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Some Theorems of Uncertain Multiple-Delay Differential Equations

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Abstract: Uncertain differential equations with a time delay, called uncertain-delay differential equations, have been successfully applied in feedback control systems. In fact, many systems have multiple delays, which can be described by uncertain differential equations with multiple delays. This paper defines uncertain differential equations with multiple delays, which are called uncertain multiple-delay differential equations (UMDDEs). Based on the linear growth condition and the Lipschitz condition, the existence and uniqueness theorem of the solutions to the UMDDEs is proven. In order to judge the stability of the solutions to the UMDDEs, the concept of the stability in measure for UMDDEs is presented. Moreover, two theorems sufficient for use as tools to identify the stability in measure for UMDDEs are proved, and some examples are also discussed in this paper.

Keywords: existence and uniqueness theorem; stability in measure; uncertain multiple-delay differential equations; uncertain theory

MSC: 34D20; 93D15

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1. Introduction

The application of multiple-delay differential equations has been observed in various feedback control systems, such as a neural network [1], an antigen-driven T-cell infection system [2], and an epidemiological system [3]. However, these feedback control systems are often influenced by "noise". When the "noise" is modeled using the Wiener process, the multiple-delay differential equations involving the Wiener process, called stochastic multiple-delay differential equations, are employed to describe a range of systems including finance [4], energy control systems [5], and neutral-type systems [6]. Nonetheless, as Liu [7] demonstrated, the Wiener process-based representation of "noise" is untenable, and the Liu process has been successfully proposed as an alternative model for "noise" descriptions.

Differential equations within the Liu process, called uncertain differential equations (UDEs), have been employed in finance [8], game theory [9], ecology [10], and heat conduction [11]. Moreover, the theoretical research on UDEs has been fruitful and includes the existence and uniqueness theorem [12], stability [13], the analytic solution [12], and the numerical solution [14]. However, some feedback control systems have a delay time, such as population ecology, chemical reaction processes, and pharmacokinetics. In these circumstances, a UDE is unsuitable for modeling a feedback control system with a time delay. Therefore, UDEs with a time delay, called uncertain-delay differential equations (UDDEs), were pioneered by Barbacioru [15] and applied to an ecology system [16]. In addition, theoretical research on UDDEs has been successful in areas such as the existence and uniqueness theorem [17], stability theorems [18–21], and parameter estimation [16]. However, some feedback control systems contain multiple delays, and uncertain differential equations with multiple delays can be used to model these systems. Therefore, uncertain differential equations with multiple delays, called uncertain multiple-delay differential equations (UMDDEs), are defined, and the existence and uniqueness theorem of the solutions for these UMDDEs is proven. In addition, the stability in measure for UMDDEs is defined. Based on different Lipschitz conditions, two sufficient theorems for UMDDEs being stable in measure are proven.

The structure of this paper is organized as follows. Section 2 gives the definition and the concept of stabling in measure for UMDDEs and proves the existence and uniqueness theorem, as well as two sufficient theorems of stabling in measure, for the solution of the UMDDEs. The conclusion is given in Section 3.

2. Uncertain Multiple-Delay Differential Equations

In this section, uncertain multiple-delay differential equations (UMDDEs) are defined, and the existence and uniqueness theorem of their solutions is proven. Moreover, the definition of stabling in measure and two sufficient theorems of stabling in measure for UMDDEs are established. In order to describe the environmental noise, Liu [22] introduced the Liu process L_t. Additionally, some correlative theorems are also demonstrated.

Theorem 1 (Yao et al. [13])**.** *Assume that $\mathcal{M}$ is the character of an uncertain measure [22] and $L_t(\zeta)$ is the sample path of the Liu process L_t for each ζ. Then,*

$$\lim_{x\to+\infty} \mathcal{M}\{L_t \le x\} = 1,$$

where $L(\zeta)$ stands for the Lipschitz constant of $L_t(\zeta)$.

Theorem 2 (Chen and Liu [12])**.** *As $L(\zeta)$ stands for the Lipschitz constant of the sample path $L_t(\zeta)$ of the Liu process L_t, and the integrable uncertain process V_t is defined over the interval $[l_1, l_2]$, then we have*

$$\left|\int_{l_1}^{l_2} V_t(\zeta)\mathrm{d}L_t(\zeta)\right| \le L(\zeta)\int_{l_1}^{l_2} |V_t(\zeta)|\mathrm{d}t.$$

Based on the Liu process, uncertain multiple-delay differential equations (UMDDEs) are defined as below.

Definition 1. *Assume that L_t is a Liu process and h_1 and h_2 are two measurable and continuous functions; then,*

$$\mathrm{d}A_t = h_1(t, A_t, A_{t-d_1}, A_{t-d_2}, \ldots, A_{t-d_m})\mathrm{d}t + h_2(t, A_t, A_{t-d_1}, A_{t-d_2}, \ldots, A_{t-d_m})\mathrm{d}L_t \quad (1)$$

is called an uncertain multiple-delay differential equation (UMDDE), where $d_1, d_2, \ldots, d_n$ stands for the delay time and $0 < d_1 < d_2 < \cdots < d_m$.

2.1. Existence and Uniqueness Theorem

This section proves the existence and uniqueness theorem of the solutions to the UMDDE (1).

