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Spectral Properties of the Laplace Operator with Variable Dependent Boundary Conditions in a Disk

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Abstract: In this work, we study the spectral properties of the Laplace operator with variable dependent boundary conditions in a disk. The boundary conditions include periodic and antiperiodic boundary conditions as well as the generalized Samarskii–Ionkin-type boundary conditions. We show eigenfunctions and eigenvalues of these problems in an explicit form. Moreover, the completeness of their eigenfunctions is investigated.

Keywords: Laplace operator; Samarskii–Ionkin-type conditions; completeness property; eigenfunctions; eigenvalues

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1. Introduction

In [1], the authors studied the Poisson equation in a context similar to classical periodic boundary value problems

$$-\Delta u = f(z),\ z \in B_1 \tag{1}$$

in the domain $B_1 = \{z = (x, y) = x + iy \in C : |z| < 1\}$ with the following imposed boundary conditions:

$$u(1,\phi) - (-1)^k u(1,\phi+\pi) = \tau(\phi),\ 0 \le \phi \le \pi, \tag{2}$$

$$\frac{\partial u}{\partial r}(1,\phi) + (-1)^k \frac{\partial u}{\partial r}(1,\phi+\pi) = \nu(\phi),\quad 0 \le \phi \le \pi, \tag{3}$$

where $r = |z|$, $\phi = \arctan(y/x)$, $f(z) \in C^\gamma(\overline{B_1})$, $\tau(\phi) \in C^{1+\gamma}[0,\pi]$ and $\nu(\phi) \in C[0,\pi]$, $0 < \gamma < 1$, $k = 1,2$. In [1], it was shown that these problems are self-adjoint, and all of their eigenvalues and eigenfunctions were explicitly calculated. The problem (1)–(3) is called antiperiodic when $k = 1$ and periodic when $k = 2$.

In [2], the authors considered the Poisson equation with the following symmetry:

$$u(1,\phi) + u(1,2\pi-\phi) = \tau(\phi),\ 0 \le \phi \le \pi, \tag{4}$$

$$\frac{\partial u}{\partial r}(1,\phi) - \frac{\partial u}{\partial r}(1,1,2\pi-\phi) = \nu(\phi),\quad 0 \le \phi \le \pi, \tag{5}$$

where $f(z) \in C^\gamma(\overline{B_1})$, $\tau(\phi) \in C^{1+\gamma}[0,\pi]$, and $\nu(\phi) \in C[0,\pi]$, $0 < \gamma < 1$. It turned out that the eigenvalues of the latter problem with $\tau = \nu = 0$ are the union of all the eigenvalues of the Dirichlet and Neumann problems and for each of them, there is only one corresponding eigenfunction. As for the problem (1)–(3), when $k = 1$, eigenvalues consist of only a part ("half") of the eigenvalues $\mu_k^{(n)}$ of the Dirichlet problem for $n = 2j$ and a part ("half") of the eigenvalues $\mu_k^{(n)}$ of the Neumann problem for $n = 2j + 1$. Moreover, it was observed that each eigenvalue has two corresponding eigenfunctions. We also refer to [2] for the multidimensional case of these problems.

In this paper, we are interested in more general nonlocal boundary value problems that include not only periodic and antiperiodic problems above but also those with Samarskii–Ionkin-type boundary conditions for the Laplace operator. The history of the latter dates back to the early 1970s, when a group of physicists at the USSR PhIAS, led by A.A. Samarskii, addressed a new problem in the nonlinear, nonstationary theory of instability in a plasma current (see, e.g., [3,4] and the review paper [5]). The model case was set up based on the heat equation:

$$u_t - u_{xx} = 0, \quad 0 < x < 1, \quad 0 < t < T,$$

along with an integral condition ensuring that the total heat remains constant:

$$\int_0^1 u(x,t)dx = \text{ const},$$

that is,

$$0 = \frac{d}{dt}\int_0^1 u(x,t)dx = \int_0^1 u_t(x,t)dx = \int_0^1 u_{xx}(x,t)dx.$$

