axioms

MDPI

Article

Arbitrary Random Variables and Wiman's Inequality

Andriy Kuryliak [1,*,†], Oleh Skaskiv [1,†] and Andriy Bandura [2,†]

[1] Faculty of Mechanics and Mathematics, Ivan Franko National University of Lviv, 79000 Lviv, Ukraine; oleh.skaskiv@lnu.edu.ua
[2] Department of Physics and Mathematics, Ivano-Frankivsk National Technical University of Oil and Gas, 76019 Ivano-Frankivsk, Ukraine; andriy.bandura@nung.edu.ua
* Correspondence: andriykuryliak@gmail.com or andriy.kurylyak@lnu.edu.ua
† These authors contributed equally to this work.

Abstract: We study the class of random entire functions given by power series, in which the coefficients are formed as the product of an arbitrary sequence of complex numbers and two sequences of random variables. One of them is the Rademacher sequence, and the other is an arbitrary complex-valued sequence from the class of sequences of random variables, determined by a certain restriction on the growth of absolute moments of a fixed degree from the maximum of the module of each finite subset of random variables. In the paper we prove sharp Wiman–Valiron's type inequality for such random entire functions, which for given $p \in (0;1)$ holds with a probability p outside some set of finite logarithmic measure. We also considered another class of random entire functions given by power series with coefficients, which, as above, are pairwise products of the elements of an arbitrary sequence of complex numbers and a sequence of complex-valued random variables described above. In this case, similar new statements about not improvable inequalities are also obtained.

Keywords: random entire function; Wiman's inequality; Levy's phenomenon; maximum modulus; maximal term; central index; dependent random variables; sub-Gaussian random variables; subexponential random variables; Pareto distribution; Cauchy distribution; maximum modulus of random variables

MSC: 30B20; 30D20

check for updates

Citation: Kuryliak, A.; Skaskiv, O.; Bandura, A. Arbitrary Random Variables and Wiman's Inequality. *Axioms* **2024**, *13*, 739. https://doi.org/10.3390/axioms13110739

Academic Editor: Hyun Soo Chung

Received: 8 October 2024
Revised: 26 October 2024
Accepted: 26 October 2024
Published: 29 October 2024

1. Introduction

Let us consider an entire functions of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{1}$$

Denote

$$M_f(r) = \max\{|f(z)|\colon |z| = r\}, \ \ \mu_f(r) = \max\{|a_n|r^n\colon n \geq 0\},$$
$$\nu_f(r) = \max\{n\colon |a_n|r^n = \mu_f(r)\}, \ \ r > 0,$$

as the maximum modulus, the maximal term, and central index of series (1), respectively.

The following Wiman–Valiron theorem is well known [1,2].

Theorem 1 ([1,2]). *For every non-constant entire function $f(z)$ of form (1) and any $\varepsilon > 0$ there exists a set $E = E(f) \subset (1, +\infty)$ of finite logarithmic measure, i.e.,*

$$\int_E d(\ln r) < +\infty,$$

such that for all $r \in (1; +\infty) \setminus E_f(\varepsilon)$ *we have*

$$M_f(r) \le \mu_f(r) \ln^{1/2+\varepsilon} \mu_f(r). \tag{2}$$

Note that the constant $1/2$ in the inequality (2) cannot be replaced in general by a smaller number. Indeed, for the entire function $f(z) = e^z$ we have ([3], p. 177)

$$\lim_{r\to+\infty} \frac{M_f(r)}{\mu_f(r)\sqrt{\ln \mu_f(r)}} = \sqrt{2\pi}. \tag{3}$$

Furthermore, from results proved in [4] for entire Dirichlet series it follows that there exists entire function $g(z)$ of the form (1) such that

$$\lim_{r\to+\infty} \frac{M_g(r)}{\mu_g(r)\sqrt{\ln \mu_g(r)}} = +\infty.$$

Therefore, inequality (2) is sharp in the class of non-constant entire functions. However, this inequality can be improved in some subclasses of entire functions, i.e., in the subclasses of:

(1) Entire functions of finite order ([3,5,6]);
(2) Entire function, which can be represented by gap power series ([7,8]);
(3) Random entire functions ([9–13]).

In this paper, we consider only random entire functions.

Let $(\Omega, \mathcal{A}, P)$ be the probability space, which allows the existence of a uniform distribution on it, where $\mathcal{A}$ is the σ-algebra of subsets of Ω, P is the probability measure on $\mathcal{A}$. In the paper, the notion "almost surely" will be used in the sense that the corresponding property holds almost everywhere with respect to the measure P on $\mathcal{A}$. We say that some relation holds almost surely if it holds for each analytic function $f(z,\omega)$ from some class of almost surely in ω.

Let $\{R_n(\omega)\}$ be the Rademacher sequence, which is a sequence of independent random variables defined on the Steinhaus probability space $(\Omega, \mathcal{A}, P)$. For any $n \in \mathbb{Z}_+$, we have

$$\mathbb{P}\{\omega\colon R_n(\omega) = -1\} = \mathbb{P}\{\omega\colon R_n(\omega) = 1\} = \frac{1}{2}.$$

Firstly, we consider random entire function of the form

$$f(z,\omega) = \sum_{n=0}^{\infty} R_n(\omega) a_n z^n. \tag{4}$$

From the results proved in [11], the following theorem can be established.

Theorem 2. *For* $f(z,\omega)$ *of the form (4) and any* $\varepsilon > 0$*, there almost surely exists a set*

$$E := E(\varepsilon, \omega, f) \subset [1; +\infty)$$

of finite logarithmic measure such that for all $r \in [1; +\infty) \setminus E$ *we have*

$$M_f(r,\omega) := \max\{|f(z,\omega)| \colon |z| = r\} \le \mu_f(r) \ln^{1/4} \mu_f(r) (\ln\ln \mu_f(r))^{1+\varepsilon}.$$

From the results proved in ([14], p. 45), the following statement can be derived. For the random entire function

$$f(z,\omega) = \sum_{n=0}^{\infty} \frac{R_n(\omega) z^n}{n!}$$

we have, almost surely,

$$\lim_{r\to+\infty}\frac{M_f(r,\omega)}{\mu_f(r)\sqrt[4]{\ln\mu_f(r)}}\geq\sqrt{\frac{\pi}{8}}. \tag{5}$$

Furthermore, from results proved in [4], it follows that there exists a random entire function $g(z)$ of the form (4) such that

$$\lim_{r\to+\infty}\frac{M_g(r,\omega)}{\mu_g(r)\sqrt[4]{\ln\mu_g(r)}}=+\infty.$$

Wiman's inequality for the most general class of random entire functions was established in [8]. Let $\{Z_n(\omega)\}$ be a sequence of real, independent, centered sub-Gaussian random variables, that is for any $n\in\mathbb{Z}_+$, we have $\mathbf{E}(Z_n)=0$, and there exist a constant $C_1>0$ such that for any $t\in[0;+\infty)$,

$$P\{\omega\colon |Z_n(\omega)|\geq t\}\leq 2\exp\Big(-\frac{t^2}{C_1}\Big).$$

Also, for such random variables (see [15]), there exists $D>0$ such that for any $k\in\mathbb{N}$ and all $\lambda_0\in\mathbb{R}$ we have

$$\mathbf{E}(e^{\lambda_0 Z_k})\leq e^{D\lambda_0^2}.$$

We denote the class of such random variables by Ξ.

For $\{Z_n(\omega)\}\in\Xi$ we have ([15], p. 81 [Exercise 7.8]) for any $k\in\mathbb{N}$: $\mathbf{E}(Z_k)=0$ and

$$\sup_{k\in\mathbb{N}}\mathbf{E}(Z_k^2)=\sup_{k\in\mathbb{N}}\mathbf{D}(Z_k)\leq 2D,$$

where $\mathbf{D}(Z_k):=\mathbf{E}(Z_k^2)-(\mathbf{E}Z_k)^2$ is the variance of random variable Z_k.

From statement established in [8], the following result can be derived (specifically for the case when $\rho=1$).

