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On Analytical Extension of Generalized Hypergeometric Function ${}_3F_2$

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Abstract: The paper considers the generalized hypergeometric function ${}_3F_2$, which is important in various fields of mathematics, physics, and economics. The method is used, according to which the domains of the analytical continuation of the special functions are the domains of convergence of their expansions into a special family of functions, namely branched continued fractions. These expansions have wide domains of convergence and better computational properties, particularly compared with series, making them effective tools for representing special functions. New domains of the analytical continuation of the generalized hypergeometric function ${}_3F_2$ with real and complex parameters have been established. The paper also includes examples of the presentation and extension of some special functions.

Keywords: generalized hypergeometric function; branched continued fraction; analytical continuation; convergence; approximation by rational functions

MSC: 33C20; 30B99; 30B40; 40A99; 41A20

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1. Introduction

Special functions are among the most intriguing functions pervasive in all fields of science and industry. Research on them has been ongoing for the past two centuries, and due to their importance, several books (see, particularly, refs. [1–4]) and websites (see, for instance, http://functions.wolfram.com) and a huge collection of papers have been devoted to these functions. Despite significant achievements in the study of special functions and their properties, this topic remains one of the most important and has many open problems.

The paper considers a generalized hypergeometric function ${}_3F_2$ defined as follows ([5], p. 8):

$$ {}_3F_2(\alpha_1,\alpha_2,\alpha_3;\beta_1,\beta_2;z)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k(\alpha_2)_k(\alpha_3)_k}{(\beta_1)_k(\beta_2)_k}\frac{z^k}{k!}, \quad (1) $$

where $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2\in\mathbb{C}$, $\beta_1,\beta_2\notin\{0,-1,-2,\ldots\}$, $(\cdot)_k$ is the Pochhammer symbol, $z\in\mathbb{C}$.

The generalized hypergeometric function ${}_3F_2$ appears, in particular, in mathematical analysis, in the problem of the asymptotic expansion of the Lauricella–Saran hypergeometric function F_K [6], and in algebra, in the problem of the quantum unique ergodicity of Eisenstein series [7]. In probability theory, this function is used to describe the hypergeometric distributions and their moments [8]. In theoretical physics, the generalized hypergeometric function ${}_3F_2$ is the solution, in particular, to the Picard-Fuchs differential equation [9,10] and Laplace's equation [11]. In string theory, it arises in the context of computing the amplitudes associated with the vibrational modes of strings, as well as in the study of interactions between strings or the analysis of gauge theories [12–14].

In quantum mechanics, the generalized hypergeometric function ${}_3F_2$ is used to describe the wave functions of quantum harmonic oscillators [15,16] and the Coulomb interaction of a system of spinless fermions [17]. In the modeling of financial processes, this function appears in financial options pricing models to compute analytical solutions [18], and in game theory, to count the number of totally mixed Nash equilibria in games of several players [19].

The paper continues to conduct research [20,21] on the representation of special functions, in particular generalized hypergeometric function ${}_3F_2$, by a special family of functions, namely branched continued fractions [22,23]. The good approximating properties of branched continued fractions (wide region of convergence, faster rate of convergence under certain conditions compared with series, and numerical stability) allow them to be an effective tool for representing special functions (see [24–28]).

Let $(ij)_0 = (i_0, j_0)$, $\mathfrak{I} = \{(1,1); (1,2); (2,1); (2,2)\}$, and

$$\mathfrak{I}^{(ij)_0} = \{(ij)_k : (ij)_k = (i_1, j_1, i_2, j_2, \ldots, i_k, j_k),\ 1 + \delta^1_{i_{k-1}} \le i_k \le 2,\ j_k \in \{1,2\},\ |i_k - j_k| \neq |i_{k-1} - j_{k-1}|,\ k \ge 1\},$$

where δ^p_k is the Kronecker symbol. In [20], it is established that, for $(ij)_0 \in \mathfrak{I}$,

$$\frac{{}_3F_2(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; -z)}{{}_3F_2(\alpha_1 + \delta^1_{i_0}\delta^1_{j_0} + \delta^2_{i_0}\delta^2_{j_0}, \alpha_2 + \delta^1_{i_0}\delta^2_{j_0} + \delta^2_{i_0}\delta^1_{j_0}, \alpha_3 + \delta^2_{i_0}; \beta_1 + \delta^1_{i_0}\delta^1_{j_0} + \delta^2_{i_0}, \beta_2 + \delta^1_{i_0}\delta^2_{j_0} + \delta^2_{i_0}; -z)}$$
$$= 1 + \sum_{\substack{i_1 = 1 + \delta^1_{i_0} \\ |i_1 - j_1| \neq |i_0 - j_0|,\ j_1 \in \{1,2\}}}^{2} \cfrac{d^{(ij)_0}_{(ij)_1} z}{1 + \sum\limits_{\substack{i_2 = 1 + \delta^1_{i_1} \\ |i_2 - j_2| \neq |i_1 - j_1|,\ j_2 \in \{1,2\}}}^{2} \cfrac{d^{(ij)_0}_{(ij)_2} z}{1 + \ddots + \sum\limits_{\substack{i_k = 1 + \delta^1_{i_{k-1}} \\ |i_k - j_k| \neq |i_{k-1} - j_{k-1}|,\ j_k \in \{1,2\}}}^{2} \cfrac{d^{(ij)_0}_{(ij)_k} z}{1 + \ddots}}}, \tag{2}$$

where for $(ij)_k \in \mathfrak{I}^{(ij)_0}$, $(ij)_0 \in \mathfrak{I}$ and $k \ge 1$,

$$d^{(ij)_0}_{(ij)_k} = \frac{\left[\beta_1 - \alpha_3 + \sum\limits_{p=0}^{k-2} \delta^1_{i_p}\delta^1_{j_p}\right]\left[\alpha_1 + \sum\limits_{p=0}^{k-2} (\delta^1_{i_p}\delta^1_{j_p} + \delta^2_{i_p}\delta^2_{j_p})\right]\left[\alpha_2 + \sum\limits_{p=0}^{k-2} (\delta^1_{i_p}\delta^2_{j_p} + \delta^2_{i_p}\delta^1_{j_p})\right]}{\left[\beta_1 + 1 + \sum\limits_{p=0}^{k-2} (\delta^1_{i_p}\delta^1_{j_p} + \delta^2_{i_p})\right]\left[\beta_1 + \sum\limits_{p=0}^{k-2} (\delta^1_{i_p}\delta^1_{j_p} + \delta^2_{i_p})\right]\left[\beta_2 + \sum\limits_{p=0}^{k-2} (\delta^1_{i_p}\delta^2_{j_p} + \delta^2_{i_p})\right]}, \tag{3}$$

if $i_{k-1} = 2$, $j_{k-1} = i_k = j_k = 1$,

$$d^{(ij)_0}_{(ij)_k} = \frac{\left[\beta_2 - \alpha_3 + \sum\limits_{p=0}^{k-2} \delta^1_{i_p}\delta^2_{j_p}\right]\left[\alpha_1 + \sum\limits_{p=0}^{k-2} (\delta^1_{i_p}\delta^1_{j_p} + \delta^2_{i_p}\delta^2_{j_p})\right]\left[\alpha_2 + \sum\limits_{p=0}^{k-2} (\delta^1_{i_p}\delta^2_{j_p} + \delta^2_{i_p}\delta^1_{j_p})\right]}{\left[\beta_2 + 1 + \sum\limits_{p=0}^{k-2} (\delta^1_{i_p}\delta^2_{j_p} + \delta^2_{i_p})\right]\left[\beta_1 + \sum\limits_{p=0}^{k-2} (\delta^1_{i_p}\delta^1_{j_p} + \delta^2_{i_p})\right]\left[\beta_2 + \sum\limits_{p=0}^{k-2} (\delta^1_{i_p}\delta^2_{j_p} + \delta^2_{i_p})\right]}, \tag{4}$$

if $i_{k-1} = j_{k-1} = j_k = 2$, $i_k = 1$,

$$d^{(ij)_0}_{(ij)_k} = \frac{\left[\beta_1 - \alpha_1 + \sum\limits_{p=0}^{k-2} \delta^2_{i_p}\delta^1_{j_p}\right]\left[\alpha_2 + \sum\limits_{p=0}^{k-2} (\delta^1_{i_p}\delta^2_{j_p} + \delta^2_{i_p}\delta^1_{j_p})\right]\left[\alpha_3 + \sum\limits_{p=0}^{k-2} \delta^2_{i_p}\right]}{\left[\beta_1 + 1 + \sum\limits_{p=0}^{k-2} (\delta^1_{i_p}\delta^1_{j_p} + \delta^2_{i_p})\right]\left[\beta_1 + \sum\limits_{p=0}^{k-2} (\delta^1_{i_p}\delta^1_{j_p} + \delta^2_{i_p})\right]\left[\beta_2 + \sum\limits_{p=0}^{k-2} (\delta^1_{i_p}\delta^2_{j_p} + \delta^2_{i_p})\right]}, \tag{5}$$

