

Article

Bounds for the Energy of Hypergraphs

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Abstract: The concept of the energy of a graph has been widely explored in the field of mathematical chemistry and is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix. The energy of a hypergraph is the trace norm of its connectivity matrices, which generalize the concept of graph energy. In this paper, we establish bounds for the adjacency energy of hypergraphs in terms of the number of vertices, maximum degree, eigenvalues, and the norm of the adjacency matrix. Additionally, we compute the sum of squares of adjacency eigenvalues of linear k -hypergraphs and derive its bounds for k -hypergraph in terms of number of vertices and uniformity of the k -hypergraph. Moreover, we determine the Nordhaus–Gaddum type bounds for the adjacency energy of k -hypergraphs.

Keywords: hypergraph; adjacency matrix; adjacency energy; Frobenius norm; singular values

MSC: 05C65; 05C50; 15A18

1. Introduction

A hypergraph $G^* = (V, E)$ consists of a pair, where $V = \{v_1, v_2, v_3, \dots, v_n\}$ is the vertex set and $E = \{e_1, e_2, e_3, \dots, e_t\}, t \in \mathbb{N}$ is the hyperedge set of G^* . Each hyperedge in $E(G^*)$ ($|e_i| \geq 2$) is a nonempty subset of the vertex set $V(G^*)$. A hypergraph is said to be a linear hypergraph if $|e_i \cap e_j| \leq 1$ for all $i \neq j$. A hypergraph G^* is a k -uniform hypergraph (or k -hypergraph) [1,2] if the cardinality of each of its hyperedges is k where $k \geq 2$. When $k = 2$, it turns into an ordinary graph. The degree of a vertex $v \in V$, $d(v)$ is defined as the number of hyperedges that contain the vertex v . A hypergraph in which every vertex $v_i \in V$ has degree r is said to be an r -regular hypergraph. If a hypergraph is both k -uniform and r -regular, we refer to it as a (k, r) -regular hypergraph (or (k, r) -hypergraph). In [2], the authors focus on the characteristics of (k, r) -hypergraphs. A k -hypergraph G^* with n vertices is said to be a complete k -hypergraph K_n^k if $E(G^*)$ is the collection of all possible k -subsets of $V(G^*)$ [3]. In such a hypergraph, there are $\binom{n}{k}$ hyperedges. The complement of k -hypergraph $G^* = (V, E)$ is the k -hypergraph $\overline{G^*} = (V, \overline{E})$, where the vertex set remains the same and the hyperedge set $\overline{E} = E(K_n^k) \setminus E$. The adjacency matrix [4] of G^* , $A(G^*) = (a_{ij})$, is a square matrix of order n such that, for all $v_i, v_j \in V$,

$$a_{ij} = \begin{cases} |\{e_k \in E(G^*) : \{v_i, v_j\} \subset e_k\}| & , v_i \neq v_j, k \in [1, n] \\ 0 & , v_i = v_j \end{cases} .$$

One of the earliest applications of spectral graph theory is the study of molecular orbital energy levels of π -electrons in conjugated hydrocarbons [5]. It involves graphs representing conjugated hydrocarbons and approximating the total π -electron energy from their eigenvalues. In 1978, Gutman [6] introduced the concept of graph energy, further expanding this area of study. Later, Nikiforov [7] extended the concept of graph energy to matrices. The energy of a matrix M is the sum of its singular values, which is also known



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as trace norm of M , $\|M\|_*$. The singular values of a matrix M are the square roots of the eigenvalues of M^*M , where M^* is the conjugate transpose of M . The singular values of a real symmetric matrix are the moduli of its eigenvalues. Graph energy is one of the most studied spectrum-based invariants in the literature. The first upper bound on energy of a graph was established by McClelland [8]. Later, Koolen and Moulton [9] improved this bound. Recent studies have indicated an increasing interest in the energy of hypergraphs, where the energy of G^* is defined based on the connectivity matrices of G^* .

In 2020, Cardoso and Trevisan [10] introduced the concept of k -hypergraph energy based on the incidence energy and the signless Laplacian energy of k -hypergraphs. In [11], the authors provided several significant results on the adjacency energy of hypergraphs. In the past few years, studies have expanded to explore Laplacian energy and distance energy [12,13] of hypergraphs. Motivated by these studies, the present paper aims to establish bounds of adjacency energy $\mathcal{E}(G^*)$ of hypergraph G^* . Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n$ be the eigenvalues of adjacency matrix of G^* , and also let $|\lambda_1^*| \geq |\lambda_2^*| \geq |\lambda_3^*| \geq \dots \geq |\lambda_n^*| \geq 0$ be the decreasing arrangement of the absolute values of the eigenvalues of $A(G^*)$, then $\mathcal{E}(G^*) = \sum_{i=1}^n |\lambda_i|$. The Frobenius norm of a matrix $M = (m_{ij})_{n \times n}$ is defined as $\|M\|_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^n m_{ij}^2} = \sqrt{(M^T M)}$, while the max norm of a matrix given by $\|M\|_{max} = \max_{i,j} |m_{ij}|$.

This paper is devoted to the study of the energy of hypergraphs. Because determining the exact values for the energy of a hypergraph is challenging, it is useful to obtain bounds based on the structural and spectral properties of hypergraphs. In Section 3, we obtain various bounds for the adjacency energy of hypergraphs in terms of the number of vertices, maximum degree, eigenvalues, and Frobenius norm of the adjacency matrix. In hypergraph theory, giving an expression for the number of closed walks of length 2 (that is, trace $(A(G^*)^2)$) is a complex task. Using graph-theoretical techniques, we determine the number of closed walks of length 2 for a linear k -hypergraph. Additionally, we compute bounds for the number of closed walks of length 2 of the k -hypergraphs; this modifies the existing bounds for the energy of hypergraphs, which makes it computationally easier. In Section 4, we derive Nordhaus–Gaddum-type inequalities for the adjacency energy of k -hypergraphs. Throughout this paper, we consider only simple hypergraphs.

2. Preliminaries

This section gives basic definitions, results, and notations used in the main results.

