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Autonomous Second-Order ODEs: A Geometric Approach

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Abstract: Given an autonomous second-order ordinary differential equation (ODE), we define a Riemannian metric on an open subset of the first-order jet bundle. A relationship is established between the solutions of the ODE and the geodesic curves with respect to the defined metric. We introduce the notion of energy foliation for autonomous ODEs and highlight its connection to the classical energy concept. Additionally, we explore the geometry of the leaves of the foliation. Finally, the results are applied to the analysis of Lagrangian mechanical systems. In particular, we provide an autonomous Lagrangian for a damped harmonic oscillator.

Keywords: second-order ODEs; Lagrangian mechanical system; Riemannian metric; curvature

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1. Introduction

Second-order ordinary differential equations (ODEs) are an essential tool for modeling a wide range of nonlinear evolutionary phenomena, especially in systems with one-dimensional dynamics. These equations arise in various fields, including physics, engineering, and biology, where they are employed to describe processes as diverse as mechanical vibrations, chemical reactions, and population dynamics. However, the study of second-order ODEs is often challenging, as no universal algorithm exists to determine their solutions. Over the past few decades, significant research has been dedicated to finding solutions and first integrals for such ODEs [1–6]. Additionally, extensive research has investigated qualitative analysis and numerical methods for solving second-order ODEs [7–9].

In most cases, researchers have explored specific families of second-order ODEs to gain meaningful insights. In this paper, we focus on autonomous second-order ODEs, the study of which remains an active area of research; see [10] and references therein. These equations are particularly interesting because many of them arise from dynamical systems governed by (not necessarily autonomous) Lagrangians. In particular, understanding the behavior of these equations can lead to deeper insights into the underlying mechanics of physical systems, making them an important topic in both theoretical and applied mathematics.

On the other hand, recent research has shown increasing interest in associating Riemannian metrics with differential equations [11–18], as this approach provides valuable insights into their behavior. Following this approach, after introducing some preliminaries in Section 2, we show in Section 3 that autonomous second-order ODEs induce a Riemannian metric on an open subset of the first-order jet bundle $J^1(\mathbb{R}, \mathbb{R})$. We then explore the geometry of the resulting Riemannian manifold and relate it to the integrability of the ODE. Specifically, we link solutions of the ODE to geodesics of the manifold. We also introduce a minimal foliation in the manifold (Section 4) for which the leaves correspond to constant-energy surfaces in the context of mechanical systems, then study the geometry of these leaves. In Section 5, we shift our focus to ODEs derived from Lagrangian systems; in particular, we illustrate how the damped harmonic oscillator can be framed within an autonomous Lagrangian formulation.

2. Preliminaries

2.1. Jet Bundles and Second-Order ODEs

A central tool in the study of ODEs is the use of jet bundles, which provide a natural geometric context for understanding the structure and solutions of these equations. In this work, we use the first-order jet bundle $J^1(\mathbb{R}, \mathbb{R})$ to study autonomous second-order ODEs that can be written in the form

$$u_2 = \phi(u, u_1), \tag{1}$$

where (x, u, u_1) stand for the standard coordinates of $J^1(\mathbb{R}, \mathbb{R})$, with x and u representing the independent and dependent variables, respectively, u_1 denoting the first derivative of u with respect to x, and ϕ denoting a smooth function defined on an open subset $U \subseteq J^1(\mathbb{R}, \mathbb{R})$. The contact form on $J^1(\mathbb{R}, \mathbb{R})$ is defined as

$$\theta = -u_1 dx + du, \tag{2}$$

and captures the first-order differential relations between u, x, and u_1 [19,20].

Recall that the vector field associated with Equation (1) is

$$\partial_x + u_1 \partial_u + \phi \partial_{u_1}, \tag{3}$$

which is defined on U and encodes all the relevant information about the equation, in the following sense. Given a smooth function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, its first-order prolongation [19] is the curve $j^1 f : I \to J^1(\mathbb{R}, \mathbb{R})$, defined by the expression

$$(j^1 f)(x) = (x, f(x), f'(x)).$$

It turns out that a smooth function f is a solution to Equation (1) if and only if its first-order prolongation $j^1 f$ is an integral curve of the vector field (3); see [20,21] for details.

2.2. Riemannian Geometry

Recall that a Riemannian manifold is a smooth manifold M equipped with a Riemannian metric g, i.e., a two-times covariant symmetric tensor field that is positive definite, and consequently non-degenerate. Every Riemannian metric gives rise to a uniquely determined torsionless metric connection ∇, called the Levi-Civita connection.

Consider a three-dimensional Riemannian manifold (M, g) with an orthonormal frame (e_1, e_2, e_3) along with its corresponding dual co-frame $(\omega^1, \omega^2, \omega^3)$. The connected form of the Levi-Civita connection is defined as the matrix of 1-forms $\Theta = (\Theta^i{}_j)$ satisfying

$$\nabla_{e_i} e_j = \sum_{k=1}^{3} e_i \lrcorner \Theta^k{}_j e_k, \quad 1 \leq i, j \leq 3, \tag{4}$$

where $\lrcorner$ denotes the interior product.

These 1-forms can be obtained from Cartan's first structural equation

$$d\omega^i = \sum_{k=1}^{3} \omega^k \wedge \Theta^i{}_k, \quad 1 \leq i \leq 3 \tag{5}$$

and the condition $\Theta^i{}_j = -\Theta^j{}_i$, which is derived from the orthonormality of the frame; details can be found in, e.g., [22–24].

