^*})_p \oplus (D_\psi)_p$, where $\psi = \frac{1+\sqrt{5}}{2}$, $\psi^* = \frac{1-\sqrt{5}}{2} = 1 - \psi$ and

$$\left(D_\psi\right)_{\mathrm{p}} = \{\xi' \in T_{\mathrm{p}}\bar{M} : \phi_{\mathrm{p}}\xi' = \psi\xi'\}$$
$$\left(D_{\psi^*}\right)_{\mathrm{p}} = \{\Phi \in T_{\mathrm{p}}\bar{M} : \phi_{\mathrm{p}}\Phi = \psi^*\Phi\}.$$

Definition 39 ([9])**.** *Let $\bar{M}$ be a differentiable manifold of dimension $(n+s)$. An almost s-golden structure on $\bar{M}$ is $\left(\phi, (\xi_\alpha, \rho_\alpha)_{\alpha=1}^{s}, \bar{g}\right)$, where ξ_α represents global vector fields (called golden vector fields); ρ_β is a differential 1-form on $\bar{M}$ such that $\rho_\beta(\xi_\alpha) = \delta_{\alpha\beta}$, where $\alpha, \beta \in \{1, \ldots, s\}$; $\bar{g}$ is a Riemannian metric such that $\bar{g}(X, \xi_\alpha) = \rho_\alpha(X)$; and ϕ is a tensor field of type $(1,1)$ satisfying*

$$\phi = \psi^* I + \sqrt{5}\sum_{\alpha=1}^{s} \rho_\alpha \otimes \xi_\alpha,$$

$\forall$ vector fields X on $\bar{M}$. In addition, if ϕ is integrable, then $\left(\phi, (\xi_\alpha, \rho_\alpha)_{\alpha=1}^{s}, \bar{g}\right)$ is an s-golden structure, and $\left(\bar{M}, \phi, (\xi_\alpha, \rho_\alpha)_{\alpha=1}^{s}, \bar{g}\right)$ is called an s-golden manifold.

Corollary 14 ([9])**.** *Any almost golden Riemannian structure (ϕ) that admits s global unit eigenvectors associated with Φ is an almost s-golden structure.*

Proposition 22 ([9])**.** *Let $\left(\bar{M}, \phi, (\xi_\alpha, \rho_\alpha)_{\alpha=1}^{s}, \bar{g}\right)$ be an s-golden manifold, U be a coordinate neighborhood on $\bar{M}$, and Φ_i be any unit vector field on U such that $\phi\Phi_i = \psi^*\Phi_i$, where $i \in \{1, \ldots, n\}$. Then, we may easily check that the set $\{\xi_\alpha, \Phi_i\}$ is a local orthonormal basis on $\bar{M}$.*

Now, a new sufficient condition of integrability for this class of structures is introduced as follows.

Theorem 95 ([9])**.** *Let $\left(\bar{M}, \Phi, (\xi_\alpha, \rho_\alpha)_{\alpha=1}^{s}, \bar{g}\right)$ be an almost s-golden structure. Then, ϕ is integrable if η_α is closed and $\left[\xi_\alpha, \xi_\beta\right] = 0\ \forall\alpha, \beta \in \{1, \ldots, s\}$.*

The author of [9] also defined two more special types of manifold, namely $\mathcal{C}$-golden manifolds and $\mathcal{G}$-golden manifolds.

Definition 40 ([9])**.** *A $\mathcal{C}$-golden manifold is an almost trans-1-golden manifold $(\bar{M}^{n+1}, \phi, \xi, \rho, \bar{g})$ (or an almost trans-s-golden manifold of type $(1,0)$) that satisfies*

$$\nabla\phi = 0.$$

Definition 41 ([9])**.** *A $\mathcal{G}$-golden manifold is an almost trans-1-golden manifold $(\bar{M}^{n+1}, \phi, \xi, \rho, \bar{g})$ (or an almost trans-s-golden manifold of type $(1,1)$) that satisfies*

$$(\nabla_X\phi)Y = \sqrt{5}(\bar{g}(X,Y)\xi + \rho(Y)X - 2\rho(X)\rho(Y)\xi),$$

$\forall X, Y$ vector fields in $\bar{M}$.