Lemma 1 ([14]). *If G^* be a k -hypergraph of n vertices and m edges. Then*

$$\sum_{i=1}^n d(v_i) = km.$$

Definition 1 ([15]). *Let $P = (p_{ij})$ and Q be two matrices of any order. Then the Kronecker product of P and Q is a block matrix,*

$$P \otimes Q = (p_{ij}Q).$$

Lemma 2 ([16]). *Let $M = \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}$ is a symmetric 2×2 block matrix. Then the spectrum of M is the union of spectra of $M_1 + M_2$ and $M_1 - M_2$.*

Lemma 3 ([17]). *If $a_1 \geq a_2 \geq \dots \geq a_n$ are real numbers such that $\sum_{i=1}^n a_i = 0$, then*

$$a_1 \leq \sqrt{\frac{n-1}{n} \sum_{i=1}^n a_i^2}.$$

Equality holds if and only if $a_2 = \dots = a_n = -\frac{a_1}{n-1}$.

Theorem 1 ([18]). Let v_i and v_j be two vertices of a hypergraph G^* . Then the number of walks of length l from v_i to v_j of G^* is $a_{ij}^{(l)}$, the $(i, j)^{th}$ entry of the matrix A^l .

Lemma 4 ([11]). Let Δ and δ be the maximum and minimum degree of a k -hypergraph with spectral radius λ_1 . Then

$$\delta(k - 1) \leq \lambda_1 \leq \Delta(k - 1).$$

Lemma 5 ([19]). Let M be any matrix of order n , $P_M(\lambda)$ be the characteristic polynomial of M , and s_t be the t -subset of $\{1, 2, 3, \dots, n\}$. Then the coefficient c_t of λ^{n-t} in $P_M(\lambda)$ is

$$c_t = (-1)^n \sum_{M_i \in \mathcal{M}} \det(M_i),$$

where \mathcal{M} is the collection of $\binom{n}{n-t}$ matrices obtained by replacing the rows and columns that correspond to the elements of s_{n-t} of M with zeros, and assigning -1 to the corresponding diagonal entries.

The following notations are used in this paper. A matrix with entries either 0 or α is referred to as a $(0, \alpha)$ matrix. We write $j \in [a, b]$ if j takes all the integer values satisfying the condition $a \leq j \leq b$. Let J_n and I_n be the all-ones matrix and identity matrix of order n , respectively, and $J_{k,n}$ denote all-ones matrix of order $k \times n$. The column vector with all entries equal to one is denoted by \mathbf{j}_n . Let M be a square matrix of order n . The characteristic polynomial of a matrix M is denoted by $P_M(\lambda) = \det(\lambda I_n - M) = \lambda^n - c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \dots + (-1)^n c_n$, where $c_t = \sum_{s_t = \{h_1, h_2, \dots, h_t\} \in S_t} \lambda_{h_1} \lambda_{h_2} \dots \lambda_{h_t}$ and S_t is the collection of all possible t -subset of $\{1, 2, \dots, n\}$.

3. Lower and Upper Bounds for Energy of Hypergraphs

This section focuses on the bounds for the energy of a hypergraph in terms of the number of vertices, maximum degree, Frobenius norm, and eigenvalues of $A(G^*)$. Recalling that $\sum_{i=1}^n \lambda_i^2 = \|A\|_F^2$, we can deduce the following results on the eigenvalues of k -hypergraphs, which helps to derive the subsequent theorems.

- $\lambda_i^2 \leq \|A\|_F^2, \forall i$, implies that $|\lambda_i| \leq \|A\|_F, \forall i$ and $|\lambda_i| |\lambda_j| \leq \|A\|_F |\lambda_j|, i \neq j$. Hence,

$$\sum_{i \neq j} |\lambda_i| |\lambda_j| \leq (n - 1) \|A\|_F \sum_{i=1}^n |\lambda_i|.$$

- Let $\lambda_{p+1}, \lambda_{p+2}, \dots, \lambda_n$ be the negative eigenvalues of G^* . Since $-|\lambda_{p+1}| - |\lambda_{p+2}| - \dots - |\lambda_n| = \lambda_{p+1} + \lambda_{p+2} + \dots + \lambda_n$,

$$\lambda_1^2 + \lambda_2^2 + \dots + \lambda_p^2 - \frac{\|A\|_F^2}{2} = \frac{1}{2} (\lambda_1 |\lambda_1| + \lambda_2 |\lambda_2| + \dots + \lambda_n |\lambda_n|).$$

Proposition 1. Let G^* be a hypergraph on n vertices. Then

$$\mathcal{E}(G^*) \leq \frac{\|A\|_F (n - 1 + \sqrt{(n - 1)^2 + 4})}{2} \tag{1}$$

Proof. Since $\sum_{i \neq j} |\lambda_i| |\lambda_j| \leq (n - 1) \|A\|_F \sum_{i=1}^n |\lambda_i|$, we obtain

$$\mathcal{E}(G^*)^2 = \sum_{i=1}^n \lambda_i^2 + \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n |\lambda_i| |\lambda_j| \leq \|A\|_F^2 + (n - 1) \|A\|_F \mathcal{E}(G^*).$$

Therefore, we have

$$\mathcal{E}(G^*)^2 - (n - 1)\|A\|_F \mathcal{E}(G^*) - \|A\|_F^2 \leq 0.$$

Consider $f(x) = x^2 - (n - 1)\|A\|_F x - \|A\|_F^2$. Let α_1 and α_2 be the roots of the quadratic equation $f(x) = 0$, then $\mathcal{E}(G^*)$ will lie in between α_1 and α_2 , since $f(\mathcal{E}(G^*)) \leq 0$. Therefore, (1) holds. \square

The following theorems present the bounds for the energy of hypergraph as functions of spectral and structural parameters such as eigenvalues, maximum degree, and Frobenius norm of the adjacency matrix of G^* , which can be identified from the corresponding hypergraph.

Theorem 2. Let G^* be a nonsingular hypergraph of order n such that $|\lambda_1^*| = \lambda_1 \geq |\lambda_2^*| \geq |\lambda_3^*| \geq \dots \geq |\lambda_n^*|$. Then

$$\mathcal{E}(G^*) \geq \lambda_1 + \frac{\|A\|_F^2 - \lambda_1^2 + (n - 1)|\lambda_n^*||\lambda_2^*|}{|\lambda_2^*| + |\lambda_n^*|}.$$

Equality holds if $|\lambda_2^*| = |\lambda_i^*|, i \in [2, n]$.