It implies the following two-point boundary condition:

$$u_x(0,t) = u_x(1,t), \quad 0 < t < T.$$

N.I. Ionkin provided a mathematical solution to this problem, which led to these boundary conditions being known as the Samarskii–Ionkin conditions. When the Fourier method is used to solve the resulting problem, it leads to the following Sturm–Liouville problem:

$$\begin{cases} -u''(x) &= \lambda u(x), \quad 0 < x < 1; \\ u'(0) &= u'(1), \quad u(0) = 0. \end{cases}$$

It has been observed that the root subspace of this boundary value problem is composed of a single eigenfunction $u_{n0} = \sin(2\pi n x)$, along with an infinite number of associated functions

$$u_{n1} = \frac{x}{4n\pi}\cos(2\pi n x) + C\sin(2\pi n x), \quad n \in \mathbb{N}.$$

This was unexpected, as prior to that, there had been no examples where infinitely many associated functions existed. Problems with such properties were referred to by V.A. Il'in as significantly non-self-adjoint. To address these, a theory of basicity was developed, based on estimates of a priori type. The key characteristic of such problems is their non-self-adjointness.

In [6], the author investigated the spectral problem for the Laplace operator with the following boundary conditions, treating it as a two-dimensional counterpart to a Samarskii–Ionkin-type problem:

$$u(1,\phi) - \alpha u(1, 2\pi - \phi) = 0, \ 0 \le \phi \le \pi, \ \alpha \ne 1, \tag{6}$$

$$\frac{\partial u}{\partial r}(1,\phi) - \frac{\partial u}{\partial r}(1, 1, 2\pi - \phi) = 0, \ \ 0 \le \phi \le \pi. \tag{7}$$

Note that this problem contains the antiperiodic problem when $\alpha = -1$ from [2] and Samarskii–Ionkin-type problem from [7] when $\alpha = 0$. In the latter case, unlike the classical Samarskii–Ionkin problem, it was shown in [6] that an associated function does exist, so its root subspace consists only of eigenfunctions. Also, observe that in general, the latter problem is non-self-adjoint when $\alpha \ne -1$. Nevertheless, the completeness of their eigenfunctions was proved in [6]. As for the studies of well-posedness with inhomogeneous conditions for the Poisson equation, we can refer to [7] when $\alpha = 0$ as well as to [8,9] with more general α in two and multidimensional cases, respectively. We also refer to [10,11] for other generalizations of (1)–(3). We can also refer to [3,12] for the Tricomi and heat equations with Samarskii–Ionkin-type boundary conditions, respectively.

Studying generalized Samarskii–Ionkin-type problems for the Laplace operator is particularly important because these boundary conditions introduce nonlocal, variable-dependent effects that are encountered in complex physical, biological, and engineering systems. These conditions generalize periodic, antiperiodic and Samarskii–Ionkin-type boundary conditions, making them more adaptable to situations where the boundary behavior depends on the global properties of the solution. For instance, in systems where energy transfer or diffusion processes are subject to feedback mechanisms, such as thermodynamics or chemical kinetics, generalized Samarskii–Ionkin boundary conditions help in modeling the collective behavior at the boundary. This could apply to scenarios like nano-scale systems, where the boundary properties are influenced by the collective behavior of particles or energy states, or quantum systems, where boundary conditions may represent integrated properties of the wave function or potential over a region.

Here, in this note, we are interested in these problems with variable dependent boundary conditions from a spectral point of view. Namely, in B_1, as described in (1)–(3) above, we investigate the following spectral problem for the Laplace operator:

$$-\Delta u(z) = \lambda u(z), |z| < 1 \tag{8}$$

with variable-dependent boundary conditions

$$u(1,\phi) - \alpha(\phi)u(1, 2\pi - \phi) = 0,\ 0 \le \phi \le \pi, \tag{9}$$

$$\frac{\partial u}{\partial r}(1,\phi) - \frac{\partial u}{\partial r}(1, 2\pi - \phi) = 0,\ \ 0 \le \phi \le \pi \tag{10}$$

or

$$u(1,\phi) - u(1, 2\pi - \phi) = 0,\ \ 0 \le \phi \le \pi, \tag{11}$$

$$\frac{\partial u}{\partial r}(1,\phi) - \beta(\phi)\frac{\partial u}{\partial r}(1, 2\pi - \phi) = 0,\ \ 0 \le \phi \le \pi, \tag{12}$$

where $\alpha(\phi) \in C^{1+\gamma}[0,\pi]$, $\beta(\phi) \in C^{\gamma}[0,\pi]$ for $0 < \gamma < 1$.