Theorem 3 ([8]). *Let $Z\in\Xi,\varepsilon>0$ and*

$$f(z,\omega)=\sum_{n=0}^{\infty}Z_n(\omega)a_nz^n. \tag{6}$$

Then there exists a set $E(\varepsilon)$ of finite logarithmic measure, such that for all $r\in(r_0(\omega),+\infty)\setminus E$, almost surely

$$M_f(r,\omega)\leq\mu_f(r)\ln^{1/4}\mu_f(r)(\ln\ln\mu_f(r))^{3/2+\varepsilon}.$$

Also in [8], there was constructed an example of random entire function of the form (6), from which it follows necessity of boundedness of sequence $\{\mathbf{D}(Z_n)\}$.

Theorem 4 ([8]). *For any $\alpha>0$ there exist a sequence of real independent random variables satisfying for all $n\in\mathbb{Z}_+$*

$$\mathbf{E}Z_n=0,\ \sup_n\mathbf{D}Z_n=+\infty,$$

with the entire function $f(z)$ of the form (6) and a constant $C>0$ such that almost surely

$$M_f(r,\omega)\geq C\mu_f(r)\ln^{1/4+\alpha}\mu_f(r),\ r>r_0(\omega).$$

It is worth noting that in the statements about random entire functions mentioned above (such as Theorem 1 from [7] and similar results), the expectation of the random variables is zero. In light of this, Professor M. M. Sheremeta, in 1996 asked whether it is possible to derive a sharper Wiman's inequality for classes of random entire functions of the form

$$f(z,\omega) = \sum_{n=0}^{\infty} Z_n(\omega) a_n z^n,$$

where $\mathbf{E}(Z_n) = \alpha \neq 0$ for $k \geq 0$. One can find a negative answer to this question in [9].

Let Δ be the class of uniformly bounded real sequences $\{X_n(\omega)\}$ such that

$$\mathbf{E}(X_{i_1} X_{i_2} \dots X_{i_k}) = \mathbf{E}(X_{i_1})\mathbf{E}(X_{i_2}) \cdot \ldots \cdot \mathbf{E}(X_{i_k})$$

for any $i_1 < i_2 < \ldots < i_k,\ i \in \mathbb{N}$.

Theorem 5 ([9])**.** *If* $\{\operatorname{Re} Z_n(\omega)\} \in \Delta, \{\operatorname{Im} Z_n(\omega)\} \in \Delta$ *and* $|\mathbf{E}Z_n| \leq O(n^{-\alpha}),\ n \to +\infty,$ $\alpha \in [0; 1/4]$, *then for any* $\varepsilon > 0$ *and* $f(z,\omega)$ *of the form (6) there exists a set* $E(\varepsilon,\omega)$ *of finite logarithmic measure, such that for all* $r \in (1; +\infty)\backslash E$*, almost surely,*

$$M_f(r,\omega) \leq \mu_f(r) \ln^{1/2-\alpha} \mu_f(r) (\ln \ln \mu_f(r))^{1+\varepsilon}. \tag{7}$$

The sharpness of inequality (7) follows from the next statement.

Theorem 6 ([9])**.** *For any sequence* $\{Z_n(\omega)\}$ *such that* $\{\operatorname{Re} Z_n(\omega)\} \in \Delta, \{\operatorname{Im} Z_n(\omega)\} \in \Delta$ *and* $|\mathbf{E}Z_n| \geq Cn^{-\alpha},\ n \to +\infty,\ \alpha \in [0; 1/4)$, *then there exists a function* $f(z,\omega)$ *of the form (6) such that, almost surely,*

$$M_f(r,\omega) \geq \frac{C}{8} \mu_f(r) \ln^{1/2-\alpha} \mu_f(r),\ r > r_0(\omega).$$

Remark that in all statements about random entire functions cited above, the inequalities were proved only with probability equal to 1 (almost surely) and only for sequences of random variables which are independent and sub-Gaussian.

The following **questions** also arise in this regard: *are we able to obtain sharp estimates of maximum modulus of random entire functions:*

(a) *with probability* $p \in (0; 1)$;
(b) *in the cases when the sequence* $\{Z_n(\omega)\}$:
    (1) *is not sub-Gaussian;*
    (2) *may not be independent.*

In this paper, we provide an answer to all these questions.

2. Additional Notations and Definitions

For two positive functions $\varphi(N)$ and $\psi(N)$, the relation $\varphi(N) \asymp \psi(N)$, as $N \to +\infty$, signifies the asymptotic equivalence of the functions up to constant factors. Specifically,

$$\varphi(N) \asymp \psi(N),\ N \to +\infty,$$

which means that there exist positive constants c, C such that the inequality

$$c\varphi(N) \leq \psi(N) \leq C\varphi(N)$$

holds for sufficiently large N.

Let us consider the random entire functions of the form

$$f(z,\omega) = f(z,\omega_1,\omega_2) = \sum_{n=0}^{\infty} R_n(\omega_1)\xi_n(\omega_2) a_n z^n, \tag{8}$$

where

$$a_n \in \mathbb{C} \colon \#\{n \colon a_n \neq 0\} = +\infty,\ \lim_{n\to+\infty} \sqrt[n]{|a_n|} = 0,$$

$\{R_n(\omega)\}$ is the Rademacher sequence, and $\{\xi_n(\omega)\}$ is a sequence of complex-valued random variables (denote by Δ_φ) such that there exist a constant $\beta > 0$ and a function

$$\varphi(N,\beta)\colon \mathbb{N}\times\mathbb{R}_+ \to [1;+\infty)$$

non-decreasing by N and β such that

$$\Big(\mathbf{E}\Big(\max_{0\le n\le N}|\xi_n|^\beta\Big)\Big)^{1/\beta} \asymp \varphi(N,\beta),\ \ N\to+\infty, \tag{9}$$

$$\alpha = \varlimsup_{N\to+\infty}\frac{\ln\varphi(N,\beta)}{\ln N} < +\infty. \tag{10}$$

We denote such a class of random entire functions with $\mathcal{E}(\varphi,\beta)$.

Remark that for any sequence $\{\xi_n(\omega)\}$ function

$$\psi(N,\beta) = \Big(\mathbf{E}\Big(\max_{0\le n\le N}|\xi_n|^\beta\Big)\Big)^{1/\beta}$$

is non-decreasing by N and β because

$$\max_{0\le n\le N}|\xi_n(\omega)|^\beta \le \max_{0\le n\le N+1}|\xi_n(\omega)|^\beta$$

and by Lyapunov's inequality for $0<\beta_1<\beta_2$ we have

$$\Big(\mathbf{E}\Big(\max_{0\le n\le N}|\xi_n|^{\beta_1}\Big)\Big)^{1/\beta_1} \le \Big(\mathbf{E}\Big(\max_{0\le n\le N}|\xi_n|^{\beta_2}\Big)\Big)^{1/\beta_2}.$$

Also, the class of random entire functions of the form

$$f(z,\omega) = \sum_{n=0}^{\infty}\xi_n(\omega)a_n z^n$$

is denoted by $\mathcal{E}_1(\varphi,\beta)$.

In this paper, we will use the following notations.

$$\mathbb{N}=\{1,2,\ldots\},\ \mathbb{Z}_+=\mathbb{N}\cup\{0\},\ \mathbb{R}=(-\infty,+\infty),$$
$$\mathbb{R}_+=(0;+\infty),\ \mathbb{C}=\{z=x+iy\colon x,y\in\mathbb{R}\},$$
$$N(r)=\min\{n_0\ge\ln\mu_f(r)\colon (\forall n\ge n_0)|a_n|r^n<1\},\quad \ln_2 x=\ln\ln x,$$
$$N_\varepsilon(r)=N(re^\varepsilon)=\min\{n_0\ge\ln\mu_f(re^\varepsilon)\colon (\forall n\ge n_0)|a_n|r^ne^{n\varepsilon}<1\}=$$
$$=\min\{n_0\ge\ln\mu_f(re^\varepsilon)\colon (\forall n\ge n_0)|a_n|r^n<e^{-n\varepsilon}\},\ \varepsilon=\frac{1}{N^\gamma(r)},\ \gamma>0,$$
$$W_N(r,\omega)=W_N(r,\omega_1,\omega_2)=\sum_{n=N_\varepsilon(r)}^{\infty}|R_n(\omega_1)||\xi_n(\omega_2)||a_n|r^n.$$

3. Auxiliary Statements

We need the following statement about upper and lower bound of $N(r)$.