Proof. Since $|\lambda_i^*| \geq |\lambda_n^*|$, we have

$$\begin{aligned} (|\lambda_i^*| - |\lambda_n^*|)(|\lambda_i^*| - |\lambda_2^*|) &\leq 0, \\ |\lambda_i^*|^2 - |\lambda_i^*|(|\lambda_2^*| + |\lambda_n^*|) + |\lambda_2^*||\lambda_n^*| &\leq 0, \end{aligned}$$

where $i \in [2, n]$. Taking summation i from 2 to n , we obtain

$$\|A\|_F^2 - \lambda_1^2 - (\mathcal{E}(G^*) - \lambda_1)(|\lambda_2^*| + |\lambda_n^*|) + (n - 1)|\lambda_2^*||\lambda_n^*| \leq 0.$$

Therefore, we obtain

$$\mathcal{E}(G^*) \geq \lambda_1 + \frac{\|A\|_F^2 - \lambda_1^2 + (n - 1)|\lambda_n^*||\lambda_2^*|}{|\lambda_2^*| + |\lambda_n^*|}.$$

Equality holds, if $(|\lambda_i^*| - |\lambda_n^*|)(|\lambda_i^*| - |\lambda_2^*|) = 0$ for all $i \in [2, n]$. Therefore, $|\lambda_2^*| = |\lambda_i^*|$, where $i \in [2, n]$. \square

Corollary 1. Let Δ be the maximum degree of the k -hypergraph G^* . Then

$$\mathcal{E}(G^*) \geq \Delta(k - 1) + \frac{\|A\|_F^2 - \Delta^2(k - 1)^2 + (n - 1)|\lambda_n^*||\lambda_2^*|}{|\lambda_2^*| + |\lambda_n^*|}.$$

Proof. Consider the function

$$f(x) = x + \frac{\|A\|_F^2 - x^2 + (n - 1)|\lambda_n^*||\lambda_2^*|}{|\lambda_2^*| + |\lambda_n^*|},$$

which is a decreasing function on $x \geq |\lambda_1|$ for $f'(x) = 1 - \frac{2x}{|\lambda_2^*| + |\lambda_n^*|} < 0$. From Lemma 4 $\lambda_1 \leq \Delta(k - 1)$, we obtain

$$\begin{aligned} \mathcal{E}(G^*) &\geq f(\lambda_1) \geq f(\Delta(k - 1)) \\ \mathcal{E}(G^*) &\geq \Delta(k - 1) + \frac{\|A\|_F^2 - \Delta^2(k - 1)^2 + (n - 1)|\lambda_n^*||\lambda_2^*|}{|\lambda_2^*| + |\lambda_n^*|} \end{aligned}$$

\square

The proof of the following proposition can be deduced by using similar arguments of Theorem 2.

Proposition 2. Let G^* be a nonsingular hypergraph of order n such that $|\lambda_1^*| \geq |\lambda_2^*| \geq \dots \geq |\lambda_n^*| > 0$. Then

$$\mathcal{E}(G^*) \leq \lambda_1 + \frac{\|A\|_F^2 - \lambda_1^2 + (n - 1)|\lambda_2^*|^2}{2|\lambda_2^*|}.$$

Equality holds if $|\lambda_2^*| = |\lambda_i^*|, i \in [2, n]$.

In [11], the authors established an upper bound $\mathcal{E}(G^*) \leq \sqrt{n \sum_{i=1}^n \lambda_i^2} = \sqrt{n} \|A\|_F$. The following theorem provides a more precise bound for the energy of hypergraphs.

Theorem 3. Let G^* be a hypergraph and the first p positive eigenvalues of G^* is denoted as $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p \geq 0$. Then

$$\mathcal{E}(G^*) \leq \sqrt{n \left(\|A\|_F^2 - \frac{4}{\|A\|_F^2} \left(\sum_{i=1}^p \lambda_i^2 - \frac{\|A\|_F^2}{2} \right)^2 \right)}.$$

Proof. First, we prove that

$$\begin{aligned} \frac{\|A\|_F^2}{\|A\|_F^4 - \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right)^2} &= \sum_{j=1}^n \left(\frac{\|A\|_F^2 |\lambda_j| - \lambda_j \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right)}{\|A\|_F^4 - \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right)^2} \right)^2 \\ &= \sum_{j=1}^n \frac{1}{\mathcal{E}(G)} \frac{\|A\|_F^2 |\lambda_j| - \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right) \lambda_j}{\|A\|_F^4 - \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right)^2}. \end{aligned}$$

Consider,

$$\begin{aligned} \frac{\|A\|_F^2}{\|A\|_F^4 - \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right)^2} &= \frac{\|A\|_F^4 \left(\sum_{i=1}^n |\lambda_i|^2\right) - 2\|A\|_F^2 \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right)^2 + \|A\|_F^2 \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right)^2}{\left(\|A\|_F^4 - \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right)^2\right)^2} \\ &= \sum_{j=1}^n \frac{\|A\|_F^4 |\lambda_j|^2 - 2\|A\|_F^2 \lambda_j |\lambda_j| \sum_{i=1}^n \lambda_i |\lambda_i| + \lambda_j^2 \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right)^2}{\left(\|A\|_F^4 - \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right)^2\right)^2} \\ &= \sum_{j=1}^n \left(\frac{\|A\|_F^2 |\lambda_j| - \lambda_j \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right)}{\|A\|_F^4 - \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right)^2} \right)^2. \end{aligned}$$

Since $\sum_{i=1}^n \lambda_i = 0$ and $\mathcal{E}(G^*) = \sum_{i=0}^n |\lambda_i|$, we obtain

$$\begin{aligned} \frac{\|A\|_F^2}{\|A\|_F^4 - \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right)^2} &= \frac{1}{\mathcal{E}(G^*)} \frac{\|A\|_F^2 \sum_{j=1}^n |\lambda_j| - \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right) \sum_{j=1}^n \lambda_j}{\|A\|_F^4 - \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right)^2} \\ &= \sum_{j=1}^n \frac{1}{\mathcal{E}(G^*)} \frac{\|A\|_F^2 |\lambda_j| - \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right) \lambda_j}{\|A\|_F^4 - \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right)^2}. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{n}{\mathcal{E}(G^*)^2} - \frac{\|A\|_F^2}{\|A\|_F^4 - \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right)^2} &= \sum_{j=1}^n \left(\frac{1}{\mathcal{E}(G^*)^2} - 2 \frac{1}{\mathcal{E}(G^*)} \frac{\|A\|_F^2 |\lambda_j| - \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right) \lambda_j}{\|A\|_F^4 - \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right)^2} \right. \\ &\quad \left. + \left(\frac{\|A\|_F^2 |\lambda_j| - \lambda_j \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right)}{\|A\|_F^4 - \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right)^2} \right)^2 \right) \\ &= \sum_{j=1}^n \left(\frac{1}{\mathcal{E}(G^*)^2} - \frac{\|A\|_F^2 |\lambda_j| - \lambda_j \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right)}{\|A\|_F^4 - \left(\sum_{i=1}^n \lambda_i |\lambda_i|\right)^2} \right)^2 \geq 0. \end{aligned}$$