On the other hand, recall that the notion of a geodesic is used in differential geometry to extend the idea of a straight line in flat spaces to arbitrary spaces. A curve $\gamma : I \subseteq \mathbb{R} \to M$ is called a geodesic if

$$\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0$$

for every $t \in I$. It turns out that if a vector field X satisfies $\nabla_X X = 0$, then its integral curves are geodesics of the manifold [25–27].

Moreover, the behavior of geodesics is influenced by the curvature of the manifold. In Riemannian geometry, the curvature plays a crucial role in determining how geodesics diverge or converge, giving insight into the local and global geometry of the space. The curvature of the manifold is encoded in the Levi-Civita connection, and can be described through the curvature 2-forms Ω^i_j, as defined by Cartan's second structure equation:

$$\Omega^i_j = d\Theta^i_j + \sum_{k=1}^{3} \Theta^i_k \wedge \Theta^k_j. \tag{6}$$

The components of the Riemann curvature tensor in the given frame are related to the curvature 2-forms by the expression [23,25]

$$R^i_{jab} = \Omega^i_j(e_a, e_b). \tag{7}$$

The sectional curvature along specific planes within the tangent space can be computed from the components of the Riemann curvature tensor; for instance, the sectional curvatures along the planes spanned by the pairs $\{e_1, e_2\}$, $\{e_1, e_3\}$, and $\{e_2, e_3\}$ are provided by the components R^1_{212}, R^1_{313}, and R^2_{323}, respectively. These components provide essential geometric information about the curvature of the manifold along the chosen planes.

Now, consider a two-dimensional manifold Σ embedded into M in such a way that the given frame (e_1, e_2, e_3) is adapted to Σ, i.e., $\omega^3|_{T\Sigma} = 0$. The surface Σ inherits a Riemannian metric from the ambient manifold, along with its corresponding Levi-Civita connection. We denote the restrictions of ω^1, ω^2 to $T\Sigma$ by $\tilde{\omega}^1, \tilde{\omega}^2$, and denote the connection forms and the curvature forms of the inherited connection by $\tilde{\Theta}$ and $\tilde{\Omega}$, respectively.

By restricting the first Cartan Equation (5) to $T\Sigma$, we have

$$\tilde{\Theta}^i_j = \Theta^i_j|_{T\Sigma}, \quad i, j = 1, 2.$$

In addition, there must exist smooth functions s_{ij} defined on Σ such that

$$\begin{aligned} \Theta^3_1|_{T\Sigma} &= s_{11}\tilde{\omega}^1 + s_{12}\tilde{\omega}^2, \\ \Theta^3_2|_{T\Sigma} &= s_{21}\tilde{\omega}^1 + s_{22}\tilde{\omega}^2, \end{aligned} \tag{8}$$

with $s_{12} = s_{21}$. Observe that we can express the functions s_{ij} in terms of the connection forms, as follows:

$$s_{ij} = \tilde{e}_j \lrcorner \Theta^3_i|_{T\Sigma}, \quad i, j = 1, 2 \tag{9}$$

where $\tilde{e}_j = e_j|_{T\Sigma}$. The reader may refer to [28] for further details.

The shape operator S is defined as

$$S = \sum_{i,j=1}^{2} s_{ij}\tilde{\omega}^i \otimes \tilde{e}_j,$$

and has two independent invariants, namely, the extrinsic Gaussian curvature

$$K_{\text{ext}} = s_{11}s_{22} - s_{12}s_{21}, \tag{10}$$

and the mean curvature

$$H = \frac{1}{2}(s_{11} + s_{22}). \tag{11}$$

Regarding the intrinsic geometry of Σ, we denote by $\tilde{R}$ the Riemann curvature tensor of Σ with respect to the inherited metric. By definition, the curvature 2-form $\tilde{\Omega}$ satisfies

$$\begin{aligned} \tilde{\Omega}^i_j &= \Omega^i_j|_{T\Sigma} + \Theta^3_1|_{T\Sigma} \wedge \Theta^3_2|_{T\Sigma} \\ &= \Omega^i_j|_{T\Sigma} + K_{\text{ext}}\tilde{\omega}^1 \wedge \tilde{\omega}^2. \end{aligned}$$

From here, and according to Equation (7), we obtain Gauss' equation

$$K_{\text{int}} = R^1{}_{212} + K_{\text{ext}}, \tag{12}$$

where $K_{\text{int}} = \tilde{R}^1{}_{212}$ is the intrinsic Gaussian curvature of Σ.

3. Riemannian Metric Associated with Autonomous Second-Order ODEs

Several recent works [11,16,17] have studied how the Riemann metric

$$g = (1 + \phi^2)dx^2 - 2\phi dx du + du^2$$

provides geometric information about the first-order ODE $u_1 = \phi(x, u)$. Here, we explore how this approach can be extended to autonomous second-order ODEs by means of the following definition.

