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Exchange Formulae for the Stieltjes–Poisson Transform over Weighted Lebesgue Spaces

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Abstract: This paper aims to develop exchange formulae for the Stieltjes–Poisson transform by using Mellin-type convolutions in the context of weighted Lebesgue spaces. A key result is the introduction of bilinear and continuous Mellin-type convolutions, expanding the scope of the analysis to include the space of weighted L^1 functions and the space of continuous functions vanishing at infinity.

Keywords: Stieltjes–Poisson transform; weighted Lebesgue spaces; Mellin-type convolutions; exchange formulae; continuous functions vanishing at infinity

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1. Introduction and Preliminaries

The Stieltjes transform is represented by the equation

$$f(x) = \int_{\mathbb{R}^+} \frac{d\alpha(t)}{x+t} = \lim_{T\to+\infty} \int_0^T \frac{d\alpha(t)}{x+t}, \tag{1}$$

where $x > 0$ and $\mathbb{R}^+ = (0, \infty)$. We consider that the function $\alpha(t)$ is a bounded variation of $(0, T)$ for any positive value of T and the limit (1) exists. If $\alpha(t)$ is the integral of a function $\phi(t)$, we arrive at the special case

$$f(x) = \int_{\mathbb{R}^+} \frac{\phi(t)}{x+t} dt, \; x > 0. \tag{2}$$

The Stieltjes transform, as discussed in prior works [1–3], results from iterating two Laplace transforms [4]. The characteristics of the integral (2) are intimately connected to those of the Laplace integral, as highlighted in the aforementioned studies. The Stieltjes transform was initially proposed in reference to the semi-infinite interval moment problem by T. S. Stieltjes [5]. Since then, it has been investigated and found to be useful in a number of areas, including operator theory, probability, continuous fractions, engineering, mathematical physics, image processing and signal processing, to name a few. On the other hand, several aspects of the generalized Stieltjes transform were investigated in [6–8] and in other relevant publications on the topic. The Stieltjes transform and its generalization have been applied to transforms of distributions in addition to classical functions [9,10]. The Stieltjes transform analysis for Boehmians over specific generalized function spaces was extensively studied in [11]. Recent research on the generalized Stieltjes transform's properties over weighted Lebesgue spaces and compact support distributions may be found in [12]. A case dealing with exchange formulae for the generalized Stieltjes transform was analyzed by the authors of [13].

Widder [14] offered a comprehensive explanation of the integral transform

$$f(x) = \int_{\mathbb{R}^+} \frac{t\phi(t)}{x^2+t^2} dt, \ x > 0, \tag{3}$$

where, through an exponential change in the variable, the expression transforms into a convolution transform with a kernel belonging to the general class discussed by Hirschman and Widder [15]. Notably, the kernel in the transform (3) is identical to the one found in the Poisson integral, which represents a harmonic function in a half-plane, as noted by Widder. He derived several inversion formulas for (3) based on this connection and subsequently applied his results to the study of harmonic functions.

Definition 1. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq 0$. For the function $f \in L^1(\mathbb{R}^+, t^{\beta-\alpha}dt)$, the Stieltjes–Poisson transform of f is defined as follows:*

$$((SP)_{\alpha,\beta}f)(x) = \int_{\mathbb{R}^+} \frac{t^\beta}{x^\alpha + t^\alpha} f(t) dt, \ x > 0. \tag{4}$$

When $\alpha = 1$ and $\beta = 0$, the result is the Stieltjes transform

$$((SP)_{1,0}f)(x) = \int_{\mathbb{R}^+} \frac{1}{x+t} f(t) dt, \ x > 0,$$

studied by Widder [3] and Goldberg [16] (amongst others).

Additionally, when $\alpha = 2$ and $\beta = 1$, the result is the Poisson transform

$$((SP)_{2,1}f)(x) = \int_{\mathbb{R}^+} \frac{t}{x^2+t^2} f(t) dt, \ x > 0,$$

studied in [14,16–18] (amongst others).

In this paper, we explore the Stieltjes–Poisson transform applied to the space $L^1(\mathbb{R}^+, t^{\beta-\alpha}dt)$ focusing on the weighted integrable functions defined on $\mathbb{R}^+$.

Standard norms and notations:

- The norm defined as $\|f\|_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}^+} |f(x)|$ characterizes the space $L^\infty(\mathbb{R}^+)$, which consists of measurable functions that are essentially bounded on the positive real numbers $\mathbb{R}^+$.
- The expression $\|f\|_{1,t^\delta dt} = \int_{\mathbb{R}^+} |f(t)| t^\delta dt$ defines a norm on the space $L^1(\mathbb{R}^+, t^\delta dt)$. This space consists of functions that are integrable with respect to the weight t^δ on the positive real numbers $\mathbb{R}^+$, where $\delta \in \mathbb{R}$.
- The notation $C_c(\mathbb{R}^+)$ represents the vector space of all continuous complex-valued functions on $\mathbb{R}^+$ that have compact support.
- The notation $C_0(\mathbb{R}^+)$ represents the vector space of all continuous complex-valued functions on $\mathbb{R}^+$ vanishing at infinity, provided with the norm $\|f\|_\infty = \sup_{x\in\mathbb{R}^+} |f(x)|$.

Definition 2. *For a fixed $\gamma \in \mathbb{R}$, we consider the next products*

$$(f \vee_\gamma \psi)(t) = \int_{\mathbb{R}^+} f\left(\frac{t}{s}\right) \psi(s) s^\gamma ds, \ t > 0,$$

and

$$(f \wedge_\gamma \psi)(t) = \int_{\mathbb{R}^+} f(s)\psi(ts)s^\gamma ds,\ t > 0,$$

provided that these integrals are well defined.

We note that $\vee_{-1} = \vee$ represents the standard Mellin convolution [19], while $\wedge_0 = \wedge$ corresponds to the second kind of Mellin-type convolution [7] [(7.4.4), p. 218].

This paper is organized into three sections. Section 1 introduces the Stieltjes–Poisson transform, starting with several definitions and notations that will be referenced later. Section 2 examines the properties of the product $\vee_\gamma$ over the space of weighted L^1 functions and the space of continuous functions vanishing at infinity and an exchange formula for the Stieltjes–Poisson transform using $\vee_\gamma$. Finally, Section 3 addresses the properties of the product $\wedge_\gamma$ over the space of weighted L^1 functions and the space of continuous functions vanishing at infinity and exchange formulae for the Stieltjes–Poisson transform via $\wedge_\gamma$.

2. Properties of the Mellin-Type Convolution $\vee_\gamma$ and Exchange Formula

This section investigates the properties of the product $\vee_\gamma$ within the space of weighted L^1 functions and continuous functions that vanish at infinity, along with an exchange formula for the Stieltjes–Poisson transform using $\vee_\gamma$.