if $i_{k-1} = j_{k-1} = j_k = 1$, $i_k = 2$,

$$d_{(ij)_k}^{(ij)_0}=\frac{\left[\beta_1-\alpha_1+\sum_{p=0}^{k-2}\delta_{i_p}^2\delta_{j_p}^1\right]\left[\alpha_3+1+\sum_{p=0}^{k-2}\delta_{i_p}^2\right]\left[\alpha_2+\sum_{p=0}^{k-2}(\delta_{i_p}^1\delta_{j_p}^2+\delta_{i_p}^2\delta_{j_p}^1)\right]}{\left[\beta_1+1+\sum_{p=0}^{k-2}(\delta_{i_p}^1\delta_{j_p}^1+\delta_{i_p}^2)\right]\left[\beta_2+1+\sum_{p=0}^{k-2}(\delta_{i_p}^1\delta_{j_p}^2+\delta_{i_p}^2)\right]\left[\beta_1+\sum_{p=0}^{k-2}(\delta_{i_p}^1\delta_{j_p}^1+\delta_{i_p}^2)\right]}, \tag{6}$$

if $i_{k-1}=j_{k-1}=i_k=2$, $j_k=1$,

$$d_{(ij)_k}^{(ij)_0}=\frac{\left[\beta_2-\alpha_2+\sum_{p=0}^{k-2}\delta_{i_p}^2\delta_{j_p}^2\right]\left[\alpha_1+\sum_{p=0}^{k-2}(\delta_{i_p}^1\delta_{j_p}^1+\delta_{i_p}^2\delta_{j_p}^2)\right]\left[\alpha_3+\sum_{p=0}^{k-2}\delta_{i_p}^2\right]}{\left[\beta_2+1+\sum_{p=0}^{k-2}(\delta_{i_p}^1\delta_{j_p}^2+\delta_{i_p}^2)\right]\left[\beta_1+\sum_{p=0}^{k-2}(\delta_{i_p}^1\delta_{j_p}^1+\delta_{i_p}^2)\right]\left[\beta_2+\sum_{p=0}^{k-2}(\delta_{i_p}^1\delta_{j_p}^2+\delta_{i_p}^2)\right]}, \tag{7}$$

if $i_{k-1}=1$, $j_{k-1}=i_k=j_k=2$,

$$d_{(ij)_k}^{(ij)_0}=\frac{\left[\beta_2-\alpha_2+\sum_{p=0}^{k-2}\delta_{i_p}^2\delta_{j_p}^2\right]\left[\alpha_3+1+\sum_{p=0}^{k-2}\delta_{i_p}^2\right]\left[\alpha_1+\sum_{p=0}^{k-2}(\delta_{i_p}^1\delta_{j_p}^1+\delta_{i_p}^2\delta_{j_p}^2)\right]}{\left[\beta_1+1+\sum_{p=0}^{k-2}(\delta_{i_p}^1\delta_{j_p}^1+\delta_{i_p}^2)\right]\left[\beta_2+1+\sum_{p=0}^{k-2}(\delta_{i_p}^1\delta_{j_p}^2+\delta_{i_p}^2)\right]\left[\beta_2+\sum_{p=0}^{k-2}(\delta_{i_p}^1\delta_{j_p}^2+\delta_{i_p}^2)\right]}, \tag{8}$$

if $j_{k-1}=1$, $i_{k-1}=i_k=j_k=2$ (see Section 5 or a description of the process of obtaining Formulas (3)–(8)), and it is shown that

$$\Psi_\varepsilon=\left\{\mathbf{z}\in\mathbb{C}:\ z\notin\left(-\infty;-\frac{1-\varepsilon}{4}\right)\right\},\quad 0<\varepsilon<1,$$

is the domain of the analytical continuation of the function on the left side of (2) under the condition that

$$\beta_k\geq\alpha_k\geq 0,\quad \beta_k\geq\alpha_3\geq 0,\quad \beta_k\neq 0,\quad k=1,2.$$

Note that for the pair $(1,1)$ (or the similar $(1,2)$), the ratio of generalized hypergeometric functions ${}_3F_2$ was considered in [21], where explicit formulas for the coefficients of the formal branched continued fraction expansion through the coefficients of the generalized hypergeometric function ${}_3F_2$ without the expansion itself are given.

The paper is organized as follows. Section 2 guarantees the union of the circular and cardioid domains as the domain of the analytic extension of the functions on the left side of (2), with complex parameters, through their branched continued fraction expansions, and, in the case of real parameters that ensure the positivity of the elements of the expansions, the domain of the analytic extension is a plane with a cut. The last result is a generalization of the corresponding result in [20]. Section 3 presents examples of the representation of special functions by their branched continued fraction expansions, while Section 4 collects important conclusions.

2. Domains of Analytical Extension

The method will be used here, according to which the domains of the analytical continuation of the special functions are the domains of convergence of their branched continued fraction expansions (see, ref. [29]).

The following is true:

Theorem 1. *Let $(ij)_0$ be an arbitrary pair in $\mathfrak{I}$. Let (1) be a generalized hypergeometric function with parameters such that*

$$\sum_{\substack{i_k=1+\delta_{i_{k-1}}^1 \\ |i_k-j_k|\neq|i_{k-1}-j_{k-1}|,\, j_k\in\{1,2\}}}^{2} \frac{|d_{(ij)_k}^{(ij)_0}|-\mathrm{Re}(d_{(ij)_k}^{(ij)_0})}{g_{(ij)_{k-1}}^{(ij)_0}(1-g_{(ij)_k}^{(ij)_0})} \le \kappa, \quad (ij)_k\in\mathfrak{I}^{(ij)_0},\ k\ge 2, \tag{9}$$

where $d_{(ij)_k}^{(ij)_0}$, $(ij)_k\in\mathfrak{I}^{(ij)_0}$, $k\ge 2$ are defined by (3)–(8) herewith $\beta_1,\beta_2\notin\{0,-1,-2,\dots\}$,

$$\kappa>0 \quad and \quad 0<g_{(ij)_k}^{(ij)_0}<1, \quad (ij)_k\in\mathfrak{I}^{(ij)_0},\ k\ge 1. \tag{10}$$

Then,

(A) The branched continued fraction (2) converges uniformly on every compact subset of the domain

$$\Omega_{\kappa,\tau}=\Omega_\kappa\bigcup\Omega_\tau, \tag{11}$$

where

$$\Omega_\kappa=\left\{z\in\mathbb{C}:\ |z|<\frac{1+\cos(\arg(z))}{\kappa}\right\} \tag{12}$$

and

$$\Omega_\tau=\left\{z\in\mathbb{C}:\ |z|<\frac{1}{8\tau}\right\}, \tag{13}$$

where

$$\tau=\sup_{(ij)_k\in\mathfrak{I}_{(ij)_0},\ k\ge 1}|d_{(ij)_k}^{(ij)_0}|, \tag{14}$$

to the function $f(z)$, holomorphic in the domain $\Omega_{\kappa,\tau}$;

(B) The function $f(z)$ is an analytic continuation of the function on the left side of (2) in the domain (11).

In our proof, we need the following:

Theorem 2. *Let $(ij)_0$ be an arbitrary pair in $\mathfrak{I}$ and let $q_{(ij)_k}^{(ij)_0}$, $(ij)_k\in\mathfrak{I}^{(ij)_0}$, $k\ge 1$ be the real numbers, such that*

$$0<q_{(ij)_k}^{(ij)_0}\le 1, \quad (ij)_k\in\mathfrak{I}^{(ij)_0},\ k\ge 1. \tag{15}$$

Then,

(A) The branched continued fraction

$$1+\sum_{\substack{i_1=1+\delta_{i_0}^1 \\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \cfrac{q_{(ij)_1}^{(ij)_0} z_{(ij)_1}^{(ij)_0}}{1+\displaystyle\sum_{\substack{i_2=1+\delta_{i_1}^1 \\ |i_2-j_2|\neq|i_1-j_1|,\, j_2\in\{1,2\}}}^{2} \cfrac{q_{(ij)_2}^{(ij)_0}(1-q_{(ij)_1}^{(ij)_0}) z_{(ij)_2}^{(ij)_0}}{1+\ddots+\displaystyle\sum_{\substack{i_k=1+\delta_{i_{k-1}}^1 \\ |i_k-j_k|\neq|i_{k-1}-j_{k-1}|,\, j_k\in\{1,2\}}}^{2} \cfrac{q_{(ij)_k}^{(ij)_0}(1-q_{(ij)_{k-1}}^{(ij)_0}) z_{(ij)_k}^{(ij)_0}}{1+\ddots}}} \tag{16}$$

converges absolutely and uniformly for

$$|z_{(ij)_k}^{(ij)_0}| \le \frac{1}{i_{k-1}}, \quad (ij)_k \in \mathfrak{I}^{(ij)_0}, \; k \ge 1; \tag{17}$$

