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On Warped Product Pointwise Pseudo-Slant Submanifolds of *LCK*-Manifolds and Their Applications

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Abstract: The concept of pointwise slant submanifolds of a Kähler manifold was presented by Chen and Garay. This research extends this notion to a more general setting, specifically in a locally conformal Kähler manifold. We study the pointwise pseudo-slant warped products of the form $\Sigma^{\theta} \times_f \Sigma^{\perp}$ in a locally conformal Kähler manifold. Using the concept of pointwise pseudo-slant, we establish the necessary and sufficient condition for it to be characterized as a warped product submanifold. In addition, we derive several results on pointwise pseudo-slant warped products that expand previously proven main ones. Further, some examples of such submanifolds and their warped products are also given.

Keywords: warped products; CR-submanifold warped product; pointwise slant warped products; locally conformal Kähler manifold

MSC: 53C40; 53C15; 53C42

Citation: Alghamdi, F. On Warped Product Pointwise Pseudo-Slant Submanifolds of *LCK*-Manifolds and Their Applications. *Axioms* **2024**, *13*, 807. https://doi.org/10.3390/axioms13110807

Academic Editor: Zhigang Wang

Received: 21 October 2024
Revised: 16 November 2024
Accepted: 18 November 2024
Published: 20 November 2024

1. Introduction

Bishop and O'Neill [1] introduced the concept of warped product manifolds in 1969, extending the notion of Riemannian product manifolds. This extension was developed to investigate manifolds with negative sectional curvature. Moreover, warped products have important applications in physics and differential geometry, especially in the context of general relativity. Warped products appear in many of the fundamental solutions of the Einstein field equations. For example, in general relativity, the models of Schwarzschild and Robertson–Walker are both warped products [2].

In the early 2000s, Chen initiated the investigation of warped products in Kähler manifolds [3,4]. He showed that the warped products in the Kähler manifold of the form $\Sigma^{\perp} \times_f \Sigma^{T}$ do not exist, where $\Sigma^{\perp}$ is an anti-invariant submanifold and Σ^{T} is an invariant submanifold. Building upon Chen's work on CR-warped product submanifolds, this topic has emerged as a major field of differential geometry research. Several researchers have extended this notion to explore various types of warped products in almost Hermitian manifolds (see [5–7]). These studies continued in more general manifolds and in almost Hermitian manifolds known as locally conformal Kähler manifolds, such as CR-warped product submanifolds [8,9], warped product semi-slant submanifolds [10,11], hemi-slant warped products [12] and pointwise slant warped products [13,14].

Locally conformal Kähler manifolds share some characteristics with Kähler geometry but provide more flexibility. They are crucial in studying non-Kähler complex manifolds and have significant applications in mathematics and theoretical physics. Specifically, they are valuable in string theory and general relativity, where their adaptable structure is useful for modeling spaces that are restrictive for standard Kähler geometry. Additionally, locally conformal Kähler manifolds help in analyzing solutions to Einstein's equations that do not have a Kähler structure [15,16].

Building on previous research, this paper extends the concept of the CR-warped product into the pointwise pseudo-slant warped product in locally conformal Kähler manifolds, leading to the discovery of several fundamental results and some generalizations. More precisely, we study pointwise pseudo-slant warped products of the form $\Sigma^{\theta} \times_f \Sigma^{\perp}$ in locally conformal Kähler manifolds, where Σ^{θ} are proper pointwise slant submanifolds and $\Sigma^{\perp}$ are anti-invariant submanifolds. We also obtained some essential results that serve as generalizations of the main findings reported in various studies in the literature.

This paper is organized as follows: Section 2 presents the essential background necessary for our study. Section 3 establishes several preliminary lemmas and results. In Section 4, we prove the main theorems, followed by Section 5, which provides various applications of our findings. Finally, we conclude with non-trivial examples of pointwise pseudo-slant warped products.

2. Basic Definitions and Lemmas

Let $\tilde{N}$ be a differentiable manifold equipped with a tensor field J of type $(1,1)$ such that $J^2 = -I$. Then, we say that $\tilde{N}$ is an almost complex manifold with an almost complex structure J if $\tilde{N}$ has even dimension ≥ 2, and there exists a Riemannian metric g on $\tilde{N}$ that is compatible with J, i.e.,

$$g(JX_1, JX_2) = g(X_1, X_2), \tag{1}$$

for all $X_1, X_2 \in \Gamma(T\tilde{N})$. If such a metric g exists, then g is called a Hermitian metric on $\tilde{N}$.