Lemma 1. *Let $\delta \in \mathbb{R}$. For a function $f \in L^1(\mathbb{R}^+, t^\delta dt)$ and a function $\psi \in C_c(\mathbb{R}^+)$, it follows that $f \vee_\gamma \psi \in L^1(\mathbb{R}^+, t^\delta dt)$ and*

$$\|f \vee_\gamma \psi\|_{1,t^\delta dt} \le \|f\|_{1,t^\delta dt} \int_{\mathbb{R}^+} |\psi(s)| s^{\gamma+\delta+1} ds < \infty.$$

Proof. We initially note that the integral mentioned above is valid for the given functions f and ψ. Additionally, we observe that if $C = \int_{\mathbb{R}^+} |\psi(s)| s^{\gamma+\delta+1} ds$, then $C < \infty$, as $\psi \in C_c(\mathbb{R}^+)$. Therefore,

$$\begin{aligned}
\|f \vee_\gamma \psi\|_{1,t^\delta dt} &= \int_{\mathbb{R}^+} |(f \vee_\gamma \psi)(t)| t^\delta dt \\
&\le \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \left| f\left(\frac{t}{s}\right) \psi(s) \right| s^\gamma ds\ t^\delta dt \\
&\le \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \left| f\left(\frac{t}{s}\right) \right| t^\delta dt\ |\psi(s)| s^\gamma ds \\
&\quad \text{(according to Fubini's theorem)} \\
&= \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} |f(u)| u^\delta du\ |\psi(s)| s^{\gamma+\delta+1} ds \\
&\quad \text{(by making a substitution of variables } u = \tfrac{t}{s}) \\
&= \|f\|_{1,t^\delta dt} \int_{\mathbb{R}^+} |\psi(s)| s^{\gamma+\delta+1} ds \\
&= C\|f\|_{1,t^\delta dt} < \infty.
\end{aligned}$$

This concludes the proof of Lemma 1. □

Remark 1. *Lemma 1 yielding to the Mellin-type convolution $\vee_\gamma$ is a continuous operation in a specific sense. In fact, let $\delta \in \mathbb{R}$. If the sequence (f_n) converges to f in $L^1(\mathbb{R}^+, t^\delta dt)$ as $n \to \infty$ and $\psi \in C_c(\mathbb{R}^+)$, it follows that the sequence $(f_n \vee_\gamma \psi)$ converges to $f \vee_\gamma \psi$ in $L^1(\mathbb{R}^+, t^\delta dt)$ as $n \to \infty$.*

The Mellin-type convolution $\vee_\gamma$ is a bilinear operation in the following sense: if $f, g \in L^1(\mathbb{R}^+, t^\delta dt)$, $\phi, \psi \in C_c(\mathbb{R}^+)$ and $\alpha, \beta \in \mathbb{C}$, then

$$(\alpha f + \beta g) \vee_\gamma \phi = \alpha(f \vee_\gamma \phi) + \beta(g \vee_\gamma \phi)$$

and

$$f \vee_\gamma (\alpha\phi + \beta\psi) = \alpha(f \vee_\gamma \phi) + \beta(f \vee_\gamma \psi).$$

Also, one obtains the following:

Proposition 1. *Let $\delta \in \mathbb{R}$. For a function $f \in L^1(\mathbb{R}^+, t^\delta dt)$, and for functions $\phi, \psi \in C_c(\mathbb{R}^+)$, we have*
(i)

$$(f \vee_\gamma \phi) \vee_\gamma \psi = f \vee_\gamma (\phi \vee \psi), \tag{5}$$

(ii)

$$f \vee_\gamma (\phi \vee_\gamma \psi) = (f \vee_\gamma \phi) \vee_{2\gamma+1} \psi. \tag{6}$$

Proof. (i) Note that the left-hand side of Equation (5) is valid according to Lemma 1. Similarly, the right-hand side of (5) is also valid, as established by Lemma 1, and because $\phi \vee \psi \in C_c(\mathbb{R}^+)$.

Now, for $t > 0$,

$$\begin{aligned}
((f \vee_\gamma \phi) \vee_\gamma \psi)(t) &= \int_{\mathbb{R}^+} (f \vee_\gamma \phi)\left(\frac{t}{s}\right)\psi(s)s^\gamma ds \\
&= \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} f\left(\frac{t}{su}\right)\phi(u)u^\gamma du\, \psi(s)s^\gamma ds \\
&= \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} f\left(\frac{t}{v}\right)\phi\left(\frac{v}{s}\right)\frac{v^\gamma}{s^\gamma}\frac{dv}{s}\, \psi(s)s^\gamma ds \\
&\quad \text{(by making a substitution of variables } v = us) \\
&= \int_{\mathbb{R}^+} f\left(\frac{t}{v}\right)\int_{\mathbb{R}^+} \phi\left(\frac{v}{s}\right)\psi(s)\frac{ds}{s}v^\gamma dv \\
&\quad \text{(according to Fubini's theorem)} \\
&= \int_{\mathbb{R}^+} f\left(\frac{t}{v}\right)(\phi \vee \psi)(v)v^\gamma dv \\
&= (f \vee_\gamma (\phi \vee \psi))(t).
\end{aligned}$$

(ii) Note that, based on Lemma 1, and given that $\phi \vee_\gamma \psi \in C_c(\mathbb{R}^+)$, the left-hand side of Equation (6) is well-defined. Similarly, from Lemma 1, we can confirm that the right-hand side of Equation (6) is also well-defined. Now, for $t > 0$,

$$\begin{aligned}
(f \vee_\gamma (\phi \vee_\gamma \psi))(t) &= \int_{\mathbb{R}^+} f\left(\frac{t}{s}\right)(\phi \vee_\gamma \psi)(s)s^\gamma ds \\
&= \int_{\mathbb{R}^+} f\left(\frac{t}{s}\right)\int_{\mathbb{R}^+} \phi\left(\frac{s}{u}\right)\psi(u)u^\gamma du\, s^\gamma ds \\
&= \int_{\mathbb{R}^+} \psi(u)\int_{\mathbb{R}^+} f\left(\frac{t}{s}\right)\phi\left(\frac{s}{u}\right)s^\gamma ds\, u^\gamma du \\
&\quad \text{(according to Fubini's theorem)} \\
&= \int_{\mathbb{R}^+} \psi(u)\int_{\mathbb{R}^+} f\left(\frac{t}{vu}\right)\phi(v)v^\gamma u^\gamma u dv\, u^\gamma du \\
&\quad \text{(by making a substitution of variables } v = \tfrac{s}{u})
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^+} \psi(u)(f \vee_\gamma \phi)\left(\frac{t}{u}\right) u^{2\gamma+1} du \\
&= ((f \vee_\gamma \phi) \vee_{2\gamma+1} \psi)(t).
\end{aligned}
$$

