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Optimal Control of Microcephaly Under Vertical Transmission of Zika

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Abstract: The Zika virus, known for its potential to induce neurological conditions such as microcephaly when transmitted vertically from infected mothers to infants, has sparked widespread concerns globally. Motivated by this, we propose an optimal control problem for the prevention of vertical Zika transmission. The novelty of this study lies in its consideration of time-dependent control functions, namely, insecticide spraying and personal protective measures taken to safeguard pregnant women from infected mosquitoes. New results provide a way to minimize the number of infected pregnant women through the implementation of control strategies while simultaneously reducing both the associated costs of control measures and the mosquito population, resulting in a decline in microcephaly cases.

Keywords: optimal control; vertical transmission; vector-borne diseases; Zika virus; microcephaly

MSC: 49M05; 92D30

Citation: Yapışkan, D.; Silva, C.J.; Torres, D.F.M. Optimal Control of Microcephaly Under Vertical Transmission of Zika. *Axioms* **2024**, *13*, 772. https://doi.org/10.3390/axioms13110772

Academic Editor: Valery Y. Glizer

Received: 23 May 2024
Revised: 22 October 2024
Accepted: 5 November 2024
Published: 6 November 2024

1. Introduction

Zika virus is a mosquito-borne disease mainly transmitted to humans through the bite of female mosquitoes *Aedes aegypti*. It presents a serious threat to public health due to its vertical transmission from pregnant women to their babies, potentially resulting in heightened occurrences of neurological disorders like microcephaly [1,2]. In February 2016, the World Health Organization (WHO) declared Zika-related microcephaly a Public Health Emergency of International Concern (PHEIC), confirming the causal link between the Zika virus and congenital malformations. The PHEIC was declared to have concluded by the WHO in November of the same year. Despite a global decline in cases of Zika virus disease from 2017 onwards, transmission continues at low levels in various countries across the Americas and other endemic regions [3].

Mathematical modeling of infectious diseases is not only an important tool in understanding the dynamics of the disease but also contributes to the process of taking necessary measures to prevent disease transmission [4–6]. Researchers have examined the vertical transmission of Zika as well as the development of microcephaly in newborn babies (for example, [7–9]). Additionally, controlled mathematical models to identify the crucial characteristics causing the transmission may be more effective in predicting the future course of the epidemic and preventing transmission [10–13]. Therefore, various perspectives on optimal control strategies for Zika transmission dynamics have been discussed in the literature [7,8,14–16].

Our study proposes an optimal control problem for the prevention of vertical Zika transmission, which is a significant public health concern. Moreover, the proposed optimal control problem differs from all the others found in the literature on Zika transmission. Indeed, we consider the uncontrolled Zika transmission model in Brazil [9] and introduce time-dependent control functions representing personal protection and insecticide spraying. The objective is to minimize the number of infected pregnant women through the implementation of control strategies while simultaneously reducing the associated costs of these control measures.

This paper is organized as follows: in Section 2, we introduce the uncontrolled Zika model; in Section 3, we formulate the optimal control problem and derive the optimality system using the Pontryagin maximum principle; in Section 4, our focus is on conducting numerical simulations to showcase the impacts of optimal control strategies; and finally, we conclude with Section 5.

2. Uncontrolled Zika Model

In this section, we recall the main assumptions of the mathematical model for the spread of Zika virus as proposed in [9]. The model considers women as the population under study. The total women population, given by N, is subdivided into four mutually exclusive compartments, according to disease status: susceptible pregnant women (S); infected pregnant women (I); women who gave birth to babies without a neurological disorder (W); and women who gave birth to babies with a neurological disorder due to microcephaly (M).

As for the mosquitoes population, there are four state variables related to the (female) mosquitoes: A_m, which corresponds to the aquatic phase, which includes the egg, larva, and pupa stages; S_m, for the mosquitoes that might contract the disease (susceptible); E_m, for the mosquitoes that are infected but are not able to transmit the Zika virus to humans (exposed); and I_m, for the mosquitoes capable of transmitting the Zika virus to humans (infected).

The following assumptions are considered in our model:

(A.1) There is no immigration of infected humans;
(A.2) The total human populations N is constant;
(A.3) The coefficient of transmission of Zika virus is constant and does not vary with seasons;
(A.4) After giving birth, pregnant women are no longer pregnant and they leave the population under study at a rate μ_h equal to the rate of humans birth;
(A.5) Death is neglected, as the period of pregnancy is much smaller than the mean humans lifespan;
(A.6) There is no resistant phase for the mosquito due to its short lifetime.

Note that male mosquitoes are not considered in this study because they do not bite humans and consequently they do not influence the dynamics of the disease. The differential system that describes the model is composed of pregnant women and women who gave birth:

$$\begin{cases} S'(t) = \Lambda - \left(\phi B \beta_{mh} \dfrac{I_m(t)}{N} + (1-\phi)\tau_1 + \mu_h\right) S(t), \\ I'(t) = \phi B \beta_{mh} \dfrac{I_m(t)}{N} S(t) - (\tau_2 + \mu_h) I(t), \\ W'(t) = (1-\phi)\tau_1 S(t) + (1-\psi)\tau_2 I(t) - \mu_h W(t), \\ M'(t) = \psi \tau_2 I(t) - \mu_h M(t), \end{cases} \tag{1}$$

where $N = S(t) + I(t) + W(t) + M(t)$ is the total population (women), with $t \in [0, t_f]$. The parameter Λ denotes the new pregnant women per week, ϕ stands for the fraction of susceptible pregnant women that become infected, B is the average daily biting (per day), β_{mh} represents the transmission probability from infected mosquitoes I_m (per bite), τ_1 is the rate at which susceptible pregnant women S give birth (in weeks), τ_2 is the rate at which infected pregnant women I give birth (in weeks), μ_h is the natural death rate for pregnant women, and ψ denotes the fraction of infected pregnant women I that give birth to babies with a neurological disorder due to microcephaly. The above system (1) is coupled with the dynamics of the mosquitoes:

$$\begin{cases} A'_m(t) = \mu_b\left(1 - \dfrac{A_m(t)}{K}\right)(S_m(t) + E_m(t) + I_m(t)) - (\mu_A + \eta_A)A_m(t), \\ S'_m(t) = \eta_A A_m(t) - \left(B\beta_{hm}\dfrac{I(t)}{N} + \mu_m\right)S_m(t), \\ E'_m(t) = \left(B\beta_{hm}\dfrac{I(t)}{N}\right)S_m(t) - (\eta_m + \mu_m)E_m(t), \\ I'_m(t) = \eta_m E_m(t) - \mu_m I_m(t), \end{cases} \tag{2}$$

where parameter β_{hm} represents the transmission probability from infected humans I_h (per bite), μ_b stands for the number of eggs at each deposit per capita (per day), μ_A is the natural mortality rate of larvae (per day), η_A is the maturation rate from larvae to adult (per day), $1/\eta_m$ represents the extrinsic incubation period (in days), $1/\mu_m$ denotes the average lifespan of adult mosquitoes (in days), and K is the maximal capacity of larvae. See Table 1 for the description of the state variables and parameters of the Zika model (1)–(2).