Definition 1. *Consider a second-order ODE in the form from* (1). *We define the associated three-dimensional Riemannian manifold as the open submanifold provided by*

$$M = \{(x, u, u_1) \in U : u_1 \neq 0\} \subseteq J^1(\mathbb{R}, \mathbb{R})$$

endowed with the Riemannian metric

$$g = (1 + u_1^2)dx^2 - 2u_1 dx du + \left(1 + \frac{\phi^2}{u_1^2}\right)du^2 - 2\frac{\phi}{u_1}du du_1 + du_1^2. \tag{13}$$

Interestingly, the vector field from (3) associated with Equation (1) has constant unit length, as can easily be verified. Moreover, the vector field ∂_{u_1} also has unit length, and is orthogonal to the associated vector field. By standard procedures, we can complete this pair of vector fields to obtain the orthonormal frame (e_1, e_2, e_3) defined by

$$
\begin{aligned}
e_1 &= \partial_x + u_1 \partial_u + \phi \partial_{u_1}, \\
e_2 &= \partial_u + \frac{\phi}{u_1}\partial_{u_1}, \\
e_3 &= \partial_{u_1}.
\end{aligned}
\tag{14}
$$

The corresponding dual co-frame $(\omega^1, \omega^2, \omega^3)$ is provided by the 1-forms

$$
\begin{aligned}
\omega^1 &= dx, \\
\omega^2 &= -u_1 dx + du, \\
\omega^3 &= -\frac{\phi}{u_1}du + du_1,
\end{aligned}
\tag{15}
$$

with ω^2 being the contact form θ of $J^1(\mathbb{R}, \mathbb{R})$ provided in Equation (2). Observe that

$$
\begin{aligned}
d\omega^1 &= 0, \\
d\omega^2 &= \frac{\phi}{u_1}\omega^1 \wedge \omega^2 + \omega^1 \wedge \omega^3, \\
d\omega^3 &= \left(\phi_{u_1} - \frac{\phi}{u_1}\right)\omega^1 \wedge \omega^3 + \frac{u_1\phi_{u_1} - \phi}{u_1^2}\omega^2 \wedge \omega^3,
\end{aligned}
\tag{16}
$$

where the subscripts denote partial derivatives (this notation is used throughout the paper). From (16), using Cartan's first structural Equation (5), we obtain the connection form

$$
\Theta = \begin{pmatrix} 0 & -\frac{\phi}{u_1}\omega^2 - \frac{1}{2}\omega^3 & -\frac{1}{2}\omega^2 - \left(\phi_{u_1} - \frac{\phi}{u_1}\right)\omega^3 \\ \frac{\phi}{u_1}\omega^2 + \frac{1}{2}\omega^3 & 0 & -\frac{1}{2}\omega^1 - \frac{u_1\phi_{u_1}-\phi}{u_1^2}\omega^3 \\ \frac{1}{2}\omega^2 + \left(\phi_{u_1} - \frac{\phi}{u_1}\right)\omega^3 & \frac{1}{2}\omega^1 + \frac{u_1\phi_{u_1}-\phi}{u_1^2}\omega^3 & 0 \end{pmatrix}. \tag{17}
$$

We are now in a position to discuss the link between the geodesic curves of the Riemannian manifold (M, g) and the solutions to Equation (1). Recall that the integral curves of e_1 are precisely the first-order prolongation of solutions of Equation (1). On the other hand, a straightforward calculation yields

$$
\nabla_{e_1} e_1 = e_1 \lrcorner \Theta^k{}_1 e_k = 0, \tag{18}
$$

meaning that the integral curves of e_1 are geodesic curves. Therefore, given a smooth function f which is a solution to Equation (1), the curve $\mathrm{j}^1 f$ is a geodesic of the manifold. The converse is not true; however, we can establish the following weaker version.

Proposition 1. *Suppose that Equation (1) satisfies $\left(\frac{\phi}{u_1}\right)_{u_1} \neq 0$. If a smooth function f is such that the curve $\mathrm{j}^1 f$ is a geodesic, then f is a solution of (1).*

Proof. Assume that $\mathrm{j}^1 f$ is a geodesic curve. We can express the vector field tangent to the curve $\mathrm{j}^1 f$ in the frame (e_1, e_2, e_3) as

$$
(\mathrm{j}^1 f)'(x) = e_1 + (f''(x) - \phi)e_3.
$$

If $\mathrm{j}^1 f$ is a geodesic curve, then

$$
\begin{aligned}
\nabla_{(\mathrm{j}^1 f)'(x)} (\mathrm{j}^1 f)'(x) = {} & (f''(x) - \phi)^2 \frac{\phi - u_1\phi_{u_1}}{u_1} e_1 \\
& + (f''(x) - \phi)^2 \frac{\phi - u_1\phi_{u_1}}{u_1^2} e_2 \\
& + \left(f'''(x) - e_1(\phi) - \frac{\phi}{u_1}(f''(x) - \phi) \right) e_3 = 0.
\end{aligned} \tag{19}
$$

Then, because $\left(\frac{\phi}{u_1}\right)_{u_1} \neq 0$, we have $\phi - u_1\phi_{u_1} \neq 0$; therefore,

$$
f''(x) - \phi = 0,
$$

completing the proof. $\square$

Remark 1. *Proposition 1 does not apply to equations satisfying $\left(\frac{\phi}{u_1}\right)_{u_1} = 0$. For example, if we consider the equation $u_2 = u_1$, we can check that the function $f(x) = x + e^x$ is not a solution; however, the prolongation $\mathrm{j}^1 f$ is a geodesic curve, as it verifies (19).*

Nevertheless, this family of equations takes the form

$$
u_2 = K(u)u_1, \tag{20}
$$

meaning that they can be fully integrated by quadratures. Indeed, we can observe that $u_1 - \int K(u) \, du$ is a first integral of Equation (20):

$$
e_1\left(u_1 - \int K(u) \, du \right) = \phi - u_1 K(u) = 0.
$$

Now, the family of first-order ODEs

$$u_1 - \int K(u)\, du = C,$$

where $C \in \mathbb{R}$, can always be solved by another quadrature, and their solutions correspond to solutions of (20).