Lemma 1 ([8]). *For any $\delta>0$ there exists a set $E(\delta)\subset(1;+\infty)$ of finite logarithmic measure such that for all $r\in(1;+\infty)\backslash E$, we have*

$$\ln\mu_f(r)\le N(r)\le\ln\mu_f(r)(\ln_2\mu_f(r))^{1+\delta},\ \ N(r)\le N_\varepsilon(r)\le(1+\gamma)N(r).$$

In order to obtain estimates, which hold outside some exception set, the next lemma is useful.

Lemma 2 ([9]). *Let $l(r)$ be a continuous increasing to $+\infty$ function on $(1;+\infty)$, $E \subset (1;+\infty)$ be a set such that its complement contains an unbounded open set. Then, there is an infinite sequence $1 < r_1 \le ... \le r_n \to +\infty$ $(n \to +\infty)$ such that*

(1) $(\forall n \in \mathbb{N}):\ r_n \notin E$;
(2) $(\forall n \in \mathbb{N}):\ \ln l(r_n) \ge \frac{n}{2}$;
(3) *if* $(r_n; r_{n+1}) \cap E \ne (r_n, r_{n+1})$, *then* $l(r_{n+1}) \le e l(r_n)$;
(4) *the set of indices, for which (3) holds, is unbounded.*

The following lemma is about the upper bound of $W_N(r,\omega)$ for random entire functions from the class $\mathcal{E}(\varphi,\beta)$.

Lemma 3. *Let $f \in \mathcal{E}(\varphi,\beta)$. For any $\delta > 0$ there exist $r_0(\omega) > 0$, a set $E(\delta) \subset (1;+\infty)$ of finite logarithmic measure such that for all $r \in (r_0(\omega);+\infty)\backslash E$, one has*

$$W_N(r,\omega) < (\ln \mu_f(r))^{\alpha+1+(2+\delta)/\beta}.$$

Proof. For $n \ge N_\varepsilon(r)$ denote $B_n = \{\omega\colon |\xi_n(\omega)|^\beta \ge n^{\alpha\beta+2+\delta_1}\}$, $\delta_1 > 0$. Using Markov's inequality and (9), we can estimate probabilities of these events. So, for some $C_1 > 0$ we have

$$\mathbb{P}(B_n) = \mathbb{P}\{\omega\colon |\xi_n(\omega)|^\beta \ge n^{\alpha\beta+2+\delta_1}\} \le \frac{\mathbf{E}|\xi_n|^\beta}{n^{\alpha\beta+2+\delta_1}} \le$$
$$\le \frac{1}{n^{\alpha\beta+2+\delta_1}}\mathbf{E}\Big(\max_{0\le k\le n}|\xi_k|^\beta\Big) \le C_1\frac{\varphi^\beta(n,\beta)}{n^{\alpha\beta+2+\delta_1}},\ \ r \to +\infty.$$

So,

$$\sum_{n=N_\varepsilon(r)}^{\infty}\mathbb{P}(B_n) \le C_1\sum_{n=N_\varepsilon(r)}^{\infty}\frac{\varphi^\beta(n,\beta)}{n^{\alpha\beta+2+\delta_1}} \le C_1\sum_{n=N_\varepsilon(r)}^{\infty}\frac{1}{n^{2+\delta_1/2}} \le \frac{1}{N_\varepsilon^{1+\delta_1/3}(r)},\quad r \to +\infty.$$

Let

$$B = \bigcup_{n=N_\varepsilon(r)}^{\infty} B_n.$$

Then, $\mathbb{P}(B) \le N_\varepsilon^{-1-\delta_1/3}(r),\ r \to +\infty$. For $\omega \notin B$, we get

$$W_N(r,\omega) = \sum_{n=N_\varepsilon(r)}^{\infty}|R_n(\omega_1)||\xi_n(\omega_2)||a_n|r^n \le \sum_{n=N_\varepsilon(r)}^{\infty} n^{\alpha+(2+\delta_1)/\beta}e^{-n\varepsilon} \le$$
$$\le C_2(N_\varepsilon(r))^{\alpha+1+(2+\delta_1)/\beta} \le (\ln \mu_f(r))^{\alpha+1+(2+2\delta_1)/\beta},\ r \to +\infty,\ (r \notin E).$$

Therefore, for $r \to +\infty$ we obtain

$$\mathbb{P}\Bigg\{\omega\colon \sum_{n=N_\varepsilon(r)}^{\infty}|R_n(\omega_1)||\xi_n(\omega_2)||a_n|r^n \ge (\ln \mu_f(r))^{\alpha+1+(2+2\delta_1)/\beta}\Bigg\} \le \frac{1}{N_\varepsilon^{1+\delta_1/3}(r)}.$$

Let us choose $l(r) = \mu_f(r)$, a set E and a sequence $\{r_k\}$ from Lemma 2. Define

$$F_k = \{\omega\colon W_N(r_k,\omega) \ge (\ln \mu_f(r_k))^{\alpha+1+(2+2\delta_1)/\beta}\}.$$

By the definition of $N_\varepsilon(r)$ we get

$$\sum_{k=1}^{\infty}P(F_k) \le \sum_{k=1}^{\infty}\frac{1}{N_\varepsilon^{1+\delta_1/3}(r_k)} \le \sum_{k=1}^{\infty}\frac{1}{\ln^{1+\delta_1/3}\mu_f(r_k)} \le \sum_{k=1}^{\infty}\frac{1}{k^{1+\delta_1/3}} < +\infty.$$

Then, by the Borel–Cantelli Lemma for almost all $\omega \in [0,1]$ for $k \geq k_0(\omega)$, we obtain

$$W_N(r_k,\omega) < (\ln \mu_f(r_k))^{\alpha+1+(2+2\delta_1)/\beta}.$$

Let $r \geq r_{k_0(\omega)}$ be an arbitrary number outside the set E, $r \in (r_p, r_{p+1})$. By Lemma 2

$$\mu_f(r_{p+1}) \leq e\mu_f(r_p) \leq e\mu_f(r).$$

Therefore, for almost all $\omega \in [0;1]$ and $r \geq r_0(\omega)$ outside of a set of finite logarithmic measure E we have

$$W_N(r,\omega) < W_N(r_{p+1},\omega) < (\ln \mu_f(r_{p+1}))^{\alpha+1+(2+2\delta_1)/\beta} \leq$$
$$\leq (\ln(e\mu_f(r)))^{\alpha+1+(2+2\delta_1)/\beta} \leq (\ln \mu_f(r))^{\alpha+1+(2+3\delta_1)/\beta}.$$

It remains to choose $\delta = 3\delta_1$. □

4. Main Results

We derive sharp asymptotic estimates for the maximum modulus of functions $f \in \mathcal{E}(\varphi,\beta)$. In this case, the elements of a sequence $\{\xi_n(\omega)\}$ may not be sub-Gaussian and could be dependent. The main result of this paper is stated in the following theorem.

Theorem 7. *Let $\delta > 0$. For $f \in \mathcal{E}(\varphi,\beta)$, there exist $r_0(\omega) > 0$ and a set $E(\delta) \subset (1;+\infty)$ of finite logarithmic measure, such that for all $r \in (r_0(\omega);+\infty)\setminus E$, we have, with probability $p \in (0;1)$,*

$$M_f(r,\omega) \leq$$
$$\leq \frac{\mu_f(r)}{(1-p)^{1/\beta}} \cdot \varphi(\ln \mu_f(r) \ln_2^{1+\delta} \mu_f(r),\beta) \cdot \ln^{1/4+\delta}\left\{\frac{\mu_f(r)}{1-p} \cdot \varphi(\ln \mu_f(r) \ln_2^{1+\delta} \mu_f(r),\beta)\right\}.$$

Remark that the exponent $1/4$ and the degree 1 of function φ cannot be replaced simultaneously by smaller numbers. This follows from the next theorem.

Theorem 8. *For any non-decreasing function $\varphi(N,\beta)$ in N and β that satisfies condition (10), there exist a sequence of random variables $\{\xi_n(\omega)\} \in \Delta_\varphi$, an entire function $f \in \mathcal{E}(\varphi,\beta)$, and a constant $c \in (0;1)$ such that, almost surely,*

$$M_f(r,\omega) \geq \mu_f(r)\varphi\left(c\frac{\ln \mu_f(r)}{\ln_2 \mu_f(r)},\beta\right) \cdot \ln^{1/4}\left\{\mu_f(r)\varphi\left(c\frac{\ln \mu_f(r)}{\ln_2 \mu_f(r)},\beta\right)\right\}, \quad r \to +\infty.$$

Also, we derive sharp asymptotic estimates for the maximum modulus of functions $f \in \mathcal{E}_1(\varphi,\beta)$. In this case, the elements of a sequence $\{\xi_n(\omega)\}$ may be dependent or not centered.