An almost complex manifold with a Hermitian metric $(\tilde{N}, J, g)$ is called an almost Hermitian manifold.

Furthermore, an almost Hermitian manifold $(\tilde{N}, J, g)$ is called a locally conformally Kähler manifold (an LCK-manifold) if a Hermitian metric g is locally conformal to a Kähler metric [16].

The LCK-manifold $\tilde{N}$ is a Hermitian manifold with structure (J, g) such that there is a global closed 1-form β (known as the Lee form) which satisfies for any $X_1, Y_1 \in T\tilde{N}$ the following [16,17]:

$$(\tilde{\nabla}_{X_1} J)Y_1 = g(\lambda, JY_1)X_1 - g(\lambda, Y_1)JX_1 + g(JX_1, Y_1)\lambda + g(X_1, Y_1)J\lambda \tag{2}$$

where $\tilde{\nabla}$ is the Levi–Civita connection on an LCK-manifold $\tilde{M}$ and λ is the dual vector field of β (i.e., $g(X_1, \lambda) = \beta(X_1)$ for $X_1 \in T\tilde{M}$), called the Lee vector field [15]. The Lee-form β of LCK-manifolds has significance for determining many geometric characteristics of their submanifolds. An LCK-manifold with an exact 1-form β is called a globally conformal Kähler manifold (GCK-manifold).

Consider N as a Riemannian manifold that is isometrically immersed in LCK-manifold $(\tilde{N}, \beta, J, g)$, with the Riemannian metric on M induced by the immersion denoted by the same symbol g. Let $\Gamma(TN)$ represent the tangent vector fields on N and $\Gamma(T^{\perp}N)$ denote the set of normal vector fields to N. Let ∇ be the Levi–Civita connection on N. The Weingarten and Gauss formulas are given as follows:

$$\tilde{\nabla}_{X_1} V = -A_V X_1 + \nabla^{\perp}_{X_1} V, \tag{3}$$

$$\tilde{\nabla}_{X_1} Y_1 = \nabla_{X_1} Y_1 + \sigma(X_1, Y_1), \tag{4}$$

for any $X_1, Y_1 \in \Gamma(TN)$ and $V \in \Gamma(T^{\perp}N)$, where σ is the second fundamental form, A_V is the Weingarten map and $\nabla^{\perp}$ is the normal connection. Also, A_V and σ are interconnected through the following relations:

$$g(\sigma(X_1, Y_1), V) = g(A_V X_1, Y_1). \tag{5}$$

A submanifold N of an LCK-manifold is totally geodesic if $\sigma(X_1, Y_1) = 0$.

Now, for any tangent vector X_1 to N and any normal vector V to N, we have

$$JX_1 = TX_1 + FX_1, \tag{6}$$

$$JV_1 = tV_1 + fV_1, \tag{7}$$

where TX_1 (tV_1) is the tangential component and FX_1 (fV_1) is the normal component of JX_1 (JV_1).

In accordance with Chen's definition [18] of a pointwise slant submanifold of an almost Hermitian manifold, we will define a pointwise slant submanifold in an LCK-manifold.

Definition 1 ([18])**.** *A Riemannian manifold N isometrically immersed into an almost Hermitian manifold $\tilde{N}$ is said to be a pointwise pseudo-slant submanifold if it has tangent bundle of orthogonal distributions $\mathfrak{D}^\theta$ and $\mathfrak{D}^\perp$ on N such that $TN = \mathfrak{D}^\theta \oplus \mathfrak{D}^\perp$, where $\mathfrak{D}^\theta$ is a pointwise slant distribution with slant function θ and $\mathfrak{D}^\perp$ is an anti-invariant distribution.*

If we denote the dimensions of $\mathfrak{D}^\theta$ and $\mathfrak{D}^\perp$ by n_1 and n_2, respectively, then CR-submanifolds and slant submanifolds represent specific cases of pseudo-slant submanifolds, corresponding to the cases where the slant angle $\theta = 0$ and $n_2 = 0$, respectively. Furthermore, invariant submanifolds correspond to pseudo-slant submanifolds with $\theta = 0$ and $n_2 = 0$, while anti-invariant submanifolds have $\theta = \frac{\pi}{2}$ or $n_1 = 0$. Moreover, a pointwise slant submanifold is proper if neither $\theta = 0, \frac{\pi}{2}$ nor θ is constant.