Theorem 3. *If the coefficients $h_1(t, a_0, a_1, \ldots, a_m)$ and $h_2(t, a_0, a_1, \ldots, a_m)$ of the UMDDE (1) with the initial states K_t in the interval $-d_m \le t \le 0$ satisfy the linear growth condition*

$$|h_1(t, \boldsymbol{a}_1)| + |h_2(t, \boldsymbol{a}_1)| \le N(1 + \sum_{i=0}^{m} |a_{i1}|)$$

and the Lipschitz condition

$$|h_1(t, \boldsymbol{a}_1) - h_1(t, \boldsymbol{a}_2)| + |h_2(t, \boldsymbol{a}_1) - h_2(t, \boldsymbol{a}_2)| \le N\sum_{i=0}^{m} |a_{i1} - a_{i2}|,$$

$\forall a_{i1}, a_{i2} \in R$ $(i = 0, 1, \ldots, m)$, $t \geq 0$, and there is a positive constant N, where $\boldsymbol{a}_1 = (a_{01}, a_{11}, \ldots, a_{m1})$ and $\boldsymbol{a}_2 = (a_{02}, a_{12}, \ldots, a_{m2})$, then the UMDDE (1) has a solution. Moreover, the solution is sample-continuous.

Proof. Assume that UMDDE (1) is in $[0, T]$, where T is any given real number. For each $\zeta \in \Theta$, we set $H_t(\zeta) = (A_t(\zeta), A_{t-d_1}(\zeta), A_{t-d_2}(\zeta), \ldots, A_{t-d_m}(\zeta))$, $H_t^n(\zeta) = (A_t^n(\zeta), A_{t-d_1}^n(\zeta)$, and $A_{t-d_2}^n(\zeta), \ldots, A_{t-d_m}^n(\zeta))$, and have

$$A_t^{(n+1)}(\zeta) = A_0 + \int_0^t h_1(s, H_s^{(n)}(\zeta))\mathrm{d}s + \int_0^t h_2(s, H_s^{(n)}(\zeta))\mathrm{d}L_s(\zeta)$$

and

$$B_t^{(n)}(\zeta) = \sup_{0 \leq s \leq t} \left| A_s^{(n+1)}(\zeta) - A_s^n(\zeta) \right|.$$

Let us use mathematical induction to prove the following formulas for any $n \in N$:

$$B_t^{(n)} \leq \left(1 + (m+1) \sup_{u \in [-d_m, 0]} |A_u|\right) \frac{N^{n+1}(1 + L(\zeta))^{n+1}(m+1)^n}{(n+1)!} t^{n+1}. \tag{2}$$

$L(\zeta)$ is the Lipschitz constant of $L_t(\zeta)$ for all $\zeta \in \Theta$ and n. Due to Formula (2), we know that it satisfies the following inequality:

$$\sum_{n=0}^{\infty} \left(1 + (m+1) \sup_{u \in [-d_m, 0]} |A_u|\right) \frac{N^{n+1}(1 + L(\zeta))^{n+1}(m+1)^n}{(n+1)!} t^{n+1} < +\infty, \forall t \in [0, T].$$

In other words, it satisfies the Weierstrass criterion. Thus, $A_t^{(n)}(\zeta)$ converges uniformly at the given time interval $[0, T]$ and the limit represents $A_t(\zeta)$. Then, we know that

$$A_t(\zeta) = A_0 + \int_0^t h_1(s, H_s(\zeta))\mathrm{d}s + \int_0^t h_2(s, H_s(\zeta))\mathrm{d}L_s(\zeta).$$

The uncertain process A_t is the only solution to the UMDDEs (1). Inequality (2) is proven as below. For any $n = 0$ and set $d_0 = 0$, we know that

$$\begin{aligned}
B_t^{(0)} &= \sup_{0 \leq s \leq t} \left\{ \left| \int_0^s h_1(u, H_0)\mathrm{d}u + \int_0^s h_2(u, H_0)\mathrm{d}L_u(\zeta) \right| \right\} \\
&\leq \sup_{0 \leq s \leq t} \left\{ \int_0^s |h_1(u, H_0)|\mathrm{d}u \right\} + L(\zeta) \sup_{0 \leq s \leq t} \left\{ \int_0^s |h_2(u, H_0)|\mathrm{d}u \right\} \\
&\leq \int_0^t |h_1(u, H_0)|\mathrm{d}u + L(\zeta) \int_0^t |h_2(u, H_0)|\mathrm{d}u \\
&\leq \left(1 + (m+1) \sup_{u \in [-d_m, 0]} |A_u|\right) N(1 + L(\zeta))t.
\end{aligned}$$