To continue our analysis of the geometry of the manifold (M, g), we can compute the curvature of the connection, i.e., the matrix of 2-forms $\Omega = (\Omega^i_j)$ provided by Equation (6), the explicit expression of which is too involved to be included here. Nevertheless, by using Equation (7), a straightforward computation yields the following components of the Riemann curvature tensor with respect to the frame (e_1, e_2, e_3):

$$R^1_{\ 212} = \Omega^1_2(e_1, e_2) = \frac{1}{4} - \frac{e_1(\phi)}{u_1}, \tag{21a}$$

$$R^1_{\ 313} = \Omega^1_3(e_1, e_3) = -\frac{3}{4} + \phi_u - \phi_{u_1}^2 - \phi\phi_{u_1 u_1} - \phi_{uu_1} u_1 + \frac{3\phi\phi_{u_1}}{u_1} - \frac{2\phi^2}{u_1^2}, \tag{21b}$$

$$R^2_{\ 323} = \Omega^2_3(e_2, e_3) =$$
$$= \frac{1}{4} - \frac{\phi\phi_{u_1} + \phi_{uu_1}}{u_1} - \frac{\phi\phi_{u_1 u_1} - \phi^2 - \phi_u + \phi_{u_1}^2}{u_1^2} + \frac{4\phi\phi_{u_1}}{u_1^3} - \frac{3\phi^2}{u_1^4}. \tag{21c}$$

Recall that these components correspond to the sectional curvatures along the planes generated by the pairs $\{e_1, e_2\}$, $\{e_1, e_3\}$, and $\{e_2, e_3\}$, respectively.

Example 1. *Consider the second-order ODE*

$$u_2 = \sqrt{1 - \kappa u_1^2}, \tag{22}$$

where κ is a nonzero real constant. In this case, the associated Riemannian manifold consists of $M = \{(x, u, u_1) \in J^1(\mathbb{R}, \mathbb{R}) : u_1 \neq 0\}$ together with the Riemannian metric

$$g = (1 + u_1^2)dx^2 - 2u_1 dx du + \left(1 - \kappa + \frac{1}{u_1^2}\right)du^2 - 2\frac{\sqrt{1 - \kappa u_1^2}}{u_1}du du_1 + du_1^2.$$

According to the above results, we have the following sectional curvatures:

$$R^1_{\ 212} = \frac{1}{4} + \kappa, \tag{23a}$$

$$R^1_{\ 313} = -\frac{3}{4} - \frac{2}{u_1^2}, \tag{23b}$$

$$R^2_{\ 323} = \frac{1}{4} + \frac{1}{u_1^2} - \frac{3}{u_1^4}. \tag{23c}$$

To illustrate, some solutions of Equation (22), *which correspond to geodesics of the associated Riemannian manifold are visualized in Figure 1a,b for $\kappa = 1$ and $\kappa = -1$, respectively.*
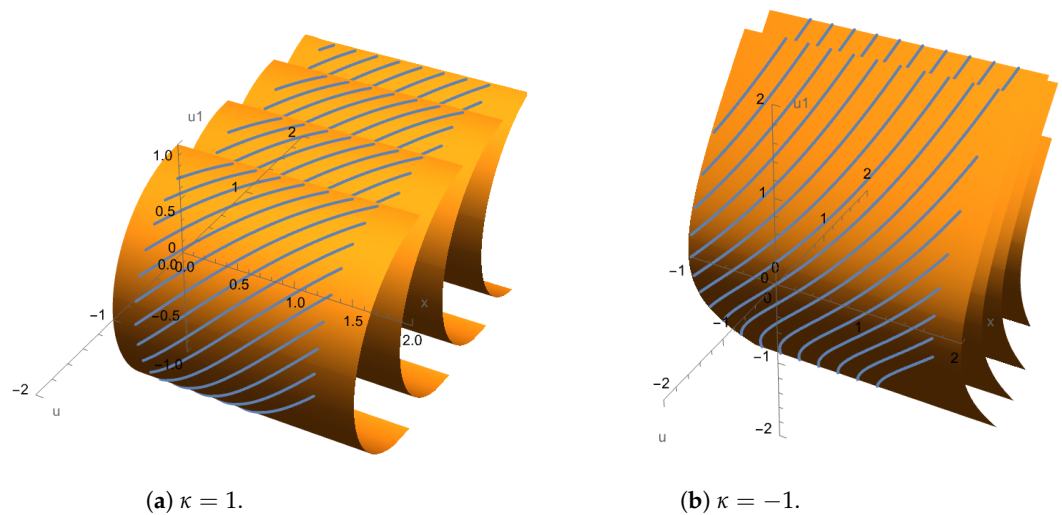
(**a**) $\kappa = 1$.

(**b**) $\kappa = -1$.

Figure 1. Visualization of the jet bundle $J^1(\mathbb{R}, \mathbb{R})$ with the energy foliation of Equation (22) for two cases: (**a**) $\kappa = 1$ and (**b**) $\kappa = -1$. The profiles of the leaves are ellipses and hyperbolas, respectively. The prolongations of several solutions, which are geodesics, are shown.

Example 2. *The autonomous second-order ODE*

$$u_2 = \frac{4u_1^2 + u^2 + u}{2u + 1} \tag{24}$$

gives rise to a Riemannian manifold which has zero sectional curvature along the planes generated by $\{e_1, e_2\}$. Indeed, from (21a), we have

$$R^1{}_{212} = \frac{1}{4} - \frac{e_1\left(\frac{4u_1^2 + u^2 + u}{2u + 1}\right)}{u_1} = 0.$$

The sectional curvature $R^1{}_{212}$ plays a significant role in understanding the geometric structure of the manifold. In particular, it is closely tied to the foliation introduced in the following section.