(B) The values of the branched continued fraction (16) and of its approximants are in the closed disk

$$|w - 1| \le 1. \tag{18}$$

Proof. In the same way as in ([30], Theorem 2), we show that the majorant of the branched continued fraction (16) is

$$1 - \sum_{\substack{i_1=1+\delta_{i_0}^1 \\ |i_1-j_1|\ne|i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \cfrac{q_{(ij)_1}^{(ij)_0}/i_0}{1 - \sum\limits_{\substack{i_2=1+\delta_{i_1}^1 \\ |i_2-j_2|\ne|i_1-j_1|,\, j_2\in\{1,2\}}}^{2} \cfrac{q_{(ij)_2}^{(ij)_0}(1-q_{(ij)_1}^{(ij)_0})/i_1}{1 - \ddots - \sum\limits_{\substack{i_k=1+\delta_{i_{k-1}}^1 \\ |i_k-j_k|\ne|i_{k-1}-j_{k-1}|,\, j_k\in\{1,2\}}}^{2} \cfrac{q_{(ij)_k}^{(ij)_0}(1-q_{(ij)_{k-1}}^{(ij)_0})/i_{k-1}}{1 - \ddots}}}. \tag{19}$$

Let $(ij)_0$ be an arbitrary pair in $\mathfrak{I}$. We set

$${}^{n}F_{(ij)_n}^{(ij)_0} = {}^{n}\widehat{F}_{(ij)_n}^{(ij)_0} = 1, \quad (ij)_n \in \mathfrak{I}^{(ij)_0}, \; n \ge 1,$$

and

$${}^{n}F_{(ij)_k}^{(ij)_0} = 1 + \sum_{\substack{i_{k+1}=1+\delta_{i_k}^1 \\ |i_{k+1}-j_{k+1}|\ne|i_k-j_k|,\, j_{k+1}\in\{1,2\}}}^{2} \cfrac{q_{(ij)_{k+1}}^{(ij)_0}(1-q_{(ij)_k}^{(ij)_0})z_{(ij)_{k+1}}^{(ij)_0}}{1 + \ddots + \sum\limits_{\substack{i_n=1+\delta_{i_{n-1}}^1 \\ |i_n-j_n|\ne|i_{n-1}-j_{n-1}|,\, j_n\in\{1,2\}}}^{2} \cfrac{q_{(ij)_n}^{(ij)_0}(1-q_{(ij)_{n-1}}^{(ij)_0})z_{(ij)_n}^{(ij)_0}}{1}},$$

$${}^{n}\widehat{F}_{(ij)_k}^{(ij)_0} = 1 - \sum_{\substack{i_{k+1}=1+\delta_{i_k}^1 \\ |i_{k+1}-j_{k+1}|\ne|i_k-j_k|,\, j_{k+1}\in\{1,2\}}}^{2} \cfrac{q_{(ij)_{k+1}}^{(ij)_0}(1-q_{(ij)_k}^{(ij)_0})/i_k}{1 - \ddots - \sum\limits_{\substack{i_n=1+\delta_{i_{n-1}}^1 \\ |i_n-j_n|\ne|i_{n-1}-j_{n-1}|,\, j_n\in\{1,2\}}}^{2} \cfrac{q_{(ij)_n}^{(ij)_0}(1-q_{(ij)_{n-1}}^{(ij)_0})/i_{n-1}}{1}},$$

where $(ij)_k \in \mathfrak{I}^{(ij)_0}$, $1 \le k \le n-1$, $n \ge 2$. Then,

$${}^{n}F_{(ij)_k}^{(ij)_0} = 1 + \sum_{\substack{i_{k+1}=1+\delta_{i_k}^1 \\ |i_{k+1}-j_{k+1}|\ne|i_k-j_k|,\, j_{k+1}\in\{1,2\}}}^{2} \frac{q_{(ij)_{k+1}}^{(ij)_0}(1-q_{(ij)_k}^{(ij)_0})z_{(ij)_{k+1}}^{(ij)_0}}{{}^{n}F_{(ij)_{k+1}}^{(ij)_0}}, \tag{20}$$

$${}^{n}\widehat{F}_{(ij)_k}^{(ij)_0} = 1 - \sum_{\substack{i_{k+1}=1+\delta_{i_k}^1 \\ |i_{k+1}-j_{k+1}|\ne|i_k-j_k|,\, j_{k+1}\in\{1,2\}}}^{2} \frac{q_{(ij)_{k+1}}^{(ij)_0}(1-q_{(ij)_k}^{(ij)_0})/i_k}{{}^{n}\widehat{F}_{(ij)_{k+1}}^{(ij)_0}}, \tag{21}$$

where $(ij)_k \in \mathfrak{I}^{(ij)_0}$, $1 \le k \le n-1$, $n \ge 2$, and, thus, for $n \ge 1$, the nth approximants of branched continued fractions (16) and (19) are written as

$$f_n^{(ij)_0} = 1 + \sum_{\substack{i_1=1+\delta^1_{i_0} \\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \cfrac{q^{(ij)_0}_{(ij)_1} z^{(ij)_0}_{(ij)_1}}{1 + \sum\limits_{\substack{i_2=1+\delta^1_{i_1} \\ |i_2-j_2|\neq|i_1-j_1|,\, j_2\in\{1,2\}}}^{2} \cfrac{q^{(ij)_0}_{(ij)_2}(1-q^{(ij)_0}_{(ij)_1}) z^{(ij)_0}_{(ij)_2}}{1 + \ddots + \sum\limits_{\substack{i_n=1+\delta^1_{i_{n-1}} \\ |i_n-j_n|\neq|i_{n-1}-j_{n-1}|,\, j_n\in\{1,2\}}}^{2} \cfrac{q^{(ij)_0}_{(ij)_n}(1-q^{(ij)_0}_{(ij)_{n-1}}) z^{(ij)_0}_{(ij)_n}}{1}}}$$

$$= 1 + \sum_{\substack{i_1=1+\delta^1_{i_0} \\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \frac{q^{(ij)_0}_{(ij)_1} z^{(ij)_0}_{(ij)_1}}{{}^nF^{(ij)_0}_{(ij)_1}}$$

and

$$\widehat{f}_n^{(ij)_0} = 1 - \sum_{\substack{i_1=1+\delta^1_{i_0} \\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \frac{q^{(ij)_0}_{(ij)_1}/i_0}{{}^n\widehat{F}^{(ij)_0}_{(ij)_1}},$$

respectively.

Let n be an arbitrary natural number. Using relations (15), (17), (20), and (21), by induction on k, $1 \le k \le n$, for $(ij)_k \in \mathfrak{I}^{(ij)_0}$, we show that

$$|{}^nF^{(ij)_0}_{(ij)_k}| \ge {}^n\widehat{F}^{(ij)_0}_{(ij)_k} \ge q^{(ij)_0}_{(ij)_k}. \tag{22}$$

For $k = n$ and $(ij)_n \in \mathfrak{I}^{(ij)_0}$, the inequalities (22) are obvious. By the induction hypothesis that (22) holds for $k = r+1$ and $(ij)_{r+1} \in \mathfrak{I}^{(ij)_0}$, where $r+1 \le n$, we prove (22) for $k = r$. Indeed,

$$\begin{aligned}
|{}^n\widehat{F}^{(ij)_0}_{(ij)_r}| &= \left| 1 + \sum_{\substack{i_{r+1}=1+\delta^1_{i_r} \\ |i_{r+1}-j_{r+1}|\neq|i_r-j_r|,\, j_{r+1}\in\{1,2\}}}^{2} \frac{q^{(ij)_0}_{(ij)_{r+1}}(1-q^{(ij)_0}_{(ij)_r}) z^{(ij)_0}_{(ij)_{r+1}}}{{}^nF^{(ij)_0}_{(ij)_{r+1}}} \right| \\
&\ge 1 - \sum_{\substack{i_{r+1}=1+\delta^1_{i_r} \\ |i_{r+1}-j_{r+1}|\neq|i_r-j_r|,\, j_{r+1}\in\{1,2\}}}^{2} \frac{q^{(ij)_0}_{(ij)_{r+1}}(1-q^{(ij)_0}_{(ij)_r}) |z^{(ij)_0}_{(ij)_{r+1}}|}{|{}^nF^{(ij)_0}_{(ij)_{r+1}}|} \\
&\ge 1 - \sum_{\substack{i_{r+1}=1+\delta^1_{i_r} \\ |i_{r+1}-j_{r+1}|\neq|i_r-j_r|,\, j_{r+1}\in\{1,2\}}}^{2} \frac{q^{(ij)_0}_{(ij)_{r+1}}(1-q^{(ij)_0}_{(ij)_r})/i_r}{{}^n\widehat{F}^{(ij)_0}_{(ij)_{r+1}}} \\
&= {}^n\widehat{F}^{(ij)_0}_{(ij)_r}.
\end{aligned}$$

It follows from (15) and (22) that ${}^n\widehat{F}^{(ij)_0}_{(ij)_{r+1}} \neq 0$. Then, replacing $q^{(ij)_0}_{(ij)_{r+1}}$ with ${}^n\widehat{F}^{(ij)_0}_{(ij)_{r+1}}$, the inequalities (22) are obtained for $k = r$.