Assuming Inequality (2) with n, we know that

$$\begin{aligned}
B_t^{(n)}(\zeta) &= \sup_{0 \leq s \leq t} \left| A_s^{(n+1)}(\zeta) - A_s^{(n)}(\zeta) \right| \\
&\leq \left(1 + (m+1) \sup_{u \in [-d_m, 0]} |A_u|\right) \frac{N^{n+1}(1 + L(\zeta))^{n+1}(m+1)^n}{(n+1)!} t^{n+1}.
\end{aligned}$$

and

$$
\begin{aligned}
B_t^{(n+1)}(\zeta) &= \sup_{0\le s\le t}\left|A_s^{(n+2)} - A_s^{(n+1)}(\zeta)\right| \\
&= \sup_{0\le s\le t}\left|\int_0^s h_1\left(u, H_u^{(n+1)}(\zeta)\right) - h_1\left(u, H_u^{(n)}(\zeta)\right)\mathrm{d}u + \int_0^s h_2\left(u, H_u^{(n+1)}(\zeta)\right) - h_2\left(u, H_u^{(n)}(\zeta)\right)\mathrm{d}L_u(\zeta)\right| \\
&\le \int_0^t \left|h_1\left(u, H_u^{(n+1)}(\zeta)\right) - h_1\left(u, H_u^{(n)}(\zeta)\right)\right|\mathrm{d}u + L(\zeta)\int_0^t \left|h_2\left(u, H_u^{(n+1)}(\zeta)\right) - h_2\left(u, H_u^{(n)}(\zeta)\right)\right|\mathrm{d}u \\
&\le \int_0^t N\sum_{i=0}^{m}\left|A_{u-d_i}^{n+1} - A_{u-\tau_i}^{n}\right|\mathrm{d}u + L(\zeta)\int_0^t N\sum_{i=0}^{m}\left|A_{u-d_i}^{n+1} - A_{u-\tau_i}^{n}\right|\mathrm{d}u \\
&\le (1+L(\zeta))N(m+1)\int_{-d_m}^{t}|A_u^{(n+1)} - A_u^{n}|\mathrm{d}u \\
&\le (1+L(\zeta))N(m+1)\int_0^t\left(1+(m+1)\sup_{u\in[-d_m,0]}|A_u|\right)\frac{N^{n+1}(1+L(\zeta))^{n+1}(m+1)^n}{(n+1)!}u^{n+1}\mathrm{d}u \\
&= \left(1+(m+1)\sup_{u\in[-d_m,0]}|A_u|\right)\frac{N^{n+2}(1+L(\zeta))^{n+2}(m+1)^{n+1}}{(n+2)!}t^{n+2}.
\end{aligned}
$$

Therefore, for any $\zeta \in \Theta$ and $n \in N$, the sample path $A_t^{(n)}(\zeta)$ is converges uniformly in $[0, T]$. If we set the limit as $A_t(\zeta)$, then

$$A_t(\zeta) = A_0 + \int_0^t h_1(s, H_s(\zeta))\mathrm{d}s + \int_0^t h_2(s, H_s(\zeta))\mathrm{d}L_s(\zeta).$$

□

2.2. Stability in Measure

This section gives the definition of the stabling in measure and two sufficient theorems for UMDDE (1) by means of the Gronwall inequality [23].

Definition 2. *For the different states A_{1s} ($s \in [-d_m, 0]$) and A_{2s} ($s \in [-d_m, 0]$), UMDDE (1) has the corresponding solutions A_{1t} and A_{2t}. For any $\varepsilon > 0$, if*

$$\lim_{\sup\limits_{s\in[-d_m,0]}|A_{1s}-A_{2s}|\to 0} \mathcal{M}\{|A_{1t} - A_{2t}| < \varepsilon, \forall t \ge 0\} = 1, \tag{3}$$

then UMDDE (1) is stable in measure.

Theorem 4. *As for its initial state, UMDDE (1) has a unique solution. Let $\boldsymbol{a}_1 = (a_{01}, a_{11}, \ldots, a_{m1})$ and $\boldsymbol{a}_2 = (a_{02}, a_{12}, \ldots, a_{m2})$; the coefficients of UMDDE (1) satisfy the condition*

$$|h_1(t, \boldsymbol{a}_1) - h_1(t, \boldsymbol{a}_2)| \vee |h_2(t, \boldsymbol{a}_1) - h_2(t, \boldsymbol{a}_2)| \le \sum_{i=0}^{m} N_{it}|a_{i1} - a_{i2}|, \tag{4}$$

where $a_i, b_i \in \Re, i = 0, 1, \ldots, m$, and the symbol $\vee$ stands for taking the minimum, and

$$\int_0^{+\infty} N_{it}\mathrm{d}t < +\infty, i = 0, 1, \ldots, m,$$

then UMDDE (1) is stable in measure.

Proof. For the different initial states $\hat{a}_{s1}$ and $\hat{a}_{s2}$, $-\tau_m \leq s \leq 0$, the corresponding solutions to UMDDE (1) are A_t and B_t. Let $\mathbf{A}_{t1} = (A_{t1}, A_{(t-d_1)1}, \ldots, A_{(t-d_m)1})$ and $\mathbf{A}_{t2} = (A_{t2}, A_{(t-d_1)2}, \ldots, A_{(t-d_m)2})$; we know that

$$\begin{cases} \mathrm{d}A_{t1} = h_1(t, \mathbf{A}_{t1})\mathrm{d}t + h_2(t, \mathbf{A}_{t1})\mathrm{d}L_t, t \in (0, +\infty) \\ A_{t1} = \hat{a}_{t1}, t \in [-d_m, 0] \end{cases}$$

and

$$\begin{cases} \mathrm{d}A_{t2} = h_1(t, \mathbf{A}_{t2})\mathrm{d}t + h_2(t, \mathbf{A}_{t2})\mathrm{d}L_t, t \in (0, +\infty) \\ A_{t2} = \hat{a}_{t2}, t \in [-d_m, 0]. \end{cases}$$