This concludes the proof of Proposition 1. □

The exchange formula for the Stieltjes–Poisson transform using $\vee_\gamma$ is given as follows

Theorem 1. *(Exchange formula). Let $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$. If $f \in L^1(\mathbb{R}^+, t^{\beta-\alpha}dt)$, $\psi \in C_c(\mathbb{R}^+)$ then*

$$
(SP)_{\alpha,\beta}(f \vee_\gamma \psi) = ((SP)_{\alpha,\beta} f) \vee_{\gamma+\beta-\alpha+1} \psi. \tag{7}
$$

Proof. First, based on Lemma 1, it follows that $f \vee_\gamma \psi \in L^1(\mathbb{R}^+, t^{\beta-\alpha}dt)$. Therefore, the left-hand side of (7) is well defined. Additionally, referencing [20] (Theorem 2.2), we obtain the inequality $\|(SP)_{\alpha,\beta} f\|_\infty \leq \|f\|_{1,t^{\beta-\alpha}dt}$. Furthermore, for every $x > 0$

$$
\begin{aligned}
\left|((SP)_{\alpha,\beta} f) \vee_{\gamma+\beta-\alpha+1} \psi(x)\right| &= \left|\int_{\mathbb{R}^+} ((SP)_{\alpha,\beta} f)\left(\frac{x}{s}\right) \psi(s) s^{\gamma+\beta-\alpha+1} ds\right| \\
&\leq \|(SP)_{\alpha,\beta} f\|_\infty \int_{\mathbb{R}^+} |\psi(s)| s^{\gamma+\beta-\alpha+1} ds \\
&\leq \|f\|_{1,t^{\beta-\alpha}dt} \int_{\mathbb{R}^+} |\psi(s)| s^{\gamma+\beta-\alpha+1} ds < \infty,
\end{aligned}
$$

then the right-hand side of (7) makes sense.

Now, for $x > 0$, we have

$$
\begin{aligned}
\big((SP)_{\alpha,\beta}(f \vee_\gamma \psi)\big)(x) &= \int_{\mathbb{R}^+} \frac{t^\beta}{x^\alpha + t^\alpha} (f \vee_\gamma \psi)(t) dt \\
&= \int_{\mathbb{R}^+} \frac{t^\beta}{x^\alpha + t^\alpha} \int_0^\infty f\left(\frac{t}{s}\right) \psi(s) s^\gamma ds dt \\
&= \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{t^\beta}{x^\alpha + t^\alpha} f\left(\frac{t}{s}\right) dt\ \psi(s) s^\gamma ds \\
&\quad \text{(according to Fubini's theorem)} \\
&= \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{(us)^\beta}{x^\alpha + (us)^\alpha} f(u) s du\ \psi(s) s^\gamma ds \\
&\quad \text{(by making a substitution of variables } u = \tfrac{t}{s}) \\
&= \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{(u)^\beta}{\left(\left(\frac{x}{s}\right)^\alpha + u^\alpha\right)} f(u) du\ \psi(s) s^{\gamma+\beta-\alpha+1} ds \\
&= \int_{\mathbb{R}^+} ((SP)_{\alpha,\beta} f)\left(\frac{x}{s}\right)\ \psi(s) s^{\gamma+\beta-\alpha+1} ds \\
&= \big(((SP)_{\alpha,\beta} f) \vee_{\gamma+\beta-\alpha+1} \psi\big)(x).
\end{aligned}
$$

This concludes the proof of Theorem 1. □

From this Theorem and taking $\beta = \alpha - 1$, $\alpha \neq 0$, one obtains

Corollary 1. *Let $\alpha \neq 0$. If $f \in L^1(\mathbb{R}^+, t^{-1}dt)$ and $\psi \in C_c(\mathbb{R}^+)$, then*

(i)

$$
(SP)_{\alpha,\alpha-1}(f \vee_\gamma \psi) = ((SP)_{\alpha,\alpha-1} f) \vee_\gamma \psi,
$$

(ii)

$$
(SP)_{\alpha,\alpha-1}(f \vee \psi) = ((SP)_{\alpha,\alpha-1} f) \vee \psi.
$$

Remark 2. *Observe that the Stieltjes transform $(SP)_{1,0}$ and the Poisson transform $(SP)_{2,1}$ are particular cases of the transform $(SP)_{\alpha,\alpha-1}$ considered in Corollary 1.*

The next result holds as follows:

Lemma 2. *If $f \in C_0(\mathbb{R}^+)$ and $\psi \in C_c(\mathbb{R}^+)$ then $f \vee_\gamma \psi \in C_0(\mathbb{R}^+)$ and $\|f \vee_\gamma \psi\|_\infty \leq \|f\|_\infty \int_{\mathbb{R}^+} |\psi(s)| s^\gamma ds < \infty$.*

Proof. We find a closed and bounded interval $[a,b] \subseteq \mathbb{R}^+$ such that the support of $\psi \subseteq [a,b]$. Let $t_n \to t$ as $n \to \infty$ in $\mathbb{R}^+$. As f is a continuous function on $\mathbb{R}^+$, we have

$$f(\frac{t_n}{s})\psi(s)s^\gamma \longrightarrow f(\frac{t}{s})\psi(s)s^\gamma \qquad \text{for each } s \in \mathbb{R}^+.$$

Since

$$\left| f(\frac{t_n}{s})\psi(s)s^\gamma \right| \leq \|f\|_\infty |\psi(s)s^\gamma|$$

and $|\psi(s)s^\gamma|$ is an integrable function on $\mathbb{R}^+$, we can apply the dominated convergence theorem, and hence

$$\begin{aligned} \lim_{n\to\infty} (f \vee_\gamma \psi)(t_n) &= \lim_{n\to\infty} \int_{\mathbb{R}^+} f(\frac{t_n}{s})\psi(s)s^\gamma ds \\ &= \int_{\mathbb{R}^+} f(\frac{t}{s})\psi(s)s^\gamma ds \\ &= (f \vee_\gamma \psi)(t). \end{aligned}$$

Thus, $f \vee_\gamma \psi$ is a continuous function on $\mathbb{R}^+$.

Let $\epsilon > 0$ be arbitrary. Since f vanishes at infinity, we can find a closed and bounded interval $[c,d] \subseteq \mathbb{R}^+$ such that

$$|f(u)| \leq \frac{\epsilon}{C+1}, \qquad \text{for all } u \notin [c,d],$$

where $C = \int_{\mathbb{R}^+} |\psi(s)| s^\gamma ds$.