Theorem 9. *Let $\delta > 0$. For $f \in \mathcal{E}_1(\varphi,\beta)$, there exist $r_0(\omega) > 0$ and a set $E(\delta) \subset (1;+\infty)$ of finite logarithmic measure, such that for all $r \in (r_0(\omega);+\infty)\setminus E$, we have, with probability $p \in (0;1)$,*

$$M_f(r,\omega) \leq$$
$$\leq \frac{\mu_f(r)}{(1-p)^{1/\beta}} \cdot \varphi(\ln \mu_f(r) \ln_2^{1+\delta} \mu_f(r),\beta) \cdot \ln^{1/2+\delta}\left\{\frac{\mu_f(r)}{1-p} \cdot \varphi(\ln \mu_f(r) \ln_2^{1+\delta} \mu_f(r),\beta)\right\}.$$

Remark that exponent $1/2$ and the degree 1 of function φ cannot be simultaneously replaced by smaller numbers.

Theorem 10. *For any non-decreasing function $\varphi(N,\beta)$ in N and β that satisfies condition (10), there exist a sequence of random variables $\{\xi_n(\omega)\}\in\Delta_\varphi$, an entire function $f\in\mathcal{E}_1(\varphi,\beta)$ and a constant $c\in(0;1)$ such that for all $\omega\in[0;1]$*

$$M_f(r,\omega)\geq \mu_f(r)\varphi\left(c\frac{\ln\mu_f(r)}{\ln_2\mu_f(r)},\beta\right)\cdot\ln^{1/2}\left\{\mu_f(r)\varphi\left(c\frac{\ln\mu_f(r)}{\ln_2\mu_f(r)},\beta\right)\right\},\ \ r\to+\infty.$$

Proof of Theorem 7. By Theorem 2, ω-almost surely there exists a set $E:=E(\varepsilon,\omega,f)\subset[1;+\infty)$ of finite logarithmic measure such that for all $r\in[1;+\infty)\setminus E$ we have

$$M_f(r,\omega)=M_f(r,\omega_1,\omega_2)\leq\mu_f(r,\omega_2)\ln^{1/4+\delta}\mu_f(r,\omega_2).$$

Then by Lemma 3 we get

$$\begin{aligned}\mu_f(r,\omega_2)&\leq\max\left\{\max_{0\leq n\leq N_\varepsilon(r)}|\xi_n(\omega_2)||a_n|r^n;\max_{N_\varepsilon(r)<n<+\infty}|\xi_n(\omega_2)||a_n|r^n\right\}\leq\\&\leq\max\left\{\max_{0\leq n\leq N_\varepsilon(r)}|\xi_n(\omega_2)|\cdot\mu_f(r);(\ln\mu_f(r))^{\alpha+1+(2+\delta)/\beta}\right\}=\\&=\max\left\{\eta(\omega_2)\mu_f(r);(\ln\mu_f(r))^{\alpha+1+(2+\delta)/\beta}\right\},\ \ r\to+\infty,\ (r\notin E),\end{aligned}$$

where

$$\eta(\omega_2)=\max_{0\leq n\leq N_\varepsilon(r)}|\xi_n(\omega_2)|$$

is the non-negative random variable. Then, by Markov's inequality, we obtain

$$P\left\{\omega\colon\eta^\beta(\omega)<\frac{\mathbf{E}\eta^\beta}{1-p}\right\}\geq p,\ \ P\left\{\omega\colon\eta(\omega)<\left(\frac{\mathbf{E}\eta^\beta}{1-p}\right)^{1/\beta}\right\}\geq p.$$

Remark that there exist $\delta>0, C>0$, a set $E\subset[1;+\infty)$ of finite logarithmic measure such that for all $r\in[1;+\infty)\setminus E$ with probability $p\in(0;1)$, we have

$$\begin{aligned}(\mathbf{E}\eta^\beta)^{1/\beta}&\leq C\varphi(N_\varepsilon,\beta)<C\varphi(\ln\mu_f(r)\ln_2^{1+\delta}\mu_f(r),\beta),\\ \mu_f(r,\omega_2)&\leq\max\left\{C\left(\frac{\mathbf{E}\eta^\beta}{1-p}\right)^{1/\beta}\mu_f(r);(\ln\mu_f(r))^{\alpha+1+(2+\delta)/\beta}\right\}\leq\\&\leq\max\left\{\frac{C\mu_f(r)}{(1-p)^{1/\beta}}\varphi(\ln\mu_f(r)\ln_2^{1+\delta}\mu_f(r),\beta);(\ln\mu_f(r))^{\alpha+1+(2+\delta)/\beta}\right\}=\\&=\frac{C\mu_f(r)}{(1-p)^{1/\beta}}\varphi(\ln\mu_f(r)\ln_2^{1+\delta}\mu_f(r),\beta).\end{aligned}\tag{11}$$

Finally, for $r\notin E$ with probability p we get

$$\begin{aligned}&M_f(r,\omega)\leq\mu_f(r,\omega_2)\ln^{1/4+\delta}\mu_f(r,\omega_2)\leq\\&\leq\frac{\mu_f(r)}{(1-p)^{1/\beta}}\cdot\varphi(\ln\mu_f(r)\ln_2^{1+\delta}\mu_f(r),\beta)\cdot\ln^{1/4+2\delta}\left\{\frac{\mu_f(r)}{(1-p)^{1/\beta}}\cdot\varphi(\ln\mu_f(r)\ln_2^{1+\delta}\mu_f(r),\beta)\right\},\end{aligned}$$

or more precisely

$$\begin{aligned}&M_f(r,\omega)\leq\mu_f(r,\omega_2)\ln^{1/4}\mu_f(r,\omega_2)\ln_2^{1+\delta}\mu_f(r,\omega_2)\leq\\&\leq\frac{\mu_f(r)}{(1-p)^{1/\beta}}\cdot\varphi(\ln\mu_f(r)\ln_2^{1+\delta}\mu_f(r),\beta)\cdot\ln^{1/4}\left\{\frac{\mu_f(r)}{(1-p)^{1/\beta}}\cdot\varphi(\ln\mu_f(r)\ln_2^{1+\delta}\mu_f(r),\beta)\right\}\times\end{aligned}$$

$$\times \ln_2^{1+2\delta}\left\{\frac{\mu_f(r)}{(1-p)^{1/\beta}}\cdot\varphi(\ln\mu_f(r)\ln_2^{1+\delta}\mu_f(r),\beta)\right\}. \tag{12}$$

□

Proof of Theorem 8. Let $\varphi(n,\beta)$ be a non-decreasing function by n and β, for which (10) holds. Suppose that

$$\xi_n(\omega_2)=\varphi(n,\beta),\;\; f(z)=\sum_{n=0}^{\infty}\frac{z^n}{\varphi(n,\beta)\cdot n!},\;\; g(z)=e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!},\;\; h(z)=e^{\varepsilon z},\;\; \varepsilon>0.$$

Remark that

$$\Big(\mathbf{E}\Big(\max_{0\le n\le N}|\xi_n|^{\beta}\Big)\Big)^{1/\beta}=\Big(\mathbf{E}\Big(\max_{0\le n\le N}(\varphi(n,\beta))^{\beta}\Big)\Big)^{1/\beta}=\Big((\varphi(N,\beta))^{\beta}\Big)^{1/\beta}=\varphi(N,\beta).$$

Then

$$f(z,\omega)=\sum_{n=0}^{\infty}R_n(\omega_1)\xi_n(\omega_2)\frac{z^n}{\varphi(n,\beta)\cdot n!}=\sum_{n=0}^{\infty}R_n(\omega_1)\frac{z^n}{n!}=g(z,\omega_1),$$
$$M_f(r,\omega)=M_g(r,\omega_1),$$
$$\mu_g(r)=\max_{n\in\mathbb{Z}_+}\frac{r^n}{n!}=\max_{n\in\mathbb{Z}_+}\Big\{\varphi(n,\beta)\frac{r^n}{\varphi(n,\beta)\cdot n!}\Big\}\ge\varphi(\nu_f(r),\beta)\mu_f(r).$$