Hence, the theorem holds. \square

Lemma 6. Let G^* be a hypergraph of order n with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Then

$$\frac{1}{n} \sum_{\substack{ij \in [1,n] \\ i \neq j}} a_{ij} \stackrel{(*)}{\leq} \lambda_1 \stackrel{(**)}{\leq} \sqrt{\frac{n-1}{n}} \|A\|_F.$$

Equality $(*)$ holds if and only if G^* is a (k, r) -hypergraph and the equality $(**)$ holds if and only if $\lambda_2 = \lambda_3 = \dots = \lambda_n = -\frac{\lambda_1}{n-1}$.

Proof. From the Rayleigh–Ritz theorem for the Hermitian matrix, we obtain

$$\lambda_1 = \max\{\mathbf{x}^T A \mathbf{x} : \mathbf{x}^T \mathbf{x} = 1\}.$$

Hence,

$$\lambda_1 \geq \frac{J_{1,n} A J_{n,1}}{J_{1,n} J_{n,1}} = \frac{1}{n} \sum_{\substack{ij \in [1,n] \\ i \neq j}} a_{ij}.$$

The adjacency matrix of a (k, r) -hypergraph has constant row sum $r(k - 1)$. Consequently, $r(k - 1)$ is an eigenvalue; its corresponding eigenvector is $J_{1,n}$. Therefore, equality $(*)$ is achieved when G^* is a (k, r) -hypergraph. Conversely, if the equality $(*)$ holds, then $A J_{n,1} = \lambda_1 J_{1,n}$. Hence, the row sum is equal to a constant. Thereby, G^* is a (k, r) -hypergraph. By Lemma 3, we have

$$\lambda_1 \leq \sqrt{\frac{n-1}{n} \sum_{i=1}^n \lambda_i^2} = \sqrt{\frac{n-1}{n}} \|A\|_F.$$

Equality $(**)$ holds if and only if $\lambda_2 = \lambda_3 = \dots = \lambda_n = -\frac{\lambda_1}{n-1}$. \square

Theorem 4. Let G^* be a k -hypergraph of order n such that $A(G^*)$ is a $(0, \alpha)$ matrix, $\alpha \leq \binom{n-2}{k-2}$. Then

$$\mathcal{E}(G^*) \leq \frac{1}{n} \sum_{\substack{ij \in [1,n] \\ i \neq j}} a_{ij} + \sqrt{(n-1) \left(\|A\|_F^2 - \left(\frac{1}{n} \sum_{\substack{ij \in [1,n] \\ i \neq j}} a_{ij} \right)^2 \right)}. \tag{2}$$

Equality holds if and only if either $G^* = \frac{n}{2}K_2^2$, G^* is a (k,r) -regular hypergraph with two distinct eigenvalues $\lambda_1 = r(k-1)$, $-\sqrt{\frac{\|A\|_F^2 - r^2(k-1)^2}{n-1}}$ or G^* is a (k,r) -regular hypergraph with three distinct eigenvalues $\lambda_1 = r(k-1)$ and $\lambda_i = \pm\sqrt{\frac{\|A\|_F^2 - r^2(k-1)^2}{n-1}}$, $i \in [2, n]$.

Proof. First, notice that

$$\sum_{i=2}^n \lambda_i^2 = \|A\|_F^2 - \lambda_1^2.$$

Now, applying Cauchy–Schwartz inequality, we obtain

$$\sum_{i=2}^n |\lambda_i| \leq \sqrt{(n-1)(\|A\|_F^2 - \lambda_1^2)}. \tag{3}$$

Hence,

$$E(G^*) \leq \lambda_1 + \sqrt{(n-1)(\|A\|_F^2 - \lambda_1^2)}.$$

Consider the function $f(x) = x + \sqrt{(n-1)(\|A\|_F^2 - x^2)}$. Then the function $f(x)$ is strictly decreasing in the interval $\left[\frac{\|A\|_F}{\sqrt{n}}, \|A\|_F\right]$. From Lemma 6 and in the view that A is a $(0, \alpha)$ matrix, we have $\frac{\|A\|_F}{\sqrt{n}} \leq \frac{1}{n} \sum_{\substack{i,j \in [1,n] \\ i \neq j}} a_{ij} \leq \lambda_1$. Since f is a decreasing function

$$f(\lambda_1) \leq f\left(\frac{1}{n} \sum_{\substack{i,j \in [1,n] \\ i \neq j}} a_{ij}\right), \tag{2} \text{ holds.}$$

Now, if the equality in (2) holds, then $\lambda_1 = \frac{1}{n} \sum_{\substack{i,j \in [1,n] \\ i \neq j}} a_{ij}$ must hold. From Lemma 6,

G^* is a (k,r) -regular hypergraph. Now, since the equality in (3) must also hold, for $i \in [2, n]$, $|\lambda_i| = \sqrt{\frac{\|A\|_F^2 - r^2(k-1)^2}{n-1}}$.

Then there are only three possible cases.

In the first case, $G^* = \frac{n}{2}K_2^2$ as it has only two distinct eigenvalues, 1 and -1 , both with multiplicity $\frac{n}{2}$.

In the second case, G^* has two distinct eigenvalues with $\lambda_1 = r(k-1)$ and $\lambda_i = -\sqrt{\frac{\|A\|_F^2 - r^2(k-1)^2}{n-1}}$, $i \in [2, n]$.