Since $\frac{t}{s} \in [c,d]$, $s \in [a,b]$ then $t \in [ac, bd]$, we find that if $t \notin [ac, bd]$ and $s \in [a,b]$ then $\frac{t}{s} \notin [c,d]$. Then, $|f(\frac{t}{s})| \leq \frac{\epsilon}{C+1}$, for $t \notin [ac, bd]$. Thus, for $t \notin [ac, bd]$, we obtain

$$\begin{aligned} |(f \vee_\gamma \psi)(t)| &\leq \int_{\mathbb{R}^+} \left| f(\frac{t}{s}) \right| |\psi(s)| s^\gamma ds \\ &\leq \frac{\epsilon}{C+1} \cdot C \leq \epsilon. \end{aligned}$$

Thus, $f \vee_\gamma \psi \in C_0(\mathbb{R}^+)$.

By direct computation, we obtain

$$\begin{aligned} \|f \vee_\gamma \psi\|_\infty &\leq \int_{\mathbb{R}^+} \left| f(\frac{t}{s}) \right| |\psi(s)| s^\gamma ds \\ &\leq \|f\|_\infty \int_{\mathbb{R}^+} |\psi(s)| s^\gamma ds < \infty. \end{aligned}$$

This concludes the proof of Lemma 2. □

Remark 3. *From this Lemma, it can be seen that the Mellin-type convolution $\vee_\gamma$ is a continuous operation in other specific senses different to that obtained in Remark 1. In fact, if the sequence*

(f_n) converges to f in $C_0(\mathbb{R}^+)$ as $n \to \infty$ and $\psi \in C_c(\mathbb{R}^+)$, it follows that the sequence $(f_n \vee_\gamma \psi)$ converges to $f \vee_\gamma \psi$ in $C_0(\mathbb{R}^+)$ as $n \to \infty$.

Observe that from Lemma 2, the Mellin-type convolution $\vee_\gamma$ is a bilinear operation in the following sense:

If $f, g \in C_0(\mathbb{R}^+)$, $\phi, \psi \in C_c(\mathbb{R}^+)$ and $\alpha, \beta \in \mathbb{C}$, then

$$(\alpha f + \beta g) \vee_\gamma \phi = \alpha(f \vee_\gamma \phi) + \beta(g \vee_\gamma \phi)$$

and

$$f \vee_\gamma (\alpha\phi + \beta\psi) = \alpha(f \vee_\gamma \phi) + \beta(f \vee_\gamma \psi).$$

Again using Lemma 2, one can obtain a corresponding result to Proposition 1.

Indeed:

Proposition 2. *For a function $f \in C_0(\mathbb{R}^+)$, and for functions $\phi, \psi \in C_c(\mathbb{R}^+)$, we have*

(i)

$$(f \vee_\gamma \phi) \vee_\gamma \psi = f \vee_\gamma (\phi \vee \psi),$$

(ii)

$$f \vee_\gamma (\phi \vee_\gamma \psi) = (f \vee_\gamma \phi) \vee_{2\gamma+1} \psi.$$

3. Properties of the Mellin-Type Convolution $\wedge_\gamma$ and Exchange Formulae

This section explores the properties of the product $\wedge_\gamma$ in the context of weighted L^1 functions and continuous functions that vanish at infinity, along with the exchange formulae for the Stieltjes–Poisson transform using $\wedge_\gamma$.

Lemma 3. *Let $\delta \in \mathbb{R}$. For a function $f \in L^1(\mathbb{R}^+, t^\delta dt)$ and a function $\psi \in C_c(\mathbb{R}^+)$, then*

(i) *$f \wedge_{1+2\delta} \psi \in L^1(\mathbb{R}^+, t^\delta dt)$ and $\|f \wedge_{1+2\delta} \psi\|_{1,t^\delta dt} \le \|f\|_{1,t^\delta dt} \int_{\mathbb{R}^+} |\psi(s)| s^\delta ds < \infty$,*

(ii) *$\psi \wedge_\gamma f \in L^1(\mathbb{R}^+, t^\delta dt)$ and $\|\psi \wedge_\gamma f\|_{1,t^\delta dt} \le \|f\|_{1,t^\delta dt} \int_{\mathbb{R}^+} |\psi(s)| s^{\gamma-\delta-1} ds < \infty$.*

Proof. (i) We initially note that the integral mentioned above is valid for the given functions f and ψ. Also,

$$\begin{aligned}
\|f \wedge_{1+2\delta} \psi\|_{1,t^\delta dt} &= \int_{\mathbb{R}^+} |(f \wedge_{1+2\delta} \psi)(t)| t^\delta dt \\
&\le \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} |f(s)\psi(ts)| s^{1+2\delta} ds \, t^\delta dt \\
&\le \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} |\psi(ts)| t^\delta dt |f(s)| \, s^{1+2\delta} ds \\
&\quad \text{(according to Fubini's theorem)} \\
&= \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} |\psi(u)| \left(\frac{u}{s}\right)^\delta \frac{1}{s} du \, |f(s)| s^{1+2\delta} ds \\
&\quad \text{(by making a substitution of variables } u = ts\text{)} \\
&= \int_{\mathbb{R}^+} |\psi(u)| u^\delta du \int_{\mathbb{R}^+} |f(s)| s^\delta ds \\
&= C\|f\|_{1,t^\delta dt} < \infty, \text{ where } C = \int_{\mathbb{R}^+} |\psi(u)| u^\delta du < \infty.
\end{aligned}$$

(ii) To ensure the integrity of the integral for the functions f and ψ, we start by noting that the integral is defined under these conditions. Additionally, let $C = \int_{\mathbb{R}^+} |\psi(s)|s^{\gamma-\delta-1}ds$. Given that $\psi \in C_c(\mathbb{R}^+)$, we can conclude that $C < \infty$. This leads us to the following observations:

$$\begin{aligned}
\|\psi \wedge_\gamma f\|_{1,t^\delta dt} &= \int_{\mathbb{R}^+} |(\psi \wedge_\gamma f)(t)|t^\delta dt \\
&\leq \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} |\psi(s)f(ts)|s^\gamma ds\, t^\delta dt \\
&\leq \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} |f(ts)|t^\delta dt\, |\psi(s)|s^\gamma ds \\
&\quad \text{(according to Fubini's theorem)} \\
&= \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} |f(u)|\left(\frac{u}{s}\right)^\delta \frac{1}{s} du\, |\psi(s)|s^\gamma ds \\
&\quad \text{(by making a substitution of variables } u = ts\text{)} \\
&= \|f\|_{1,t^\delta dt} \int_{\mathbb{R}^+} |\psi(s)|s^{\gamma-\delta-1}ds \\
&= C\|f\|_{1,t^\delta dt} < \infty.
\end{aligned}$$