Remark 42. *The geometric properties and examples of $\mathcal{C}$-golden manifolds and $\mathcal{G}$-golden manifolds are also discussed in [9].*

13.3. Golden* Manifold

The concept of a *golden* manifold* was introduced in [9] under the name "$\mathcal{G}$-golden manifold" for a manifold of any dimension. But here, the dimension is odd.

Definition 42 ([8])**.** *A golden* manifold is an almost golden Riemannian contact manifold* $(\bar{M}^{n+1}, \phi, \xi, \rho, \bar{g})$ *that satisfies*

$$(\nabla_X \phi)Y = \sqrt{5}(\bar{g}(X,Y)\xi + \rho(Y)X - 2\rho(X)\rho(Y)\xi), \quad \forall X, Y \in \Gamma(T\bar{M}).$$

Lemma 7 ([8])**.** *Any golden* manifold is a golden Riemannian manifold.*

Proposition 23 ([8])**.** *On golden* manifolds, the sectional curvature of all plane sections containing* ξ *is 1.*

Proposition 24 ([8])**.** *If the sectional curvature of any golden* manifold is a constant (c), then* $c = -1$.

Proposition 25 ([8])**.** *On a golden* manifold, the golden sectional curvature is 2.*

13.4. The (α, p)-Golden Metric Manifolds

Consider an even-dimensional manifold $(\bar{M})$ that has an α structure (F_α). A Riemannian metric $\bar{g}$ is fixed such that

$$\bar{g}(F_\alpha X, Y) = \alpha \bar{g}(X, F_\alpha Y), \tag{74}$$

which is equivalent to

$$\bar{g}(F_\alpha X, F_\alpha Y) = \bar{g}(X, Y) \tag{75}$$

for any vector fields $(X, Y \in \Gamma(T\bar{M})$, where $\Gamma(T\bar{M})$ is the set of smooth sections of $T\bar{M})$ [10].

Definition 43 ([10])**.** *The Riemannian metric($\bar{g}$) defined on an even-dimensional manifold* $(\bar{M})$ *endowed with an* α *structure* (F_α) *that verifies equivalent identities (74) and (75) is called an* (α, F_α)*-compatible metric. Thus, one can conclude that the Riemannian metric* $(\bar{g})$ *verifies the following identity for any* $X, Y \in \Gamma(T\bar{M})$.*:*

$$\bar{g}(\phi_{\alpha,p} X, Y) - \alpha \bar{g}(X, \phi_{\alpha,p} Y) = \frac{p}{2}(1-\alpha)\bar{g}(X,Y), \tag{76}$$

Moreover, $\bar{g}$ *and* $(\phi_{\alpha,p})$ *are related by*

$$\bar{g}(\phi_{\alpha,p} X, \Phi_{\alpha,p} Y) = \frac{p}{2}(\bar{g}(\phi_{\alpha,p} X, Y) + \bar{g}(X, \phi_{\alpha,p} Y)) + p^2 \bar{g}(X,Y) \tag{77}$$

for any $X, Y \in \Gamma(T\bar{M})$.

Definition 44 ([10])**.** *An almost* (α, p)*-golden Riemannian manifold is a triple* $(\bar{M}, \phi_{\alpha,p}, \bar{g})$*, where* $\bar{M}$ *is an even-dimensional manifold,* $\phi_{\alpha,p}$ *is an almost* (α, p)*-golden structure, and* $\bar{g}$ *is a Riemannian metric that verifies identities (76) and (77).*

Proposition 26 ([10])**.** *If* $(\bar{M}, \phi_{\alpha,p}, \bar{g})$ *is an almost* (α, p)*-golden Riemannian manifold of dimension* $2m$*, then the Trace of the* $\phi_{\alpha,p}$ *structure satisfies*

$$\mathrm{Trace}\left(\phi^2_{\alpha,p}\right) = p \cdot \mathrm{Trace}(\phi_{\alpha,p}) + \frac{5\alpha - 1}{2} m p^2.$$

Definition 45 ([10])**.** *If ∇ is the Levi–Civita connection on $(\bar{M}, \bar{g})$, then the covariant derivative (∇F_α) is a tensor field of type $(1,2)$ defined by*

$$(\nabla_X F_\alpha)Y := \nabla_X F_\alpha Y - F_\alpha \nabla_X Y,$$

for any $X, Y \in \Gamma(T\bar{M})$.