It is established in [18] that a submanifold N of an almost Hermitian manifold $\tilde{N}$ is called a pointwise slant if and only if it satisfies the condition

$$T^2 = -(\cos^2\theta)I, \tag{8}$$

where θ is a real-valued function defined on N and I is the identity map on the tangent bundle TN of N. A pointwise slant submanifold is termed proper if it contains neither totally real nor complex points, meaning $0 < \cos^2\theta < 1$.

Similarly, if N is a submanifold of a locally conformal Kähler (LCK-manifold) $\tilde{N}$, then N is pointwise slant if and only if it satisfies condition (8) as well.

Moreover, from (8), for a pointwise slant submanifold of an LCK-manifold $\tilde{N}$, the following useful relation was obtained:

$$g(TX_1, TY_1) = (\cos^2\theta)\, g(X_1, Y_1), \tag{9}$$

Also,

$$g(FX_1, FY_1) = (\sin^2\theta)\, g(X_1, Y_1) \tag{10}$$

$$tFX_1 = -\sin^2\theta\, X_1, \quad fFX_1 = -FTX \tag{11}$$

for any $X_1, Y_1 \in \Gamma(TN)$.

The following results on pointwise pseudo-slant submanifolds of LCK-manifolds were established in [14] and are instrumental in proving the main theorems.

Lemma 1 ([14])**.** *On the pointwise pseudo-slant submanifold N of $\tilde{N}$, the following relations hold for any $V_1 \in \Gamma(\mathfrak{D}^\perp)$, $X_1, X_2 \in \Gamma(\mathfrak{D}^\theta)$:*

$$\begin{aligned}\cos^2\theta g(\nabla_{X1}X_2, V_1) = {} & [g(A_{JV_1}TX_2 - A_{FTX_2}V_1, X_1) + g(X_1, TX_2)g(\lambda, JV_1)] \\ & - g(X_1, X_2)g(\lambda, V_1).\end{aligned}$$

Lemma 2 ([14])**.** *On the pointwise pseudo-slant submanifold N of $\tilde{N}$, the following relations hold for any $V_1, V_2 \in \Gamma(\mathfrak{D}^\perp)$, $X_1 \in \Gamma(\mathfrak{D}^\theta)$:*

$$\cos^2\theta g(\nabla_{V2}V_1, X_1) = [g(A_{FTX_1}V_1 - A_{JV_1}TX_1, V_2) - g(V_1, V_2)g(\lambda, X_1)].$$

Next, we present the following result for the leaves of the anti-invariant distribution $\mathfrak{D}^{\perp}$ and pointwise slant distribution $\mathfrak{D}^{\theta}$.

Theorem 1 ([14]). *On the pointwise pseudo-slant submanifold N of LCK-manifold $\tilde{N}$, the following hold for any $V_1 \in \Gamma(\mathfrak{D}^{\perp})$, $X_1 \in \Gamma(\mathfrak{D}^{\theta})$:*

(i) *The totally real distribution $\mathfrak{D}^{\perp}$ defines a totally geodesic foliation in N if and only if*

$$A_{FTX_1}V_1 - A_{JV_1}TX_1 = g(\lambda, X_1)V_1.$$

(ii) *The proper pointwise slant distribution $\mathfrak{D}^{\theta}$ defines a totally geodesic foliation if and only if*

$$A_{JV_1}TX_1 - A_{FTX_1}V_1 = \cos^2\theta g(\lambda, V_1)X_1 - g(\lambda, JV_1)TX_1.$$

The integrability theorem is introduced as follows.