Nonlocal boundary value problems for elliptic equations, where the boundary conditions are expressed as relationships between the values of an unknown function and its derivatives at different points on the domain boundary, are known as Bitsadze–Samarskii-type problems [13]. Extensive applications of these nonlocal boundary value problems in physics, technology, and various other scientific fields are thoroughly discussed in review papers [14,15].

Studying spectral problems for the Laplace operator with variable-dependent boundary conditions is important due to its broad mathematical significance and practical applications across various fields. For instance, in physics, studying the spectral properties of the Laplace operator with variable boundaries allows for better modeling of quantum systems with dynamic constraints or acoustic systems where the boundary conditions (such as material properties) vary spatially. In engineering, understanding the spectral shifts due to variable boundary conditions can lead to improved design in structures, where boundary properties, like support stiffness, vary. For biological and medical applications, variable boundary conditions can model dynamic environments such as changing tissue properties or brain waves, leading to more accurate simulations.

In the special cases $\alpha(\phi) = -1$ and $\beta(\phi) = -1$, the problems (8)–(10) and (8), (11) and (12) reduce to the antiperiodic and periodic boundary value problems, respectively, from [2]. Moreover, when $\alpha = 0$, it covers the Samarskii–Ionkin-type problem from [7], and its generalized version from [6] when $\alpha(\phi) = \alpha$ is constant.

Our main motivation in these spectral problems is twofold: the non-self-adjointness of the problems and the appearance of a variable-dependent coefficient in the boundary conditions. These two properties make the problems more subtle. For example, because of them, the direct use of the method of separation of variables is impossible. Here, in this note, we propose another method that reduces the solution of the problems to a sequential solution of two classical local boundary value problems. By this method, we calculate eigenfunctions and eigenvalues of the problems (8)–(10) and (8), (11) and (12) in explicit forms. Furthermore, we prove the completeness of their system of eigenfunctions.

2. Main Results

In this section, we discuss our main results. Before stating them, let us introduce necessary notations.

Let L_α denote the closure in $L^2(B_1)$ of the operator corresponding to the differential expression $\ell_1 u = -\Delta u(z)$, acting on the linear manifold of functions $u(z) \in C^{2+\gamma}(B_1)$ that satisfy the following variable-dependent boundary conditions

$$u(1,\phi) - \alpha(\phi)u(1,2\pi-\phi) = 0, \quad \frac{\partial u}{\partial r}(1,\phi) - \frac{\partial u}{\partial r}(1,2\pi-\phi) = 0, \;\; 0 \le \phi \le \pi,$$

where $\alpha(\phi) \in C^{1+\gamma}[0,\pi]$ for $0 < \gamma < 1$.

Similarly, we use L_β to denote the closure in $L^2(B_1)$ of the operator given by $\ell_2 u = -\Delta u(z)$, acting on the linear manifold of functions $u(z) \in C^{2+\gamma}(B_1)$, with the following variable-dependent boundary conditions:

$$u(1,\phi) - u(1,2\pi-\phi) = 0, \quad \frac{\partial u}{\partial r}(1,\phi) - \beta(\phi)\frac{\partial u}{\partial r}(1,2\pi-\phi) = 0, \;\; 0 \le \phi \le \pi,$$

where $\beta(\phi) \in C^{\gamma}[0,\pi]$ for $0 < \gamma < 1$.