Remark 43. *The necessary and sufficient criteria for a submanifold in an almost (α, p)-golden Riemannian manifold to be an invariant submanifold were also determined by Hretcanu and Crasmareanu in [10]. For further results on golden Riemannian manifolds, we refer readers to [95–98].*

Author Contributions: Conceptualization, B.-Y.C., M.A.C. and A.P.; methodology, B.-Y.C., M.A.C. and A.P.; software, M.A.C. and A.P.; validation, B.-Y.C. and M.A.C.; formal analysis, B.-Y.C., M.A.C. and A.P.; investigation, M.A.C. and A.P.; resources, A.P.; data curation, B.-Y.C., M.A.C. and A.P.; methodology, B.-Y.C., M.A.C. and A.P.; software and writing—original draft preparation, B.-Y.C., M.A.C. and A.P.; writing—review and editing, B.-Y.C.; visualization, M.A.C. and A.P.; supervision, B.-Y.C. and M.A.C.; project administration, M.A.C. and A.P.; funding acquisition, B.-Y.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in this study are included in the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Livio, M. *The Golden Ratio: The Story of phi, the World's Most Astonishing Number*; Random House, Inc. Crown: New York, NY, USA, 2008.
2. Goldberg, S.I.; Yano, K. Polynomial structures on manifolds. *Kodai Math. Sem. Rep.* **1970**, *22*, 199–218. [CrossRef]
3. Hretcanu, C.E.; Crasmareanu, M. On some invariant submanifolds in a Riemannian manifold with golden structure. *An. Ştiinţ Univ. Al. I. Cuza Iaşi. Mat. (N.S.)* **2007**, *53*, 199–211.
4. Crasmareanu, M.; Hretcanu, C. Golden differential geometry. *Chaos Solitons Fractals* **2008**, *38*, 1229–1238. [CrossRef]
5. Hretcanu, C.E.; Crasmareanu, M. Applications of the golden ratio on Riemannian manifolds. *Turkish J. Math.* **2009**, *33*, 179–191. [CrossRef]
6. Gezer, A.; Cengiz, N.; Salimov, A. On integrability of golden Riemannian structures. *Turkish J. Math.* **2013**, *37*, 693–703. [CrossRef]
7. Özkan, M. Prolongations of golden structures to tangent bundles. *Differ. Geom. Dyn. Syst.* **2014**, *16*, 227–238.
8. Beldjilali, G. A new class of golden Riemannian manifold. *Int. Electron. J. Geom.* **2020**, *13*, 1–8. [CrossRef]
9. Gherici, B. s-golden manifolds. *Mediter. J. Math.* **2019**, *16*, 56. [CrossRef]
10. Hretcanu, C.E.; Crasmareanu, M. The (α, p)-golden metric manifolds and their submanifolds. *Mathematics* **2023**, *11*, 3046. [CrossRef]
11. Erdoğan, F.E.; Yildirim, C. On a study of the totally umbilical semi-invariant submanifolds of golden Riemannian manifolds. *Politeknik Dergisi* **2018**, *21*, 967–970. [CrossRef]
12. Gök, M.; Keleş, S.; Kılıç, E. Invariant submanifolds in golden Riemannian manifolds. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **2020**, *69*, 1119–1132.
13. Hretcanu, C.E.; Blaga, A.M. Warped product submanifolds in locally golden Riemannian manifolds with a slant factor. *Mathematics* **2021**, *9*, 2125. [CrossRef]
14. Bishop, R.L.; O'Neill, B. Manifolds of negative curvature. *Trans. Am. Math. Soc.* **1969**, *145*, 1–49. [CrossRef]
15. Chen, B.-Y. Geometry of warped product CR-submanifolds in Kaehler manifolds, *Monatsh. Math.* **2001**, *133*, 177–195.
16. Chen, B.-Y. Geometry of warped products as Riemannian submanifolds and related problems. *Soochow J. Math.* **2002**, *28*, 125–156.
17. Chen, B.-Y. Another general inequality for CR-warped products in complex space forms. *Hokkaido Math. J.* **2003**, *32*, 415–444. [CrossRef]
18. Chen, B.-Y. CR-warped products in complex projective spaces with compact holomorphic factor. *Monatsh. Math.* **2004**, *141*, 177–186. [CrossRef]
19. Chen, B.-Y. *Differential Geometry of Warped Product Manifolds and Submanifolds*; World Scientific Publ.: Hackensack, NJ, USA, 2017.
20. Blaga, A.M.; Hretcanu, C.E. Golden warped product Riemannian manifolds. *Lib. Math. (N.S.)* **2018**, *37*, 39–50.
21. Duggal, K.L.; Bejancu, A. *Lightlike Submanifold of Semi-Riemannian Manifolds and Applications*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1996.
22. Poyraz, N.; Yaşar, E. Lightlike hypersurfaces of a golden semi-Riemannian manifold. *Mediterr. J. Math.* **2017**, *14*, 204. [CrossRef]
23. Poyraz, N.; Yaşar, E. Lightlike submanifolds of golden semi-Riemannian manifolds. *J. Geom. Phys.* **2019**, *141*, 92–104. [CrossRef]
24. Ahmad, M.; Qayyoom, M.A. CR-lightlike submanifolds of a golden semi-Riemannian manifold. *arXiv* **2022**, arXiv:2210.10445.
25. Erdoğan, F.E. Screen transversal lightlike submanifolds of golden semi-Riemannian manifolds. *arXiv* **2018**, arXiv:1805.06783.