Table 1. Variables and parameters of the Zika model (1)–(2), as given in [9].

Variable/Symbol	Description
$S(t)$	susceptible pregnant women
$I(t)$	infected pregnant women
$W(t)$	women who gave birth to babies without a neurological disorder
$M(t)$	women who gave birth to babies with a neurological disorder due to microcephaly
$A_m(t)$	mosquitoes in the aquatic phase
$S_m(t)$	susceptible mosquitoes
$E_m(t)$	exposed mosquitoes
$I_m(t)$	infected mosquitoes
Λ	new pregnant women (per week)
ϕ	fraction of S that become infected
B	average daily biting (per day)
β_{mh}	transmission probability from I_m (per bite)
τ_1	rate at which S give birth (in weeks)
τ_2	rate at which I give birth (in weeks)
μ_h	natural death rate
ψ	fraction of I that gives birth to babies with a neurological disorder
β_{hm}	transmission probability from I_h (per bite)
μ_b	number of eggs at each deposit per capita (per day)
μ_A	natural mortality rate of larvae (per day)
η_A	maturation rate from larvae to adult (per day)
$1/\eta_m$	extrinsic incubation period (in days)
$1/\mu_m$	average lifespan of adult mosquitoes (in days)
K	maximal capacity of larvae

We consider system (1)–(2) with given initial conditions:

$$\begin{aligned} &S(0) = S_0, \quad I(0) = I_0, \quad W(0) = W_0, \quad M(0) = M_0, \\ &A_m(0) = A_{m0}, \quad S_m(0) = S_{m0}, \quad E_m(0) = E_{m0}, \quad I_m(0) = I_{m0}, \end{aligned}$$

with $(S_0, I_0, W_0, M_0, A_{m0}, S_{m0}, E_{m0}, I_{m0}) \in \mathbb{R}^8_+$. In what follows, we assume $\beta_{mh} = \beta_{hm}$.

The positivity and boundedness of solutions, as well as the existence and stability analysis of both disease-free and endemic equilibria, were studied in [9]. Additionally, the basic reproduction number and its sensitivity was also analyzed in [9]. Considering the data collected by the WHO between 4 February 2016 and 10 November 2016 in Brazil, the authors of [9] concluded that the parameters most sensitive to interventions are B and β_{mh}. Hence, to mitigate the transmission of the Zika virus, it is imperative to implement control measures aimed at reducing the number of daily mosquito bites, B, and the transmission probability from the infected mosquitoes, β_{mh}. Furthermore, the fraction ϕ, representing susceptible pregnant women (S) who contract the virus, exhibits a sensitivity index very close to $+1$. This underscores the critical importance of preventive measures aimed at safeguarding susceptible pregnant women from infection.

These conclusions encourage us to pursue the identification of optimal strategies for mitigating the transmission of the Zika virus. In the upcoming section, we address this by introducing an optimal control problem.

3. Optimal Control Problem

In this section, we formulate an optimal control problem for Zika transmission. Our objective is to minimize the number of infected pregnant women, reduce the mosquito population, and minimize the cost associated with the implementation of the control measures.

According to the Centers for Disease Control and Prevention, the best way to prevent Zika is to be protected from mosquito bites [17]. Moreover, everyone, including pregnant and breastfeeding women, should take steps to prevent mosquito bites [17]. When used as directed, EPA-registered insect repellents are proven safe and effective, even for pregnant and breastfeeding women. Taking this into account, we propose a controlled model by introducing in the Zika model (1)–(2) two control functions $u_1(\cdot)$ and $u_2(\cdot)$. The control u_1 represents protective clothing, insect repellent, and bed-nets to protect pregnant women from infected mosquitoes; while control u_2 refers to the insecticide spray applied to the mosquito population. The dynamical control system for Zika transmission that we propose is then given by

$$\begin{cases} S'(t) = \Lambda - (1-u_1(t))\phi B\beta_{mh}\frac{I_m}{N}S(t) - (1-\phi)\tau_1 S(t) - \mu_h S(t), \\ I'(t) = (1-u_1(t))\phi B\beta_{mh}\frac{I_m(t)}{N}S(t) - (\tau_2+\mu_h)I(t), \\ W'(t) = (1-\phi)\tau_1 S(t) + (1-\psi)\tau_2 I(t) - \mu_h W(t), \\ M'(t) = \psi\tau_2 I - \mu_h M(t), \\ A'_m(t) = \mu_b\left(1 - \frac{A_m(t)}{K}\right)(S_m(t)+E_m(t)+I_m(t)) - (\mu_A+\eta_A)A_m(t), \\ S'_m(t) = \eta_A A_m(t) - B\beta_{hm}\frac{I(t)}{N}S_m(t) + \mu_m S_m(t) - u_2(t)S_m(t), \\ E'_m(t) = B\beta_{hm}\frac{I(t)}{N}S_m(t) - (\eta_m+\mu_m)E_m(t) - u_2(t)E_m(t), \\ I'_m(t) = \eta_m E_m(t) - \mu_m I_m(t) - u_2(t)I_m(t)\,. \end{cases} \tag{3}$$

The set U of admissible control functions is defined by

$$U = \left\{ (u_1(\cdot), u_2(\cdot)) \in \left(L^\infty(0,t_f)\right)^2 \,\middle|\, 0 \le u_1(t), u_2(t) \le u_{\max} \le 0.5,\ \left[0, t_f\right] \right\}.$$

Our objective is twofold: first, to minimize the incidence of Zika virus infection among pregnant women, thereby reducing the risk of microcephaly in newborns; second, to decrease the mosquito population responsible for transmission. Achieving these goals involves implementing preventative measures while optimizing budget allocation to minimize costs. To achieve this, we consider the objective functional of the optimal control problem as follows:

$$J(I(\cdot), N_m(\cdot), u_{1,}(\cdot), u_2(\cdot)) = \int_0^{t_f} \left(w_1 I(t) + w_2 N_m(t) + w_3 u_1^2(t) + w_4 u_2^2(t) \right) dt, \tag{4}$$

where the weight coefficients w_1 and w_2 are the weights for infected pregnant women and the mosquito population, respectively. Also, the coefficients w_3 and w_4 are measures of the cost of preventive interventions related to the controls u_1 and u_2, respectively.