Theorem 2 ([14]). *Let $(\tilde{N}, J, g)$ be an LCK-manifold and N a pointwise pseudo-slant submanifold of $\tilde{N}$. Then, we have the following:*

(i) *The distribution of anti-invariant $\mathfrak{D}^{\perp}$ is integrable if and only if*

$$A_{JV_2}V_1 = A_{JV_1}V_2.$$

(ii) *The distribution $\mathfrak{D}^{\theta}$ of pointwise slant is integrable if and only if*

$$g(A_{JV_1}TX_2 - A_{FTX_2}V_1, X_1) = g(A_{JV_1}TX_1 - A_{FTX_1}V_1, X_2) + 2g(X_1, TX_2)g(\lambda, JV_1).$$

3. Pointwise Pseudo-Slant Warped Products $\Sigma^{\theta} \times_f \Sigma^{\perp}$

In this study on LCK-manifold $\tilde{N}$, we investigate the warped product submanifold $N = \Sigma^{\theta} \times_f \Sigma^{\perp}$ when Σ^{θ} is a proper pointwise slant submanifold and $\Sigma^{\perp}$ is an anti-invariant submanifold under the condition that the Lee vector field λ is tangent to N. This new type of warped product is called the pointwise pseudo-slant warped product of an LCK-manifold.

To begin, we need to know some definitions. A warped product $\Sigma_1 \times \Sigma_2$ of two pseudo-Riemannian manifolds (Σ_1, g_1) and (Σ_2, g_2) with their metrics g_1 and g_2 is the product manifold $N = \Sigma_1 \times_f \Sigma_2$ equipped with the warped product metric g defined by

$$g = g_1 + f^2 g_2$$

where f is a positive differential function on Σ_1. The function f is called the warping function of the warped product [3]. A warped product manifold $\Sigma_1 \times_f \Sigma_2$ is called a Riemannian product if its warping function f is constant.

Definition 2. *In an LCK-manifold $\tilde{N}$, a warped product pointwise pseudo-slant submanifold $\Sigma^{\theta} \times_f \Sigma^{\perp}$ is a warped product, where Σ^{θ} is a pointwise slant submanifold and $\Sigma^{\perp}$ is an anti-invariant submanifold. A warped product $\Sigma^{\theta} \times_f \Sigma^{\perp}$ is a proper pointwise pseudo-slant submanifold if Σ^{θ} is a proper pointwise slant. If not, it is classified as non-proper.*

First, we remember the well-known lemma that follows.

Lemma 3 ([1]). *For any $X_1, X_2 \in T(\Sigma_1)$ and $V_1, V_2 \in T(\Sigma_2)$, we have, for a warped product manifold $N = \Sigma_1 \times {}_f\Sigma_2$ and the warping function f, the following:*

(i) $\nabla_{X1}X_2 \in T(\Sigma_1)$;

(ii) $\nabla_{X1}V_1 = \nabla_{V_1}X_1 = X_1(\ln f)(V_1)$;

(iii) $\nabla_{V1}X_2 = \nabla^{\Sigma_2}_{V_1}V_2 - g(V_1, V_2)\vec{\nabla}\ln f$.

Here, the gradient $\vec{\nabla} \ln f$ of the function $\ln f$ is given by $g(\vec{\nabla} f, X_1) = X_1(f)$, and both ∇ and ∇^{Σ_2} denote the Levi–Civita connections on N and Σ_2, respectively.

To simplify, we refer to the tangent spaces of $\Sigma^{\perp}$ and Σ^{θ} as $\mathfrak{D}^{\perp}$ and $\mathfrak{D}^{\theta}$, respectively. Note that, in an LCK-manifold $\tilde{N}$, the normal bundle for warped product pointwise pseudo-slant $N = \Sigma^{\theta} \times_f \Sigma^{\perp}$ is given by

$$T^{\perp}N = \nu \oplus J\mathfrak{D}^{\perp} \oplus F\mathfrak{D}^{\theta}. \tag{12}$$

such that ν is an orthogonal complementary distribution of $J\mathfrak{D}^{\perp} \oplus F\mathfrak{D}^{\theta}$ in $T^{\perp}N$.

Proposition 1. *Let $\tilde{N}$ be an LCK-manifold and $N = \Sigma^{\theta} \times {}_f\Sigma^{\perp}$ a proper warped product pointwise pseudo-slant in $\tilde{N}$; then, the Lee vector field λ is normal to $\mathfrak{D}^{\perp}$.*

Proof. We have, for any $X_1, X_2 \in \Gamma(\mathfrak{D}^{\theta})$ and $V_1 \in \Gamma(\mathfrak{D}^{\perp})$,

$$g(\sigma(X_1, X_2), JV_1) = g(\tilde{\nabla}_{X1}X_2, JV_1)$$

By the structure of a complex manifold, we derive

$$g(\sigma(X_1, X_2), JV_1) = -g(J\tilde{\nabla}_{X1}X_2, V_1)$$

Then, we have from the definition of J

$$g(\sigma(X_1, X_2), JV_1) = g((\tilde{\nabla}_{X1}J)X_2), V_1) - g(\tilde{\nabla}_{X1}JX_2, V_1). \tag{13}$$