Then, we have the following result for the problem L_α:

Theorem 1. *Let $\alpha(\phi) \in C^{1+\gamma}[0,\pi]$ for $0 < \gamma < 1$ with $\alpha(\phi) \neq 1$. Then, we have the following system of eigenfunctions for the operator L_α:*

$$u_k^1(z) = J_k(r\sqrt{\lambda_D})\cos k\phi, \; k = 0,1,2,..., \tag{13}$$

$$u_m^2(z) = J_m(r\sqrt{\lambda_N})\sin m\phi + \frac{a_0}{2}J_0(r\sqrt{\lambda_N}) + \sum_{n=1}^{\infty} a_n J_n(r\sqrt{\lambda_N})\cos n\phi, \tag{14}$$

for all $0 \le \phi \le 2\pi$, $m = 1,2,...$, where a_n is defined by

$$a_n = -\frac{2J_m(\sqrt{\lambda_N})}{\pi J_n(\sqrt{\lambda_N})}\int_0^\pi \frac{1+\alpha(\psi)}{1-\alpha(\psi)}\sin m\psi \cos n\psi d\psi, \; n = 0,1,....$$

Here, $J_i(x)$, for $i = 0,1,\ldots$, denote the Bessel functions, and λ_N and λ_D represent the eigenvalues corresponding to the Neumann and Dirichlet problems for the Laplace equation in the unit disk, respectively.

Moreover, the system of eigenfunctions given by (13) and (14) for the operator L_α is complete in $L^2(B_1)$.

Remark 1. *Note that when $\alpha(\phi) = const$, our result implies Theorem 3.1 [6]. Moreover, the special cases $\alpha(\phi) = 0$ and $\alpha(\phi) = -1$ were considered in [2,16], respectively. Recall that the case $\alpha(\phi) = -1$ for the operator L_α corresponds to the antiperiodic problem from [1,2], which is a self-adjoint problem. Additionally, since L_α has a complete system of eigenfunctions, as given by (13) and (14), in $L^2(B_1)$, we have found all eigenfunctions of the problem. In particular, this means that, compared to the classical Samarskii–Ionkin problem, there are no associated functions. As for the general variable-dependent case, we can refer to the recent work [17] for a similar problem to L_α with an angular derivative instead of the radial derivative. As for applications, this type of nonlocal boundary value problem for elliptic equations is referred to as a Bitsadze–Samarskii-type problem in some references (see, e.g., [13]), and possible applications are discussed in [14,15].*

Now, let us state our result for the problem L_β:

Theorem 2. *Let $\beta(\phi) \in C^\gamma[0,\pi]$ for $0 < \gamma < 1$ with $\beta(\phi) \neq 1$. Then, we have the following system of eigenfunctions for the operator L_β:*

$$u_k^1(z) = J_k(r\sqrt{\lambda_N})\cos k\phi, \; k = 0,1,2,..., \tag{15}$$

$$u_m^2(z) = J_m(r\sqrt{\lambda_D})\sin m\phi + \frac{a_0}{2}J_0(r\sqrt{\lambda_D}) + \sum_{n=1}^{\infty} a_n J_n(r\sqrt{\lambda_D})\cos n\phi, \tag{16}$$

for all $0 \leq \phi \leq 2\pi$, $m = 1,2,...$, where

$$a_n = -\frac{2\sqrt{\lambda_D}J_m'(\sqrt{\lambda_D})}{\pi J_n(\sqrt{\lambda_D})}\int_0^\pi \frac{1+\beta(\psi)}{1-\beta(\psi)}\sin m\psi \cos n\psi d\psi, \; n = 0,1,....$$

Here, $J_i(x)$, λ_N and λ_D are defined as in Theorem 1.

Furthermore, the system of the eigenfunctions given by (15) and (16) for the operator L_β is complete in $L^2(B_1)$.