Assuming that $L_t(\zeta)$ represents the Lipschitz continuous sample of L_t, we know that

$$A_{t1}(\zeta) = A_{01} + \int_0^t h_1(u, \mathbf{A}_{u1}(\zeta))\mathrm{d}u + \int_0^t h_2(u, \mathbf{A}_{u1}(\zeta))\mathrm{d}L_u(\zeta)$$

and

$$A_{t2}(\zeta) = A_{02} + \int_0^t h_1(u, \mathbf{A}_{u2}(\zeta))\mathrm{d}u + \int_0^t h_2(u, \mathbf{A}_{u2}(\zeta))\mathrm{d}L_u(\zeta).$$

According to the Lipschitz condition (4) and Theorem 2, $L(\zeta)$ is the Lipschitz constant $L_t(\zeta)$ and

$$\begin{aligned} |A_{t1}(\zeta) - A_{t2}(\zeta)| =& \left| \left\{ A_{01} + \int_0^t h_1(u, \mathbf{A}_{u1}(\zeta))\mathrm{d}u + \int_0^t h_2(u, \mathbf{A}_{u1}(\zeta))\mathrm{d}L_u(\zeta) \right\} \right. \\ & \left. - \left\{ A_{02} + \int_0^t h_1(u, \mathbf{A}_{u2}(\zeta))\mathrm{d}u + \int_0^t h_2(u, \mathbf{A}_{u2}(\zeta))\mathrm{d}L_u(\zeta) \right\} \right| \\ \leq& |A_{01} - A_{02}| + \int_0^t \left\{ \sum_{i=0}^m N_{iu} |A_{(u-d_i)1}(\zeta) - A_{(u-d_i)2}(\zeta)| \right\} \mathrm{d}u \\ & + \int_0^t L(\zeta) \left\{ \sum_{i=0}^m N_{iu} |A_{(u-d_i)1}(\zeta) - A_{(u-d_i)2}(\zeta)| \right\} \mathrm{d}u \\ =& |A_{01} - A_{02}| + (1 + L(\zeta)) \left\{ \sum_{i=0}^m \int_0^t N_{iu} |A_{(u-d_i)1}(\zeta) - A_{(u-d_i)2}(\zeta)| \mathrm{d}u \right\}. \end{aligned}$$

Based on Condition (4),

$$\int_0^{+\infty} N_{jt}\mathrm{d}t < +\infty, j = 0, 1, \ldots, m.$$

So, $M_1 > 0$ exists and we know that

$$\int_0^{+\infty} N_{jt}\mathrm{d}t < M_1, j = 0, 1, \ldots, m.$$

By setting $\eta = u - \tau_i$, we know that

$$\begin{aligned} &\int_0^t N_{iu} |A_{(u-d_i)1}(\zeta) - A_{(u-d_i)2}(\zeta)| \mathrm{d}u \\ &\quad = \int_{-d_i}^{t-d_i} N_{i(\eta+d_i)} |A_{\eta 1}(\zeta) - A_{\eta 2}(\zeta)| \mathrm{d}\eta \\ &\quad \leq \sup_{s \in [-d_i, 0]} \{|A_{s1}(\zeta) - A_{s2}(\zeta)|\} \int_0^{\tau_i} N_{i\eta}\mathrm{d}\eta + \int_0^t N_{i(\eta+d_i)} |A_{\eta 1}(\zeta) - A_{\eta 2}(\zeta)| \mathrm{d}\eta \\ &\quad \leq M_1 \sup_{s \in [-d_i, 0]} \{|A_{s1}(\zeta) - A_{s2}(\zeta)|\} + \int_0^t N_{i(\eta+d_i)} |A_{\eta 1}(\zeta) - A_{\eta 2}(\zeta)| \mathrm{d}\eta. \end{aligned}$$

and

$$
\begin{aligned}
|A_{t1}(\zeta)-A_{t2}(\zeta)| \leq & |A_{01}-A_{02}|+(1+L(\zeta))\left\{\sum_{i=0}^{m}\int_0^t N_{iu}|A_{(u-d_i)1}(\zeta)-A_{(u-d_i)2}(\zeta)|\mathrm{d}u\right\} \\
\leq & |A_{01}-A_{02}|+(1+L(\zeta))\left\{\sum_{i=0}^{m}\left(M_1 \sup_{s\in[-d_i,0]}\{|A_{s1}(\zeta)-A_{s2}(\zeta)|\}\right.\right. \\
& \left.\left.+\int_0^t N_{i(\eta+d_i)}|A_{\eta 1}(\zeta)-A_{\eta 2}(\zeta)|\mathrm{d}\eta\right)\right\} \\
\leq & |A_{01}-A_{02}|+(1+L(\zeta))\left\{M_1(m+1)\sup_{s\in[-d_m,0]}\{|A_{s1}(\zeta)-A_{s2}(\zeta)|\}\right. \\
& \left.+\int_0^t\left(\sum_{i=0}^{m}N_{i(\eta+d_i)}\right)|A_{\eta 1}(\zeta)-A_{\eta 2}(\zeta)|\mathrm{d}\eta\right\} \\
\leq & \{(1+L(\zeta))M_1(m+1)+1\}\sup_{s\in[-\tau_m,0]}\{|A_{s1}(\zeta)-A_{s2}(\zeta)|\} \\
& +(1+L(\zeta))\int_0^t\left(\sum_{i=0}^{m}N_{i(\eta+d_i)}\right)|A_{\eta 1}(\zeta)-A_{\eta 2}(\zeta)|\mathrm{d}\eta.
\end{aligned}
$$