Thus, from Formula (2),

$$g(\sigma(X_1, X_2), JV_1) = g(X_1, X_2)g(J\lambda, V_1) + g(TX_1, X_2)g(\lambda, V_1) + g(\tilde{\nabla}_{X1}V_1, JX_2). \tag{14}$$

Then, we have from Lemma 3 (ii)

$$g(\sigma(X_1, X_2), JV_1) = g(X_1, X_2)g(J\lambda, V_1) + g(TX_1, X_2)g(\lambda, V_1) + X_1(\ln f)g(V_1, TX_2). \tag{15}$$

By the orthogonality of the vector fields V_1 and TX_2, it follows that

$$g(\sigma(X_1, X_2), JV_1) = g(X_1, X_2)g(J\lambda, V_1) + g(TX_1, X_2)g(\lambda, V_1). \tag{16}$$

Then, using the symmetry of σ and g, we obtain that

$$g(TX_1, X_2)g(\lambda, V_1) = 0 \tag{17}$$

Thus, for any $V \in \Gamma(\mathfrak{D}^{\perp})$, we have $g(\lambda, V_1) = 0$, which proves the proposition. □

Remark 1. *In our research, proposition 1 indicates the Lee vector field λ is in $(\mathfrak{D}^{\theta})$.*

Remark 2. *The Lee vector field λ is tangent to $\mathfrak{D}$ for a CR-warped product in an LCK-manifold [9].*

Next, the following lemmas are preparatory.

Lemma 4. *Let $\tilde{N}$ be an LCK-manifold and $N = \Sigma^{\theta} \times {}_f\Sigma^{\perp}$ be a warped product submanifold of $\tilde{N}$, with tangent Lee vector field λ to N. Then, for any $X_1, X_2 \in \Gamma(\mathfrak{D}^{\theta})$ and $V_1, V_2 \in \Gamma(\mathfrak{D}^{\perp})$, we have the following:*

(i) $g(\sigma(X_1, X_2), JV_1) = 0$;

(ii) $g(\sigma(X_1, V_1), JV_2) = [g(\lambda, TX_1) - TX_1(\ln f)]g(V_1, V_2) + g(\sigma(V_1, V_2), FX_1)$, *where Σ^{θ} are proper pointwise slant and $\Sigma^{\perp}$ are anti-invariant submanifolds of N.*

Proof. By interchanging X_1 with X_2 in Equation (16), we obtain

$$g(\sigma(X_1, X_2), JV_1) = g(X_1, X_2)g(J\lambda, V_1) + g(X_1, TX_2)g(\lambda, V_1), \quad (18)$$

Now, using (16) and (18) with the symmetry of σ and g, we conclude that

$$g(\sigma(X_1, X_2), JV_1) = g(X_1, X_2)g(\lambda, JV_1).$$

Applying Proposition 1 and the condition that λ is tangent to N, which implies $g(\lambda, JV_1) = 0$, we then derive statement (i) of the lemma.

Now, for statement (ii), we obtain

$$g(\sigma(X_1, V_1), JV_2) = -g(J\tilde{\nabla}_{V1}X_1, V_2),$$

Using the definition of the covariant derivative of J,

$$g(\sigma(X_1, V_1), JV_2) = g((\tilde{\nabla}_{V1}J)X_1, V_2) - g(\tilde{\nabla}_{V1}JX_1, V_2).$$

Then, from Lemma 3 (ii) and Equation (6),

$$g(\sigma(X_1, V_1), JV_2) = g((\tilde{\nabla}_{V1}J)X_1, V_2) - TX_1(\ln f)g(V_1, V_2) + g(A_{FX_1}V_1, V_2).$$

Next, from the definition of the structure of LCK-manifolds (2),

$$g(\sigma(X_1, V_1), JV_2) = [g(\lambda, TX_1) - TX_1(\ln f)]g(V_1, V_2) + g(\sigma(V_1, V_2), FX_1),$$

Therefore, the second statement (ii) of the lemma is given by the last equation. □

Thus, we can give the subsequent result.