Remark 2. *Note that the case $\beta(\phi) = 1$ of the operator L_β reduces to the periodic problem from [1,2], which is a self-adjoint problem. As in the previous remark, the completeness of the system of eigenfunctions of the operator L_α implies that the obtained eigenfunctions (15) and (16) constitute all the eigenfunctions of the problem. In particular, this means that there are no associated functions.*

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. We begin the proof by introducing the following auxiliary functions:

$$c(r,\phi) = \frac{1}{2}(u(r,\phi) + u(r,2\pi-\phi)), \quad s(r,\phi) = \frac{1}{2}(u(r,\phi) - u(r,2\pi-\phi)). \tag{17}$$

Taking into account the representation (17) and substituting into the equation and boundary conditions of L_β, we derive the following spectral Neumann problem for the function $s(z)$:

$$\begin{cases} -\Delta s(z) = \lambda s(z), \; z \in B_1; \\ \frac{\partial s}{\partial r}(1,\phi) = 0, \;\; 0 \leq \phi \leq 2\pi. \end{cases} \tag{18}$$

whereas for $c(z)$, one obtains the following spectral Dirichlet problem, which depends on $s(z)$:

$$\begin{cases} -\Delta c(z) = \lambda c(z), \; z \in B_1; \\ c(1,\phi) = \begin{cases} -\frac{1+\alpha(\phi)}{1-\alpha(\phi)}s(1,\phi), & 0 \leq \phi \leq \pi, \\ \frac{1+\alpha(2\pi-\phi)}{1-\alpha(2\pi-\phi)}s(1,\phi), & \pi \leq \phi \leq 2\pi. \end{cases} \end{cases} \tag{19}$$

Now, we split the rest of the proof into the following two cases: $\lambda \neq \lambda_N$ and $\lambda = \lambda_N$.

In the first case, it is easy to note that we have $s(z) = 0$ in $\overline{B_1}$. Then, the spectral Dirichlet problem (19) can be written as follows:

$$\begin{cases} -\Delta c(z) = \lambda c(z), \; z \in B_1; \\ c(1,\phi) = 0, \;\; 0 \leq \phi \leq 2\pi. \end{cases} \tag{20}$$

Due to the property $c(r,\phi) = c(r,2\pi-\phi)$ and the representation (17), we have

$$u_k(z) = c_k(z) = J_k(r\sqrt{\lambda_D})\cos k\phi, k = 0,1,.... \tag{21}$$

In the latter case, $\lambda = \lambda_N$, by taking into account the symmetry $s(r,\phi) = -s(r,2\pi-\phi)$, for the spectral problem (18), we obtain

$$s_m(z) = J_m(r\sqrt{\lambda_N})\sin m\phi, m = 1,2,.... \tag{22}$$

Hence, the spectral Dirichlet problem (19) takes the form

$$\begin{cases} -\Delta c(z) = \lambda_N c(z), \ z \in B_1; \\ c(1,\phi) = \begin{cases} -\frac{1+\alpha(\phi)}{1-\alpha(\phi)} J_m(\sqrt{\lambda_N}) \sin m\phi, \ 0 \le \phi \le \pi, \\ \frac{1+\alpha(2\pi-\phi)}{1-\alpha(2\pi-\phi)} J_m(\sqrt{\lambda_N}) \sin m\phi, \ \pi \le \phi \le 2\pi. \end{cases} \end{cases} \quad (23)$$

Then, we can look for $c(r,\phi)$ in the form

$$c(r,\phi) = \frac{a_0}{2} J_0(r\sqrt{\lambda_N}) + \sum_{n=1}^{\infty} a_n J_n(r\sqrt{\lambda_N}) \cos n\phi \quad (24)$$

since we have $c(r,\phi) = c(r, 2\pi - \phi)$.

Substituting this representation into the boundary condition (23) implies that

$$\begin{aligned} a_n J_n(\sqrt{\lambda_N}) = &-\int_0^{\pi} \frac{1+\alpha(\psi)}{\pi(1-\alpha(\psi))} J_m(\sqrt{\lambda_N}) \sin m\psi \cos n\psi d\psi \\ &+ \int_{\pi}^{2\pi} \frac{1+\alpha(2\pi-\psi)}{\pi(1-\alpha(2\pi-\psi))} J_m(\sqrt{\lambda_N}) \sin m\psi \cos n\psi d\psi \\ &= -\int_0^{\pi} \frac{2(1+\alpha(\psi))}{\pi(1-\alpha(\psi))} J_m(\sqrt{\lambda_N}) \sin m\psi \cos n\psi d\psi, \ n = 0,1,..., \end{aligned}$$

that is,

$$a_n = -\frac{2J_m(\sqrt{\lambda_N})}{\pi J_n(\sqrt{\lambda_N})} \int_0^{\pi} \frac{1+\alpha(\psi)}{1-\alpha(\psi)} \sin m\psi \cos n\psi d\psi, \ n = 0,1,....$$