In the third case, $\lambda_1 = r(k-1)$ and $\lambda_i = \pm\sqrt{\frac{\|A\|_F^2 - r^2(k-1)^2}{n-1}}$, $i \in [2, n]$. \square

Lemma 7. Let G^* be a connected k -hypergraph on n vertices and m edges such that any two edges of G^* share at most 1 vertex. And let $a_{ii}^{(2)}$ be the diagonal entry of $A(G^*)^2$ corresponding to vertex v_i . Then,

$$a_{ii}^{(2)} = d(v_i)(k-1).$$

Proof. By Theorem 1, $a_{ii}^{(2)}$ gives the number of walks of length two starting and ending on vertex v_i . Suppose that $a_{ii}^{(2)} \neq d(v_i)(k-1)$.

Case I: $a_{ii}^{(2)} > d(v_i)(k - 1)$. Then either v_i must be adjacent to more than $d(v_i)(k - 1)$ vertices or v_i can traverse to the neighboring edge through edge e_i and traverse back to another edge e_{i+1} . This is a contradiction since the edges of G^* share at most one vertex.

Case II: $a_{ii}^{(2)} < d(v_i)(k - 1)$. That is, v_i does not have $d(v_i)(k - 1)$ vertices at a distance of length 1. This contradicts the definition of the degree of a vertex in the k -hypergraph. This contradicts our assumption. Hence the theorem holds. \square

The following theorem allows us to reformulate the bounds in Proposition 1 and Theorem 3 of linear k -hypergraph in terms of uniformity and number of edges. By determining the bounds for the sum of the squares of the adjacency eigenvalues of a k -hypergraph, we establish the bounds for the Frobenius norm, which is utilized in most of the theorems. The proof of the following theorem is the direct consequence of Lemmas 1 and 7.

Theorem 5. *Let G^* be a connected k -hypergraph on n vertices and m edges such that any two vertices of G^* share at most 1 vertex. Then*

$$\sum_{i=1}^n \lambda_i^2 = k(k - 1)m.$$

Theorem 6. *Let G^* be a connected k -hypergraph with n vertices and m edges, and $\lambda_i, i \in [1, n]$ be the adjacency eigenvalues of G^* :*

$$k(k - 1)m \leq \sum_{i=1}^n \lambda_i^2 \leq (k - 1)m(m(k - 2) + 2).$$

Equality in the upper bounds holds if and only if the intersection of any two edges must contain $(k - 1)$ vertices.

Proof. Let $E = \{e_1, e_2, e_3, \dots, e_m\}$ be the edge set of G^* . If $|e_i \cap e_j| = s, s \geq 1, i \neq j, i, j \in [1, m]$, then consider that the closed walks of length 2 starting from vertex $v_i \in e_i$ has to traverse back and forth through the same edge. From Theorem 5, the total number of such walk is $k(k - 1)m$. For $s > 1$, let $v_i, v_j \in e_i \cap e_j$. Then there are two additional walks starting from v_i , say $v_i e_i v_j e_j v_i$ and $v_i e_j v_j e_i v_i$. Therefore, for each such intersection, there are $2(s - 1)$ additional walks starting from v_i .

Since G^* is a simple k -hypergraph, the cardinality of the intersection of any two edges must be less than or equal to $k - 1$. Hence, the maximum number of possible walks for a vertex is $2(k - 2)$. Then the total number of possible walks from all vertices in the intersection is $2(k - 1)(k - 2)$. If there are m edges, there are only $\frac{m(m-1)}{2}$ ways in which the intersection of two edges can be taken. Therefore,

$$\sum_{i=1}^n \lambda_i^2 \leq k(k - 1)m + 2(k - 1)(k - 2) \frac{m(m - 1)}{2} = (k - 1)m(m(k - 2) + 2).$$

\square

Example 1. *From Figure 1a, it is clear that the intersection of any two edges of G_1^* contains $k - 1 = 3$ vertices. Then*

$$\sum_{i=1}^n \lambda_i^2(G_1^*) = 72 = m(k - 1)(m(k - 2) + 2).$$

For G_2^* (Figure 1b), the intersection of the edges $\{v_1, v_2, v_3, v_4\}$ and $\{v_1, v_4, v_5, v_6\}$ contains only $2(\neq k - 1)$ vertices. Then

$$\sum_{i=1}^n \lambda_i^2(G_2^*) = 56 \leq m(k - 1)(m(k - 2) + 2) = 72.$$

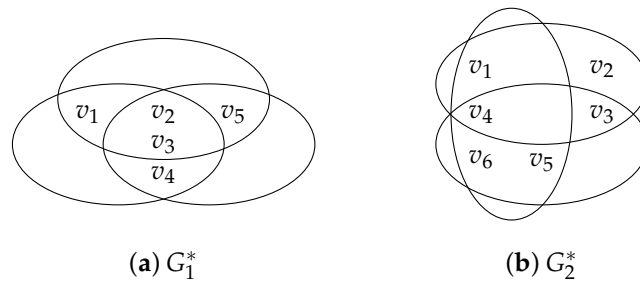


Figure 1. Example for hypergraphs with $\|A\|_F^2 = (k - 1)m(m(k - 2) + 2)$ and $\|A\|_F^2 < (k - 1)m(m(k - 2) + 2)$.

Lemma 8. Let $M = (m_{ij})_{n \times n}$ be a symmetric matrix with diagonal entries as 0 and s_{n-2} be the $(n - 2)$ -subset of $\{1, 2, 3, \dots, n\}$. Then, the determinant of the matrix obtained by replacing the rows and columns that correspond to the elements of s_{n-2} of M with zeros, and assigning -1 to the corresponding diagonal entries, is $(-1)^{n-1}m_{pq}^2$, where $p, q \in [1, n] \setminus s_{n-2}$.

Proof. For any arbitrary $p, q \notin s_{n-2}$, the required matrix will be of the form $-I_n + C$, where $C = (c_{ij})$ is an $n \times n$ matrix whose p -th and q -th diagonal is 1, $c_{pq} = c_{qp} = m_{pq} \neq 0$, and all the other entries are zero. Note that we can obtain a diagonal matrix by interchanging the p -th and q -th rows of $-J_n + C$. As a result, the determinant of the matrix is equal to $(-1)^{n-1}m_{pq}^2$. \square

The following proposition gives the relation between energy, coefficient c_2 of the characteristic polynomial, and the Frobenius norm of the adjacency matrix of G^* . It directly follows from Lemmas 5 and 8.