26. Erdoğan, F.E.; Yildirim, C.; Karatas, E. Transversal lightlike submanifolds of golden semi-Riemannian manifolds. *arXiv* **2018**, arXiv:1804.03600.
27. Kumar, S.; Yadav, A. Screen slant lightlike submanifolds of golden semi-Riemannian manifolds. *Novi Sad J. Math.* **2024**, *54*, 13–27. [CrossRef]
28. Poyraz, N.; Doğan, B. Semi-invariant lightlike submanifolds of golden semi-Riemannian manifolds. *BalıKesir Üniversitesi Fen Bilim. Enstitüsü Dergisi* **2020**, *22*, 106–121. [CrossRef]
29. Shanker, G.; Kaur, R.; Yadav, A. Screen semi-invariant lightlike submanifolds of a golden semi-Riemannian manifold. *Bingöl Üniversitesi Tek. Bilim. Derg.* **2022**, *3*, 1–11.
30. Erdoğan, F.E.; Perktaş, S.Y.; Bozdağ, Ş.N.; Acet, B.E. Lightlike hypersurfaces of meta-golden semi-Riemannian manifolds. *Mathematics* **2023**, *11*, 4798. [CrossRef]
31. Chen, B.-Y. Some pinching and classification theorems for minimal submanifolds. *Arch. Math.* **1993**, *60*, 568–578. [CrossRef]
32. Chen, B.-Y. *Pseudo-Riemannian Geometry, δ-Invariants and Applications*; World Scientific Publ.: Hackensack, NJ, USA, 2011.
33. Aydin, M.E.; Mihai, A.; Mihai, I. Some inequalities on submanifolds in statistical manifolds of constant curvature. *Filomat* **2015**, *29*, 465–477. [CrossRef]
34. Mihai, I.; Mohammed, M. Optimal inequalities for submanifolds in trans-Sasakian manifolds endowed with a semi-symmetric metric connection. *Symmetry* **2023**, *15*, 877. [CrossRef]
35. Choudhary, M.A.; Uddin, S. Some basic inequalities on golden Riemannian product manifolds with constant curvatures. *Filomat* **2023**, *37*, 1155–1166. [CrossRef]
36. Lone, M.S.; Bahadir, O.; Park, C.; Hwang, I. Basic inequalities for statistical submanifolds in golden-like statistical manifolds. *Open Math.* **2022**, *20*, 153–166. [CrossRef]
37. Casorati, F. Mesure de la courbure des surfaces suivant l'idée commune. *Acta Math.* **1890**, *14*, 95–110. [CrossRef]
38. Almekhildy, A.J.; Bahadir, O. Some Inequalities for golden Riemannian space forms. *Palest. J. Math.* **2023**, *12*, 732–744.
39. Chen, B.-Y. Recent developments in δ-Casorati curvature invariants. *Turkish J. Math.* **2021**, *45*, 1–46. [CrossRef]
40. Choudhary, M.A.; Mihai, I. Inequalities for the generalized normalized δ-Casorati curvatures of submanifolds in golden Riemannian manifolds. *Axioms* **2023**, *12*, 952. [CrossRef]
41. Choudhary, M.A.; Park, K.S. Optimization on slant submanifolds of golden Riemannian manifolds using generalized normalized δ-Casorati curvatures. *J. Geometry* **2020**, *111*, 31. [CrossRef]
42. Wintgen, P. Sur l'inégalité de Chen-Willmore. *C. R. Acad. Sci. Paris Sér. A-B* **1979**, *288*, A993–A995.
43. Chen, B.-Y. Recent developments in Wintgen inequality and Wintgen ideal submanifolds. *Int. Electron. J. Geom.* **2021**, *14*, 6–45. [CrossRef]