From (15) and (22), it also follows that ${}^{n}F^{(ij)_0}_{(ij)_r} \neq 0$ and ${}^{n}\widehat{F}^{(ij)_0}_{(ij)_r} > 0$ for $n \geq 1$, $(ij)_k \in \mathfrak{I}^{(ij)_0}$, and $1 \leq k \leq n$. Applying ([23], p. 28), (20) and (21), we have, for $n \geq 1$ and $k \geq 1$,

$$f^{(ij)_0}_{n+k} - f^{(ij)_0}_{n} = 1 + \sum_{\substack{i_1=1+\delta^1_{i_0} \\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \frac{q^{(ij)_0}_{(ij)_1} z^{(ij)_0}_{(ij)_1}}{{}^{n+k}F^{(ij)_0}_{(ij)_1}} - \left(1 + \sum_{\substack{i_1=1+\delta^1_{i_0} \\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \frac{q^{(ij)_0}_{(ij)_1} z^{(ij)_0}_{(ij)_1}}{{}^{n}F^{(ij)_0}_{(ij)_1}}\right)$$
$$= -\sum_{\substack{i_1=1+\delta^1_{i_0} \\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \frac{q^{(ij)_0}_{(ij)_1} z^{(ij)_0}_{(ij)_1}}{{}^{n+k}F^{(ij)_0}_{(ij)_1}\, {}^{n}F^{(ij)_0}_{(ij)_1}} \left({}^{n+k}F^{(ij)_0}_{(ij)_1} - {}^{n}F^{(ij)_0}_{(ij)_1}\right).$$

Let r be an arbitrary natural number such that $1 \leq r \leq n-1$ and $n \geq 2$. Let $(ij)_r \in \mathfrak{I}^{(ij)_0}$. Then, we obtain, for $n \geq 2$ and $k \geq 1$,

$$\begin{aligned} {}^{n+k}F^{(ij)_0}_{(ij)_r} - {}^{n}F^{(ij)_0}_{(ij)_r} &= 1 + \sum_{\substack{i_{r+1}=1+\delta^1_{i_r} \\ |i_{r+1}-j_{r+1}|\neq|i_r-j_r|,\, j_{r+1}\in\{1,2\}}}^{2} \frac{q^{(ij)_0}_{(ij)_{r+1}} (1 - q^{(ij)_0}_{(ij)_r}) z^{(ij)_0}_{(ij)_{r+1}}}{{}^{n+k}F^{(ij)_0}_{(ij)_{r+1}}} \\ &\quad - \left(1 + \sum_{\substack{i_{r+1}=1+\delta^1_{i_r} \\ |i_{r+1}-j_{r+1}|\neq|i_r-j_r|,\, j_{r+1}\in\{1,2\}}}^{2} \frac{q^{(ij)_0}_{(ij)_{r+1}} (1 - q^{(ij)_0}_{(ij)_r}) z^{(ij)_0}_{(ij)_{r+1}}}{{}^{n}F^{(ij)_0}_{(ij)_{r+1}}}\right) \\ &= -\sum_{\substack{i_{r+1}=1+\delta^1_{i_r} \\ |i_{r+1}-j_{r+1}|\neq|i_r-j_r|,\, j_{r+1}\in\{1,2\}}}^{2} \frac{q^{(ij)_0}_{(ij)_{r+1}} (1 - q^{(ij)_0}_{(ij)_r}) z^{(ij)_0}_{(ij)_{r+1}}}{{}^{n+k}F^{(ij)_0}_{(ij)_{r+1}}\ {}^{n}F^{(ij)_0}_{(ij)_{r+1}}} \left({}^{n+k}F^{(ij)_0}_{(ij)_{r+1}} - {}^{n}F^{(ij)_0}_{(ij)_{r+1}}\right). \end{aligned} \tag{23}$$

Since

$${}^{n+k}F^{(ij)_0}_{(ij)_n} - {}^{n}F^{(ij)_0}_{(ij)_n} = \sum_{\substack{i_{n+1}=1+\delta^1_{i_n} \\ |i_{n+1}-j_{n+1}|\neq|i_n-j_n|,\, j_{n+1}\in\{1,2\}}}^{2} \frac{q^{(ij)_0}_{(ij)_{n+1}} (1 - q^{(ij)_0}_{(ij)_n}) z^{(ij)_0}_{(ij)_{n+1}}}{{}^{n+k}F^{(ij)_0}_{(ij)_{n+1}}},$$

after the $(n-1)$th application of the recurrence relation (23), we get

$$f^{(ij)_0}_{n+k} - f^{(ij)_0}_{n} = (-1)^{n+1} \sum_{\substack{i_1=1+\delta^1_{i_0} \\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \sum_{\substack{i_2=1+\delta^1_{i_1} \\ |i_2-j_2|\neq|i_1-j_1|,\, j_2\in\{1,2\}}}^{2} \cdots \sum_{\substack{i_{n+1}=1+\delta^1_{i_n} \\ |i_{n+1}-j_{n+1}|\neq|i_n-j_n|,\, j_{n+1}\in\{1,2\}}}^{2} \frac{\prod_{r=1}^{n+1} q^{(ij)_0}_{(ij)_r} (1 - q^{(ij)_0}_{(ij)_{r-1}}) z^{(ij)_0}_{(ij)_r}}{\prod_{r=1}^{n+1} {}^{n+k}F^{(ij)_0}_{(ij)_r} \prod_{r=1}^{n} {}^{n}F^{(ij)_0}_{(ij)_r}},$$

where $q^{(ij)_0}_{(ij)_0} = 0$.

Using (17) and (22), we get the following:

$$
\begin{aligned}
|f_{n+k}^{(ij)_0}-f_n^{(ij)_0}| &\le \sum_{\substack{i_1=1+\delta_{i_0}^1\\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2}\ \sum_{\substack{i_2=1+\delta_{i_1}^1\\ |i_2-j_2|\neq|i_1-j_1|,\, j_2\in\{1,2\}}}^{2}\dots\sum_{\substack{i_{n+1}=1+\delta_{i_n}^1\\ |i_{n+1}-j_{n+1}|\neq|i_n-j_n|,\, j_{n+1}\in\{1,2\}}}^{2}\frac{\prod\limits_{r=1}^{n+1}q_{(ij)_r}^{(ij)_0}(1-q_{(ij)_{r-1}}^{(ij)_0})|z_{(ij)_r}^{(ij)_0}|}{\prod\limits_{r=1}^{n+1}|^{n+k}F_{(ij)_r}^{(ij)_0}|\prod\limits_{r=1}^{n}|^{n}F_{(ij)_r}^{(ij)_0}|}\\
&\le \sum_{\substack{i_1=1+\delta_{i_0}^1\\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2}\ \sum_{\substack{i_2=1+\delta_{i_1}^1\\ |i_2-j_2|\neq|i_1-j_1|,\, j_2\in\{1,2\}}}^{2}\dots\sum_{\substack{i_{n+1}=1+\delta_{i_n}^1\\ |i_{n+1}-j_{n+1}|\neq|i_n-j_n|,\, j_{n+1}\in\{1,2\}}}^{2}\frac{\prod\limits_{r=1}^{n+1}q_{(ij)_r}^{(ij)_0}(1-q_{(ij)_{r-1}}^{(ij)_0})/i_{r-1}}{\prod\limits_{r=1}^{n+1}{}^{n+k}\widehat{F}_{(ij)_r}^{(ij)_0}\prod\limits_{r=1}^{n}{}^{n}\widehat{F}_{(ij)_r}^{(ij)_0}}\\
&= -(\widehat{f}_{n+k}^{(ij)_0}-\widehat{f}_n^{(ij)_0}),
\end{aligned}
$$

where $n \ge 1$, $k \ge 1$, and $q_{(ij)_0}^{(ij)_0} = 0$. Thus,

$$|f_{n+k}^{(ij)_0}-f_n^{(ij)_0}| \le \widehat{f}_n^{(ij)_0}-\widehat{f}_{n+k}^{(ij)_0}, \quad n\ge 1,\ k\ge 1, \tag{24}$$

that is, the sequence $\{\widehat{f}_n^{(ij)_0}\}$ is monotonically decreasing. In addition, from (22), we have, for $n \ge 1$,

$$
\begin{aligned}
\widehat{f}_n^{(ij)_0} &= 1-\sum_{\substack{i_1=1+\delta_{i_0}^1\\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2}\frac{q_{(ij)_1}^{(ij)_0}/i_0}{{}^{n}\widehat{F}_{(ij)_1}^{(ij)_0}}\\
&\ge 1-\sum_{\substack{i_1=1+\delta_{i_0}^1\\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2}\frac{1}{i_0}\\
&= 0.
\end{aligned}
$$

Thus, there exists a limit

$$\widehat{f}^{(ij)_0} = \lim_{n\to\infty}\widehat{f}_n^{(ij)_0}.$$

Now, by relation (24), we obtain, for $k \ge 1$,

$$
\begin{aligned}
\sum_{n=1}^{k}|f_{n+1}^{(ij)_0}-f_n^{(ij)_0}| &\le -\sum_{n=1}^{k}(\widehat{f}_{n+1}^{(ij)_0}-\widehat{f}_n^{(ij)_0})\\
&= \sum_{\substack{i_1=1+\delta_{i_0}^1\\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2}\frac{q_{(ij)_1}^{(ij)_0}}{i_0}-\widehat{f}_{k+1}^{(ij)_0}.
\end{aligned}
$$

It follows that as $k \to \infty$, the branched continued fraction (16) converges absolutely and uniformly for $|z_{(ij)_k}^{(ij)_0}| \le 1/i_{k-1}$, $(ij)_k \in \mathfrak{I}^{(ij)_0}$, $k \ge 1$. This proves (A).