This concludes the proof of Lemma 3. □

Remark 4. *Lemma 3 yielding to the Mellin-type convolution $\wedge_\gamma$ is a continuous operation in a specific sense. In fact, let $\delta \in \mathbb{R}$. If a sequence (f_n) converges to f in $L^1(\mathbb{R}^+, t^\delta dt)$ as $n \to \infty$ and $\psi \in C_c(\mathbb{R}^+)$, then:*

(i) *the sequence $(f_n \wedge_{1+2\delta} \psi)$ converges to $f \wedge_{1+2\delta} \psi$ in $L^1(\mathbb{R}^+, t^\delta dt)$ as $n \to \infty$,*

(ii) *the sequence $(\psi \wedge_\gamma f_n)$ converges to $\psi \wedge_\gamma f$ in $L^1(\mathbb{R}^+, t^\delta dt)$ as $n \to \infty$.*

The Mellin-type convolution $\wedge_\gamma$ is a bilinear operation in the following sense:
If $f, g \in L^1(\mathbb{R}^+, t^\delta dt)$, $\phi, \psi \in C_c(\mathbb{R}^+)$ and $\alpha, \beta \in \mathbb{C}$, then

$$(\alpha f + \beta g) \wedge_{1+2\delta} \phi = \alpha(f \wedge_{1+2\delta} \phi) + \beta(g \wedge_{1+2\delta} \phi)$$

and

$$f \wedge_{1+2\delta} (\alpha\phi + \beta\psi) = \alpha(f \wedge_{1+2\delta} \phi) + \beta(f \wedge_{1+2\delta} \psi).$$

Also,

$$(\alpha\phi + \beta\psi) \wedge_\gamma f = \alpha(\phi \wedge_\gamma f) + \beta(\psi \wedge_\gamma f)$$

and

$$\phi \wedge_\gamma (\alpha f + \beta g) = \alpha(\phi \wedge_\gamma f) + \beta(\phi \wedge_\gamma g).$$

Proposition 3. *Let $\delta \in \mathbb{R}$. For a function $f \in L^1(\mathbb{R}^+, t^\delta dt)$, and for functions $\phi, \psi \in C_c(\mathbb{R}^+)$, we have*

(i)

$$(f \wedge_{1+2\delta} \phi) \wedge_{1+2\delta} \psi = f \vee (\phi \wedge_{1+2\delta} \psi), \tag{8}$$

(ii)

$$f \wedge_{1+2\delta} (\phi \wedge_{1+2\delta} \psi) = \phi \wedge_{1+2\delta} (f \wedge_{1+2\delta} \psi), \tag{9}$$

(iii)

$$(\psi \wedge_\gamma \phi) \wedge_\gamma f = \psi \vee (\phi \wedge_\gamma f), \tag{10}$$

(iv)

$$\psi \wedge_\gamma (\phi \wedge_\gamma f) = (\phi \vee \psi) \wedge_\gamma f. \tag{11}$$

Proof. (i) Note that, according to Lemma 3(i), the left side of (8) is well defined. Similarly, the right side of (8) is well defined based on Lemma 1 and the fact that $\phi \wedge_{1+2\delta} \psi \in C_c(\mathbb{R}^+)$. Now, for $t > 0$

$$\begin{aligned}
((f \wedge_{1+2\delta} \phi) \wedge_{1+2\delta} \psi)(t) &= \int_{\mathbb{R}^+} (f \wedge_{1+2\delta} \phi)(s)\psi(ts)s^{1+2\delta}ds \\
&= \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} f(u)\phi(us)u^{1+2\delta}du\, \psi(ts)s^{1+2\delta}ds \\
&= \int_{\mathbb{R}^+} f(u) \int_{\mathbb{R}^+} \phi(us)\psi(ts)s^{1+2\delta}ds u^{1+2\delta}du \\
&\quad \text{(according to Fubini's theorem)} \\
&= \int_{\mathbb{R}^+} f(u) \int_{\mathbb{R}^+} \phi(v)\, \psi(\frac{t}{u}v)\left(\frac{v}{u}\right)^{1+2\delta}\frac{1}{u}dv u^{1+2\delta}du \\
&\quad \text{(by making a substitution of variables } v = us) \\
&= \int_{\mathbb{R}^+} (\phi \wedge_{1+2\delta} \psi)\left(\frac{t}{u}\right)f(u)\frac{du}{u} \\
&= (f \vee (\phi \wedge_{1+2\delta} \psi))(t).
\end{aligned}$$

(ii) Note that, based on Lemma 3(i), and given that $\phi \wedge_{1+2\delta} \psi \in C_c(\mathbb{R}^+)$, the left-hand side of Equation (9) is well defined. Furthermore, referring to Lemma 3(i) and (ii), we see that the right-hand side of (9) is also well defined. Now, for $t > 0$

$$\begin{aligned}
(f \wedge_{1+2\delta} (\phi \wedge_{1+2\delta} \psi))(t) &= \int_{\mathbb{R}^+} f(s)(\phi \wedge_{1+2\delta} \psi)(ts)s^{1+2\delta}ds \\
&= \int_{\mathbb{R}^+} f(s) \int_{\mathbb{R}^+} \phi(u)\psi(tsu)u^{1+2\delta}du s^{1+2\delta}ds \\
&= \int_{\mathbb{R}^+} \phi(u) \int_{\mathbb{R}^+} f(s)\psi(tsu)s^{1+2\delta}ds u^{1+2\delta}du \\
&\quad \text{(according to Fubini's theorem)} \\
&= \int_{\mathbb{R}^+} \phi(u)(f \wedge_{1+2\delta} \psi)(tu)u^{1+2\delta}du \\
&= (\phi \wedge_{1+2\delta} (f \wedge_{1+2\delta} \psi))(t).
\end{aligned}$$

(iii) Note that based on Lemma 3(ii) and the fact that $\psi \wedge_\gamma \phi \in C_c(\mathbb{R}^+)$, the left-hand side of (10) is well defined. Additionally, using Lemma 3(ii) in conjunction with Lemma 1, we can conclude that the right-hand side of (10) is also well defined. Now, consider $t > 0$

$$\begin{aligned}
((\psi \wedge_\gamma \phi) \wedge_\gamma f)(t) &= \int_{\mathbb{R}^+} (\psi \wedge_\gamma \phi)(s)f(st)s^\gamma ds \\
&= \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \psi(u)\phi(us)u^\gamma du f(st)s^\gamma ds \\
&= \int_{\mathbb{R}^+} \psi(u) \int_{\mathbb{R}^+} \phi(us)f(st)s^\gamma ds u^\gamma du \\
&\quad \text{(according to Fubini's theorem)} \\
&= \int_{\mathbb{R}^+} \psi(u) \int_{\mathbb{R}^+} \phi(v)f\left(\frac{v}{u}t\right)\frac{v^\gamma}{u^\gamma}\frac{1}{u}dv u^\gamma du \\
&\quad \text{(by making a substitution of variables } v = us) \\
&= \int_{\mathbb{R}^+} \psi(u)(\phi \wedge_\gamma f)\left(\frac{t}{u}\right)\frac{1}{u}du \\
&= (\psi \vee (\phi \wedge_\gamma f))(t).
\end{aligned}$$