4. Energy Foliation

We now turn our attention to a key geometric feature of the manifold (M, g), namely, the existence of a foliation, referred to here as the energy foliation, that is related to the integrability of the differential equation in (1).

Consider the distribution provided by

$$D_p := \{v \in T_p M : v \lrcorner (\omega^3)_p = 0\}$$

for each $p \in M$. This distribution is spanned by the vector fields $\{e_1, e_2\}$. Thus, it is involutive, as

$$[e_1, e_2] = -\frac{\phi}{u_1} e_2.$$

By Frobenius' theorem [29], there exists a foliation $\mathcal{E}$ on M such that the tangent space to the leaf at $p \in M$ is D_p. The leaves are provided locally by the level sets of a certain smooth function $E : M \to \mathbb{R}$, i.e., in a neighborhood V of p, the leaves are described by

$$\Sigma_C = \{p \in V : E(p) = C\}, \ C \in \mathbb{R}.$$

Because $T_p \Sigma_C = D_p$, such a function E must satisfy

$$dE = \mu \omega^3 \tag{25}$$

for a non-vanishing smooth function μ defined on V. Equivalently, E must be a solution to the following homogeneous linear partial differential equation (PDE):

$$E_u + \frac{\phi}{u_1} E_{u_1} = 0. \tag{26}$$

Definition 2. *We call the foliation $\mathcal{E}$ on M the energy foliation of Equation (1), or simply the energy of the equation.*

Remark 2. *Note that while the energy foliation $\mathcal{E}$ of an autonomous second-order ODE can be locally defined by a function E, it can be described equally well by any other function that is functionally dependent on E. As we will see in Theorem 1, in classical examples arising in mechanics, one of these functions is precisely the classical notion of energy of the system, thereby justifying the terminology; however, for a general autonomous second-order ODE it is important to clarify that we cannot refer to any specific function E as* the energy, *as there are many such functions and no single one is a natural or preferred choice.*

Remark 3. *Given any function E satisfying (26), we can solve the differential equation*

$$u_1 L_{u_1} - L = E \tag{27}$$

to determine a Lagrangian $L = L(u, u_1)$ suitable for Equation (1). This Lagrangian may be considered nonstandard; see [30] and references therein. Indeed, a particular solution to (27) is provided by

$$L(u, u_1) = u_1 \int \frac{E(u, u_1)}{u_1^2} \, du_1. \tag{28}$$

We can check that the corresponding Euler–Lagrange equation for this Lagrangian is

$$-E_u - \frac{u_2}{u_1} E_{u_1} = 0,$$

which by virtue of (25) is equivalent to Equation (1). This establishes the existence of local solutions to the inverse problem of the calculus of variations for autonomous second-order ODEs, which is a topic of ongoing relevance in mathematical physics [30–33]. Importantly, the obtained Lagrangians are themselves autonomous.

Example 3. *In the case of Equation (22) in Example 1, we can find a function for determining its corresponding energy foliation by solving the following PDE (26):*

$$u_1 E_u + \sqrt{1 - \kappa u_1^2} E_{u_1} = 0. \tag{29}$$

We can check that a solution to Equation (29) is the smooth function

$$E = u + \frac{1}{\kappa} \sqrt{1 - \kappa u_1^2}.$$

Thus, the leaves of the energy foliation $\mathcal{E}$ of Equation (22) are the surfaces in M provided by

$$\Sigma_C = \left\{ (x, u, u_1) \in M : u + \frac{1}{\kappa} \sqrt{1 - \kappa u_1^2} = C \right\}, \quad C \in \mathbb{R}.$$

Figure 1a,b illustrates these energy foliation leaves for the cases of $\kappa = +1$ and $\kappa = -1$, respectively. According to Remark 3 and taking into account (28), either the smooth function

$$L(u, u_1) = -u - \frac{1}{\kappa} \left(\sqrt{1 - \kappa u_1^2} + \sqrt{\kappa} u_1 \arcsin(\sqrt{\kappa} u_1) \right) \tag{30}$$

in case $\kappa > 0$ or the smooth function

$$L(u, u_1) = -u - \frac{1}{\kappa} \left(\sqrt{1 - \kappa u_1^2} + \sqrt{-\kappa} u_1 arcsinh(\sqrt{-\kappa} u_1) \right) \tag{31}$$

for $\kappa < 0$ is a solution to Equation (27), meaning that it is a nonstandard autonomous Lagrangian for which the Euler–Lagrange equation is (22).

Example 4. *To find the energy foliation for the second-order ODE (24) in Example 2, we solve the following PDE:*

$$4(2u + 1)u_1 E_u + (4u_1^2 + u^2 + u)E_{u_1} = 0. \tag{32}$$

We can check that the smooth function

$$E = \frac{1}{2u + 1} u_1^2 - \frac{4u^2 + 2u + 1}{32u + 16}$$

is a particular solution to (32) and that its level sets define the energy foliation of (24).

On the other hand, we can use the expression (28) in Remark 3 to obtain

$$L = \frac{1}{2u + 1} u_1^2 + \frac{4u^2 + 2u + 1}{32u + 16},$$

which is an autonomous Lagrangian for which the corresponding Euler–Lagrange equation is (24). In this case, the Lagrangian takes the form of a kinetic energy term plus a potential energy term, and the function E can be considered as a mechanical energy term.