The optimal control problem consists of determining

$$X^* = (S^*, I^*, W^*, M^*, A_m^*, S_m^*, E_m^*, I_m^*)$$

associated with an admissible control pair $u^* = \left(u_1^*, u_2^*\right) \in U$ on the time interval $\left[0, t_f\right]$, satisfying (3), given initial conditions $S(0)$, $I(0)$, $W(0)$, $M(0)$, $A_m(0)$, $S_m(0)$, $E_m(0)$, and $I_m(0)$ and minimizing the objective functional (4), i.e.,

$$J(X^*, u^*) = \min_{(X,u)\in\mathcal{X}\times U} J(X, u), \tag{5}$$

where $\mathcal{X}$ is the set of admissible trajectories. We achieve the necessary optimality conditions with the help of Pontryagin's maximum principle (PMP) [18]. Note that the existence of optimal controls is ensured by the convexity of the integrand of the objective functional (4) with regard to the control functions (u_1, u_2) and the fact that the control system (3) satisfies a Lipschitz condition with respect to the state variables S, I, W, M, A_m, S_m, E_m, and I_m [19,20].

Theorem 1. *Let* $\left(u_1^*, u_2^*\right) \in U$ *be the optimal controls that minimize the objective functional* (4) *and* $(S^*, I^*, W^*, M^*, A_m^*, S_m^*, E_m^*, I_m^*)$ *be the optimal solution for the dynamical system* (3)*. Thus, there are costate variables* $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8)$ *satisfying*

$$\begin{cases} \lambda_1'(t) &=& (\lambda_1'(t) - \lambda_2'(t))(1 - u_1^*(t))\phi B\beta_{mh}\frac{I_m^{*\prime}(t)}{(N^{*\prime}(t))^2}(I^{*\prime}(t) + W^{*\prime}(t) + M^{*\prime}(t)) \\ &+& (\lambda_1'(t) - \lambda_3(t))(1-\phi)\tau_1 + \lambda_1(t)\mu_h, -(\lambda_6(t) - \lambda_7(t))B\beta_{hm}\frac{I(t)}{(N^*(t))^2}S_m^*(t) \\ \lambda_2'(t) &=& -w_1 + (\lambda_2(t) - \lambda_1(t))(1 - u_1^*(t))\phi B\beta_{mh}\frac{I_m^*(t)}{(N^*(t))^2}S^*(t) + \lambda_2(t)(\tau_2 + \mu_h) \\ &-& \lambda_3(t)(1-\psi)\tau_2 - \lambda_4(t)\psi\tau_2 + (\lambda_6(t) - \lambda_7(t))B\beta_{hm}\frac{(S^*(t)+W^*(t)+M^*(t))}{(N^*(t))^2}S_m^*(t) \\ \lambda_3'(t) &=& (\lambda_2(t) - \lambda_1(t))(1 - u_1^*(t))\phi B\beta_{mh}\frac{I_m^*(t)}{(N^*(t))^2}S^*(t) + \lambda_3(t)\mu_h \\ &-& (\lambda_6(t) - \lambda_7(t))B\beta_{hm}\frac{I^*(t)}{(N^*(t))^2}S_m^*(t) \\ \lambda_4'(t) &=& (\lambda_2(t) - \lambda_1(t))(1 - u_1^*(t))\phi B\beta_{mh}\frac{I_m^*(t)}{(N^*(t))^2}S^*(t) + \lambda_4(t)\mu_h \\ &-& (\lambda_6(t) - \lambda_7(t))B\beta_{hm}\frac{I}{(N^*(t))^2}S_m^*(t) \\ \lambda_5'(t) &=& -\lambda_5(t)\left(\mu_b\frac{1}{K}(S_m^*(t) + E_m^*(t) + I_m^*(t)) + (\mu_A + \eta_A)\right) - \lambda_6(t)\eta_A \\ \lambda_6'(t) &=& -w_2 - \lambda_5(t)\mu_b\left(1 - \frac{A_m^*(t)}{K}\right) + (\lambda_6(t) - \lambda_7(t))B\beta_{hm}\frac{I^*(t)}{N^*(t)} + \lambda_6(t)(\mu_m + u_2^*(t)) \\ \lambda_7'(t) &=& -w_2 - \lambda_5(t)\mu_b\left(1 - \frac{A_m^*(t)}{K}\right) + \lambda_7(t)(\eta_m + \mu_m + u_2^*(t)) - \lambda_7(t)\eta_m \\ \lambda_8'(t) &=& -w_2 + (\lambda_1(t) - \lambda_2(t))(1 - u_1^*(t))\phi B\beta_{mh}\frac{S^*(t)}{N^*(t)} - \lambda_5(t)\mu_b\left(1 - \frac{A_m^*(t)}{K}\right) \\ &+& \lambda_8(t)(\mu_m + u_2^*(t)) \end{cases} \tag{6}$$

with transversality conditions

$$\lambda_i\left(t_f\right) = 0, \;\; i = 1, 2, \ldots, 8. \tag{7}$$

In addition,

$$\begin{cases} u_1^*(t) = \min\left(\max\left(\frac{(\lambda_1(t)-\lambda_2(t))\phi B\beta_{mh}\frac{I_m^*(t)}{N^*(t)}S^*(t)}{w_3}, 0\right), u_{\max}\right), \\ u_2^*(t) = \min\left(\max\left(-\frac{\lambda_6 S_m^*(t)+\lambda_7 I_m^*(t)+\lambda_8 E_m^*(t)}{w_4}, 0\right), u_{\max}\right). \end{cases} \tag{8}$$

Proof. Using PMP [18], we obtain the necessary optimality conditions (6)–(8) that an optimal solution must provide. We introduce the Hamiltonian $\mathcal{H}$ to form the necessary optimality conditions:

$$\begin{aligned}\mathcal{H}(t,X,u,\lambda) &= w_1 I + w_2 N_m + w_3 u_1^2 + w_4 u_2^2\\ &+ \lambda_1\Big(\Lambda-(1-u_1)\phi B\beta_{mh}\tfrac{I_m}{N}S-(1-\phi)\tau_1 S-\mu_h S\Big)\\ &+ \lambda_2\Big((1-u_1)\phi B\beta_{mh}\tfrac{I_m}{N}S-(\tau_2+\mu_h)I\Big)\\ &+ \lambda_3((1-\phi)\tau_1 S+(1-\psi)\tau_2 I-\mu_h W)\\ &+ \lambda_4(\psi\tau_2 I-\mu_h M)\\ &+ \lambda_5\Big(\mu_b\Big(1-\tfrac{A_m}{K}\Big)(S_m+E_m+I_m)-(\mu_A+\eta_A)A_m\Big)\\ &+ \lambda_6\Big(\eta_A A_m-B\beta_{hm}\tfrac{I}{N}S_m+\mu_m S_m-u_2 S_m\Big)\\ &+ \lambda_7\Big(B\beta_{hm}\tfrac{I}{N}S_m-(\eta_m+\mu_m)E_m-u_2 E_m\Big)\\ &+ \lambda_8(\eta_m E_m-\mu_m I_m-u_2 I_m).\end{aligned} \tag{9}$$