According to the Gronwall inequality [23], we know that

$$|A_{t1}(\zeta)-A_{t2}(\zeta)| \leq \sup_{s\in[-d_m,0]}\{|A_{s1}(\zeta)-A_{s2}(\zeta)|\}\exp(2M_1(m+1)(1+L(\zeta))).$$

With the help of Theorem 1, $\forall\varepsilon>0$, and $R_1>0$, it follows that

$$\mathcal{M}\{\zeta|L(\zeta)\leq R_1\}\geq 1-\varepsilon.$$

Set

$$\theta_1=\exp(-2M_1(1+m)(1+R_1))\varepsilon.$$

If

$$\sup_{s\in[-d_m,0]}\{|A_{s1}-A_{s2}|\}\leq\theta_1,$$

then we know that

$$
\begin{aligned}
\mathcal{M}\{|A_{t1}-A_{t2}|\leq\varepsilon\} & \geq \mathcal{M}\left\{\sup_{s\in[-d_m,0]}\{|A_{s1}-A_{s2}|\}\exp(2M_1(1+m)(1+L(\zeta)))\leq\varepsilon\right\} \\
& \geq \mathcal{M}\{\zeta|L(\zeta)\leq R_1\} \\
& \geq 1-\varepsilon.
\end{aligned}
$$

If

$$\sup_{s\in[-d_m,0]}\{|A_{s1}-A_{s2}|\}\to 0,$$

we know that

$$\mathcal{M}\{|A_{t1}-A_{t2}|\leq\varepsilon\}\to 1, \forall t>0.$$

Thus

$$\lim_{\sup\limits_{s\in[-d_m,0]}\{|A_{s1}-A_{s2}|\}\to 0}\mathcal{M}\{|A_{t1}-A_{t2}|\leq\varepsilon,\forall t\geq 0\}=1.$$

□

Example 1. *Consider the UMDDE*

$$\mathrm{d}A_t = \cos(x)\exp(-\frac{x}{2})A_{t-1}\mathrm{d}t + \frac{\sin^2(x)}{t(1+(\ln t)^2)}A_{t-2}\mathrm{d}L_t, \tag{5}$$

where A_{t-1} and A_{t-2} stand for the delay term. By setting $h_1 = \cos(x)\exp(-\frac{x}{2})b_1$ and $h_2 = \frac{\sin^2(x)}{t(1+(\ln t)^2)}a_2$, we obtain

$$N_{1t} = \cos(x)\exp(-\frac{x}{2}), N_{2t} = \frac{\sin^2(x)}{t(1+(\ln t)^2)},$$

then

$$\left|\int_0^{+\infty} N_{1t}\mathrm{d}t\right| \le \int_0^{+\infty}\left|\cos(x)\exp(-\frac{x}{2})\right|\mathrm{d}t \le 2 < +\infty$$

and

$$\left|\int_0^{+\infty} N_{2t}\mathrm{d}t\right| \le \int_0^{+\infty}\left|\frac{\sin^2(x)}{t(1+(\ln t)^2)}\right|\mathrm{d}t \le \frac{\pi}{2} < +\infty.$$

Based on Theorem 4, UMDDE (5) is stable in measure.

Corollary 1. *Consider the UMDDE*

$$\mathrm{d}A_t = (n_{0t}A_t + n_{1t}A_{t-d_1} + \cdots + n_{mt}A_{t-d_m} + \hat{l}_t)\mathrm{d}t + (n_{0t}A_t + n_{1t}A_{t-d_1} + \cdots + n_{mt}A_{t-d_m} + \check{l}_t)\mathrm{d}L_t \tag{6}$$

satisfying

$$\int_0^{+\infty} n_{it}\mathrm{d}t < +\infty, i = 0, 1, \ldots, m, \tag{7}$$

where $\hat{l}_t$ and $\check{l}_t$ are the real functions; if ao, then UMDDE (6) is stable in measure.