In the rest of this section, we are explore the geometry of the leaves of the energy foliation $\mathcal{E}$. The metric provided by (13) can be written as

$$g = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3,$$

as the co-frame (15) is orthonormal. Because $\omega^3|_{\Sigma_C} \equiv 0$, the induced metric on the surface Σ_C is

$$\tilde{g} = \tilde{\omega}^1 \otimes \tilde{\omega}^1 + \tilde{\omega}^2 \otimes \tilde{\omega}^2, \tag{33}$$

where $\tilde{\omega}^i$ denotes the restriction of ω^i to Σ_C for $i = 1, 2$.

Correspondingly, the shape operator of Σ_C is provided by

$$S = \frac{1}{2} \tilde{\omega}^1 \otimes \tilde{e}_2 + \frac{1}{2} \tilde{\omega}^2 \otimes \tilde{e}^1,$$

as from Equation (9) together with (17) we have

$$s_{11} = 0, s_{12} = s_{21} = \frac{1}{2}, s_{22} = 0.$$

Then, we have the following result regarding the extrinsic geometry of Σ_C within M.

Proposition 2. *The leaves of the energy foliation are minimal surfaces with constant extrinsic curvature.*

Proof. The mean curvature for each of these surfaces is provided by

$$H = \frac{1}{2}(s_{11} + s_{22}) = 0,$$

meaning that they are minimal surfaces.

On the other hand, the extrinsic Gauss curvature is

$$K_{ext} = s_{11}s_{22} - s_{12}s_{21} = -\frac{1}{4}.$$

□

Remark 4. *According to this result, the energy of an autonomous second-order ODE forms a minimal foliation of the associated manifold (M, g). The study of this kind of foliation is an area of significant interest in differential geometry [34–38].*

On the other hand, with respect to the intrinsic geometry of Σ_C, we have the following immediate consequence of Equations (12) and (21a).

Corollary 1. *The intrinsic Gauss curvature of Σ_C is*

$$K_{int} = -\frac{e_1(\phi)}{u_1}. \tag{34}$$

Example 5. *We now return to Example 3. The Riemannian metric (33) induced on the surfaces Σ_C obtained by means of the local parameterization of Σ_C is provided by*

$$\iota(x, u) = \left(x, u, \sqrt{\frac{1}{\kappa} - \kappa(C - u)^2} \right),$$

is

$$\tilde{g} = \left(\frac{\kappa + 1}{\kappa} - \kappa(C - u)^2 \right) dx^2 - 2\sqrt{\frac{1}{\kappa} - \kappa(C - u)^2} dx du + du^2.$$

According to (34), the intrinsic Gauss curvature is,

$$K_{int} = -\frac{e_1(\sqrt{1 - \kappa u_1^2})}{u_1} = \kappa. \tag{35}$$

On the other hand, the shape operator of Σ_C can be represented in coordinate frame by the matrix

$$S = \begin{pmatrix} -\frac{1}{2}\sqrt{\frac{1}{\kappa} - \kappa(C - u)^2} & \frac{1}{2} - \frac{1}{2\kappa} + \frac{\kappa}{2}(C - u)^2 \\ \frac{1}{2} & \frac{1}{2}\sqrt{\frac{1}{\kappa} - \kappa(C - u)^2} \end{pmatrix}.$$

We can check that the extrinsic Gauss curvature of leaf Σ_C is

$$K_{ext} = \det(S) = -\frac{1}{4},$$

while the mean curvature is $H = \frac{1}{2}tr(S) = 0$; thus, it is a minimal surface, as stated in Proposition 2.

Example 6. *In the case of Equation (24) of Example 2, similar computations show that the leaves of its energy foliation satisfy*

$$K_{int} = K_{ext} = -\frac{1}{4}.$$

5. Lagrangian Mechanical Systems

In this section, we focus on second-order ODEs arising as the Euler–Lagrange equations of one-dimensional mechanical systems defined by a Lagrangian function. The variable x represents time, u denotes the generalized coordinate, and u_1 represents the generalized velocity.

We begin by providing a justification for the terminology introduced in Definition 2. Consider a mechanical system governed by a Lagrangian $L = L(x, u, u_1)$. For such a system, recall that the function

$$h(u, u_1) := u_1 L_{u_1} - L \tag{36}$$

is known as the *energy function*. Under standard assumptions for L, this function coincides with the total mechanical energy of the system, which is typically expressed as $h(u, u_1) = T(u, u_1) + V(u)$, where T and V represent the kinetic and potential energies, respectively.

In the case of an autonomous Lagrangian system, where $L = L(u, u_1)$, the energy function (36) remains conserved along the solutions of the equation of motion of the system. For further details regarding the energy function and its conservation in autonomous systems, the reader is referred to [39] (Section 2.7).

Having established these preliminaries, we now state the following result.