Finally, by the inequalities (17) and (22), we get, for $n \geq 1$ (see (ref. [31], Theorem 1)),

$$\begin{aligned} |f_n^{(ij)_0} - 1| &\leq \sum_{\substack{i_1=1+\delta_{i_0}^1 \\ |i_1-j_1| \neq |i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \frac{q_{(ij)_1}^{(ij)_0} |z_{(ij)_1}^{(ij)_0}|}{|{}^{n+k}F_{(ij)_1}^{(ij)_0}|} \\ &\leq \sum_{\substack{i_1=1+\delta_{i_0}^1 \\ |i_1-j_1| \neq |i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \frac{1}{i_0} \\ &= 1, \end{aligned}$$

which proves (B). □

Note that the assumption that $d_{(ij)_k}^{(ij)_0}$, $(ij)_k \in \mathfrak{I}^{(ij)_0}$, $k \geq 1$ in Theorem 1 involves κ and that the domain of the analytic continuation also depends on this κ; and the smaller is κ, the larger the domain.

Proof Theorem 1. Let $(ij)_0$ be an arbitrary pair in $\mathfrak{I}$. We set

$$ {}^nF_{(ij)_n}^{(ij)_0}(z) = 1, \quad (ij)_n \in \mathfrak{I}^{(ij)_0},\ n \geq 1, \tag{25}$$

and

$$ {}^nF_{(ij)_k}^{(ij)_0}(z) = 1 + \sum_{\substack{i_{k+1}=1+\delta_{i_k}^1 \\ |i_{k+1}-j_{k+1}| \neq |i_k-j_k|,\, j_{k+1}\in\{1,2\}}}^{2} \cfrac{d_{(ij)_{k+1}}^{(ij)_0} z}{1 + \sum\limits_{\substack{i_{k+2}=1+\delta_{i_{k+1}}^1 \\ |i_{k+2}-j_{k+2}| \neq |i_{k+1}-j_{k+1}|,\, j_{k+2}\in\{1,2\}}}^{2} \cfrac{d_{(ij)_{k+2}}^{(ij)_0} z}{1 + \ddots + \sum\limits_{\substack{i_n=1+\delta_{i_{n-1}}^1 \\ |i_n-j_n| \neq |i_{n-1}-j_{n-1}|,\, j_n\in\{1,2\}}}^{2} \cfrac{d_{(ij)_n}^{(ij)_0} z}{1}}}, $$

where $(ij)_k \in \mathfrak{I}^{(ij)_0}$, $1 \leq k \leq n-1$, $n \geq 2$. Then,

$$ {}^nF_{(ij)_k}^{(ij)_0}(z) = 1 + \sum_{\substack{i_{k+1}=1+\delta_{i_k}^1 \\ |i_{k+1}-j_{k+1}| \neq |i_k-j_k|,\, j_{k+1}\in\{1,2\}}}^{2} \frac{d_{(ij)_{k+1}}^{(ij)_0} z}{{}^nF_{(ij)_{k+1}}^{(ij)_0}(z)}, \quad (ij)_k \in \mathfrak{I}^{(ij)_0},\ 1 \leq k \leq n-1,\ n \geq 2. \tag{26}$$

and, therefore, for $n \geq 1$,

$$ f_n^{(ij)_0}(z) = 1 + \sum_{\substack{i_1=1+\delta_{i_0}^1 \\ |i_1-j_1| \neq |i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \frac{d_{(ij)_1}^{(ij)_0} z}{{}^nF_{(ij)_1}^{(ij)_0}(z)}, $$

where $f_n^{(ij)_0}(z)$ denotes the nth approximant of the branched continued fraction (2).
We set

$$ z = \rho e^{i\varphi}, \quad \rho = |z|. \tag{27}$$

Let n be an arbitrary natural number and z be an arbitrary fixed point in (12). By induction on k, $1 \leq k \leq n$, for $(ij)_k \in \mathfrak{I}^{(ij)_0}$, we prove that

$$ \mathrm{Re}({}^nF_{(ij)_k}^{(ij)_0}(z) e^{-i\varphi/2}) > (1 - g_{(ij)_k}^{(ij)_0}) \cos(\varphi/2) \geq c > 0. \tag{28}$$

Note that from an arbitrary fixed point z, $z \in \Omega_\kappa$, it follows that for anywhere in its neighborhood, there exists a positive number δ such that $0 < \delta \leq \pi/2$, $|\varphi/2| \leq \pi/2 - \delta$, and, therefore,

$$\begin{aligned}(1 - g_{(ij)_k}^{(ij)_0}) \cos(\varphi/2) &\geq (1 - g_{(ij)_k}^{(ij)_0}) \cos(\pi/2 - \delta) \\ &= (1 - g_{(ij)_k}^{(ij)_0}) \sin(\delta) \\ &= c \\ &> 0.\end{aligned}$$

From (25), it is clear that for $k = n$ and for $(ij)_k \in \mathfrak{I}^{(ij)_0}$, the inequalities (28) hold. By the induction hypothesis that (28) holds for $k = r+1$ and $(ij)_{r+1} \in \mathfrak{I}^{(ij)_0}$ such that $r+1 \leq n$, we show (28) for $k = r$ and $(ij)_r \in \mathfrak{I}^{(ij)_0}$. The use of (26) and (27) for $(ij)_r \in \mathfrak{I}^{(ij)_0}$ leads to

$${}^{n}F_{(ij)_r}^{(ij)_0}(z) e^{-\varphi/2} = e^{-\varphi/2} + \sum_{\substack{i_{r+1}=1+\delta_{i_r}^1 \\ |i_{r+1}-j_{r+1}| \neq |i_r - j_r|,\, j_{r+1} \in \{1,2\}}}^{2} \frac{d_{(ij)_{r+1}}^{(ij)_0} \rho}{{}^{n}F_{(ij)_{r+1}}^{(ij)_0}(z) e^{-\varphi/2}}.$$

Then, for an arbitrary i_{r+1}, $1 + \delta_{i_r}^1 \leq i_{r+1} \leq 2$, $|i_{r+1} - j_{r+1}| \neq |i_r - j_r|$, $j_{r+1} \in \{1,2\}$, it follows from (9) that

$$|d_{(ij)_{r+1}}^{(ij)_0}| - \mathrm{Re}(d_{(ij)_{r+1}}^{(ij)_0}) \leq \kappa g_{(ij)_r}^{(ij)_0} (1 - g_{(ij)_{r+1}}^{(ij)_0}).$$

From this inequality, it is easy to show that

$$\left(\mathrm{Im}(d_{(ij)_{r+1}}^{(ij)_0} \rho)\right)^2 \leq 4\,\mathrm{Re}(d_{(ij)_{r+1}}^{(ij)_0} \rho) + 4,$$

since from (10) and (12), it follows that

$$\rho g_{(ij)_r}^{(ij)_0} (1 - g_{(ij)_{r+1}}^{(ij)_0}) < \frac{2}{\kappa}.$$

Now, using ([32], Corollary 2), (9), (10), (12), and the induction hypothesis, we obtain

$$\begin{aligned}\mathrm{Re}({}^{n}F_{(ij)_k}^{(ij)_0}(z) e^{-i\varphi/2} &\geq \cos(\varphi/2) - \frac{\rho}{2} \sum_{\substack{i_{r+1}=1+\delta_{i_r}^1 \\ |i_{r+1}-j_{r+1}| \neq |i_r - j_r|,\, j_{r+1} \in \{1,2\}}}^{2} \frac{|d_{(ij)_{r+1}}^{(ij)_0}| - \mathrm{Re}(d_{(ij)_{r+1}}^{(ij)_0})}{\mathrm{Re}({}^{n}F_{(ij)_{r+1}}^{(ij)_0}(z) e^{-\varphi/2})} \\ &> \cos(\varphi/2) - \frac{\cos(\varphi/2)}{\kappa} \sum_{\substack{i_{r+1}=1+\delta_{i_r}^1 \\ |i_{r+1}-j_{r+1}| \neq |i_r - j_r|,\, j_{r+1} \in \{1,2\}}}^{2} \frac{|d_{(ij)_{r+1}}^{(ij)_0}| - \mathrm{Re}(d_{(ij)_{r+1}}^{(ij)_0})}{1 - g_{(ij)_{r+1}}^{(ij)_0}} \\ &\geq \cos(\varphi/2) - g_{(ij)_r}^{(ij)_0} \cos(\varphi/2) \\ &= (1 - g_{(ij)_r}^{(ij)_0}) \cos(\varphi/2).\end{aligned}$$

It follows from (28) that ${}^{n}F_{(ij)_1}^{(ij)_0}(z) \neq 0$ for $(ij)_1 \in \mathfrak{I}^{(ij)_0}$, $n \geq 1$, and $z \in \Omega_\kappa$. Thus, the approximants $f_n^{(ij)_0}(z)$, $n \geq 1$, of (2) form a sequence of holomorphic functions in Ω_κ.