It follows from (10) that there exists $\varepsilon\in(0;1)$ such that we have

$$\varphi(n,\beta)\le\Big(\frac{1}{\varepsilon}\Big)^n,\;\; \max_{n\in\mathbb{Z}_+}\frac{r^n}{\varphi(n,\beta)\cdot n!}\ge\max_{n\in\mathbb{Z}_+}\frac{(\varepsilon r)^n}{n!},$$
$$\ln\mu_f(r)\ge\ln\mu_h(r)=\ln\mu_g(\varepsilon r)\ge\varepsilon r-\ln(\varepsilon r)\ge\frac{\varepsilon}{2}r,$$
$$\ln\mu_f(r)-\ln\mu_f(1)=\int_1^r\frac{\nu_f(t)}{t}dt\le\nu_f(r)\int_1^r\frac{dt}{t}=\nu_f(r)\ln r,$$
$$\ln\mu_f(r)-\ln\mu_f(1)\le\nu_f(r)\ln r,$$
$$\nu_f(r)\ge\frac{\ln\mu_f(r)-\ln\mu_f(1)}{\ln r}\ge\frac{1}{2}\cdot\frac{\ln\mu_f(r)}{\ln r}\ge\frac{\varepsilon}{4}\cdot\frac{r}{\ln r},$$
$$\frac{\ln\mu_f(r)}{\ln_2\mu_f(r)}\le\frac{\ln\mu_g(r)}{\ln_2\mu_g(r)}\le\frac{\ln M_g(r)}{\ln_2 M_g(r)}=\frac{r}{\ln r}\le\frac{4\nu_f(r)}{\varepsilon},$$
$$\nu_f(r)\ge\frac{\varepsilon}{4}\cdot\frac{\ln\mu_f(r)}{\ln_2\mu_f(r)},\;\; r\to+\infty.$$

Therefore, by inequality (5) we get ω-almost surely

$$M_f(r,\omega)=M_g(r,\omega_1)\ge\frac{\sqrt{\pi}}{3}\mu_g(r)\ln^{1/4}\mu_g(r)\ge$$
$$\ge\frac{\sqrt{\pi}}{3}\varphi(\nu_f(r),\beta)\mu_f(r)\ln^{1/4}(\varphi(\nu_f(r),\beta)\mu_f(r))\ge$$
$$\ge\frac{\sqrt{\pi}}{3}\varphi\Big(\frac{\varepsilon}{4}\cdot\frac{\ln\mu_f(r)}{\ln_2\mu_f(r)},\beta\Big)\mu_f(r)\ln^{1/4}\Big(\varphi\Big(\frac{\varepsilon}{4}\cdot\frac{\ln\mu_f(r)}{\ln_2\mu_f(r)},\beta\Big)\mu_f(r)\Big),\;\; r\to+\infty.$$

□

Proof of Theorem 9. By Theorem 1, there exists a set $E:=E(\varepsilon,f)\subset[1;+\infty)$ of finite logarithmic measure such that for all $r\in[1;+\infty)\setminus E$ and for almost all $\omega\in[0;1]$, we have

$$M_f(r,\omega)\le\mu_f(r,\omega)\ln^{1/2+\delta}\mu_f(r,\omega).$$

Finally, using (11) for $r \notin E$ with probability p we obtain

$$M_f(r,\omega) \le \mu_f(r,\omega_2) \ln^{1/2+\delta} \mu_f(r,\omega_2) \le$$
$$\le \frac{\mu_f(r)}{(1-p)^{1/\beta}} \cdot \varphi(\ln \mu_f(r) \ln_2^{1+\delta} \mu_f(r), \beta) \cdot \ln^{1/2+\delta} \left\{ \frac{\mu_f(r)}{(1-p)^{1/\beta}} \cdot \varphi(\ln \mu_f(r) \ln_2^{1+\delta} \mu_f(r), \beta) \right\}.$$

□

Proof of Theorem 10. Let $\varphi(n,\beta)$ be a function which satisfies (10) and does not decrease by n and β.

Suppose that

$$\xi_n(\omega) = \varphi(n,\beta), \;\; f(z) = \sum_{n=0}^{\infty} \frac{z^n}{\varphi(n,\beta) \cdot n!}, \;\; g(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Then

$$f(z,\omega) = \sum_{n=0}^{\infty} \xi_n(\omega) \frac{z^n}{\varphi(n,\beta) \cdot n!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = g(z), \;\; M_f(r,\omega) = M_g(r).$$

As in proof of Theorem 8, for some $\varepsilon > 0$, we get

$$\mu_g(r) \ge \varphi(\nu_f(r),\beta)\mu_f(r), \;\; \nu_f(r) \ge \frac{\varepsilon}{4} \cdot \frac{\ln \mu_f(r)}{\ln_2 \mu_f(r)}, \;\; r \to +\infty.$$

Therefore, by inequality (3) for $r \to +\infty$ we have for all $\omega \in [0;1]$

$$M_f(r,\omega) = M_g(r) \ge \sqrt{\pi}\mu_g(r) \ln^{1/2} \mu_g(r) \ge$$
$$\ge \sqrt{\pi}\varphi(\nu_f(r),\beta)\mu_f(r) \ln^{1/2}(\varphi(\nu_f(r),\beta)\mu_f(r)) \ge$$
$$\ge \sqrt{\pi}\varphi\left(\frac{\varepsilon}{4} \cdot \frac{\ln \mu_f(r)}{\ln_2 \mu_f(r)}, \beta\right)\mu_f(r) \ln^{1/2}\left(\varphi\left(\frac{\varepsilon}{4} \cdot \frac{\ln \mu_f(r)}{\ln_2 \mu_f(r)}, \beta\right)\mu_f(r)\right).$$

□

5. Some Corollaries

First, we consider the case of sequence $\{\xi_n(\omega)\}$ is an almost surely bounded, i.e., for almost all $\omega \in [0;1]$

$$\exists C_0 > 0: \; \max_{n \in \mathbb{N}} |\xi_n(\omega)| \le C_0.$$

Then, we can choose $\beta = 1$ and

$$\varphi(N,1) = \mathbf{E}\left(\max_{0 \le n \le N} |\xi_n|\right) \le C_0.$$

Corollary 1. *Let $\delta > 0$ and a sequence $\{\xi_n(\omega)\}$ be almost surely bounded. Then, for each function $f \in \mathcal{E}(\varphi,\beta)$, there exist $r_0(\omega) > 0$ and a set $E(\delta) \subset (1;+\infty)$ of finite logarithmic measure, such that for all $r \in (r_0(\omega);+\infty)\backslash E$, we have, with probability $p \in (0;1)$,*

$$M_f(r,\omega) \le \frac{\mu_f(r)}{1-p} \cdot \ln^{1/4} \frac{\mu_f(r)}{1-p} \cdot \ln_2^{1+\delta} \frac{\mu_f(r)}{1-p}.$$

Let Ξ_ρ be the class of random variables $\{\xi_n(\omega)\}$ such that there exist a constant $C_1 > 0$ such that for every $n \in \mathbb{Z}_+$ and any $t \in [0;+\infty)$, we have

$$P\{\omega\colon |\xi_n(\omega)| \geq t\} \leq 2\exp\Big(-\frac{t^\rho}{C_1}\Big),\ C_1 > 0,\ \rho > 0. \tag{13}$$

Remark, that if $\rho = 2$ then Ξ_ρ is the class of sub-Gaussian random variables and if $\rho = 1$ then Ξ_ρ is the class of subexponential random variables.

We prove that for any $N \in \mathbb{N}$

$$\varphi(N,1) = \mathbf{E}\Big(\max_{1\leq n\leq N}|\xi_n|\Big) \leq C_2(\rho, C_1)\cdot(1+\ln N)^{1/\rho}. \tag{14}$$

Inequality (14) is sharp in the case of $\rho = 2$. Indeed ([16], p. 28) [Ex.2.5.11], in the case of $\{\xi_n(\omega)\}$ is a sequence of independent real standard Gaussian random variables there exists a constant $c_1 > 0$ such that

$$\mathbf{E}\Big(\max_{1\leq n\leq N}|\xi_n|\Big) \geq c_1\sqrt{\ln N},\ N\to+\infty.$$

We will prove that the degree $1/\rho$ in inequality (14) is sharp for the class of random variables Ξ_ρ for any $\rho > 0$. This follows from such a statement.