Thus, the eigenfunctions of the problem L_α are

$$u_k^1(z) = J_k(r\sqrt{\lambda_D}) \cos k\phi, 0 \le \phi \le 2\pi, k = 0,1,2,..., \quad (25)$$

$$u_m^2(z) = J_m(r\sqrt{\lambda_N}) \sin m\phi + \frac{a_0}{2} J_0(r\sqrt{\lambda_N}) + \sum_{n=1}^{\infty} a_n J_n(r\sqrt{\lambda_N}) \cos n\phi \quad (26)$$

for all $0 \le \phi \le 2\pi, m = 1,2,....$

The convergence of the series in (26) can be verified by leveraging the asymptotic properties of the Bessel function and applying the Leibniz criterion.

Now, to complete the proof, it remains to verify that $\{u_k^1\}_{k=0}^{\infty}$ and $\{u_m^2\}_{m=1}^{\infty}$ are complete in $L^2(B_1)$. For this, we observe that

$$\int_{B_1} u_k^1(z) f(z) dz = \int_0^1 \int_0^{\pi} r J_k(r\sqrt{\lambda_D})(f(r,\phi) + f(r, 2\pi - \phi)) \cos k\phi dr d\phi = 0.$$

The completeness of $\{rJ_k(r\sqrt{\lambda_D}) \cos k\phi\}_{k=0}^{\infty}$ in $L^2(B_1^+)$, with $B_1^+ = B_1 \cap \{y > 0\}$, yields

$$f(r,\phi) = -f(r, 2\pi - \phi), \ 0 \le \phi \le \pi. \quad (27)$$

Using this observation, we obtain

$$\begin{aligned} \int_{B_1} u_m^2(z) f(z) dz &= \int_{B_1} \left(J_m(r\sqrt{\lambda_N}) \sin m\phi \right) f(z) dz \\ &+ \int_{B_1} \left(\frac{a_0}{2} J_0(r\sqrt{\lambda_N}) + \sum_{n=1}^{\infty} a_n J_n(r\sqrt{\lambda_N}) \cos n\phi \right) f(z) dz \\ &= \int_0^1 \int_0^{\pi} (f(r,\phi) - f(r, 2\pi - \phi)) r J_m(r\sqrt{\lambda_N}) \sin m\phi dr d\phi = 0. \end{aligned}$$

Now, taking into account the completeness of $\{rJ_m(r\sqrt{\lambda_N})\sin m\phi\}_{m=1}^{m=\infty}$ in $L^2(B_1^+)$, we conclude that

$$f(r,\phi) = f(r, 2\pi - \phi),\ 0 \le \phi \le \pi. \tag{28}$$

A combination of (27) and (28) yields $f(z) = 0$ in B_1, implying that the obtained eigenfunctions (25) and (26) are complete in $L^2(B_1)$. □

Proof of Theorem 2. Similar to the proof of Theorem 1, we will employ the auxiliary functions from (17). A direct calculation shows that these auxiliary functions satisfy the following spectral problems: for the function $s(z)$, we obtain the spectral Dirichlet problem

$$\begin{cases} -\Delta s(z) = \lambda s(z),\ z \in B_1; \\ s(1,\phi) = 0,\ \ 0 \le \phi \le 2\pi \end{cases} \tag{29}$$

and for the function $c(z)$, we have the spectral Neumann problem

$$\begin{cases} -\Delta c(z) = \lambda c(z),\ z \in B_1; \\ \frac{\partial c}{\partial r}(1,\phi) = \begin{cases} -\frac{1+\beta(\phi)}{1-\beta(\phi)}\frac{\partial s}{\partial r}(1,\phi), & 0 \le \phi \le \pi, \\ \frac{1+\beta(2\pi-\phi)}{1-\beta(2\pi-\phi)}\frac{\partial s}{\partial r}(1,\phi), & \pi \le \phi \le 2\pi, \end{cases} \end{cases} \tag{30}$$

which depends on $s(z)$.