Using (28), we obtain, for $z \in \Omega_\kappa$ and $n \geq 1$,

$$
\begin{aligned}
|f_n^{(ij)_0}(z)| &\leq 1 + \sum_{\substack{i_1=1+\delta_{i_0}^1 \\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \frac{d_{(ij)_1}^{(ij)_0}|z|}{\mathrm{Re}({}^nF_{(ij)_1}^{(ij)_0}(z)e^{-i\varphi/2})} \\
&< 1 + \sum_{\substack{i_1=1+\delta_{i_0}^1 \\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \frac{2d_{(ij)_1}^{(ij)_0}}{(1-g_{(ij)_1}^{(ij)_0})\kappa\cos(\varphi/2)} \\
&= C(\Omega_\kappa),
\end{aligned}
$$

i.e., the sequence $\{f_n^{(ij)_0}(z)\}$ is uniformly bounded on the domain (12) and, at the same time, is uniformly bounded on every compact subset of this domain.

Let

$$L = \min\left\{\frac{1}{2\tau}, \frac{1}{\kappa}\right\},$$

and assume that the domain

$$\Xi_R = \{\mathbf{z} \in \mathbb{R}:\ 0 < z < R < L\}$$

is contained in Ω_κ for each $0 < R < 1/L$, in particular $\Xi_{1/(2L)} \subset \Omega_\kappa$. Using (12)–(14), it is easily shown that for arbitrary $z \in \Xi_R$, $\Xi_R \subset \Omega_\kappa$, the inequalities

$$|d_{(ij)_k}^{(ij)_0} z| < \tau L \leq \frac{1}{2} \leq \frac{1}{i_{k-1}} \quad (ij)_k \in \mathfrak{I}^{(ij)_0},\ k \geq 1,$$

are valid, i.e., the elements of (2) satisfy the conditions of Theorem 2, with

$$
\begin{aligned}
q_{(ij)_k}^{(ij)_0} &= \frac{1}{2}, \quad (ij)_k \in \mathfrak{I}^{(ij)_0},\ k \geq 1, \\
z_{(ij)_1}^{(ij)_0} &= 2d_{(ij)_1}^{(ij)_0} z, \quad (ij)_1 \in \mathfrak{I}^{(ij)_0}, \\
z_{(ij)_k}^{(ij)_0} &= 4d_{(ij)_k}^{(ij)_0} z, \quad (ij)_k \in \mathfrak{I}^{(ij)_0},\ k \geq 2.
\end{aligned}
$$

It follows from Theorem 2 that the branched continued fraction (2) converges in Ξ_R, $\Xi_R \subset \Omega_\kappa$. Thus, by ([33], Theorem 24.2), the convergence of this branched continued fraction is uniform on compact subsets of Ω_κ.

By Theorem 2, with

$$q_{(ij)_k}^{(ij)_0} = \frac{1}{2}, \quad (ij)_k \in \mathfrak{I}^{(ij)_0},\ k \geq 1,$$

the branched continued fraction (2) converges for $z \in \Omega_\tau$, where Ω_τ is defined by (13), and all its approximants lie in the closed disk (18) if $z \in \Omega_\tau$. It follows from ([33], Theorem 24.2) that the convergence is uniform on compact subsets of (13). Thus, this and the above prove (A).

The proof of (B) is similar to the proof of ([20], Theorem 2), hence it is omitted. □

Corollary 1. *Let $i_0 = j_0 = 1$. Suppose that α_2, α_3, β_1, and β_2 are complex numbers that satisfy the inequality*

$$\sum_{\substack{i_k=1+\delta_{i_{k-1}}^1 \\ |i_k-j_k|\neq|i_{k-1}-j_{k-1}|,\, j_k\in\{1,2\}}}^{2} \frac{|d_{(ij)_k}^{(ij)_0}| - \mathrm{Re}(d_{(ij)_k}^{(ij)_0})}{g_{(ij)_{k-1}}^{(ij)_0}(1-g_{(ij)_k}^{(ij)_0})} \leq \kappa, \quad (ij)_k \in \mathfrak{I}^{(ij)_0},\ k \geq 1,$$

where $d^{(ij)_0}_{(ij)_k}$, $(ij)_k \in \mathfrak{I}^{(ij)_0}$, $k \geq 1$ *are defined by (3)–(8), where* $\alpha_1 = 0$, β_1 *is replaced by* $\beta_1 - 1$ *and* $\beta_1 \notin \{1, 0, -1, -2, \ldots\}$, $\beta_2 \notin \{0, -1, -2, \ldots\}$, *and* κ *is a positive number, and* $g^{(ij)_0}_{(ij)_0}$, $g^{(ij)_0}_{(ij)_k}$, $(ij)_k \in \mathfrak{I}^{(ij)_0}$, $k \geq 1$ *are the real numbers, such that*

$$0 < g^{(ij)_0}_{(ij)_0} < 1, \quad 0 < g^{(ij)_0}_{(ij)_k} < 1, \quad (ij)_k \in \mathfrak{I}^{(ij)_0},\ k \geq 1.$$

Then, the branched continued fraction

$$\cfrac{1}{1 + \displaystyle\sum_{\substack{i_1=1+\delta^1_{i_0} \\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \cfrac{d^{(ij)_0}_{(ij)_1} z}{1 + \displaystyle\sum_{\substack{i_2=1+\delta^1_{i_1} \\ |i_2-j_2|\neq|i_1-j_1|,\, j_2\in\{1,2\}}}^{2} \cfrac{d^{(ij)_0}_{(ij)_2} z}{1 + \ddots + \displaystyle\sum_{\substack{i_k=1+\delta^1_{i_{k-1}} \\ |i_k-j_k|\neq|i_{k-1}-j_{k-1}|,\, j_k\in\{1,2\}}}^{2} \cfrac{d^{(ij)_0}_{(ij)_k} z}{1 + \ddots}}}} \tag{29}$$

converges uniformly on every compact subset of (11) to the function $f(\mathbf{z})$, *holomorphic in (11); in addition,* $f(\mathbf{z})$ *is an analytic continuation of the function* ${}_3F_2(1, \alpha_2, \alpha_3; \beta_1, \beta_2; -z)$ *in (11).*

Note that the similar consequences are valid if

(i) $i_0 = 1$, $j_0 = 2$, $\alpha_2 = 0$ and β_2 is replaced by $\beta_2 - 1$;
(ii) $i_0 = 2$, $j_0 = 1$, $\alpha_2 = 0$ (or $\alpha_3 = 0$) and α_3 (or α_2), β_1, β_2 are replaced by $\alpha_3 - 1$ (or $\alpha_2 - 1$), $\beta_1 - 1$, $\beta_2 - 1$, respectively;
(iii) $i_0 = 2$, $j_0 = 2$, $\alpha_1 = 0$ (or $\alpha_3 = 0$) and α_1 (or α_3), β_1, β_2 are replaced by $\alpha_1 - 1$ (or $\alpha_3 - 1$), $\beta_1 - 1$, $\beta_2 - 1$, respectively.

The following result is a generalization of ([20], Theorem 2):

Theorem 3. *Let* $(ij)_0$ *be an arbitrary pair in* $\mathfrak{I}$. *Let (1) be a generalized hypergeometric function with parameters such that*

$$0 < d^{(ij)_0}_{(ij)_k} \leq \tau, \quad (ij)_k \in \mathfrak{I}^{(ij)_0},\ k \geq 2,$$

where $d^{(ij)_0}_{(ij)_k}$, $(ij)_k \in \mathfrak{I}^{(ij)_0}$, $k \geq 2$ *are defined by (3)–(8),* $\beta_1, \beta_2 \notin \{0, -1, -2, \ldots\}$, τ *is a positive number. Then,*

(A) The branched continued fraction (2) converges uniformly on every compact subset of the domain

$$\Pi_\tau = \left\{ z \in \mathbb{C} : \ z \notin \left(-\infty, -\frac{1}{8\tau} \right] \right\} \tag{30}$$

to the function $f(z)$ *holomorphic in* Π_τ;

(B) The function $f(z)$ *is an analytic continuation of the function on the left side of (2) in (30).*