We consider the following equations based on the Hamiltonian to obtain the necessary optimality conditions of the problem:

- State equations:

$$\begin{aligned}&\frac{dS}{dt}=\frac{\partial\mathcal{H}}{\partial\lambda_1},\ \frac{dI}{dt}=\frac{\partial\mathcal{H}}{\partial\lambda_2},\ \frac{dW}{dt}=\frac{\partial\mathcal{H}}{\partial\lambda_3},\ \frac{dM}{dt}=\frac{\partial\mathcal{H}}{\partial\lambda_4},\\ &\frac{dA_m}{dt}=\frac{\partial\mathcal{H}}{\partial\lambda_5},\ \frac{dS_m}{dt}=\frac{\partial\mathcal{H}}{\partial\lambda_6},\ \frac{dE_m}{dt}=\frac{\partial\mathcal{H}}{\partial\lambda_7},\ \frac{dI_m}{dt}=\frac{\partial\mathcal{H}}{\partial\lambda_8};\end{aligned} \tag{10}$$

- Adjoint equations:

$$\begin{aligned}&\frac{d\lambda_1}{dt}=-\frac{\partial\mathcal{H}}{\partial S},\ \frac{d\lambda_2}{dt}=-\frac{\partial\mathcal{H}}{\partial I},\ \frac{d\lambda_3}{dt}=-\frac{\partial\mathcal{H}}{\partial W},\ \frac{d\lambda_4}{dt}=-\frac{\partial\mathcal{H}}{\partial M},\\ &\frac{d\lambda_5}{dt}=-\frac{\partial\mathcal{H}}{\partial A_m},\ \frac{d\lambda_6}{dt}=-\frac{\partial\mathcal{H}}{\partial S_m},\ \frac{d\lambda_7}{dt}=-\frac{\partial\mathcal{H}}{\partial E_m},\ \frac{d\lambda_8}{dt}=-\frac{\partial\mathcal{H}}{\partial I_m},\end{aligned} \tag{11}$$

  subject to transversality conditions $\lambda_i(t_f)=0$, $i=1,2,\ldots,8$;
- Minimality condition:

$$\mathcal{H}(t,X^*(t),u^*(t),\lambda(t))=\min_{v\in[0,u_{\max}]\times[0,u_{\max}]}\mathcal{H}(t,X^*(t),v,\lambda(t)). \tag{12}$$

We enforce these conditions to the Hamiltonian and see that the state Equation (10) correspond to the dynamical system (3); we obtain the costate system (6)–(7) from the costate Equation (11) and transversality conditions; while we obtain the control functions (8) from the minimality condition (12) of the PMP [18]. The proof is complete. □

4. Numerical Simulations: Case Study in Brazil

In this section, we solve numerically the suggested optimal control problem, defined by (3)–(5). For this, we use the fourth-order Runge–Kutta method (see, e.g., [21] for details). We consider the real data publicly available at the WHO, considered in [9], of the confirmed cases of Zika in Brazil between 4 February 2016 and 10 November 2016. According to [9], we consider as initial values S_0 = 2,180,686 (S_0 is the number of newborns corresponding to the simulation period) and the number of births in the period, $I_0=1$, $M_0=0$, and $W_0=0$ for the human female populations, and $A_{m0}=S_{m0}=I_{m0}=1.0903\times10^6$, and $E_{m0}=6.5421\times10^6$ for the mosquitoes populations. We obtain numerical simulations for a final time $t_f=160$ (weeks). Also, the parameters of the model (3) are listed in Table 2 (see [9]), and the maximum value of the control u_1 is assumed to be $u_{\max}=0.5$.

Table 2. Parameter values for system (1)–(2) and weight coefficients for (4).

Symbol	Description	Value	References
Λ	new pregnant women (per week)	3,000,000/52	[9]
ϕ	fraction of S that become infected	0.459	[9]
B	average daily biting (per day)	1	[9]
β_{mh}	transmission probability from I_m (per bite)	0.6	[9]
τ_1	rate at which S give birth (in weeks)	37	[9]
τ_2	rate at which I give birth (in weeks)	1/25	[9]
μ_h	natural death rate	1/50	[9]
ψ	fraction of I that gives birth to babies with a neurological disorder	0.133	[9]
β_{hm}	transmission probability from I_h (per bite)	0.6	[9]
μ_b	number of eggs at each deposit per capita (per day)	80	[9]
μ_A	natural mortality rate of larvae (per day)	1/4	[9]
η_A	maturation rate from larvae to adult (per day)	0.5	[9]
$1/\eta_m$	extrinsic incubation period (in days)	125	[9]
$1/\mu_m$	average lifespan of adult mosquitoes (in days)	125	[9]
K	maximal capacity of larvae	1.09034×10^6	[9,22]
w_1	the weight coefficient for infected pregnant women	10	assummed
w_2	the weight coefficient for the mosquito population	10	assummed
w_3	the weight coefficient for the cost of protective measure	100	assummed
w_4	the weight coefficient for the cost of spraying insecticide	100	assummed

In Figures 1 and 2, graphical representations depict the transmission dynamics of Zika within women and mosquito populations, with and without implementation of control measures. These visualizations illustrate the efficacy of control measures in averting cases of microcephaly by diminishing the count of infected pregnant women and eradicating the mosquito population responsible for Zika transmission. In Figures 3 and 4, we observe that employing the suggested combination of control measures robustly hampers Zika transmission. The mosquito population is essentially eradicated by the 10th week through the implementation of insecticide spraying control measures. Furthermore, the combined effect of both control measures resulted in a notable reduction in infection cases in pregnant women by the 10th week. A comparison of the control strategies reveals that the two control strategies are the most effective. However, the results also demonstrate that the spraying insecticide measure has a notable impact. Lastly, in Figure 5, while control u_1 remains in effect during all time window, control u_2 ceases to be effective around the 40th week due to its near-complete elimination of the mosquito population. Additionally, Figures 6–11 show comparative results of control strategies, considering different weight coefficients. As illustrated in Figures 6 and 7, an increase in the weight coefficients of the two control strategies is associated with a rise in the number of babies born with microcephaly. This phenomenon occurs concurrently with no discernible impact on the mosquito population. As shown in Figure 8, this phenomenon can be attributed to the diminished rate of exertion associated with the protective measure control in response to an elevated weight coefficient. Although a similar scenario would be expected for only one control strategy, it affects only the behavior of the spraying insecticide control rather than the behavior of the controlled system (see Figures 9–11). As a result, it reveals that in all cases, the two control strategies are the most robust for vertical Zika transmission.