Theorem 1. *Given an autonomous one-dimensional Lagrangian system, its energy function (36) defines the energy foliation of the corresponding equation of motion.*

Proof. The Euler–Lagrange equation for a mechanical system defined by the autonomous Lagrangian $L = L(u, u_1)$ is

$$\frac{d}{dx} L_{u_1} - L_u = 0,$$

which can be expanded as

$$u_1 L_{uu_1} + u_2 L_{u_1 u_1} - L_u = 0.$$

This is an autonomous second-order ODE, for which the corresponding ω^3 (see Equation (15)) is provided by

$$\omega^3 = -\frac{L_u - u_1 L_{uu_1}}{u_1 L_{u_1 u_1}} du + du_1.$$

In order to check whether the energy function (36) satisfies condition (25), we compute the exterior derivative

$$dh = (u_1 L_{uu_1} - L_u) du + u_1 L_{u_1 u_1} du_1.$$

As we observe that $dh = u_1 L_{u_1 u_1} \omega^3$, the result is proven. □

In the following subsections, we analyze the applicability of the results presented in this paper to specific examples of Lagrangian mechanical systems.

5.1. Lagrangian for a Particle in a Gravitational Field

A classical problem in the context of Newtonian gravity involves understanding the behavior of a particle under the influence of the gravitational field created by a mass distribution. Consider a one-dimensional universe with spatial coordinate u and a mass distribution defined by the smooth function $\rho(u)$. This distribution generates a gravitational field that exerts a force on a test particle of mass m located at position u. The gravitational potential at position u due to the mass distribution, denoted by $\Phi(u)$, is derived from the Poisson equation [40], which in one dimension is provided by

$$\Phi_{uu} = 4\pi G \rho(u),$$

where G is the gravitational constant.

The Lagrangian for the test particle moving in this gravitational field is then provided by

$$L = \frac{1}{2} m u_1^2 - m \Phi(u),$$

while the corresponding equation of motion is

$$u_2 = -\Phi_u. \tag{37}$$

Within our framework, according to Equation (13), the metric for the ODE (37) is

$$g = (1 + u_1^2)dx^2 - 2u_1 dxdu + \left(1 + \frac{\Phi_u^2}{u_1^2}\right)du^2 + 2\frac{\Phi_u}{u_1}dudu_1 + du_1^2, \tag{38}$$

while the orthonormal co-frame determined in (15) is

$$\begin{aligned}
\omega^1 &= dx, \\
\omega^2 &= -u_1 dx + du, \\
\omega^3 &= \frac{\Phi_u}{u_1}du + du_1.
\end{aligned} \tag{39}$$

To find the energy foliation, we do not need to identify smooth functions E and μ satisfying $dE = \mu\omega^3$. According to Theorem 1, the energy foliation is described by the energy function of the system, which in this case coincides with the total mechanical energy:

$$h = \frac{1}{2}mu_1^2 + m\Phi(u).$$

According to Proposition 2, the surfaces of constant energy defined by $h(u, u_1) = C, C \in \mathbb{R}$ are minimal. Their intrinsic Gauss curvature is provided by Equation (34) in Corollary 1:

$$K_{\text{int}} = -\frac{e_1(-\Phi_u)}{u_1} = \Phi_{uu} = 4\pi G\rho(u). \tag{40}$$

Thus, it can be seen that the intrinsic Gauss curvature of the leaves of the energy foliation is directly tied to the mass density generating the gravitational field. Equation (40) reflects the fact that the curvature is proportional to the amount of matter present in the system.

Two distinct examples corresponding to the mass distributions $\rho(u) = \frac{3}{\pi u^5}$ and $\rho(u) = \frac{2u^2 - 1}{2\pi}e^{-u^2}$ are shown in Figure 2a and Figure 2b, respectively.

(a) $\rho(u) = \frac{3}{\pi u^5}$ **(b)** $\rho(u) = \frac{2u^2 - 1}{2\pi}e^{-u^2}$
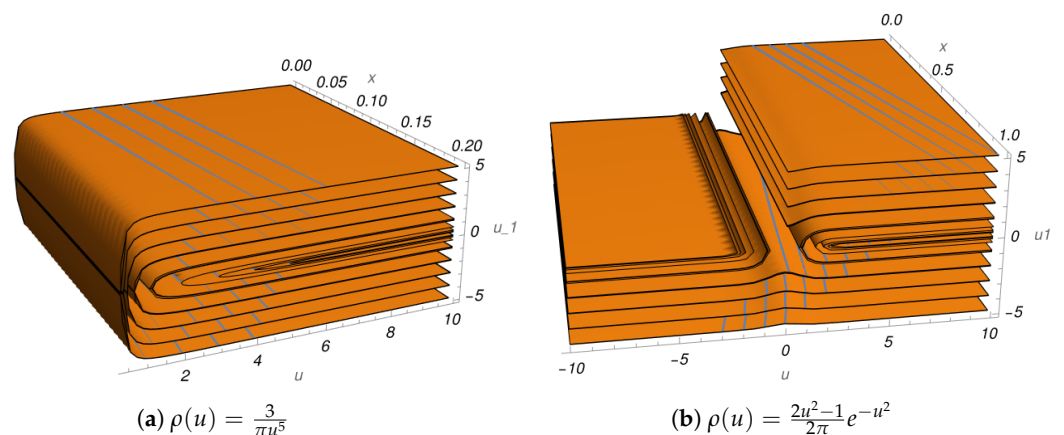
Figure 2. Energy foliations induced by Equation (37) in the jet bundle $J^1(\mathbb{R}, \mathbb{R})$ along with the prolongations of several solutions.

5.2. Damped Harmonic Oscillator

In this subsection, we examine the well studied damped harmonic oscillator, a fundamental system in classical mechanics that is central to understanding a wide variety of physical systems from mechanical vibrations to electrical circuits. It describes the motion

of an oscillating object subject to a restoring force proportional to its displacement along with a damping force that opposes its velocity.