(iv) Note that according to Lemma 3(ii), the left-hand side of Equation (11) is well defined. Similarly, the right-hand side of (11) is also well defined, as indicated by Lemma 3(ii) and the fact that $\phi \vee \psi \in C_c(\mathbb{R}^+)$. Now, let us consider $t > 0$

$$\begin{aligned}(\psi \wedge_\gamma (\phi \wedge_\gamma f))(t) &= \int_{\mathbb{R}^+} \psi(s)(\phi \wedge_\gamma f)(ts)s^\gamma ds \\ &= \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} \phi(u)f(uts)u^\gamma du\, \psi(s)s^\gamma ds \\ &= \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} f(tv)\phi\Big(\frac{v}{s}\Big)\frac{v^\gamma}{s^\gamma}\frac{dv}{s}\, \psi(s)s^\gamma ds \\ &\quad \text{(by making a substitution of variables } v = us) \\ &= \int_{\mathbb{R}^+} f(tv)\int_{\mathbb{R}^+} \phi\Big(\frac{v}{s}\Big)\psi(s)\frac{ds}{s}v^\gamma dv \\ &\quad \text{(according to Fubini's theorem)} \\ &= \int_{\mathbb{R}^+} f(tv)(\phi \vee \psi)(v)v^\gamma dv \\ &= ((\phi \vee \psi) \wedge_\gamma f)(t).\end{aligned}$$

This concludes the proof of Proposition 3. □

The exchange formulae for the Stieltjes–Poisson transform using $\wedge_\gamma$ are given as follows:

Theorem 2. *(Exchange formulae). Let $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$. If $f \in L^1(\mathbb{R}^+, t^{\beta-\alpha}dt)$ and $\psi \in C_c(\mathbb{R}^+)$, then we have the following result:*
(i)

$$(SP)_{\alpha,\beta}(f \wedge_{1+2(\beta-\alpha)} \psi) = f \wedge_{\beta-\alpha} ((SP)_{\alpha,\beta}\psi), \tag{12}$$

(ii)

$$(SP)_{\alpha,\beta}(\psi \wedge_\gamma f) = \psi \wedge_{\gamma+\alpha-\beta-1} ((SP)_{\alpha,\beta} f). \tag{13}$$

Proof. (i) Consider $f \in L^1(\mathbb{R}^+, t^{\beta-\alpha}dt)$ and $\psi \in C_c(\mathbb{R}^+)$. By applying Lemma 3(i), we can deduce that $f \wedge_{1+2(\beta-\alpha)} \psi \in L^1(\mathbb{R}^+, t^{\beta-\alpha}dt)$. This confirms the existence of the left-hand side of Equation (12). Furthermore, the right-hand side of (12) also exists, as it holds true for every $x > 0$

$$\begin{aligned}\big|\big(f \wedge_{\beta-\alpha} ((SP)_{\alpha,\beta}\psi)\big)(x)\big| &= \left|\int_{\mathbb{R}^+} f(s)((SP)_{\alpha,\beta}f)(xs)s^{\beta-\alpha}ds\right| \\ &\leq \|(SP)_{\alpha,\beta}f\|_\infty \int_{\mathbb{R}^+} |f(s)|s^{\beta-\alpha}ds \\ &\leq \|f\|^2_{1,t^{\beta-\alpha}dt},\end{aligned}$$

using $\|(SP)_{\alpha,\beta}f\|_\infty \leq \|f\|_{1,t^{\beta-\alpha}dt}$ [20] (Theorem 2.2).

Now, for $x > 0$, we have

$$\begin{aligned}\Big((SP)_{\alpha,\beta}(f \wedge_{1+2(\beta-\alpha)} \psi)\Big)(x) &= \int_{\mathbb{R}^+} \frac{t^\beta}{x^\alpha + t^\alpha}(f \wedge_{1+2(\beta-\alpha)} \psi)(t)dt \\ &= \int_{\mathbb{R}^+} \frac{t^\beta}{x^\alpha + t^\alpha}\int_{\mathbb{R}^+} f(s)\psi(ts)s^{1+2(\beta-\alpha)}dsdt \\ &= \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} \frac{t^\beta}{x^\alpha + t^\alpha}\psi(ts)dtf(s)s^{1+2(\beta-\alpha)}ds \\ &\quad \text{(according to Fubini's theorem)} \\ &= \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} \frac{\big(\frac{u}{s}\big)^\beta}{x^\alpha + \big(\frac{u}{s}\big)^\alpha}\psi(u)\frac{1}{s}duf(s)s^{1+2(\beta-\alpha)}ds\end{aligned}$$

$$
\begin{aligned}
&\text{(by making a substitution of variables } u = ts\text{)}\\
&= \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} \frac{u^\beta}{(xs)^\alpha + u^\alpha}\psi(u)du f(s)s^{\beta-\alpha}ds\\
&= \int_{\mathbb{R}^+} ((SP)_{\alpha,\beta}\psi)(xs)f(s)s^{\beta-\alpha}ds\\
&= \big(f \wedge_{\beta-\alpha} ((SP)_{\alpha,\beta}\psi)\big)(x).
\end{aligned}
$$

(ii) Initially, based on Lemma 3(ii), it can be concluded that $\psi \wedge_\gamma f \in L^1(\mathbb{R}^+, t^\delta dt)$. This implies that the left-hand side of (13) is well defined. Additionally, as stated in [20] (Theorem 2.2), we find that $\|(SP)_{\alpha,\beta}f\|_\infty \leq \|f\|_{1,t^{\beta-\alpha}dt}$. Thus, for every $x > 0$

$$
\begin{aligned}
\big|\big(\psi \wedge_{\gamma+\alpha-\beta-1} ((SP)_{\alpha,\beta}f)\big)(x)\big| &= \left|\int_{\mathbb{R}^+} \psi(s)((SP)_{\alpha,\beta}f)(xs)s^{\gamma+\alpha-\beta-1}ds\right|\\
&\leq \|(SP)_{\alpha,\beta}f\|_\infty \int_{\mathbb{R}^+} |\psi(s)|s^{\gamma+\alpha-\beta-1}ds\\
&\leq \|f\|_{1,t^{\beta-\alpha}dt} \int_{\mathbb{R}^+} |\psi(s)|s^{\gamma+\alpha-\beta-1}ds < \infty,
\end{aligned}
$$

then the right-hand side of (13) makes sense.