Proposition 3. Let G^* be a k -hypergraph with eigenvalues $\lambda_i, i \in [1, n]$. Then the coefficient c_2 of λ^{n-2} of $P_A(\lambda)$ is given by

$$c_2 = -\|A\|_F^2 = \sum_{i=1}^n \lambda_i^2.$$

4. Nordhaus–Gaddum Problem for Energy of k -Hypergraph

In [20], the authors provide an upper bound for the sum of the energy of a graph and its complement. In this section, we extended it to the k -hypergraph by giving bounds in terms of number of vertices.

Theorem 7. Let A be a square matrix of order n , with $\|A\|_{max} = \binom{n-2}{t-2} = \alpha$ (say), $t \in [1, n]$ and with zero diagonal, then

$$\|A + \frac{\alpha}{2}I_n\|_* + \|\alpha J - A - \frac{\alpha}{2}I_n\|_* \leq \alpha(n + (n - 1)\sqrt{n}).$$

Equality holds when A is a $(0, \alpha)$ matrix with row sum equal to $\frac{\alpha(n - 1)}{2}$ and singular values

$$\sigma_1\left(A + \frac{\alpha}{2}I_n\right) = \frac{\alpha n}{2}, \sigma_i\left(A + \frac{\alpha}{2}I_n\right) = \frac{\alpha\sqrt{n}}{2}, i \in [2, n].$$

Proof. Let σ_i and $\bar{\sigma}_i, i \in [1, n]$ be the singular values of $B = (b_{ij})$ and $\bar{B} = (\bar{b}_{ij})$, respectively, where $B = A + \frac{\alpha}{2}I_n$ and $\bar{B} = \alpha J - A - \frac{\alpha}{2}I_n$. By AM-QM inequality, $\sum_{i=2}^n(\sigma_i + \bar{\sigma}_i) \leq \sqrt{2(n-1)\sum_{i=2}^n(\sigma_i^2 + \bar{\sigma}_i^2)}$. Hence,

$$\|B\|_* + \|\bar{B}\|_* \leq \sigma_1 + \bar{\sigma}_1 + \sqrt{2(n-1)\sum_{i=2}^n(\sigma_i^2 + \bar{\sigma}_i^2)}.$$

Since $\sum_{i,j}(b_{ij}^2 + \bar{b}_{ij}^2) = n^2\alpha^2 - \frac{n\alpha^2}{2} + \sum_{i,j}2a_{ij}(a_{ij} - \alpha)$, we have

$$\sum_{i=1}^n(\sigma_i^2 + \bar{\sigma}_i^2) = \text{tr}(BB^T) + \text{tr}(\bar{B}\bar{B}^T) = \sum_{i,j}(b_{ij}^2 + \bar{b}_{ij}^2) \leq \alpha^2\left(n^2 - \frac{n}{2}\right).$$

Hence, we obtain

$$\|B\|_* + \|\bar{B}\|_* \leq \sigma_1 + \bar{\sigma}_1 + \sqrt{2(n-1)\left(\alpha^2\left(n^2 - \frac{n}{2}\right) - \sigma_1^2 - \bar{\sigma}_1^2\right)}.$$

By using GM-AM inequality, we obtain

$$\|B\|_* + \|\bar{B}\|_* \leq \sigma_1 + \bar{\sigma}_1 + \sqrt{2(n-1)\left(\alpha^2\left(n^2 - \frac{n}{2}\right) - \frac{(\sigma_1 + \bar{\sigma}_1)^2}{2}\right)}. \tag{4}$$

Consider the function

$$f(x) = x + \sqrt{2(n-1)\left(\alpha^2\left(n^2 - \frac{n}{2}\right) - \frac{x^2}{2}\right)},$$

which is decreasing in $x \geq n\alpha$. By the definition of operator norm and Cauchy’s inequality,

$$\sigma_1 + \bar{\sigma}_1 \geq \frac{1}{n}\langle (B + \bar{B})\mathbf{j}_n, \mathbf{j}_n \rangle \geq \alpha n.$$

Thus, we obtain

$$f(\sigma_1 + \bar{\sigma}_1) \leq f(\alpha n).$$

Using this fact and (4), we obtain $\|B\|_* + \|\bar{B}\|_* \leq f(\alpha n)$, which immediately gives the result.

It is obvious that $\sum_{i,j}(b_{ij}^2 + \bar{b}_{ij}^2) = \alpha^2\left(n^2 - \frac{n}{2}\right)$ only when the entries of A are either 0 or α . From the equality condition of GM-AM inequality and $\sigma_1 + \bar{\sigma}_1 = \alpha n$, we obtain $\sigma_1 = \bar{\sigma}_1 = \frac{\alpha n}{2}$. If the equality condition on σ_1 as an operator norm is attained, that is, $\sigma_1 = \frac{1}{n}\langle B\mathbf{j}_n, \mathbf{j}_n \rangle$, then all the row sums and column sums of B are equal to $\frac{\alpha n}{2}$. On the other hand,

$$\bar{B}\bar{B}^T = (\alpha J_n - B)(\alpha J_n - B)^T = BB^T,$$

and, therefore, $\sigma_i = \bar{\sigma}_i = \frac{\alpha\sqrt{n}}{2}, i \in [2, n]$. \square

Theorem 8. If A is a symmetric non-negative matrix of order $n \geq 8$ with $\|A\|_{\max} \leq \binom{n-2}{t-2} = \alpha$ (say), $t \in [1, n]$ and zero diagonal, then

$$\|A\|_* + \|\alpha J_n - \alpha I_n - A\|_* \leq \alpha(n-1)(1 + \sqrt{n})$$

with equality holding if and only if A is a $(0, \alpha)$ matrix, with all row and column sums equal to $\frac{\alpha(n-1)}{2}$ and singular values $\sigma_1(A + \frac{\alpha}{2}I_n) = \frac{\alpha n}{2}, \sigma_i(A + \frac{\alpha}{2}I_n) = \frac{\alpha\sqrt{n}}{2}$ for $i \in [2, n]$.