44. Choudhary, M.A.; Bahadir, O.; Alsulami, H. Generalized Wintgen inequality for some submanifolds in golden Riemannian space forms. *Balkan J. Geom. Appl.* **2020**, *25*, 1–11.
45. Bahadır, O.; Uddin, S. Slant submanifolds of golden Riemannian manifolds. *J. Math. Ext.* **2019**, *13*, 23–39.
46. Hretcanu, C. Submanifolds in Riemannian manifold with golden structure. In Proceedings of the Workshop on Finsler Geometry and Its Applications, Balatonföldvár, Hungary, 28 May–2 June 2007.
47. Opozda, B. A sectional curvature for statistical structures. *Linear Algebra Appl.* **2016**, *497*, 134–161. [CrossRef]
48. Opozda, B. Bochner's technique for statistical structures. *Ann. Glob. Anal. Geom.* **2015**, *48*, 357–395. [CrossRef]
49. Choudhary, M.A.; Blaga, A.M. Generalized Wintgen inequality for slant submanifolds in metallic Riemannian space forms. *J. Geometry* **2021**, *112*, 26. [CrossRef]
50. Choudhary, M.A.; Khedher, K.M.; Bahadır, O.; Siddiqi, M.D. On golden Lorentzian manifolds equipped with generalized symmetric metric connection. *Mathematics* **2021**, *9*, 2430. [CrossRef]
51. Huylebrouck, D. The meta-golden ratio chi. In Proceedings of the Bridges 2014: Mathematics, Music, Art, Architecture, Culture, Seoul, Republic of Korea, 14–19 August 2014; pp. 151–158.
52. Şahin, F.; Şahin, B. Meta-golden Riemannian manifolds. *Math. Methods Appl. Sci.* **2022**, *45*, 10491–10501. [CrossRef]
53. Etayo, F.; deFrancisco, A.; Santamaría, R. Classification of almost Norden golden manifolds. *Bull. Malays. Math. Sci. Soc.* **2020**, *43*, 3941–3961. [CrossRef]
54. Gök, M.; Kılıç, E.; Keles, E. Anti-invariant submanifolds of locally decomposable golden Riemannian manifolds. *Balkan J. Geom. Appl.* **2020**, *25*, 47–60.
55. Gök, M.; Kılıç, E. Non-invariant submanifolds of locally decomposable golden Riemannian manifolds. *Arab. J. Math.* **2021**, *10*, 77–89. [CrossRef]
56. Gök, M.; Kılıç, E. The Stability problem of certain anti-Invariant submanifolds in golden Riemannian manifolds. *Int. Electron. J. Geom.* **2023**, *16*, 343–348. [CrossRef]
57. Ahmad, M.; Qayyoom, M.A.On submanifolds in a Riemannian manifold with golden structure. *Turkish J. Math. Comp. Sci.* **2019**, *11*, 8–23.
58. Erdoğan, F.E.; Yıldırım, C. Semi-invariant submanifolds of golden Riemannian manifolds. *AIP Conf. Proc.* **2017**, *20044*, 1833.
59. Gök, M.; Keleş, S.; Kılıç, E. Some characterizations of semi-invariant submanifolds of golden Riemannian manifolds. *Mathematics* **2019**, *7*, 1209. [CrossRef]
60. Şahin, F.; Şahin, B.; Erdoğan, F.E. Golden Riemannian manifolds having constant sectional curvatures and their submanifolds. *Mediterr. J. Math.* **2022**, *19*, 171. [CrossRef]