Now, for $x > 0$, we have

$$
\begin{aligned}
\big((SP)_{\alpha,\beta}(\psi \wedge_\gamma f)\big)(x) &= \int_{\mathbb{R}^+} \frac{t^\beta}{x^\alpha + t^\alpha}(\psi \wedge_\gamma f)(t)dt\\
&= \int_{\mathbb{R}^+} \frac{t^\beta}{x^\alpha + t^\alpha}\int_{\mathbb{R}^+} \psi(s)f(ts)s^\gamma ds dt\\
&= \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} \frac{t^\beta}{x^\alpha + t^\alpha} f(ts)dt\, \psi(s)s^\gamma ds\\
&\quad \text{(according to Fubini's theorem)}\\
&= \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} \frac{\left(\frac{u}{s}\right)^\beta}{x^\alpha + \left(\frac{u}{s}\right)^\alpha} f(u)\frac{1}{s}du\, \psi(s)s^\gamma ds\\
&\quad \text{(by making a substitution of variables } u = ts\text{)}\\
&= \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} \frac{u^\beta}{(xs)^\alpha + u^\alpha} f(u)du\, \psi(s)s^{\gamma+\alpha-\beta-1}ds\\
&= \int_{\mathbb{R}^+} \psi(s)((SP)_{\alpha,\beta}f)(xs)s^{\gamma+\alpha-\beta-1}ds\\
&= \big(\psi \wedge_{\gamma+\alpha-\beta-1} ((SP)_{\alpha,\beta}f)\big)(x).
\end{aligned}
$$

This concludes the proof of Theorem 2. □

From this Theorem and taking $\beta = \alpha - 1$, $\alpha \neq 0$, one obtains

Corollary 2. *Let $\alpha \neq 0$. If $f \in L^1(\mathbb{R}^+, t^{-1}dt)$ and $\psi \in C_c(\mathbb{R}^+)$, then*
(i)

$$(SP)_{\alpha,\alpha-1}(f \wedge_{-1} \psi) = f \wedge_{-1} ((SP)_{\alpha,\alpha-1}\psi),$$

(ii)

$$(SP)_{\alpha,\alpha-1}(\psi \wedge_\gamma f) = \psi \wedge_\gamma ((SP)_{\alpha,\alpha-1}f),$$

(iii)

$$(SP)_{\alpha,\alpha-1}(\psi \wedge f) = \psi \wedge ((SP)_{\alpha,\alpha-1}f),$$

Remark 5. *Observe that the Stieltjes transform* $(SP)_{1,0}$ *and the Poisson transform* $(SP)_{2,1}$ *are particular cases of the transform* $(SP)_{\alpha,\alpha-1}$ *considered in Corollary 2.*

The next result holds as follows:

Lemma 4. *If* $f \in C_0(\mathbb{R}^+)$ *and* $\psi \in C_c(\mathbb{R}^+)$, *then*
(i)

$$f \wedge_{-1} \psi \in C_0(\mathbb{R}^+) \text{ and } \|f \wedge_{-1} \psi\|_\infty \leq \|f\|_\infty \int_{\mathbb{R}^+} |\psi(s)| s^{-1} ds < \infty,$$

(ii)

$$\psi \wedge_\gamma f \in C_0(\mathbb{R}^+) \text{ and } \|\psi \wedge_\gamma f\|_\infty \leq \|f\|_\infty \int_{\mathbb{R}^+} |\psi(s)| s^{\gamma} ds < \infty.$$

Proof. (i) We find a closed and bounded interval $[a, b] \subseteq \mathbb{R}^+$ such that the support of $\psi \subseteq [a, b]$. Let $t_n \to t$ as $n \to \infty$ in $\mathbb{R}^+$. As f is a continuous function on $\mathbb{R}^+$, we have

$$f(\frac{s}{t_n})\psi(s)s^{-1} \longrightarrow f(\frac{s}{t})\psi(s)s^{-1} \qquad \text{for each } s \in \mathbb{R}^+.$$

Since

$$\left| f(\frac{s}{t_n})\psi(s)s^{-1} \right| \leq \|f\|_\infty \left| \psi(s)s^{-1} \right|$$

and $\left|\psi(s)s^{-1}\right|$ is an integrable function on $\mathbb{R}^+$, we can apply the dominated convergence theorem, and hence

$$\begin{aligned} \lim_{n\to\infty} (f \wedge_{-1} \psi)(t_n) &= \lim_{n\to\infty} \int_{\mathbb{R}^+} f(s)\psi(t_n s)s^{-1} ds \\ &= \int_{\mathbb{R}^+} f(\frac{s}{t_n})\psi(s)s^{-1} ds \\ &= \int_{\mathbb{R}^+} f(\frac{s}{t})\psi(s)s^{-1} ds \\ &= \int_{\mathbb{R}^+} f(s)\psi(st)s^{-1} ds \\ &= (f \wedge_{-1} \psi)(t). \end{aligned}$$

Thus, $f \wedge_{-1} \psi$ is a continuous function on $\mathbb{R}^+$.

Let $\epsilon > 0$ be arbitrary. Since f vanishes at infinity, we can find a closed and bounded interval $[c, d] \subseteq \mathbb{R}^+$ such that

$$|f(u)| \leq \frac{\epsilon}{C+1}, \qquad \text{for all } u \notin [c, d],$$

where $C = \int_{\mathbb{R}^+} |\psi(s)| s^{-1} ds$.

Since $\frac{s}{t} \in [c, d]$, $s \in [a, b]$ then $t \in [\frac{a}{d}, \frac{b}{c}]$; we find that if $t \notin [\frac{a}{d}, \frac{b}{c}]$ and $s \in [a, b]$ then $\frac{s}{t} \notin [c, d]$. Then, $|f(\frac{s}{t})| \leq \frac{\epsilon}{C+1}$, for $t \notin [\frac{a}{d}, \frac{b}{c}]$. Thus, for $t \notin [\frac{a}{d}, \frac{b}{c}]$ one obtains

$$\begin{aligned} |(f \wedge_{-1} \psi)(t)| &\leq \int_{\mathbb{R}^+} \left| f(\frac{s}{t}) \right| |\psi(s)| s^{-1} ds \\ &\leq \frac{\epsilon}{C+1} \cdot C \leq \epsilon. \end{aligned}$$

Thus, $f \wedge_{-1} \psi \in C_0(\mathbb{R}^+)$.