Proof. For simplicity, we set $\bar{A} = \alpha J_n - \alpha I_n - A$ and $\lambda_i, \bar{\lambda}_i, i \in [1, n]$ to be the eigenvalues of A and \bar{A} , respectively. By Weyl’s inequality,

$$\lambda_i + \bar{\lambda}_{n-i+2} \leq \lambda_2(\alpha(J - I)) = -\alpha, i \in [2, n]. \tag{5}$$

Let $n^+(A)$ be the non-negative eigenvalues of A . We define a set P such that

$$P = \{s | 2 \leq s \leq n, \lambda_s \geq 0 \text{ or } \bar{\lambda}_{n-s+2} \geq 0\} \text{ and } p = |P|.$$

Since $\lambda_1, \bar{\lambda}_1 \geq 0$, the total number of positive eigenvalues of A is, $n^+(A) + n^+(\bar{A}) = p + 2$. Now, for $s \in P$, we have $\lambda_s \geq 0$, then by (5) $\bar{\lambda}_{n-s+2} < 0$. Next, we prove the theorem for different values of $p, p \leq n - 1$.

Case I: $p = n - 1$. As in Theorem 7, we set $B = A + \frac{\alpha}{2}I_n$ and $\bar{B} = \alpha J_n - A - \frac{\alpha}{2}I_n = \bar{A} + \frac{\alpha}{2}I_n$. Recalling $\sigma_i(A) = |\lambda_i(A)| = |\lambda_i|$ and $\sum_{i=1}^n \lambda_i(B) = tr(B) = \frac{\alpha n}{2}$, we obtain

$$\sum_{\lambda_i(B) \leq 0} |\lambda_i(B)| = \sum_{\lambda_i(B) \geq 0} \lambda_i(B) - \frac{\alpha n}{2}.$$

Again, we have

$$\|B\|_* + \|\bar{B}\|_* = \sum_{i=1}^n |\lambda_i(B)| + \sum_{i=1}^n |\lambda_i(\bar{B})| = 2 \sum_{\lambda_i(B) \geq 0} \lambda_i(B) + \sum_{\lambda_i(B) \geq 0} \lambda_i(B) - \alpha n.$$

Also, we obtain

$$\begin{aligned} \sum_{\lambda_i(B) \geq 0} \lambda_i(B) + \sum_{\lambda_i(\bar{B}) \geq 0} \lambda_i(\bar{B}) &= \sum_{\lambda_i \geq -\frac{\alpha}{2}} |\lambda_i + \frac{\alpha}{2}| + \sum_{\bar{\lambda}_i \geq -\frac{\alpha}{2}} |\bar{\lambda}_i + \frac{\alpha}{2}| \\ &\geq \sum_{\lambda_i \geq 0} |\lambda_i + \frac{\alpha}{2}| + \sum_{\bar{\lambda}_i \geq 0} |\bar{\lambda}_i + \frac{\alpha}{2}| \\ &= \frac{1}{2} (\|A\|_* + \|\bar{A}\|_* + (p + 2)\alpha), \end{aligned}$$

which imply that

$$\|B\|_* + \|\bar{B}\|_* \geq \|A\|_* + \|\bar{A}\|_* + (p + 2)\alpha - \alpha n = \|A\|_* + \|\bar{A}\|_* + \alpha.$$

From Theorem 7, we obtain

$$\alpha[n + (n - 1)\sqrt{n}] \geq \|B\|_* + \|\bar{B}\|_* \geq \|A\|_* + \|\bar{A}\|_* + \alpha.$$

Case II: $p = n - 2$. Thus, there exists precisely one $s \in [2, n]$ such that $\lambda_s < 0$ and $\bar{\lambda}_{n-s+2} < 0$. For the sake of computational simplicity, let $x = \lambda_1 + \bar{\lambda}_1$ and $y = |\lambda_s| + |\bar{\lambda}_{n-s+2}|$. By applying $\sum_{i=1}^n (\lambda_i + \bar{\lambda}_{n-i+2}) = 0$, we obtain

$$x = \lambda_1 + \bar{\lambda}_1 = -(\lambda_s + \bar{\lambda}_{n-s+2}) - \sum_{\substack{i=2 \\ i \neq s}}^n (\lambda_i + \bar{\lambda}_{n-i+2}) = y - \sum_{\substack{i=2 \\ i \neq s}}^n (\lambda_i + \bar{\lambda}_{n-i+2}).$$

From (5), we have $\sum_{\substack{i=2 \\ i \neq s}}^n (\lambda_i + \bar{\lambda}_{n-i+2}) \leq -(n - 2)\alpha$ and $y = |\lambda_k| + |\lambda_{n-k+2}| \geq \alpha$. Thus,

$$x \geq y + (n - 2)\alpha \geq (n - 1)\alpha. \tag{6}$$

Also,

$$\lambda_i^2 + \bar{\lambda}_{n-i+2}^2 = \frac{(|\lambda_i| + |\bar{\lambda}_{n-i+2}|)^2 + (|\lambda_i| - |\bar{\lambda}_{n-i+2}|)^2}{2} \geq \frac{(|\lambda_i| + |\bar{\lambda}_{n-i+2}|)^2}{2} + \frac{\alpha^2}{2}.$$

On the other hand, we obtain

$$\begin{aligned} \alpha^2 n(n-1) &\geq \sum_{i,j} (a_{i,j}^2 + \bar{a}_{i,j}^2) = \sum_{i=1}^n (\lambda_i^2 + \bar{\lambda}_i^2) \\ &= \lambda_1^2 + \bar{\lambda}_1^2 + \lambda_s + \bar{\lambda}_{n-s+2}^2 + \sum_{i \in P} (\lambda_i^2 + \bar{\lambda}_{n-i+2}^2) \\ &\geq \frac{x^2}{2} + \frac{y^2}{2} + \sum_{i \in P} \frac{(|\lambda_i| + |\bar{\lambda}_{n-i+2}|)^2}{2} + \frac{\alpha^2 p}{2}. \end{aligned}$$

From *A.M – Q.M* inequality, we obtain $\frac{(\sum_{i \in P} |\lambda_i| + |\lambda_{n-i+2}|)^2}{p^2} \leq \frac{\sum_{i \in P} (|\lambda_i| + |\lambda_{n-i+2}|)^2}{p}$. Therefore,

$$\alpha^2 n(n-1) \geq \frac{x^2}{2} + \frac{y^2}{2} + \frac{1}{2p} \left(\sum_{i \in P} |\lambda_i| + |\lambda_{n-i+2}| \right)^2 + \frac{\alpha^2 p}{2}.$$