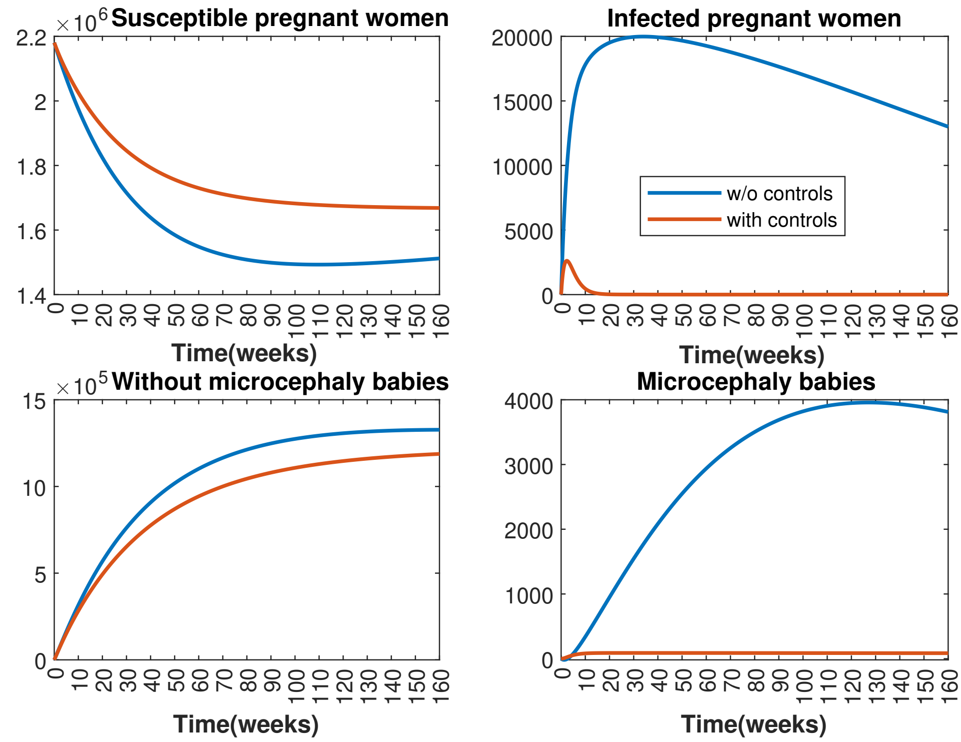

Figure 1. Effect of control strategies on women population, with and without controls. (**Top left**): susceptible pregnant women S; (**Top right**): infected pregnant women I; (**Bottom left**): women who gave birth to babies without microcephaly W; (**Bottom right**): women who gave birth to babies with microcephaly M.

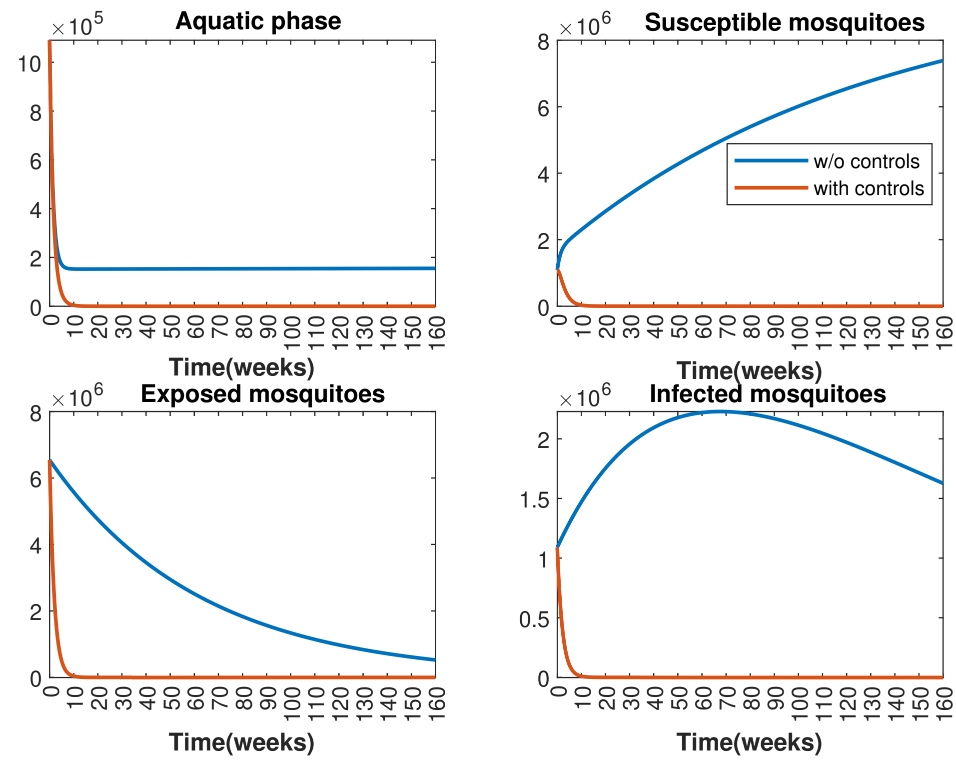

Figure 2. Effect of control strategies on mosquitoes, with and without controls. (**Top left**): mosquitoes in the aquatic phase A_m; (**Top right**): susceptible mosquitoes S_m; (**Bottom left**): exposed mosquitoes E_m; (**Bottom right**): infected mosquitoes I_m.

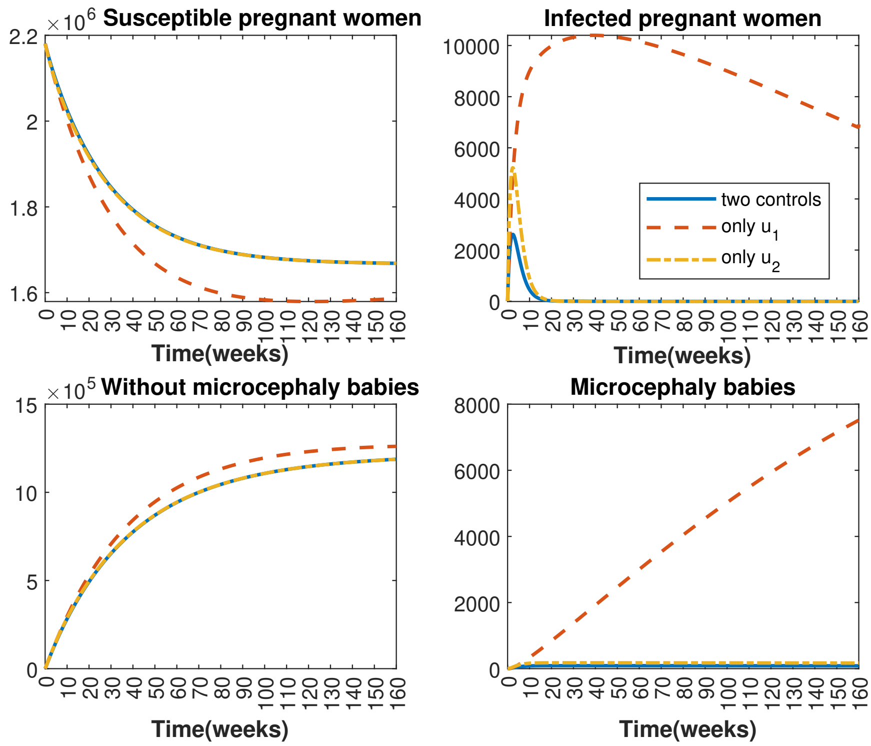

Figure 3. Comparative impact of the control strategies on women population, considering both controls and one single control u_1 or u_2. (**Top left**): susceptible pregnant women S; (**Top right**): infected pregnant women I; (**Bottom left**): women who gave birth to babies without microcephaly W; (**Bottom right**): women who gave birth to babies with microcephaly M.