By direct computation, we obtain

$$\begin{aligned}\|f \wedge_{-1} \psi\|_\infty &\leq \int_{\mathbb{R}^+} \left|f(\frac{s}{t})\right| |\psi(s)| s^{-1} ds \\ &\leq \|f\|_\infty \int_{\mathbb{R}^+} |\psi(s)| s^{-1} ds < \infty.\end{aligned}$$

Therefore, (i) holds.

(ii) We find a closed and bounded interval $[a, b] \subseteq \mathbb{R}^+$ such that the support of $\psi \subseteq [a, b]$. Let $t_n \to t$ as $n \to \infty$ in $\mathbb{R}^+$. As f is a continuous function on $\mathbb{R}^+$, we have

$$f(t_n s)\psi(s)s^\gamma \longrightarrow f(ts)\psi(s)s^\gamma \qquad \text{for each } s \in \mathbb{R}^+.$$

Since

$$|f(t_n s)\psi(s)s^\gamma| \leq \|f\|_\infty |\psi(s)s^\gamma|$$

and $|\psi(s)s^\gamma|$ is an integrable function on $\mathbb{R}^+$, we can apply the dominated convergence theorem, and hence

$$\begin{aligned}\lim_{n\to\infty} (\psi \wedge_\gamma f)(t_n) &= \lim_{n\to\infty} \int_{\mathbb{R}^+} \psi(s) f(t_n s) s^\gamma ds \\ &= \int_{\mathbb{R}^+} \psi(s) f(st) s^\gamma ds \\ &= (\psi \wedge_\gamma f)(t).\end{aligned}$$

Thus, $\psi \wedge_\gamma f$ is a continuous function on $\mathbb{R}^+$.

Let $\epsilon > 0$ be arbitrary. Since f vanishes at infinity, we can find a closed and bounded interval $[c, d] \subseteq \mathbb{R}^+$ such that

$$|f(u)| \leq \frac{\epsilon}{C+1}, \qquad \text{for all } u \notin [c, d],$$

where $C = \int_{\mathbb{R}^+} |\psi(s)| s^\gamma ds$.

Since $ts \in [c, d]$, $s \in [a, b]$ then $t \in [\frac{c}{b}, \frac{d}{a}]$, we find that if $t \notin [\frac{c}{b}, \frac{d}{a}]$ and $s \in [a, b]$ then $ts \notin [c, d]$. Then, $|f(\frac{s}{t})| \leq \frac{\epsilon}{C+1}$, for $t \notin [\frac{c}{b}, \frac{d}{a}]$. Thus, for $t \notin [\frac{c}{b}, \frac{d}{a}]$ we obtain

$$\begin{aligned}|(\psi \wedge_\gamma f)(t)| &\leq \int_{\mathbb{R}^+} |\psi(s)||f(ts)| s^\gamma ds \\ &\leq \frac{\epsilon}{C+1} \cdot C \leq \epsilon.\end{aligned}$$

Thus, $\psi \wedge_\gamma f \in C_0(\mathbb{R}^+)$.

By direct computation, we obtain

$$\begin{aligned}\|\psi \wedge_\gamma f\|_\infty &\leq \int_{\mathbb{R}^+} |\psi(s)||f(ts)| s^\gamma ds \\ &\leq \|f\|_\infty \int_{\mathbb{R}^+} |\psi(s)| s^\gamma ds < \infty.\end{aligned}$$

Therefore, (ii) holds.

This concludes the proof of Lemma 4. □

Remark 6. *From this Lemma, one can see that the Mellin-type convolution $\wedge_\gamma$ is a continuous operation in another specific sense different to that obtained in Remark 4. In fact, if the sequence (f_n) converges to f in $C_0(\mathbb{R}^+)$ as $n \to \infty$ and $\psi \in C_c(\mathbb{R}^+)$, then:*

(i) *the sequence $(f_n \wedge_{-1} \psi)$ converges to $f \wedge_{-1} \psi$ in $C_0(\mathbb{R}^+)$ as $n \to \infty$,*

(ii) *the sequence $(\psi \wedge_\gamma f_n)$ converges to $\psi \wedge_\gamma f$ in $C_0(\mathbb{R}^+)$ as $n \to \infty$.*

Observe that from Lemma 4, the Mellin-type convolution denoted as $\wedge_\gamma$ represents a bilinear operation characterized by the following properties:

If $f, g \in C_0(\mathbb{R}^+)$, $\phi, \psi \in C_c(\mathbb{R}^+)$ and $\alpha, \beta \in \mathbb{C}$, then

$$(\alpha f + \beta g) \wedge_{-1} \phi = \alpha(f \wedge_{-1} \phi) + \beta(g \wedge_{-1} \phi)$$

and

$$f \wedge_{-1} (\alpha\phi + \beta\psi) = \alpha(f \wedge_{-1} \phi) + \beta(f \wedge_{-1} \psi).$$

Also,

$$(\alpha\phi + \beta\psi) \wedge_\gamma f = \alpha(\phi \wedge_\gamma f) + \beta(\psi \wedge_\gamma f)$$

and

$$\phi \wedge_\gamma (\alpha f + \beta g) = \alpha(\phi \wedge_\gamma f) + \beta(\phi \wedge_\gamma g).$$

Also, again using Lemma 4, one obtains a corresponding result to Proposition 3.

Indeed

Proposition 4. *For a function $f \in C_0(\mathbb{R}^+)$, and for functions $\phi, \psi \in C_c(\mathbb{R}^+)$, we have*

(i)

$$(f \wedge_{-1} \phi) \wedge_{-1} \psi = f \vee (\phi \wedge_{-1} \psi),$$

(ii)

$$f \wedge_{-1} (\phi \wedge_{-1} \psi) = \phi \wedge_{-1} (f \wedge_{-1} \psi),$$

(iii)

$$(\psi \wedge_\gamma \phi) \wedge_\gamma f = \psi \vee (\phi \wedge_\gamma f),$$

(iv)

$$\psi \wedge_\gamma (\phi \wedge_\gamma f) = (\phi \vee \psi) \wedge_\gamma f.$$

4. Final Observations and Conclusions

This paper develops exchange formulae for the Stieltjes–Poisson transform using Mellin-type convolutions in weighted Lebesgue spaces. A key contribution of this paper is the inclusion of a more general Mellin-type convolution product, which is bilinear and continuous. While a broader exchange formula was proven in [21], this paper provides novel insights, particularly with the incorporation of the space of continuous functions vanishing at infinity. This inclusion enlarges the applicability of the results, offering a more versatile framework for understanding the Stieltjes-Poisson transform.

In conclusion, this research expands upon existing work by providing new exchange formulae for the Stieltjes–Poisson transform and extending the function spaces considered, thereby opening new avenues for further exploration and application in the field of harmonic analysis and related areas. The results obtained in this paper can be used as a foundation for future studies on the interaction between Mellin-type convolutions and other integral transforms in weighted function spaces.

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