This system is governed by the following second-order differential equation:

$$u_2 = -\alpha u_1 - \lambda u \tag{41}$$

where $u = u(x)$ represents the displacement as a function of time, $u_1 = u_1(x)$ is the velocity, and $u_2 = u_2(x)$ is the acceleration. The parameter α represents the damping coefficient, which quantifies the resistance to motion (such as friction or air resistance), while λ is the spring constant, which characterizes the strength of the restoring force.

It is well known that this system does not present energy conservation in the classical sense; indeed, this system is typically described using the time-dependent Lagrangian [39]:

$$L = \frac{e^{\alpha x}}{2}\left(u_1^2 - \lambda u^2\right)$$

meaning that the corresponding energy function (36) is

$$h = \frac{e^{\alpha x}}{2}\left(u_1^2 + \lambda u^2\right), \tag{42}$$

which is clearly not conserved over time.

However, as Equation (41) does not explicitly depend on time, we can apply the ideas introduced in the previous sections. First, we can consider the Riemannian metric associated with Equation (41), which is provided by

$$g = (1 + u_1^2)dx^2 - 2u_1 dx du + \left(1 + \left(\alpha + \lambda\frac{u}{u_1}\right)^2\right)du^2 - 2\left(\alpha + \lambda\frac{u}{u_1}\right)du du_1 + du_1^2.$$

The orthonormal co-frame introduced in (15) is

$$\begin{aligned} \omega^1 &= dx, \\ \omega^2 &= -u_1 dx + du, \\ \omega^3 &= \left(\alpha + \lambda\frac{u}{u_1}\right)du + du_1. \end{aligned} \tag{43}$$

To find the energy foliation $\mathcal{E}$, we take condition (26), which in this case corresponds to the following PDE:

$$u_1 E_u - (\alpha u_1 + \lambda u)E_{u_1} = 0. \tag{44}$$

In what follows, we focus on the underdamped case, i.e., $\alpha^2 < 4\lambda$; however, the remaining cases can be developed in an analogous manner. We can check whether a particular solution for Equation (44) in this case is provided by

$$E = \frac{e^{\frac{\alpha}{\omega}\arctan\left(\frac{\alpha u_1 + 2\lambda u}{2\omega u_1}\right)}}{2}(\alpha u u_1 + u_1^2 + \lambda u^2), \tag{45}$$

where $\omega := \frac{1}{2}\sqrt{4\lambda - \alpha^2}$. The level sets of this function define the energy foliation $\mathcal{E}$ of the second-order ODE (41).

It is interesting to highlight that the function E defined in (45) exhibits a structural resemblance to the energy function presented in Equation (42). Moreover, by setting $\alpha = 0$ in Equation (45), which corresponds to the undamped harmonic oscillator, we recover the classical expression for the mechanical energy of the harmonic oscillator:

$$E = \frac{1}{2}(u_1^2 + \lambda u^2).$$

Consequently, (45) can be regarded as a non-mechanical form of energy that remains conserved along the solutions.

On the other hand, according to Proposition 2 the leaves of the foliation $\mathcal{E}$ are minimal surfaces with $K_{\text{ext}} = -\frac{1}{4}$. Moreover, by Corollary 1, their intrinsic curvature is

$$K_{\text{int}} = \lambda - \alpha^2 - \alpha\lambda\frac{u}{u_1}.$$

It is worth noting that the intrinsic curvature of this system is closely related to the damping coefficient and the spring constant. In particular, for a harmonic oscillator without damping, the intrinsic curvature simplifies to the spring constant: $K_{\text{int}} = \lambda$.

Finally, observe that the energy foliation of the second-order ODE (41) can also be expressed as the level sets of the smooth function

$$\tilde{E} := \ln(2E) = \frac{\alpha}{\omega}\arctan\left(\frac{\alpha u_1 + 2\lambda u}{2\omega u_1}\right) + \ln\left(\alpha u u_1 + u_1^2 + \lambda u^2\right). \tag{46}$$

By solving the differential equation in (27) using $\tilde{E}$, we find the following nonstandard autonomous Lagrangian:

$$L = \frac{2u_1}{\omega u}\arctan\left(\frac{\alpha u + 2u_1}{2u\omega}\right) - \frac{\alpha}{\omega}\arctan\left(\frac{\alpha u_1 + 2\lambda u}{2\omega u_1}\right) - \ln\left(\alpha u u_1 + \lambda u^2 + u_1^2\right) \tag{47}$$

which has previously appeared in the literature [41], though derived through a different methodology.

6. Concluding Remarks

In this work, we introduce a Riemannian metric on an open subset of the first-order jet bundle $J^1(\mathbb{R}, \mathbb{R})$ via an autonomous second-order ODE. We study the geometry of the resulting Riemannian manifold in relation to the integrability of the second-order ODE. Moreover, the notion of energy foliation for autonomous second-order ODEs is defined, and we show its connection to the classical notion of energy in the case of ODEs arising from mechanical systems. Furthermore, we explore the geometric properties of this foliation, proving that it constitutes a minimal foliation.

We apply our framework to ODEs derived from Lagrangian mechanics, providing a conservative approach to certain systems that may initially seem to lack conserved energy. In particular, we examine the damped harmonic oscillator as a key example to illustrate the applicability of our results.

This research not only highlights a link between mathematical analysis, differential geometry, and physics, but also opens interesting directions for further investigation. Future work could explore higher-order ODEs or extend the analysis to Lagrangian systems in higher dimensions. These extensions hold significant potential for a better understanding of both the geometric structure of differential equations and their physical interpretations.

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