Proof. Let

$$h_1 = n_{0t}b_0 + n_{1t}b_1 + \cdots + n_{mt}b_m + \hat{l}_t$$
$$h_2 = n_{0t}b_0 + n_{1t}b_1 + \cdots + n_{mt}b_m + \check{l}_t,$$

we obtain

$$|h_1(t,\mathbf{b}_1) - h_1(t,\mathbf{b}_2)| \vee |h_2(t,\mathbf{b}_1) - h_2(t,\mathbf{b}_2)| \le \sum_{i=0}^{m} n_{it}|b_{i1} - b_{i2}|,$$

where $\mathbf{b}_i = (b_{1i}, b_{2i}$ and $\ldots, b_{ni}), i = 1, 2$. Based on the condition (7), this satisfies the condition (4). Therefore, UMDDE (6) is stable in measure. □

Example 2. *Consider the UMDDE*

$$\mathrm{d}A_t = (\cos(x)A_{t-1} + \sin(x)\exp(-x)A_{t-4})\mathrm{d}t + (\cos(x)A_{t-1} + \sin(x)\exp(-x)A_{t-4})\mathrm{d}L_t, \tag{8}$$

where A_{t-1} and A_{t-4} stand for the delay term. Let

$$h_1 = \cos(x)a + \sin(x)\exp(-x)b, \quad h_2 = \cos(x)a + \sin(x)\exp(-x)b.$$

Setting

$$n_{1t} = \cos(x), n_{2t} = \sin(x)\exp(-x),$$

we find that

$$\left|\int_0^{+\infty} n_{1t}\mathrm{d}t\right| = \left|\int_0^{+\infty}\cos(x)\mathrm{d}t\right| \le 2 < +\infty,$$

$$\left|\int_0^{+\infty} n_{2t}\mathrm{d}t\right| \leq \int_0^{+\infty} |\sin(x)\exp(-x)|\mathrm{d}t \leq \int_0^{+\infty} |\exp(-x)|\mathrm{d}t = 1 < +\infty.$$

By Corollary 1, UMDDE (8) is then stable in measure.

Theorem 5. *If UMDDE (1) with a given initial condition has a unique solution, by setting $\boldsymbol{a} = (a_0, a_1, \ldots, a_m)$ and $\boldsymbol{b} = (b_0, b_1, \ldots, b_m)$, the coefficients of UMDDE (1) satisfy the conditions*

$$\begin{aligned} |h_1(t,\boldsymbol{a}) - h_1(t,\boldsymbol{b})| &\leq \sum_{i=0}^{m} F_{it}|a_i - b_i| \\ |h_2(t,\boldsymbol{a}) - h_2(t,\boldsymbol{b})| &\leq \sum_{i=0}^{m} G_{it}|a_i - b_i|, \end{aligned} \tag{9}$$

where $a_i, b_i \in \Re, i = 0, 1, \ldots, m$, and the symbol $\vee$ stands for taking the minimum,

$$\int_0^{+\infty} N_{it}\mathrm{d}t < +\infty$$

$$\int_0^{+\infty} F_{jt}\mathrm{d}t < +\infty, \int_0^{+\infty} G_{jt}\mathrm{d}t < +\infty, j = 0, 1, \ldots, m,$$

then UMDDE (1) is stable in measure.

Proof. According to the initial states a_{1s} and a_{2s}, $-d_m \leq s \leq 0$, the corresponding solutions for UMDDE (1) are A_{1t} and B_{1t}. By setting $\mathbf{A}_{1t} = (A_{1t}, A_{1(t-d_1)}, A_{1(t-d_2)}, \ldots, A_{1(t-d_m)})$ and $\mathbf{A}_{2t} = (A_{2t}, A_{2(t-d_1)}, A_{2(t-d_2)}, \ldots, A_{2(t-d_m)})$, we find that

$$\begin{cases} \mathrm{d}A_{1t} = h_1(t, \mathbf{A}_{1t})\mathrm{d}t + h_2(t, \mathbf{A}_{1t})\mathrm{d}L_t, t \in (0, +\infty) \\ A_{1t} = a_{1t}, t \in [-d_m, 0] \end{cases}$$

and

$$\begin{cases} \mathrm{d}A_{2t} = h_1(t, \mathbf{A}_{2t})\mathrm{d}t + h_2(t, \mathbf{A}_{2t})\mathrm{d}L_t, t \in (0, +\infty) \\ A_{2t} = a_{2t}, t \in [-d_m, 0]. \end{cases}$$

Assuming that $L_t(\zeta)$ is the continuous Lipschitz sample of L_t, we obtain

$$A_{1t}(\zeta) = A_{10} + \int_0^t h_1(u, \mathbf{A}_{1u}(\zeta))\mathrm{d}u + \int_0^t h_2(u, \mathbf{A}_{1u}(\zeta))\mathrm{d}L_u(\zeta)$$

and

$$A_{2t}(\zeta) = A_{20} + \int_0^t h_1(u, \mathbf{A}_{2u}(\zeta))\mathrm{d}u + \int_0^t h_2(u, \mathbf{A}_{2u}(\zeta))\mathrm{d}L_u(\zeta).$$

Moreover, $L(\zeta)$ represents the Lipschitz constants of $L_t(\zeta)$. By applying the Lipschitz condition (9) and Theorem 2,