Lemma 4. *There exists a sequence $\{\xi_n(\omega)\} \in \Xi_\rho$ such that for any $N \in \mathbb{N}$, we have*

$$\mathbf{E}\Big(\max_{1\leq n\leq N}|\xi_n|\Big) \geq \Big(1-\frac{1}{e}\Big)(\ln N)^{1/\rho}.$$

Proof. Let $\{\xi_n(\omega)\}$ be a sequence of independent non-negative random variables such that for any $n \in \mathbb{Z}_+$, we have

$$\mathbb{P}\{\omega\colon \xi_n(\omega) \geq t\} = \exp(-t^\rho),\ t > 0.$$

Then, $\{\xi_n(\omega)\} \in \Xi_\rho$. In this case, we have

$$\begin{aligned}
\mathbf{E}\Big(\max_{1\leq n\leq N}|\xi_n|\Big) &= \int_0^{+\infty}\mathbb{P}\Big\{\omega\colon \max_{1\leq n\leq N}\xi_n(\omega)\geq t\Big\}dt = \\
&= \int_0^{+\infty}\Big(1-\mathbb{P}\Big\{\omega\colon \max_{1\leq n\leq N}\xi_n(\omega) < t\Big\}\Big)dt = \\
&= \int_0^{+\infty}\Big(1-\prod_{n=1}^N\mathbb{P}\Big\{\omega\colon \xi_n(\omega)<t\Big\}\Big)dt = \int_0^{+\infty}(1-(1-\mathbb{P}\{\omega\colon \xi_1(\omega)\geq t\})^N)dt = \\
&= \int_0^{+\infty}(1-(1-\exp(-t^\rho))^N)dt.
\end{aligned}$$

One can make the substitution $t = (\ln N)^{1/\rho}\cdot y$. Then, we obtain

$$\begin{aligned}
\mathbf{E}\Big(\max_{1\leq n\leq N}|\xi_n|\Big) &= \int_0^{+\infty}(1-(1-\exp(-y^\rho\ln N))^N)(\ln N)^{1/\rho}dy \geq \\
&\geq (\ln N)^{1/\rho}\int_0^1(1-(1-N^{-y^\rho})^N)dy \geq (\ln N)^{1/\rho}\int_0^1\Big(1-\Big(1-\frac{1}{N}\Big)^N\Big)dy = \\
&= \Big(1-\Big(1-\frac{1}{N}\Big)^N\Big)(\ln N)^{1/\rho} \geq \Big(1-\frac{1}{e}\Big)(\ln N)^{1/\rho}.
\end{aligned}$$

□

The following statement holds without the condition of independence of sequence $\{\xi_n(\omega)\}$.

Theorem 11. *Let $\delta > 0$ and $\{\xi_n(\omega)\} \in \Xi_\rho$. Then, for $f \in \mathcal{E}(\varphi, \beta)$, there exist $r_0(\omega) > 0$ and a set $E(\delta) \subset (1; +\infty)$ of finite logarithmic measure, such that for all $r \in (r_0(\omega); +\infty)\backslash E$, we have, with probability $p \in (0; 1)$,*

$$M_f(r,\omega) \leq \frac{\mu_f(r)}{1-p} \cdot \ln^{1/4} \frac{\mu_f(r)}{1-p} \cdot \left(\ln_2 \frac{\mu_f(r)}{1-p}\right)^{1+1/\rho+\delta}. \tag{15}$$

Proof. Firstly, we prove (14). Let $b = (2C_1)^{1/\rho} > 0$. Then, using (13) we get

$$\frac{\mathbf{E}\left(\max\limits_{1\leq n\leq N} |\xi_n|\right)}{(1+\ln N)^{1/\rho}} = \mathbf{E}\left(\max_{1\leq n\leq N} \frac{|\xi_n|}{(1+\ln N)^{1/\rho}}\right) \leq \mathbf{E}\left(\max_{1\leq n\leq N} \frac{|\xi_n|}{(1+\ln n)^{1/\rho}}\right) =$$

$$= \int\limits_0^{+\infty} P\left\{\omega\colon \max_{1\leq n\leq N} \frac{|\xi_n(\omega)|}{(1+\ln n)^{1/\rho}} > t\right\} dt =$$

$$= \int\limits_0^{b} P\left\{\omega\colon \max_{1\leq n\leq N} \frac{|\xi_n(\omega)|}{(1+\ln n)^{1/\rho}} > t\right\} dt + \int\limits_b^{+\infty} P\left\{\omega\colon \max_{1\leq n\leq N} \frac{|\xi_n(\omega)|}{(1+\ln n)^{1/\rho}} > t\right\} dt \leq$$

$$\leq b + \int\limits_b^{+\infty} \sum_{n=1}^{N} P\left\{\omega\colon \frac{|\xi_n(\omega)|}{(1+\ln n)^{1/\rho}} > t\right\} dt =$$

$$= b + \int\limits_b^{+\infty} \sum_{n=1}^{N} P\{\omega\colon |\xi_n(\omega)| > t(1+\ln n)^{1/\rho}\} dt \leq$$

$$\leq b + \int\limits_b^{+\infty} \sum_{n=1}^{N} 2\exp\left(-\frac{t^\rho(1+\ln n)}{C_1}\right) dt = b + \int\limits_b^{+\infty} \sum_{n=1}^{N} 2\exp\left(-\frac{t^\rho}{C_1}\right) n^{-t^\rho/C_1} dt \leq$$

$$\leq b + 2\sum_{n=1}^{N} \int\limits_b^{+\infty} \exp\left(-\frac{t^\rho}{C_1}\right) n^{-2} dt < b + 2\sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \int\limits_b^{+\infty} \exp\left(-\frac{t^\rho}{C_1}\right) dt =$$

$$= (2C_1)^{1/\rho} + \frac{\pi}{3} \cdot \int\limits_{(2C_1)^{1/\rho}}^{+\infty} \exp\left(-\frac{t^\rho}{C_1}\right) dt = C_2(\rho) < +\infty.$$

Therefore, (14) holds. It continues to use (12). □

Using Lemma 4, we deduce the following statement.

Theorem 12. *There exist a sequence of random variables $\{\xi_n(\omega)\} \in \Xi_\rho$, an entire function $f \in \mathcal{E}(\varphi, \beta)$ and a constant $c \in (0; 1)$ such that, almost surely,*

$$M_f(r,\omega) \geq c\mu_f(r) \ln^{1/4} \mu_f(r) (\ln_2 \mu_f(r))^{1/\rho}, \ \ r \to +\infty.$$

Proof. By Lemma 4, we can choose $\beta = 1$ and $\varphi(N, 1) = (\ln N)^{1/\rho}$ and by Theorem 8 we get

$$M_f(r,\omega) \geq \mu_f(r) \ln^{1/\rho}\left(c\frac{\ln \mu_f(r)}{\ln_2 \mu_f(r)}\right) \cdot \ln^{1/4}\left\{\mu_f(r) \ln^{1/\rho}\left(c\frac{\ln \mu_f(r)}{\ln_2 \mu_f(r)}\right)\right\} \geq$$

$$\geq c\mu_f(r) \ln^{1/4} \mu_f(r) (\ln_2 \mu_f(r))^{1/\rho}, \ \ r \to +\infty.$$

□

If $\{\xi_n(\omega)\}$ satisfies

$$\exists a > 0 \colon \sup_{n\in\mathbb{N}} \mathbf{E}|\xi_n|^a < +\infty, \tag{16}$$

then we obtain

$$\varphi(N,a) \le C_3(a)N^{1/a}, \ \ C_3(a) > 0.$$

Corollary 2. *Let $\delta > 0$ and $\{\xi_n(\omega)\}$ satisfies condition (16). Then, for a random entire function f of form (8), there exist $r_0(\omega) > 0$ and a set $E(\delta) \subset (1;+\infty)$ of finite logarithmic measure, such that for all $r \in (r_0(\omega);+\infty)\backslash E$, we have, with probability $p \in (0;1)$,*

$$M_f(r,\omega) \le \frac{\mu_f(r)}{(1-p)^{1/a}} \cdot \Big(\ln \frac{\mu_f(r)}{1-p}\Big)^{1/4+1/a+\delta}.$$