61. Ahmad, M.; Qayyoom, M.A. Warped product skew semi-invariant submanifolds of locally golden Riemannian manifolds. *Honam Math. J.* **2022**, *44*, 1–16.
62. Chen, B.-Y. Slant immersions. *Bull. Austral. Math. Soc.* **1990**, *41*, 135–147. [CrossRef]
63. Chen, B.-Y. *Geometry of Slant Submanifolds*; Katholieke Universiteit Leuven: Leuven, Belgium, 1990.
64. Chen, B.-Y.; Tazawa, Y. Slant surfaces with codimension two. *Ann. Fac. Sci. Toulouse Math.* **1990**, *11*, 29–43. [CrossRef]
65. Chen, B.-Y.; Garay, O.J. Pointwise slant submanifolds in almost Hermitian manifolds. *Turkish J. Math.* **2012**, *36*, 630–640. [CrossRef]
66. Ahmad, M.; Qayyoom, M.A. CR-submanifolds of a golden Riemannian manifold. *Palest. J. Math.* **2023**, *12*, 689–696.
67. Hretcanu, C.E.; Blaga, A.M. Warped product submanifolds in metallic Riemannian manifolds. *Tamkang J. Math.* **2020**, *51*, 161–186. [CrossRef]
68. Kupeli, D.N. *Singular Semi-Riemannian Geometry*; Kluwer Academic Publishers: Dordrecht, The Netherland, 1996.
69. Gupta, G.; Kumar, R.; Sachdeva, R. Geometry of lightlike submanifolds of golden semi-Riemannian manifolds. *J. Int. Acad. Phys. Sci.* **2019**, *23*, 211–227.
70. Poyraz, N.Ṡcreen semi-invariant lightlike submanifolds of golden semi-Riemannian manifolds. *Int. Electron. J. Geom.* **2021**, *14*, 207–216. [CrossRef]
71. Erdoğan, F.E. On some types of lightlike submanifolds of golden semi-Riemannian manifolds. *Filomat* **2019**, *33*, 3231–3242. [CrossRef]
72. Qayyoom, M.A.; Bossly, R.; Ahmad, M. On CR-lightlike submanifolds in a golden semi-Riemannian manifold. *AIMS Math.* **2024**, *9*, 13043–13057. [CrossRef]
73. Duggal, K.L.; Jin, D.H. Half lightlike submanifolds of codimension 2. *Math. J. Toyama Univ.* **1999**, *22*, 121–161.
74. Poyraz, N.; Yaşar, E.; Dönmez, D. Half lightlike submanifolds of a golden semi-Riemannian manifold. *Kragujevac J. Math.* **2024**, *48*, 99–122. [CrossRef]
75. Pruthi, M.; Kumar, S. Golden generic lightlike submanifolds of a golden semi-Riemannian manifold. *Balkan J. Geom. Appl.* **2022**, *27*, 118–129.
76. Yadav, A.; Kumar, S. Screen generic lightlike submanifolds of golden semi-Riemannian manifolds. *Mediterr. J. Math.* **2022**, *19*, 248. [CrossRef]
77. Poyraz, N.Ġolden GCR-lightlike submanifolds of golden semi-Riemannian manifolds. *Mediterr. J. Math.* **2020**, *17*, 1–16. [CrossRef]
78. Poyraz, N.Ġeometry of golden GCR-lightlike submanifolds of golden semi-Riemannian manifolds, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)* **2022**, *68*, 107–122. [CrossRef]
79. Acet, B.E. Screen pseudo slant lightlike submanifolds of golden semi-Riemannian manifolds. *Hacet. J. Math. Stat.* **2020**, *49*, 2037–2045. [CrossRef]
80. Yadav, A.; Kumar, S. Screen semi slant lightlike submanifolds of golden semi-Riemannian manifolds. *Jordan J. Math. Stat.* **2022**, *15*, 823–841.