Here, we again consider the following two cases: $\lambda \ne \lambda_D$ and $\lambda = \lambda_D$.

In the first case, since (29) is a spectral Dirichlet problem, we obtain $s(z) = 0$ in $\overline{B_1}$. Substituting it into the spectral Neumann problem (30) implies

$$\begin{cases} -\Delta c(z) = \lambda c(z),\ z \in B_1; \\ \frac{\partial c}{\partial r}(1,\phi) = 0,\ \ 0 \le \phi \le 2\pi \end{cases} \tag{31}$$

Taking into account $s(z) = 0$ in $\overline{B_1}$ and $c(r,\phi) = c(r, 2\pi - \phi)$ due to the representation (17), we obtain

$$u_k(z) = c_k(z) = J_k(r\sqrt{\lambda_N})\cos k\phi, k = 0, 1, \ldots. \tag{32}$$

In the latter case, $\lambda = \lambda_D$, now using the symmetry $s(r,\phi) = -s(r, 2\pi - \phi)$, we obtain for the spectral Dirichlet problem (29) that

$$s_m(z) = J_m(r\sqrt{\lambda_D})\sin m\phi, m = 1, 2, \ldots. \tag{33}$$

Therefore, the Neumann problem (30) can be rewritten as

$$\begin{cases} -\Delta c(z) = \lambda_D c(z),\ z \in B_1; \\ \frac{\partial c}{\partial r}(1,\phi) = \begin{cases} -\frac{1+\beta(\phi)}{1-\beta(\phi)}\sqrt{\lambda_D}J_m'(\sqrt{\lambda_D})\sin m\phi,\ 0 \le \phi \le \pi; \\ \frac{1+\beta(2\pi-\phi)}{1-\beta(2\pi-\phi)}\sqrt{\lambda_D}J_m'(\sqrt{\lambda_D})\sin m\phi,\ \pi \le \phi \le 2\pi. \end{cases} \end{cases} \tag{34}$$

Because of the symmetry $c(r,\phi) = c(r, 2\pi - \phi)$, as in (24), we seek $c(r,\phi)$ in the form

$$c(r,\phi) = \frac{a_0}{2}J_0(r\sqrt{\lambda_D}) + \sum_{n=1}^{\infty} a_n J_n(r\sqrt{\lambda_D})\cos n\phi. \tag{35}$$

Plugging this representation into the boundary condition of the problem (34), we obtain

$$a_n J_n(\sqrt{\lambda_D}) = -\int_0^{\pi} \frac{\sqrt{\lambda_D}(1+\beta(\psi))}{\pi(1-\beta(\psi))} J'_m(\sqrt{\lambda_D}) \sin m\psi \cos n\psi d\psi$$
$$+ \int_{\pi}^{2\pi} \frac{\sqrt{\lambda_D}(1+\beta(2\pi-\psi))}{\pi(1-\beta(2\pi-\psi))} J'_m(\sqrt{\lambda_D}) \sin m\psi \cos n\psi d\psi$$
$$= -\int_0^{\pi} \frac{2\sqrt{\lambda_D}(1+\beta(\psi))}{\pi(1-\beta(\psi))} J'_m(\sqrt{\lambda_D}) \sin m\psi \cos n\psi d\psi,\ n = 0,1,...,$$

yielding

$$a_n = -\frac{2\sqrt{\lambda_D} J'_m(\sqrt{\lambda_D})}{\pi J_n(\sqrt{\lambda_D})} \int_0^{\pi} \frac{1+\beta(\psi)}{1-\beta(\psi)} \sin m\psi \cos n\psi d\psi,\ n = 0,1,....$$