Proof. Here, we can choose $\beta = a$. Then

$$\Big(\mathbf{E}\Big(\max_{0\le n\le N} |\xi_n|^a\Big)\Big)^{1/a} \le \left(\mathbf{E}\left(\sum_{n=0}^{N} |\xi_n|^a\right)\right)^{1/a} = \left(\sum_{n=0}^{N} \mathbf{E}|\xi_n|^a\right)^{1/a} \le$$
$$\le (N+1)^{1/a}\Big(\sup_{n\in\mathbb{N}} \mathbf{E}|\xi_n|^a\Big)^{1/a} \le C_3(a)N^{1/a}, \ \ C_3(a) > 0, \ \ N \to +\infty.$$

which continues to use Theorem 7. There exist $r_0(\omega) > 0$ and a set $E(\delta) \subset (1;+\infty)$ of finite logarithmic measure such that for all $r \in (r_0(\omega);+\infty)\backslash E$ we have with probability $p \in (0;1)$

$$M_f(r,\omega) \le$$
$$\le \frac{C(a)\mu_f(r)}{(1-p)^{1/a}} \cdot (\ln \mu_f(r) \ln_2^{1+\delta} \mu_f(r))^{1/a} \cdot \ln^{1/4+\delta}\left\{\frac{\mu_f(r)}{1-p} \cdot (\ln \mu_f(r) \ln_2^{1+\delta} \mu_f(r))^{1/a}\right\}.$$

□

Let $\{\xi_n(\omega)\}$ be a sequence of independent Pareto distributed random variables with parameter $\gamma > 0$, which is the density function of $\xi_n(\omega)$

$$f_{\xi_n}(x) = \begin{cases} \frac{\gamma}{x^{1+\gamma}}, & x \ge 1; \\ 0, & x < 1. \end{cases}$$

Corollary 3. *Let $\delta > 0$ and $\{\xi_n(\omega)\}$ be Pareto distributed random variables with parameter $\gamma > 0$. Then, for a random entire function f of the form (8), there exist $r_0(\omega) > 0$ and a set $E(\delta) \subset (1;+\infty)$ of finite logarithmic measure, such that for all $r \in (r_0(\omega);+\infty)\backslash E$, we have, with probability $p \in (0;1)$,*

$$M_f(r,\omega) \le \frac{\mu_f(r)}{(1-p)^{1/\delta}} \cdot \Big(\ln \frac{\mu_f(r)}{1-p}\Big)^{1/4+1/\gamma+\delta}.$$

Proof. It is enough to remark that $\{\xi_n(\omega)\}$ satisfies Corollary 2 with $a = \gamma - \delta$ for any $\delta \in (0;\gamma)$. □

Remark that exponents $1/4 + 1/a$ and $1/4 + 1/\gamma$ in the inequalities of Corollaries 2 and 3, respectively, cannot be replaced by smaller numbers. This follows from the next statement.

Theorem 13. *Let $\{\xi_n(\omega)\}$ be a sequence of independent Pareto distributed random variables with parameter $\gamma > 0$. For any $\delta > 0$ there exist an entire function $f \in \mathcal{E}(\varphi, \beta)$ and a constant $c \in (0;1)$ such that, almost surely,*

$$M_f(r,\omega) \geq c\mu_f(r)\frac{(\ln \mu_f(r))^{1/4+1/\gamma}}{(\ln_2 \mu_f(r))^{1/\gamma}} \geq \mu_f(r)(\ln \mu_f(r))^{1/4+1/a-\delta}, \ \ r \to +\infty.$$

Proof. Let $\{\xi_n(\omega)\}$ be a sequence of independent random variables having a Pareto distribution with parameter $\gamma > 0$. Then, for any $\beta > 0$ we get

$$\varphi(N,\beta) = N^{1/\gamma}, \ N \to +\infty.$$

Firstly, we calculate expectation

$$\mathbf{E}\Big(\max_{1\leq n\leq N} |\xi_n|^{\beta}\Big) = \int_1^{+\infty} x^{\beta} d\Big(1 - \frac{1}{x^{\gamma}}\Big)^N = \int_1^{+\infty} x^{\beta} N\Big(1 - \frac{1}{x^{\gamma}}\Big)^{N-1} \frac{\gamma}{x^{\gamma+1}}dx =$$
$$= \gamma N \int_1^{+\infty} \Big(1 - \frac{1}{x^{\gamma}}\Big)^{N-1} \frac{1}{x^{\gamma-\beta+1}}dx.$$

One can make the substitution $t = 1 - x^{-\gamma}$. Then

$$\mathbf{E}\Big(\max_{1\leq n\leq N} |\xi_n|^{\beta}\Big) = N\int_0^1 t^{N-1}(1-t)^{-\beta/\gamma}dt = NB\Big(N, 1 - \frac{\beta}{\gamma}\Big) = N\frac{\Gamma(N)\Gamma(1-\frac{\beta}{\gamma})}{\Gamma(N+1-\frac{\beta}{\gamma})} \sim$$
$$\sim N\frac{\Gamma(1-\frac{\beta}{\gamma})}{N^{1-\frac{\beta}{\gamma}}} = \Gamma\Big(1-\frac{\beta}{\gamma}\Big)N^{\frac{\beta}{\gamma}}, \ \ N \to +\infty.$$

Therefore, $\varphi(N,\beta) = 2(\Gamma(1-\frac{\beta}{\gamma}))^{1/\beta}N^{1/\gamma}$. We can choose $\beta = \delta$.
By Theorem 8, there exist $c > 0$ and $\varepsilon > 0$ such that we have almost surely

$$M_f(r,\omega) \geq \mu_f(r)\left(c\frac{\ln \mu_f(r)}{\ln_2 \mu_f(r)}\right)^{1/\gamma} \cdot \ln^{1/4}\left\{\mu_f(r)\left(c\frac{\ln \mu_f(r)}{\ln_2 \mu_f(r)}\right)^{1/\gamma}\right\} >$$
$$> \mu_f(r)(\ln \mu_f(r))^{1/\gamma+1/4}(\ln_2 \mu_f(r))^{-1/\gamma}, \ \ r \to +\infty.$$

Also, for any $\delta_1 \in (0;\gamma)$, we choose $a = \gamma - \delta_1$. Then, almost surely, one has

$$M_f(r,\omega) \geq \mu_f(r)(\ln \mu_f(r))^{1/\gamma+1/4}(\ln_2 \mu_f(r))^{-1/\gamma} \geq$$
$$\geq \mu_f(r)(\ln \mu_f(r))^{1/(a+\delta_1)+1/4}(\ln_2 \mu_f(r))^{-1/(a+\delta_1)} \geq$$
$$\geq \mu_f(r)(\ln \mu_f(r))^{1/a+1/4-\delta}, \ r \to +\infty,$$

where $\delta > \frac{\delta_1}{a(a+\delta_1)}$. □

If $\xi_n(\omega)$ has Cauchy distribution for all $n \geq 0$, i.e., density function of $\xi_n(\omega)$

$$f_{\xi_n}(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, \ x \in \mathbb{R},$$

we obtain such a statement.

Corollary 4. *Let $\delta > 0$ and $\{\xi_n(\omega)\}$ be a sequence of Cauchy distributed random variables. Then, for random entire function f of form (8) there exist $r_0(\omega) > 0$ and a set $E(\delta) \subset (1;+\infty)$ of finite logarithmic measure, such that for all $r \in (r_0(\omega);+\infty)\setminus E$, we obtain, with probability $p \in (0;1)$,*

$$M_f(r,\omega) \le \frac{\mu_f(r)}{(1-p)^{1+\delta}} \cdot \ln^{5/4+\delta} \frac{\mu_f(r)}{1-p}.$$

Proof. Let $\delta_2 \in (0;1/3)$. Remark that

$$\mathbf{E}|\xi_n|^{1-\delta_2} = \frac{2}{\pi} \int\limits_0^{+\infty} \frac{x^{1-\delta_2}}{1+x^2} dx < +\infty.$$

Therefore, we can choose $a = 1-\delta_2$ in Corollary 2. So, by Corollary 2, there exist $r_0(\omega) > 0$ and a set $E(\delta_2) \subset (1;+\infty)$ of finite logarithmic measure such that for all $r \in (r_0(\omega);+\infty)\setminus E$ we have with probability $p \in (0;1)$

$$M_f(r,\omega) \le \frac{\mu_f(r)}{(1-p)^{1/(1-\delta_2)}} \cdot \ln^{1/4+1/(1-\delta_2)+\delta_2} \frac{\mu_f(r)}{1-p} \le \frac{\mu_f(r)}{(1-p)^{1+2\delta_2}} \cdot \ln^{5/4+3\delta_2} \frac{\mu_f(r)}{1-p}.$$

It continues to choose $\delta = 3\delta_2$. □

Remark, that exponent $5/4$ in Corollary 4 cannot be replaced by smaller number. This follows from the next statement.