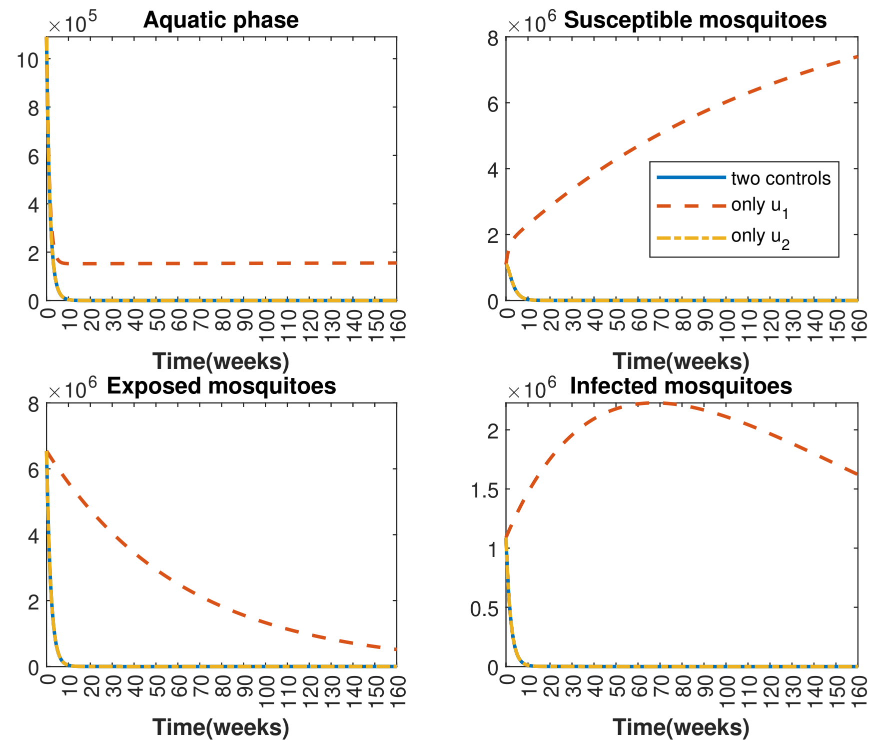

Figure 4. Comparative impact of the control strategies on mosquitoes, considering both controls and one single control u_1 or u_2. (**Top left**): mosquitoes in the aquatic phase A_m; (**Top right**): susceptible mosquitoes S_m; (**Bottom left**): exposed mosquitoes E_m; (**Bottom right**): infected mosquitoes I_m.

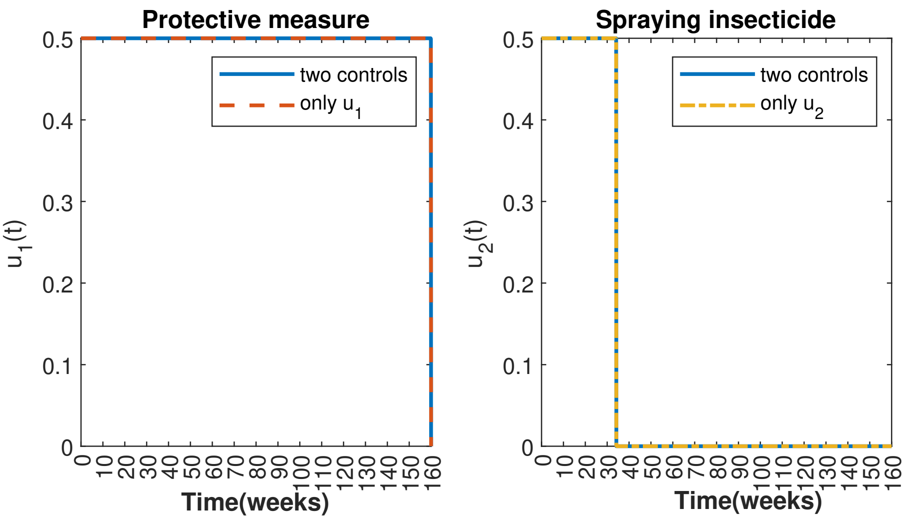

Figure 5. Control strategies u_1 and u_2.

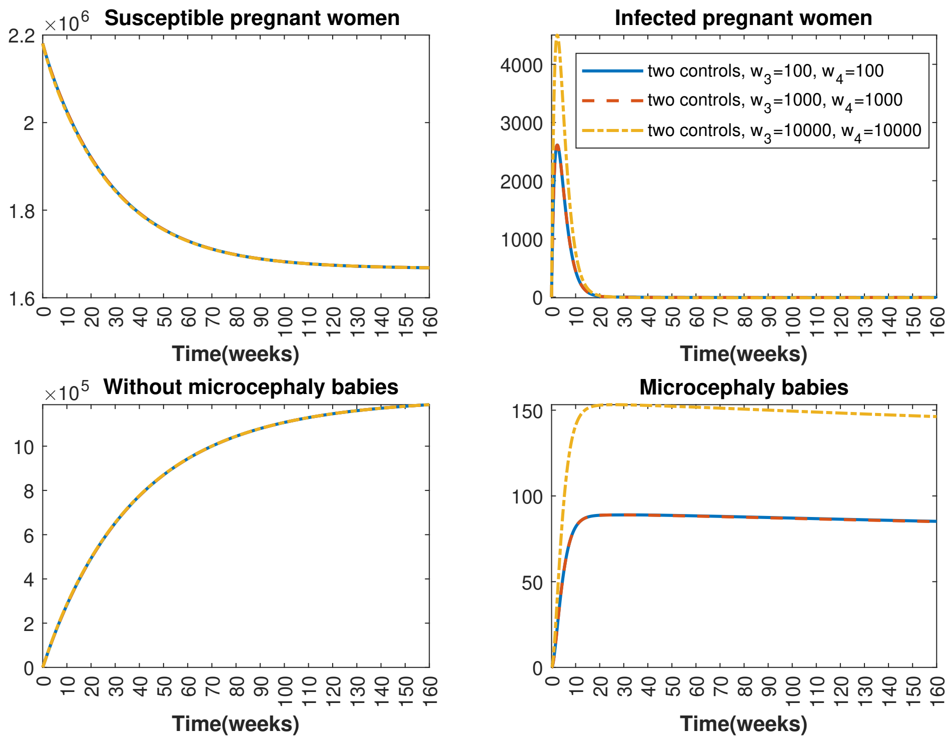

Figure 6. Comparative impact of control strategies with different weight coefficients on women population, considering two controls. (**Top left**): susceptible pregnant women S; (**Top right**): infected pregnant women I; (**Bottom left**): women who gave birth to babies without microcephaly W; (**Bottom right**): women who gave birth to babies with microcephaly M.