$$\begin{aligned} |A_{2t}(\zeta) - A_{2t}(\zeta)| =& \Bigg|\left\{A_{10} + \int_0^t h_1(u, \mathbf{A}_{1u}(\zeta))\mathrm{d}u + \int_0^t h_2(u, \mathbf{A}_{1u}(\zeta))\mathrm{d}L_u(\zeta)\right\} \\ &- \left\{A_{20} + \int_0^t h_1(u, \mathbf{A}_{2u}(\zeta))\mathrm{d}u + \int_0^t h_2(u, \mathbf{A}_{2u}(\zeta))\mathrm{d}L_u(\zeta)\right\}\Bigg| \\ \leq& |A_{10} - A_{20}| + \int_0^t \left\{\sum_{i=0}^{m} F_{iu}|A_{1(u-d_i)}(\zeta) - A_{2(u-d_i)}(\zeta)|\right\}\mathrm{d}u \\ &+ \int_0^t L(\zeta)\left\{\sum_{i=0}^{m} G_{iu}|A_{1(u-d_i)}(\zeta) - A_{2(u-d_i)}(\zeta)|\right\}\mathrm{d}u \\ \leq& |A_{10} - A_{20}| + (1 + L(\zeta))\left\{\sum_{i=0}^{m} \int_0^t (F_{iu} + G_{iu})|A_{1(u-d_i)}(\zeta) - A_{2(u-d_i)}(\zeta)|\mathrm{d}u\right\}. \end{aligned}$$

Based on Condition (9),

$$\int_0^{+\infty} F_{it}\mathrm{d}t < +\infty, \int_0^{+\infty} G_{it}\mathrm{d}t < +\infty, i = 0, 1, \ldots, m.$$

Then, as $M_2 > 0$,

$$\int_0^{+\infty} F_{it}\mathrm{d}t < M_2, \int_0^{+\infty} G_{it}\mathrm{d}t < M_2, i = 0, 1, \ldots, m.$$

Setting $\eta = u - \tau_i$, we obtain

$$\begin{aligned}
&\int_0^t F_{iu}|A_{1(u-d_i)}(\zeta) - A_{2(u-d_i)}(\zeta)|\mathrm{d}u \\
&\quad = \int_{-d_i}^{t-d_i} F_{i(\eta+d_i)}|A_{1\eta}(\zeta) - A_{2\eta}(\zeta)|\mathrm{d}\eta \\
&\quad \le \sup_{s\in[-d_i,0]}\{|A_{1s}(\zeta) - A_{2s}(\zeta)|\} \int_0^{d_i} F_{i\eta}\mathrm{d}\eta + \int_0^t F_{i(\eta+d_i)}|A_{1\eta}(\zeta) - A_{2\eta}(\zeta)|\mathrm{d}\eta \\
&\quad \le M_2 \sup_{s\in[-d_i,0]}\{|A_{1s}(\zeta) - A_{2s}(\zeta)|\} + \int_0^t F_{i(\eta+d_i)}|A_{1\eta}(\zeta) - A_{2\eta}(\zeta)|\mathrm{d}\eta.
\end{aligned}$$

Similarly,

$$\int_0^t M_{iu}|A_{1(u-d_i)}(\zeta) - A_{2(u-d_i)}(\zeta)|\mathrm{d}u \le M_2 \sup_{s\in[-d_i,0]}\{|A_{1s}(\zeta) - A_{2s}(\zeta)|\} + \int_0^t G_{i(\eta+d_i)}|A_{1\eta}(\zeta) - A_{2\eta}(\zeta)|\mathrm{d}\eta.$$

Therefore,

$$\begin{aligned}
|A_{1t}(\zeta) - A_{2t}(\zeta)| &\le |A_{10} - B_{20}| + (1 + L(\zeta))\left\{\sum_{i=0}^m \int_0^t (F_{iu} + G_{iu})|A_{1(u-d_i)}(\zeta) - A_{2(u-d_i)}(\zeta)|\mathrm{d}u\right\} \\
&\le |A_{10} - B_{20}| + (1 + L(\zeta))\left\{2M_2(m+1) \sup_{s\in[-d_m,0]}\{|A_{1s}(\zeta) - A_{2s}(\zeta)|\}\right. \\
&\qquad \left. + \int_0^t \left(\sum_{i=0}^m (F_{i(\eta+d_i)} + G_{i(\eta+d_i)})\right)|A_{1\eta}(\zeta) - A_{2\eta}(\zeta)|\mathrm{d}\eta\right\} \\
&\le \{2(1 + L(\zeta))M_2(m+1) + 1\} \sup_{s\in[-d_m,0]}\{|A_{1s}(\zeta) - A_{2s}(\zeta)|\} \\
&\qquad + (1 + L(\zeta)) \int_0^t \left(\sum_{i=0}^m (F_{i(\eta+d_i)} + G_{i(\eta+d_i)})\right)|A_{1\eta}(\zeta) - A_{2\eta}(\zeta)|\mathrm{d}\eta.
\end{aligned}$$

According the Gronwall inequality [23],

$$\begin{aligned}
|A_{1t}(\zeta) - A_{2t}(\zeta)| &\le \{2(1 + L(\zeta))M_2(m+1) + 1\} \sup_{s\in[-d_m,0]}\{|A_{1s}(\zeta) - A_{2s}(\zeta)|\} \\
&\qquad \cdot \exp\left((1 + L(\zeta)) \int_0^t \sum_{i=0}^m (F_{i(\eta+\tau_i)} + G_{i(\eta+d_i)})\mathrm{d}\eta\right) \\
&\le \{2(1 + L(\zeta))M_2(m+1) + 1\} \sup_{s\in[-d_m,0]}\{|A_{1s}(\zeta) - A_{2s}(\zeta)|\} \\
&\qquad \cdot \exp((m+1)(1 + L(\zeta))M_2) \\
&\le \sup_{s\in[-d_m,0]}\{|A_{1s}(\zeta) - A_{2s}(\zeta)|\} \exp(3M_2(m+1)(1 + L(\zeta))).
\end{aligned}$$