Proof. If

$$g^{(ij)_0}_{(ij)_k} = \frac{1}{2}, \quad (ij)_k \in \mathfrak{I}^{(ij)_0},\ k \geq 1,$$

and

$$d^{(ij)_0}_{(ij)_k} > 0, \quad (ij)_k \in \mathfrak{I}^{(ij)_0},\ k \geq 2,$$

then the condition (9) holds for all $\kappa > 0$. Let Υ be an arbitrary compact subset of (30). Then, $\Upsilon \subseteq \Omega_{\kappa,\tau} \subseteq \Pi_\tau$ for some sufficiently small κ whose $\Omega_{\kappa,\tau}$ is the domain (11). Thus, Theorem 3 is a direct consequence of Theorem 2. □

Corollary 2. *Let $i_0 = j_0 = 1$. Suppose that α_2, α_3, β_1, and β_2 are complex numbers that*

$$0 < d^{(ij)_0}_{(ij)_k} \le \tau, \quad (ij)_k \in \mathfrak{I}^{(ij)_0},\ k \ge 1,$$

where $d^{(ij)_0}_{(ij)_k}$, $(ij)_k \in \mathfrak{I}^{(ij)_0}$, $k \ge 1$ are defined by (3)–(8), where $\alpha_1 = 0$, β_1 is replaced by $\beta_1 - 1$, and $\beta_1 \notin \{1, 0, -1, -2, \ldots\}$, $\beta_2 \notin \{0, -1, -2, \ldots\}$, τ is a positive number. Then, the branched continued fraction (29) converges uniformly on every compact subset of (30) to the function $f(\mathbf{z})$ holomorphic in (30); in addition, $f(\mathbf{z})$ is an analytic continuation of the function ${}_3F_2(1, \alpha_2, \alpha_3; \beta_1, \beta_2; -z)$ in (30).

Note that similar consequences also hold for cases (i)–(iii).

3. Examples

Consider the dilogarithm function (see, for example, ref. [34]):

$$\begin{aligned}\mathrm{Li}_2(z) &= z\,{}_3F_2(1,1,1;2,2;z)\\ &= \sum_{k=1}^{\infty} \frac{z^k}{k^2}.\end{aligned}$$

It follows from Corollary 2 that the branched continued fraction

$$\cfrac{z}{1 - \cfrac{c^{1,1}_{2,1} z}{1 - \cfrac{c^{1,1}_{2,1,1,1} z}{1 - \cfrac{c^{1,1}_{2,1,1,1,2,1} z}{1 - \ddots}} - \cfrac{c^{1,1}_{2,1,2,2} z}{1 - \cfrac{c^{1,1}_{2,1,2,2,1,2} z}{1 - \ddots} - \cfrac{c^{1,1}_{2,1,2,2,2,1} z}{1 - \ddots}}}},$$

where $d^{1,1}_{(ij)_k}$, $(ij)_k \in \mathfrak{I}_{1,1}$, defined by Formulas (3)–(8), where $i_0 = j_0 = 1$, $\alpha_1 = 0$, and β_1 is replaced by $\beta_1 - 1$, is an analytic continuation of the dilogarithm function $\mathrm{Li}_2(z)$ in the domain

$$\Xi_\tau = \left\{ z \in \mathbb{C} :\ z \notin \left[\frac{1}{8\tau}, +\infty\right) \right\},$$

where τ is defined by (14).

In ([35], Formula 7.4.3.17a), it is given that

$$\begin{aligned}\operatorname{arcsinh}^2 \sqrt{z} &= z\,{}_3F_2(1,1,1;3/2,2;-z)\\ &= \sum_{n=0}^{\infty} \frac{(-1)^n ((1)_n)^3}{(3/2)_n (2)_n} \frac{z^{n+1}}{n!}.\end{aligned}$$

(here, the principal branch of the square root is assumed). Thus, by Corollary 2, the branched continued fraction

$$\cfrac{z}{1 + \cfrac{z/3}{1 + \cfrac{z/15}{1 + \cfrac{2z/15}{1 + \ddots}} + \cfrac{8z/21}{1 + \cfrac{4z/75}{1 + \ddots} + \cfrac{3z/7}{1 + \ddots}}}}$$

is an analytic continuation of function $\operatorname{arcsinh}^2 \sqrt{z}$ in the domain

$$\Pi = \{ z \in \mathbb{C} :\ |\arg(z)| < \pi \}.$$

4. Discussion and Conclusions

The article considers the generalized hypergeometric functions in $\mathbb{C}$, which include the Gaussian hypergeometric function [36] and its confluent cases, which in turn have many special functions such as elementary functions, Bessel functions, and classical orthogonal polynomials. The main examples are the ratios of generalized hypergeometric functions ${}_3F_2$, which have a representation in the form of the branched continued fractions [22]. We proved that these ratios in the case of complex parameters have analytical extensions in the domain that is the union of the circular and cardioid domains, and in the case of real parameters, it is a plane with a cut. In the real case, the result is a generalization ([20], Theorem 2). However, we cannot extend the ratios of generalized hypergeometric functions ${}_3F_2$ to wider domains in the case of complex parameters, while numerical experiments indicate their existence.

Further investigations can be continued in several directions. First, we can try to replace branched continued fractions with equivalent branched continued fractions with partial numerators equal to 1 and study their convergence. Note that there are other parabolic and angular domains of convergence [37–39]. However, we know almost nothing about the behavior of the partial denominators of such equivalent branched continued fractions. Another direction is the truncation error analysis. It was proved in ([20], Theorem 1) that branched continued fraction expansions converge for $z \leq 0$ at least as fast as geometric series with a ratio of q, $0 < q < 1$. Note that the obtained estimates of the rate of convergence in [40–43] can be applied to branched continued fraction expansions in which ratios of generalized hypergeometric functions ${}_3F_2$ are expanded. In [44], a new approach to the study of the numerical stability of branched continued fractions is proposed, which is an important direction in the aspect of computations. We can also study other functions, including discrete matrix hypergeometric functions [45]. In [17], possible applications of the generalized hypergeometric function ${}_3F_2$ in the Coulomb interaction of the system of spinless fermions were considered. Our further investigation will be devoted to the development of this approach for the above-mentioned expansions.

5. Formulas of the Coefficients of the Branched Continued Fraction Expansions

The process of obtaining the explicit Formulas (2)–(8) begins with with four three-term recurrence relations (for details, see refs. [20,21]),

$$
\begin{aligned}
{}_3F_2(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; z) &= {}_3F_2(\alpha_1 + 1, \alpha_2, \alpha_3; \beta_1 + 1, \beta_2; z) \\
&\quad - \frac{(\beta_1 - \alpha_1)\alpha_2\alpha_3}{(\beta_1 + 1)\beta_1\beta_2} z \, {}_3F_2(\alpha_1 + 1, \alpha_2 + 1, \alpha_3 + 1; \beta_1 + 2, \beta_2 + 1; z), \\
{}_3F_2(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; z) &= {}_3F_2(\alpha_1, \alpha_2 + 1, \alpha_3; \beta_1, \beta_2 + 1; z) \\
&\quad - \frac{(\beta_2 - \alpha_2)\alpha_1\alpha_3}{(\beta_2 + 1)\beta_1\beta_2} z \, {}_3F_2(\alpha_1 + 1, \alpha_2 + 1, \alpha_3 + 1; \beta_1 + 1, \beta_2 + 2; z), \\
{}_3F_2(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; z) &= {}_3F_2(\alpha_1, \alpha_2, \alpha_3 + 1; \beta_1 + 1, \beta_2; z) \\
&\quad - \frac{(\beta_1 - \alpha_3)\alpha_1\alpha_2}{(\beta_1 + 1)\beta_1\beta_2} z \, {}_3F_2(\alpha_1 + 1, \alpha_2 + 1, \alpha_3 + 1; \beta_1 + 2, \beta_2 + 1; z), \\
{}_3F_2(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; z) &= {}_3F_2(\alpha_1, \alpha_2, \alpha_3 + 1; \beta_1, \beta_2 + 1; z) \\
&\quad - \frac{(\beta_2 - \alpha_3)\alpha_1\alpha_2}{(\beta_2 + 1)\beta_1\beta_2} z \, {}_3F_2(\alpha_1 + 1, \alpha_2 + 1, \alpha_3 + 1; \beta_1 + 1, \beta_2 + 2; z),
\end{aligned}
$$

and two four-term recurrence relations,

$$\begin{aligned}
{}_3F_2(\alpha_1,\alpha_2,\alpha_3;\beta_1,\beta_2;z) &= {}_3F_2(\alpha_1,\alpha_2+1,\alpha_3+1;\beta_1+1,\beta_2+1;z) \\
&\quad - \frac{(\beta_1-\alpha_3)\alpha_1\alpha_2}{(\beta_1+1)\beta_1\beta_2} z\, {}_3F_2(\alpha_1+1,\alpha_2+1,\alpha_3+1;\beta_1+2,\beta_2+1;z) \\
&\quad - \frac{(\beta_2-\alpha_2)(\alpha_3+1)\alpha_1}{(\beta_1+1)(\beta_2+1)\beta_2} z\, {}_3F_2(\alpha_1+1,\alpha_2+1,\alpha_3+2;\beta_1+2,\beta_2+2;z), \\
{}_3F_2(\alpha_1,\alpha_2,\alpha_3;\beta_1,\beta_2;z) &= {}_3F_2(\alpha_1+1,\alpha_2,\alpha_3+1;\beta_1+1,\beta_2+1;z) \\
&\quad - \frac{(\beta_2-\alpha_3)\alpha_1\alpha_2}{(\beta_2+1)\beta_1\beta_2} z\, {}_3F_2(\alpha_1+1,\alpha_2+1,\alpha_3+1;\beta_1+1,\beta_2+2;z) \\
&\quad - \frac{(\beta_1-\alpha_1)(\alpha_3+1)\alpha_2}{(\beta_1+1)(\beta_2+1)\beta_2} z\, {}_3F_2(\alpha_1+1,\alpha_2+1,\alpha_3+2;\beta_1+2,\beta_2+2;z).
\end{aligned}$$