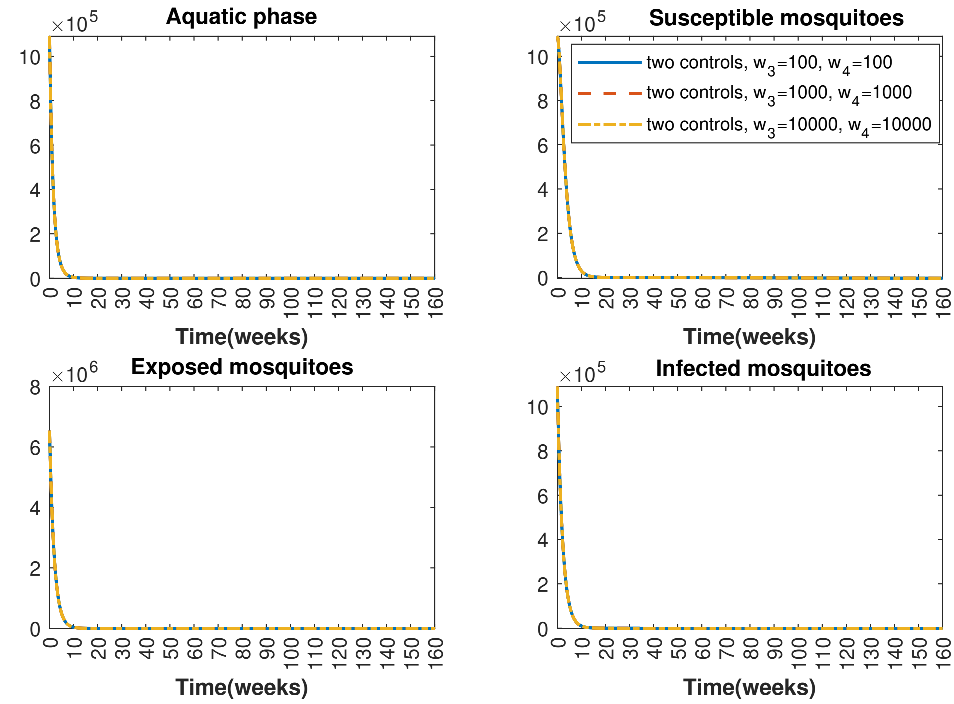

Figure 7. Comparative impact of control strategies with different weight coefficients on mosquitoes, considering two controls. (**Top left**): mosquitoes in the aquatic phase A_m; (**Top right**): susceptible mosquitoes S_m; (**Bottom left**): exposed mosquitoes E_m; (**Bottom right**): infected mosquitoes I_m.

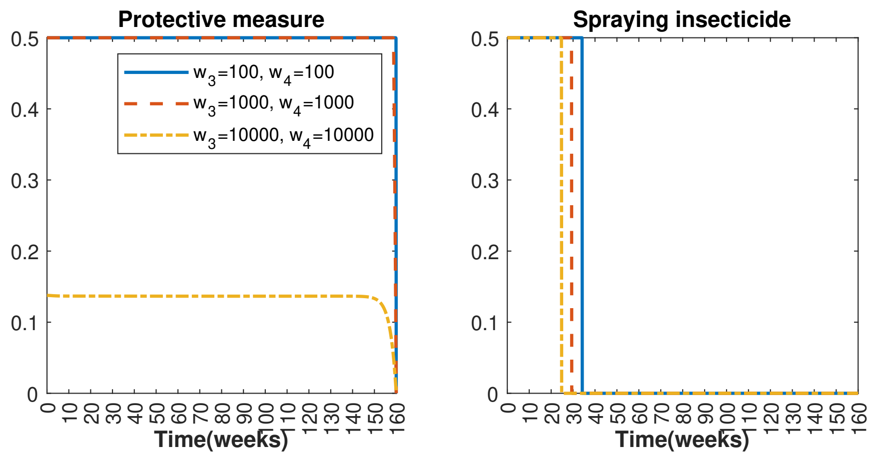

Figure 8. Control strategies u_1 and u_2 with weight coefficients $w_1 = w_2 = 10$ and $w_3 = w_4$ with $w_3, w_4 \in \{100, 1000, 10,000\}$.

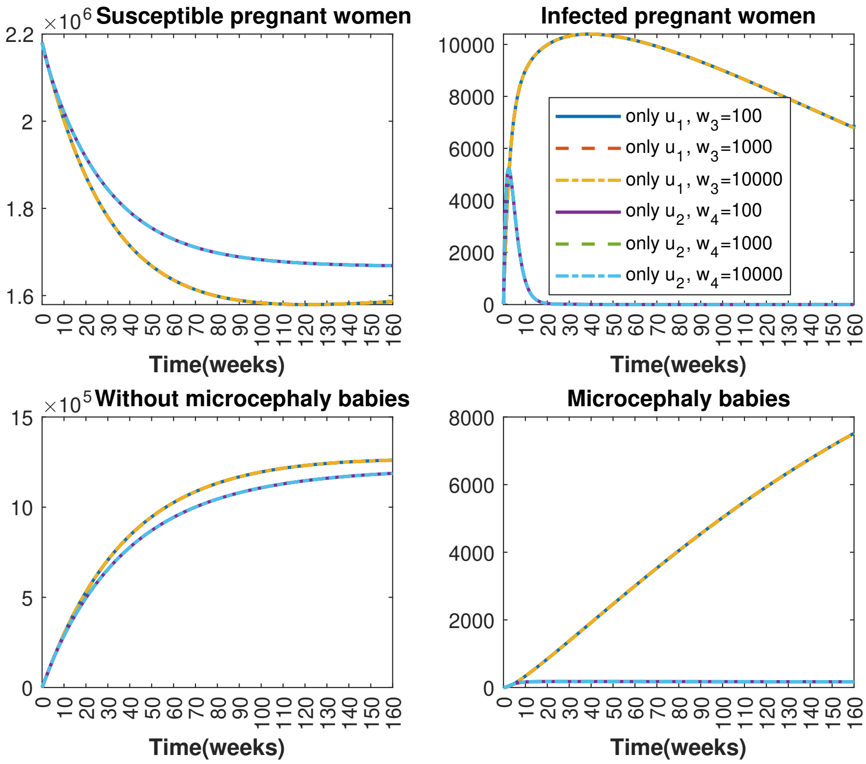

Figure 9. Comparative impact of control strategies with different weight coefficients on women population, considering one single control u_1 or u_2. (**Top left**): susceptible pregnant women S; (**Top right**): infected pregnant women I; (**Bottom left**): women who gave birth to babies without microcephaly W; (**Bottom right**): women who gave birth to babies with microcephaly M.