With the help of Theorem 1, $\forall\varepsilon > 0$, $P > 0$ and

$$\mathcal{M}\{\zeta | L(\zeta) \le P\} \ge 1 - \varepsilon.$$

Set

$$\theta = \exp(-3M_2(1+m)(1+P))\varepsilon.$$

If

$$\sup_{s\in[-d_m,0]} \{|A_{1s} - A_{2s}|\} \le \theta,$$

we know that

$$\begin{aligned}\mathcal{M}\{|A_{1t} - A_{2t}| \le \varepsilon\} &\ge \mathcal{M}\left\{ \sup_{s\in[-d_m,0]} \{|A_{1s} - A_{2s}|\} \exp(3M_2(1+m)(1+L(\zeta))) \le \varepsilon \right\} \\ &\ge \mathcal{M}\{\zeta | L(\zeta) \le P\} \ge 1 - \varepsilon.\end{aligned}$$

In other words, if

$$\sup_{s\in[-d_m,0]} \{|A_{1s} - A_{2s}|\} \to 0,$$

we obtain

$$\mathcal{M}\{|A_{1t} - B_{1t}| \le \varepsilon\} \to 1, \forall t > 0.$$

Therefore,

$$\lim_{\sup_{s\in[-d_m,0]} \{|A_{1s} - A_{2s}|\} \to 0} \mathcal{M}\{|A_{1t} - A_{2t}| \le \varepsilon, \forall t \ge 0\} = 1.$$

□

Remark 1. *Actually, Theorem 4 and Theorem 5 are equivalent. If Inequality (4) is established, we can set $F_{it} = G_{it} = N_{it}$ and $i = 0, 1, 2, \ldots, m$, and then Inequality (9) is established. But, if Inequality (9) is established, we set $N_{jt} = F_{jt} + G_{jt}$ and $j = 0, 1, 2, \ldots, m$, and then Inequality (4) is established.*

Corollary 2. *Assume the UMDDE*

$$\mathrm{d}A_t = (a_{0t}A_t + a_{1t}A_{t-d_1} + \cdots + a_{mt}A_{t-d_m} + a_t)\mathrm{d}t + (b_{0t}A_t + b_{1t}A_{t-d_1} + \cdots + b_{mt}A_{t-d_m} + b_t)\mathrm{d}L_t \tag{10}$$

satisfies

$$\int_0^{+\infty} a_{it}\mathrm{d}t < +\infty, \int_0^{+\infty} b_{it}\mathrm{d}t < +\infty, i = 0, 1, \ldots, m, \tag{11}$$

where a_{it} and b_{it} are real-valued functions and $i = 0, 1, \ldots, m$; then, UMDDE (10) is stable in measure.

Proof. Set

$$h_1 = a_{0t}a_0 + a_{1t}a_1 + \cdots + a_{mt}a_m + a_t, h_2 = b_{0t}b_0 + b_{1t}b_1 + \cdots + b_{mt}b_m + b_t,$$

then

$$N_{it} = a_{it}, M_{it} = b_{it}, i = 0, 1, 2, \ldots, m.$$

By applying Condition (11) and Theorem 5, UMDDE (10) is stable in measure. □

Example 3. *Consider the UMDDE*

$$\mathrm{d}A_t = \left(t^2 \exp(-t^3)A_{t-1} + \frac{t}{1+t^4}A_{t-2}\right)\mathrm{d}t + (\exp(-t)\cos(t)A_{t-1} + \sin(-t)A_{t-2})\mathrm{d}L_t, \tag{12}$$

where A_{t-1} and A_{t-2} stand for the delay term. Set

$$h_1(t, a_1, a_2) = t^2 \exp(-t^3)a + \frac{t}{1+t^4}b$$

and

$$h_2(t, a_1, a_2) = \exp(-t)\cos(t)a + \sin(-t)b,$$

and we obtain

$$\int_0^{+\infty} t^2 \exp(-t^3)\mathrm{d}t = \frac{1}{3} < +\infty, \int_0^{+\infty} \frac{t}{1+t^4}\mathrm{d}t = \frac{\pi}{4} < +\infty,$$

$$\left|\int_0^{+\infty} \exp(-t)\cos(t)\mathrm{d}t\right| \leq 1 < +\infty, \left|\int_0^{+\infty} \sin(-t)\mathrm{d}t\right| \leq 2 < +\infty.$$

According to Corollary 2, UMDDE (12) is then stable in measure.

3. Conclusions

In order to model a feedback control system with multiple delays, uncertain multiple-delay differential equations (UMDDEs) were defined in this paper. Moreover, the existence and uniqueness theorem for the solutions to these UMDDEs was proven. In order to judge the stability of the solutions to these UMDDEs, the concept of stabling in measure was provided. Meanwhile, two sufficient theorems were proven to testify the stability in measure of the solutions to the UMDDEs.

Based on these uncertain multiple-delay differential equations, the stability in mean, stability in p-moment, numerical methods, uncertain multiple-delay Logistic models, parameter estimations, and numerical simulations can be investigated in the future.

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