On simplification, we obtain

$$\|A\|_* + \|\bar{A}\|_* \leq x + y + \sqrt{(n-2) \left(2\alpha^2 n(n-1) - x^2 - y^2 - (n-2)\alpha^2 \right)}.$$

Consider the function

$$f(x, y) = x + y + \sqrt{(n-1)(2\alpha^2 n(n-1) - x^2 - y^2 - \alpha^2(n-\alpha))},$$

which is decreasing in x for $n \geq 4$, $x \geq \alpha(n-2)$ and $y \geq \alpha$. From (6), $x \geq y + (n-2)\alpha$ then $f(x, y) \leq f(y + (n-2)\alpha, \alpha)$. Let

$$\begin{aligned} g(y) &= f(y + (n-2)\alpha, \alpha) \\ &= 2y + (n-2)\alpha \\ &\quad + \sqrt{(n-2) \left(2\alpha^2 n(n-1) - y^2 - (n-2)^2 \alpha^2 - 2y(n-2)\alpha - y^2 - \alpha^2(n-2) \right)} \end{aligned}$$

is a decreasing function in y for $n \geq 6$ and $y \geq \alpha$. Since $((n-1)\alpha, \alpha) \leq (y + n - 2, y)$, we obtain

$$f(y + n - 2, y) \leq f((n-1)\alpha, \alpha) = \alpha(n + \sqrt{(n-2)(n-1)n}).$$

Therefore,

$$\|A\|_* + \|\bar{A}\|_* \leq f(x, y) \leq f(y + (n-2)\alpha, y) \leq \alpha(n + \sqrt{(n-2)(n-1)n}).$$

Hence, for $n \geq 6$

$$\|A\|_* + \|\bar{A}\|_* \leq \alpha(n + \sqrt{(n-2)(n-1)n}) < \alpha((n-1)(1 + \sqrt{n})).$$

Case III: $p \leq n - 3$. Then there exist distinct $s, t \in \{2, 3, \dots, n\} \setminus P$. Let $x = \lambda_1 + \bar{\lambda}_1$, $y = |\lambda_s| + |\bar{\lambda}_{n-s+2}| + |\lambda_t| + |\bar{\lambda}_{n-t+2}|$. Since $y \geq \alpha$, we obtain

$$x = \lambda_1 + \bar{\lambda}_1 \geq y + (n-3)\alpha \geq (n-1)\alpha.$$

Also,

$$\alpha^2 n(n-1) \geq \frac{x^2}{2} + \frac{y^2}{4} + \frac{1}{2p} \left(\sum_{i \in P} |\lambda_i| + |\bar{\lambda}_{n-i+2}| \right)^2 + \frac{\alpha^2 p}{2}.$$

Using similar arguments in Cases I and II, we obtain

$$\|A\|_* + \|\bar{A}\|_* \leq x + y + \sqrt{2(n-3) \left(\alpha^2 n(n-1) - \frac{x^2}{2} - \frac{y^2}{4} - \frac{\alpha^2}{2} \right)}.$$

Consider the function

$$f(x, y) = x + y + \sqrt{2(n-3) \left(\alpha^2 n(n-1) - \frac{x^2}{2} - \frac{y^2}{4} - \frac{\alpha^2}{2} \right)},$$

which is decreasing in x for $n \geq 5$, $x \geq (n-1)\alpha$ and $y \geq 2\alpha$. Therefore, if $x \geq y + (n-3)\alpha$ then $f(x, y) \leq f(y + (n-3)\alpha, y)$. Thus,

$$f(y + (n-3)\alpha, y) = 2y + (n-3)\alpha + \sqrt{(n-3) \left(2\alpha^2 n(n-1) - \alpha^2 - (y + (n-3)\alpha)^2 - \frac{y^2}{2} \right)}$$

is a decreasing function in y for $n \geq 8$ and $y \geq 2\alpha$. Therefore,

$$\begin{aligned} \|A\|_* + \|\bar{A}\|_* &\leq f(y + (n-3)\alpha, y) \leq f((n-1)\alpha, 2\alpha) = \alpha \left(n + 1 + \sqrt{(n-3)(n^2 - 4)} \right) \\ &< \alpha \left(n - 1 + (n-1)\sqrt{n} \right), \end{aligned}$$

where $n \geq 8$ which completes the proof. \square

Next, we provide the Nordhaus–Gaddum-type bounds for the adjacency energy of a k -hypergraph. The following corollary is an immediate consequence of Theorem 8.

Corollary 2. Let G^* be a k -hypergraph on n vertices and A be the adjacency matrix of G^* . If \bar{G}^* is its complement, then

$$\mathcal{E}(G^*) + \mathcal{E}(\bar{G}^*) \leq \alpha(n-1)(1 + \sqrt{n}),$$

where $\alpha = \binom{n-2}{k-2}$. Equality holds if and only if A is a $(0, \alpha)$ matrix, with all rows and columns sum equal to $\frac{\alpha(n-1)}{2}$ and $|\lambda_i(A + \frac{1}{2}I_n)| = \frac{\alpha\sqrt{n}}{2}$ for $i \in [2, n]$.

The following proposition shows that the energy of a k -hypergraph and its complement differs only by $2\alpha(n-1)$.

Proposition 4. Let G^* be a k -hypergraph on n vertices and A be the adjacency matrix of G^* . Then

$$|\mathcal{E}(G^*) - \mathcal{E}(\bar{G}^*)| \leq 2\alpha(n-1),$$

where $\alpha = \binom{n-2}{k-2}$.

Proof. By using triangle inequality for trace norm, we obtain

$$\|\bar{A}\|_* \leq \|\alpha(J - I)\|_* + \|A\|_* \text{ and } \|A\|_* \leq \|\alpha(J - I)\|_* + \|\bar{A}\|_*.$$

Hence, $|\|\bar{A}\|_* - \|A\|_*| \leq 2\alpha(n-1)$. \square

5. Conclusions

The bounds for adjacency energy of hypergraphs in terms of different parameters, such as eigenvalues, norm, number of vertices, and maximum degree of the hypergraph

are obtained. Also, bounds for the Frobenius norm of the adjacency matrix are established, which in turn modify the bounds for the adjacency energy of k -hypergraphs. In addition, a relation of the energy of the k -hypergraph and its complement is established.

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