Further, using these formulas, one obtains the following four relations:

$$\begin{aligned}
\frac{{}_3F_2(\alpha_1,\alpha_2,\alpha_3;\beta_1,\beta_2;z)}{{}_3F_2(\alpha_1+1,\alpha_2,\alpha_3;\beta_1+1,\beta_2;z)} &= 1 - \frac{\dfrac{(\beta_1-\alpha_1)\alpha_2\alpha_3}{(\beta_1+1)\beta_1\beta_2}z}{\dfrac{{}_3F_2(\alpha_1+1,\alpha_2,\alpha_3;\beta_1+1,\beta_2;z)}{{}_3F_2(\alpha_1+1,\alpha_2+1,\alpha_3+1;\beta_1+2,\beta_2+1;z)}}, \\
\frac{{}_3F_2(\alpha_1,\alpha_2,\alpha_3;\beta_1,\beta_2;z)}{{}_3F_2(\alpha_1,\alpha_2+1,\alpha_3;\beta_1,\beta_2+1;z)} &= 1 - \frac{\dfrac{(\beta_2-\alpha_2)\alpha_1\alpha_3}{(\beta_2+1)\beta_1\beta_2}z}{\dfrac{{}_3F_2(\alpha_1,\alpha_2+1,\alpha_3;\beta_1,\beta_2+1;z)}{{}_3F_2(\alpha_1+1,\alpha_2+1,\alpha_3+1;\beta_1+1,\beta_2+2;z)}},
\end{aligned}$$

and

$$\begin{aligned}
\frac{{}_3F_2(\alpha_1,\alpha_2,\alpha_3;\beta_1,\beta_2;z)}{{}_3F_2(\alpha_1,\alpha_2+1,\alpha_3+1;\beta_1+1,\beta_2+1;z)} &= 1 - \frac{\dfrac{(\beta_1-\alpha_3)\alpha_1\alpha_2}{(\beta_1+1)\beta_1\beta_2}z}{\dfrac{{}_3F_2(\alpha_1,\alpha_2+1,\alpha_3+1;\beta_1+1,\beta_2+1;z)}{{}_3F_2(\alpha_1+1,\alpha_2+1,\alpha_3+1;\beta_1+2,\beta_2+1;z)}} \\
&\quad - \frac{\dfrac{(\beta_2-\alpha_2)(\alpha_3+1)\alpha_1}{(\beta_1+1)(\beta_2+1)\beta_2}z}{\dfrac{{}_3F_2(\alpha_1,\alpha_2+1,\alpha_3+1;\beta_1+1,\beta_2+1;z)}{{}_3F_2(\alpha_1+1,\alpha_2+1,\alpha_3+2;\beta_1+2,\beta_2+2;z)}}, \\
\frac{{}_3F_2(\alpha_1,\alpha_2,\alpha_3;\beta_1,\beta_2;z)}{{}_3F_2(\alpha_1+1,\alpha_2,\alpha_3+1;\beta_1+1,\beta_2+1;z)} &= 1 - \frac{\dfrac{(\beta_2-\alpha_3)\alpha_1\alpha_2}{(\beta_2+1)\beta_1\beta_2}z}{\dfrac{{}_3F_2(\alpha_1+1,\alpha_2,\alpha_3+1;\beta_1+1,\beta_2+1;z)}{{}_3F_2(\alpha_1+1,\alpha_2+1,\alpha_3+1;\beta_1+1,\beta_2+2;z)}} \\
&\quad - \frac{\dfrac{(\beta_1-\alpha_1)(\alpha_3+1)\alpha_2}{(\beta_1+1)(\beta_2+1)\beta_1}z}{\dfrac{{}_3F_2(\alpha_1+1,\alpha_2,\alpha_3+1;\beta_1+1,\beta_2+1;z)}{{}_3F_2(\alpha_1+1,\alpha_2+1,\alpha_3+2;\beta_1+2,\beta_2+2;z)}}.
\end{aligned}$$

Let, for $(ij)_0 \in \mathfrak{I}$, $(ij)_k \in \mathfrak{I}_{(ij)_0}$ and $k \geq 1$,

$$\mathfrak{a}^{(ij)_0}_{(ij)_{k-1}} = \left(\alpha_1 + \sum_{p=0}^{k-1}(\delta^1_{i_p}\delta^1_{j_p} + \delta^2_{i_p}\delta^2_{j_p}), \alpha_2 + \sum_{p=0}^{k-1}(\delta^1_{i_p}\delta^2_{j_p} + \delta^2_{i_p}\delta^1_{j_p}), \alpha_3 + \sum_{p=0}^{k-1}\delta^2_{i_p}\right),$$

and

$$\mathfrak{b}^{(ij)_0}_{(ij)_{k-1}} = \left(\beta_1 + \sum_{p=0}^{k-1}(\delta^1_{i_p}\delta^1_{j_p} + \delta^2_{i_p}), \beta_2 + \sum_{p=0}^{k-1}(\delta^1_{i_p}\delta^2_{j_p} + \delta^2_{i_p})\right).$$

Then, these relations can be written as follows:

$$\frac{{}_3F_2(\alpha_1,\alpha_2,\alpha_3;\beta_1,\beta_2;z)}{{}_3F_2(\mathfrak{a}^{(ij)_0}_{(ij)_0};\mathfrak{b}^{(ij)_0}_{(ij)_0};z)} = 1 - \sum_{\substack{i_1=1+\delta^1_{i_0} \\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \frac{d^{(ij)_0}_{(ij)_1} z}{\dfrac{{}_3F_2(\mathfrak{a}^{(ij)_0}_{(ij)_0};\mathfrak{b}^{(ij)_0}_{(ij)_0};z)}{{}_3F_2(\mathfrak{a}^{(ij)_0}_{(ij)_1};\mathfrak{b}^{(ij)_0}_{(ij)_1};z)}},$$

where $(ij)_0 \in \mathfrak{I}$, $d^{(ij)_0}_{(ij)_1}$, $(ij)_1 \in \mathcal{I}_{(ij)_0}$, $(ij)_0 \in \mathcal{I}$ are defined by Formulas (3)–(8) for $k = 1$.

By analogy, it is clear that for $k \geq 2$ and $(ij)_{k-1} \in \mathcal{I}_{(ij)_0}$, $(ij)_0 \in \mathcal{I}$, the following recurrence relation holds:

$$\frac{{}_3F_2(\mathfrak{a}^{(ij)_0}_{(ij)_{k-2}};\mathfrak{b}^{(ij)_0}_{(ij)_{k-2}};z)}{{}_3F_2(\mathfrak{a}^{(ij)_0}_{(ij)_{k-1}};\mathfrak{b}^{(ij)_0}_{(ij)_{k-1}};z)} = 1 - \sum_{\substack{i_k=1+\delta^1_{i_{k-1}} \\ |i_k-j_k|\neq|i_{k-1}-j_{k-1}|,\, j_k\in\{1,2\}}}^{2} \frac{d^{(ij)_0}_{(ij)_k} z}{\dfrac{{}_3F_2(\mathfrak{a}^{(ij)_0}_{(ij)_{k-1}};\mathfrak{b}^{(ij)_0}_{(ij)_{k-1}};z)}{{}_3F_2(\mathfrak{a}^{(ij)_0}_{(ij)_k};\mathfrak{b}^{(ij)_0}_{(ij)_k};z)}},$$

where $d^{(ij)_0}_{(ij)_k}$, $(ij)_k \in \mathcal{I}_{(ij)_0}$, $(ij)_0 \in \mathcal{I}$, $k \geq 2$ are defined by Formulas (3)–(8).

Note that the explicit formulas for the vectors $\mathfrak{a}^{(ij)_0}_{(ij)_{k-1}}$, $\mathfrak{b}^{(ij)_0}_{(ij)_{k-1}}$, $(ij)_0 \in \mathcal{I}$, $(ij)_k \in \mathcal{I}_{(ij)_0}$, $k \geq 1$ and the coefficients $d^{(ij)_0}_{(ij)_k}$, $(ij)_0 \in \mathcal{I}$, $(ij)_k \in \mathcal{I}_{(ij)_0}$, $k \geq 1$ of the branched continued fraction expansions (2) are obtained by selecting Kronecker symbols without using computer algebra or artificial intelligence.

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