Thus, for the spectral problem L_β, we have obtained the following eigenfunctions:

$$u_k^1(z) = J_k(r\sqrt{\lambda_N}) \cos k\phi, 0 \le \phi \le 2\pi, k = 0,1,2,..., \tag{36}$$

$$u_m^2(z) = J_m(r\sqrt{\lambda_D}) \sin m\phi + \frac{a_0}{2} J_0(r\sqrt{\lambda_D}) + \sum_{n=1}^{\infty} a_n J_n(r\sqrt{\lambda_D}) \cos n\phi \tag{37}$$

for all $0 \le r \le 1, 0 \le \phi \le 2\pi, m = 1,2,....$

By using asymptotic forms of the Bessel function and applying the Leibniz criterion, one can show the convergence of the series in (37).

Now, it remains to show that $\{u_k^1(z)\}_{k=0}^{\infty}$ and $\{u_m^2(z)\}_{m=1}^{\infty}$ are complete in $L^2(B_1)$. For this, we note that

$$\int_{B_1} u_k^1(z) f(z) dz = \int_0^1 \int_0^{\pi} r J_k(r\sqrt{\lambda_N})(f(r,\phi) + f(r,2\pi-\phi)) \cos k\phi dr d\phi = 0.$$

Using the completeness of $\{rJ_k(r\sqrt{\lambda_N}) \cos k\phi\}_{k=0}^{k=\infty}$ in $L^2(B_1^+)$, with $B_1^+ = B_1 \cap \{y > 0\}$, one can derive from above that

$$f(r,\phi) = -f(r,2\pi-\phi),\ 0 \le \phi \le \pi. \tag{38}$$

Taking into this account, we calculate

$$\int_{B_1} u_m^2(z) f(z) dz = \int_{B_1} \Big(J_m(r\sqrt{\lambda_D}) \sin m\phi\Big) f(z) dz$$
$$+ \int_{B_1} \left(\frac{a_0}{2} J_0(r\sqrt{\lambda_D}) + \sum_{n=1}^{\infty} a_n J_n(r\sqrt{\lambda_D}) \cos n\phi\right) f(z) dz$$
$$= \int_0^1 \int_0^{\pi} r J_m(r\sqrt{\lambda_D}) \sin m\phi (f(r,\phi) - f(r,2\pi-\phi)) dr d\phi = 0.$$

The completeness of $\{rJ_m(r\sqrt{\lambda_D}) \sin m\phi\}_{m=1}^{m=\infty}$ in $L^2(B_1^+)$ yields

$$f(r,\phi) = f(r,2\pi-\phi),\ 0 \le \phi \le \pi. \tag{39}$$

A combination of (38) and (39) implies $f(z) = 0$ in B_1, concluding that the obtained eigenfunctions (36) and (37) are complete in $L^2(B_1)$, as desired. □

4. Discussion and Conclusions

In this paper, we have studied the spectral properties of the Laplace operator in a disk. More specifically, the paper aims to investigate the non-self-adjointness of the problems and the appearance of a variable-dependent coefficient in a variety of boundary conditions, which include periodic and antiperiodic boundary conditions and Samarskii–Ionkin-type boundary conditions. Due to the non-self-adjoint nature of our problems and the presence of variable-dependent coefficients in the boundary conditions, direct application of the separation of variables is impossible. Therefore, we have proposed a method that reduces the solution of the problems to a sequential solution of two local boundary value problems. By this approach, we have managed to calculate the eigenfunctions and eigenvalues of the problems in explicit forms. Moreover, we have proved the completeness of their system of eigenfunctions, which means we have found all eigenfunctions of the problems. In particular, compared to the classical Samarskii–Ionkin problem, we have shown that there are no associated functions for the problems L_α and L_β. Our results imply the spectral results of [1,2,6–11] when the variable-dependent coefficient is constant. As for applications, such nonlocal boundary value problems for elliptic equations are referred to as Bitsadze–Samarskii-type problems in some references (see, e.g., [13]), and possible applications can be found in [14,15].

As for possible future work, it would be interesting to investigate whether the system of eigenfunctions we have constructed forms an unconditional basis in $L^2(B_1)$. Additionally, considering inverse problems for such problems could also be of interest.

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