Theorem 14. *Let $\{\xi_n(\omega)\}$ be a sequence of independent Cauchy distributed random variables. There exist an entire function $f \in \mathcal{E}(\varphi,\beta)$ and a constant $c \in (0;1)$ such that, almost surely,*

$$M_f(r,\omega) \ge c\mu_f(r) \frac{\ln^{5/4} \mu_f(r)}{\ln_2 \mu_f(r)}, \ \ r \to +\infty.$$

Proof. Firstly, we remark that for $\beta < 1$

$$\mathbf{E}\Big(\max_{1\le n\le N} |\xi_n|^\beta\Big) = \int\limits_0^{+\infty} x^\beta d\Big(\frac{2}{\pi}\arctan x\Big)^N =$$

$$= \int\limits_0^{+\infty} x^\beta N\Big(\frac{2}{\pi}\arctan x\Big)^{N-1} \frac{2}{\pi(1+x^2)} dx \le \frac{2N}{\pi} \int\limits_0^{+\infty} \frac{x^\beta}{1+x^2} dx = C_1(\beta)N.$$

On the other hand, one has

$$\mathbf{E}\Big(\max_{1\le n\le N} |\xi_n|^\beta\Big) > \int\limits_1^{+\infty} x^\beta d\Big(\frac{2}{\pi}\arctan x\Big)^N = \int\limits_1^{+\infty} x^\beta N\Big(\frac{2}{\pi}\arctan x\Big)^{N-1} \frac{1}{\pi(1+x^2)} dx \ge$$

$$\ge \frac{1}{2\pi} \int\limits_1^{+\infty} x^\beta N\Big(1-\frac{1}{x}\Big)^{N-1} \frac{1}{x^2} dx = \frac{N}{2\pi} \int\limits_1^{+\infty} \Big(1-\frac{1}{x}\Big)^{N-1} \frac{1}{x^{2-\beta}} dx.$$

Now, one can make the substitution $t = 1 - x^{-1}$. Then

$$\mathbf{E}\Big(\max_{1\le n\le N} |\xi_n|^\beta\Big) \ge \frac{N}{2\pi} \int\limits_0^1 t^{N-1}(1-t)^{-\beta} dt = \frac{N}{2\pi} B\Big(N, 1-\beta\Big) = \frac{N}{2\pi} \cdot \frac{\Gamma(N)\Gamma(1-\beta)}{\Gamma(N+1-\beta)} \sim$$

$$\sim \frac{N}{2\pi} \cdot \frac{\Gamma(1-\beta)}{N^{1-\beta}} = \frac{\Gamma(1-\beta)}{2\pi} N^\beta, \ \ \varphi(N,\beta) \ge C_2(\beta)N, \ \ N \to +\infty.$$

It continues to use Theorem 8. Then, there exist an entire function $f \in \mathcal{E}(\varphi, \beta)$ and constant $c \in (0;1)$ such that we have, almost surely,

$$M_f(r,\omega) \geq \mu_f(r) c \frac{\ln \mu_f(r)}{\ln_2 \mu_f(r)} \cdot \ln^{1/4} \left\{ \mu_f(r) c \frac{\ln \mu_f(r)}{\ln_2 \mu_f(r)} \right\} \geq \frac{c}{2} \mu_f(r) \frac{\ln^{5/4} \mu_f(r)}{\ln_2 \mu_f(r)}, \quad r \to +\infty.$$

□

6. Discussion

Remark 1. *For the random entire function* $f \in \mathcal{E}_1(\varphi, \beta)$ *Theorem 13 and Corollaries 1–3 hold when we replace exponent* $1/4$ *by* $1/2$*, and Corollary 4 and Theorem 14 also hold if we replace the exponent* $5/4$ *by* $3/2$*, respectively.*

It is obvious that the classical Wiman's inequality (2) cannot be improved for $f \in \mathcal{E}_1(\varphi, \beta)$. Finally, we formulate the following open problem

Problem 1. *Is the degree* $1 + 1/\rho$ *of* $\ln_2 \frac{\mu_f(r)}{1-p}$ *sharp in the inequality (15)?*

7. Conclusions

We prove sharp Wiman–Valiron's type inequality for random entire functions, which holds with a probability $p \in (0;1)$ outside the set of finite logarithmic measure. Random variables, which are multipliers of Taylor's coefficients of entire functions, may not be sub-Gaussian and may not be independent.

Author Contributions: Conceptualization, O.S.; investigation, A.K. and O.S.; writing—original draft preparation, A.K. and O.S.; Writing—Review and Editing, A.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Valiron, G. *Fonctions Analytiques*; Presses Universitaires de France: Paris, France, 1954.
2. Wittich, H. *Neuere Untersuchungen über Eindeutige Analytische Funktionen*; Springer: Berlin/Heidelberg, Germany; Göttingen, Germany, 1955.
3. Pólya, G.; Szegő, G. *Aufgaben und Lehrsätze aus der Analysis*, 2nd ed.; Springer: Berlin, Germany, 1925.
4. Skaskiv, O.B. On the classical Wiman inequality for entire Dirichlet series. *Visn. L'viv. Univ. Ser. Mekh.-Mat.* **1999**, *54*, 180–182.
5. Hayman, W.K. The local growth of power series: A survey of the Wiman-Valiron method. *Canad. Math. Bull.* **1974**, *17*, 317–358. [CrossRef]
6. Filevych, P.V. Wiman-Valiron type inequalities for entire and random entire functions of finite logarithmic order. *Siberian Math. J.* **2003**, *42*, 579–586. [CrossRef]
7. Skaskiv, O.B. Random gap power series and Wiman's inequality. *Mat. Stud.* **2008**, *30*, 101–106. Available online: http://matstud.org.ua/texts/2008/30_1/101-106.pdf (accessed on 20 October 2024). (In Ukrainian)
8. Kuryliak, A. Subnormal independent random variables and Levy's phenomenon for entire functions. *Mat. Stud.* **2017**, *47*, 10–19. [CrossRef]
9. Filevych, P.V. Correlation between the maximum modulus and maximal term of random entire functions. *Mat. Stud.* **1997**, *7*, 157–166. Available online: http://matstud.org.ua/texts/1997/7_2/7_2_157-166.pdf (accessed on 20 October 2024).
10. Filevych, P. Some classes of entire functions in which the Wiman-Valiron inequality can be almost certainly improved. *Mat. Stud.* **1996**, *6*, 59–66. Available online: http://matstud.org.ua/texts/1996/6/6_059-066.pdf (accessed on 20 October 2024).
11. Erdős, P.; Rényi, A. On random entire function. *Zastos. Mat.* **1969**, *10*, 47–55. [CrossRef]
12. Lévy, P. Sur la croissance de fonctions entière. *Bull. Soc. Math. Fr.* **1930**, *58*, 127–149. [CrossRef]
13. Steele, J.M. Wiman's inequality for entire functions with rapidly oscilating coefficients. *J. Math. Anal. Appl.* **1987**, *123*, 550–558. [CrossRef]
14. Hayman, W.K. On the characteristic of functions meromorphic in the plane and of their integrals. *Proc. Lond. Math. Soc.* **1965**, *14*, 93–128. [CrossRef]

15. Kahane, J.P. *Some Random Series of Functions*; Cambridge University Press: Cambridge, UK, 1985.
16. Vershynin, R. *High-Dimensional Probability*; V.47; Cambridge Series in Statistical and Probabilistic Mathematics; Cambridge University Press: Cambridge, UK, 2024.