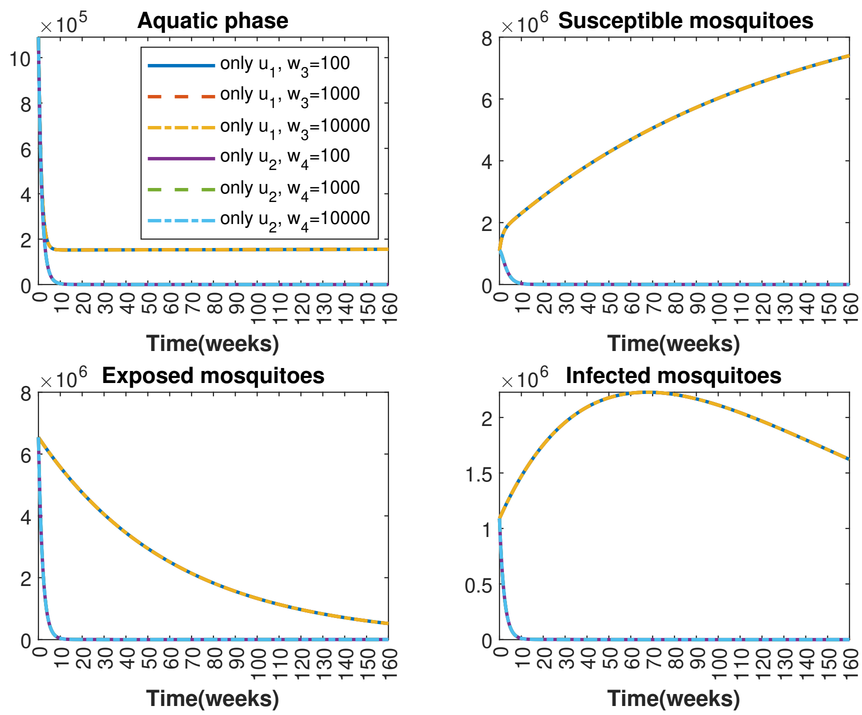

Figure 10. Comparative impact of control strategies with different weight coefficients on mosquitoes, considering one single control u_1 or u_2. (**Top left**): mosquitoes in the aquatic phase A_m; (**Top right**): susceptible mosquitoes S_m; (**Bottom left**): exposed mosquitoes E_m; (**Bottom right**): infected mosquitoes I_m.

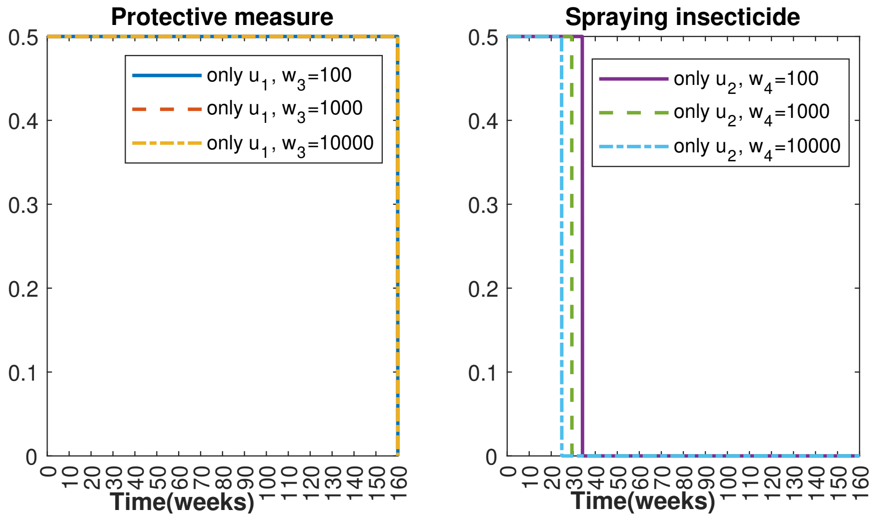

Figure 11. Control strategy u_1 with $w_1 = w_2 = 10$ and $w_3 \in \{100, 1000, 10,000\}$ ($u_2 \equiv 0$); control strategy u_2 with $w_1 = w_2 = 10$ and $w_4 \in \{100, 1000, 10,000\}$ ($u_1 \equiv 0$).

5. Conclusions

In this study, we have proposed control strategies aimed at thwarting the transmission of Zika virus, known for precipitating neurological disorders like microcephaly. For this purpose, an optimal control problem has been formulated for a model representing the vertical transmission of the Zika virus from infected mothers to infants in Brazil. Our primary objective has been to curtail the incidence of infection among pregnant women and diminish the mosquito population, all while ensuring cost-effectiveness in implementing these strategies. The optimal control problem has been solved via Pontryagin's maximum principle. Finally, the numerical results have been obtained using the fourth-order Runge–Kutta method with the help of the MATLAB (2021b) numeric computing environment. Through numerical simulations, we have demonstrated that the adoption of these control measures has led to a consistent reduction in the count of infected pregnant women from the outset of the intervention, consequently resulting in a decline in cases of microcephaly.

Author Contributions: Conceptualization, C.J.S. and D.F.M.T.; methodology, D.Y., C.J.S., and D.F.M.T.; software, D.Y.; validation, C.J.S. and D.F.M.T.; formal analysis, D.Y., C.J.S., and D.F.M.T.; investigation, D.Y., C.J.S., and D.F.M.T.; writing—original draft preparation, D.Y., C.J.S., and D.F.M.T.; writing—review and editing, D.Y., C.J.S., and D.F.M.T.; visualization, D.Y.; supervision, D.F.M.T. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Funda ção para a Ciência e a Tecnologia (FCT), grant numbers: UIDB/04106/2020 (https://doi.org/10.54499/UIDB/04106/2020) and UIDP/04106/2020 (https://doi.org/10.54499/UIDP/04106/2020); and project Mathematical Modelling of Multiscale Control Systems: Applications to Human Diseases (CoSysM3), reference 2022.03091.PTDC (https://doi.org/10.54499/2022.03091.PTDC).

Data Availability Statement: Data are contained within the article.

Acknowledgments: D.Y. is grateful to the 2211/A General Domestic Doctoral Scholarship supported by TUBITAK and an Erasmus+ Grant. The authors would like to sincerely thank the two referees for contributing to the initially submitted version with several constructive questions, comments, and suggestions.

Conflicts of Interest: The authors declare no conflicts of interest.

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