81. Kumar, S.; Yadav, A. Semi-slant lightlike submanifolds of golden semi-Riemannian manifolds. *Kragujevac. J. Math.* **2025**, *49*, 141–158. [CrossRef]
82. Ahmad, M.; Ahmad, M.; Mofarreh, F. Bi-slant lightlike submanifolds of golden semi-Riemannian manifolds. *AIMS Math.* **2023**, *8*, 19526–19545. [CrossRef]
83. Yadav, A.; Kumar, S. Slant lightlike submanifolds of golden semi-Riemannian manifolds. *Mat. Vesnik* **2023**, *75*, 175–188.
84. Erdoğan, F.E.; Perktaş, S.Y.; Acet, B.E.; Bozdağ, Ş.N. Lightlike hypersurfaces of almost Norden golden semi-Riemannian manifolds. *Facta Univ. Ser. Math. Inform.* **2022**, *37*, 813–828.
85. Shanker, G.; Kaur, R.; Yadav, A. Study on warped product of screen real lightlike submanifolds of a golden semi-Riemannian manifold. *Balkan J. Geom. Appl.* **2022**, *27*, 152–161.
86. Chen, B.-Y. Strings of Riemannian invariants, inequalities, ideal immersions and their applications. In *The Third Pacific Rim Geometry Conference (Seoul, 1996)*; International Press: Cambridge, MA, USA, 1998; pp. 7–60.
87. Chen, B.-Y. Some new obstructions to minimal and Lagrangian isometric immersions. *Jpn. J. Math. (N.S.)* **2000**, *26*, 105–127. [CrossRef]
88. Chen, B.-Y. A general optimal inequality for arbitrary Riemannian submanifolds. *J. Inequal. Pure Appl. Math.* **2005**, *6*, 77.
89. Decu, S. Optimal inequalities involving Casorati curvature of slant submanifolds in quaternionic space forms. In Proceedings of the Riemannian Geometry and Applications–Proceedings RIGA, Bucharest, Romania, 19–21 May 2014; pp. 87–96.
90. Decu, S.; Haesen, S.; Verstraelen, L. Optimal inequalities characterizing quasi-umbilical submanifolds. *J. Inequal. Pure Appl. Math.* **2008**, *9*, 79.
91. Lee, J.W.; Choudhary, M.A.; Siddiqui, A.N. Some bounds for Casorati curvatures on Golden Riemannian space forms with SSM connection. *Hacet. J. Math. Stat.* **2024**, *53*, 1024–2038.
92. Guadalupe, I.V.; Rodriguez, L. Normal curvature of surfaces in space forms. *Pacific J. Math.* **1983**, *106*, 95–103. [CrossRef]
93. Smet, P.J.D.; Dillen, F.; Verstraelen, L.; Vrancken, L. A pointwise inequality in submanifold theory. *Arch. Math. (Brno)* **1999**, 35, 115–128.
94. Ge, J.; Tang, Z. A proof of the DDVV conjecture and its equality case. *Pacific J. Math.* **2008**, *237*, 87–95. [CrossRef]
95. Etayo, F.; Santamaría, R.; Upadhyay, A. On the geometry of almost golden Riemannian manifolds. *Mediter. J. Math.* **2017**, *14*, 1–14. [CrossRef]
96. Siddiqi, M.D. Generic submanifolds of golden Riemannian manifolds. *Uzbek Math. J.* **2019**, *4*, 123–131. [CrossRef]

97. Verma, S.; Ahmad, M. CR-lightlike warped product in golden semi-Riemannian manifolds. *South East Asian J. Math. Math. Sci.* **2024**, *20*, 181–196. [CrossRef]
98. Rani, S.; Gupta, R.S. Ricci solitons on golden Riemannian manifolds. *Mediterr. J. Math.* **2023**, *20*, 145. [CrossRef]