Lemma 5. *Let $N = \Sigma^\theta \times_f \Sigma^\perp$ be a non-trivial warped product pointwise pseudo-slant submanifold of an LCK-manifold $\tilde{N}$ such that λ Lee field is tangent to N. Then, we have*

$$g(\sigma(TX_1, V_1), JV_2) - g(\sigma(V_1, V_2), FTX_1) = \cos^2\theta[X_1(\ln f) - g(\lambda, X_1)]g(V_1, V_2)$$

for any $X \in \Gamma(\mathfrak{D}^\theta)$ and $V_1, V_2 \in \Gamma(\mathfrak{D}^\perp)$.

Proof. The proof is derived from Lemma 4 (ii) by interchanging X_1 with TX_1. Then, we obtain the result by using Formula (8). □

By considering both Lemma 4 and Lemma 5, we can present the subsequent outcome:

Theorem 3. *Let $\tilde{N}$ be an LCK-manifold. Then, a warped product pointwise pseudo-slant $N = \Sigma^\theta \times_f \Sigma^\perp$, with the Lee vector field λ tangent to N, satisfies the following:*

(i) $g(A_{JV_1}X_1, X_2) = 0.$

(ii) $g(A_{JV_2}TX_1 - A_{FTX_1}V_2, V_1) = \cos^2\theta[X_1(\ln f) - \beta(X_1)]g(V_1, V_2).$

This holds for any $X_1, X_2 \in \Gamma(\mathfrak{D}^\theta)$ and $V_1, V_2 \in \Gamma(\mathfrak{D}^\perp)$.

Proof. Statement (i) is just Lemma 4 (i). Statement (ii) follows from Lemma 5. □

4. Main Results

This section establishes several properties of pointwise pseudo-slant submanifolds and presents the essential result related to their characterization as warped products.

Theorem 4. *Let $\tilde{N}$ be an LCK-manifold and $N = \Sigma^\theta \times_f \Sigma^\perp$ be a proper pointwise pseudo-slant warped product submanifold. Then, if $\sigma(X_1, V_1) \in \nu$, then we have*

$$g(A_{FTX_1}V_1, V_2) = \cos^2\theta[g(\lambda, X_1) - X_1(\ln f)]g(V_1, V_2),$$

for any $X_1 \in \Gamma(\mathfrak{D}^\theta)$ and $V_1, V_2 \in \Gamma(\mathfrak{D}^\perp)$.

Proof. From Lemma 5, we have

$$g(\sigma(TX_1, V_1), JV_2) - g(\sigma(V_1, V_2), FTX_1) = \cos^2\theta[X_1(\ln f) - g(\lambda, X_1)]g(V_1, V_2).$$

Using the hypothesis that $\sigma(X_1, V_1) \in \nu$, it follows that

$$g(\sigma(V_1, V_2), FTX_1) = \cos^2\theta[g(\lambda, X_1) - X_1(\ln f)]g(V_1, V_2).$$

Since ν and $JD^\perp$ are orthogonal distributions, we conclude that $g(\sigma(TX_1, V_1), JV_2) = 0$. Finally, applying Formula (5), we obtain the desired result. □

For a proper pointwise pseudo-slant submanifold N in an LCK-manifold, N is called mixed totally geodesic if the second fundamental form σ of N satisfies $\sigma(X_1, V_1) = 0$, for any $X_1 \in \Gamma(\mathfrak{D}^\theta)$ and $V_1 \in \Gamma(\mathfrak{D}^\perp)$. As a consequence of Theorem 4, we arrive at the subsequent outcome.

Theorem 5. *Let N be a mixed totally geodesic pointwise pseudo-slant warped product $N = \Sigma^\theta \times_f \Sigma^\perp$ of an LCK-manifold $\tilde{N}$. Then, for any $X_1 \in \Gamma(\mathfrak{D}^\theta)$ and $V_1 \in \Gamma(\mathfrak{D}^\perp)$,*

$$A_{FTX_1}V_1 = \cos^2\theta[\beta(X_1) - X_1(\ln f)]V_1.$$

Moreover, we can also establish the next result.

Theorem 6. *Let $\tilde{N}$ be an LCK-manifold and $N = \Sigma^\theta \times_f \Sigma^\perp$ be a proper pointwise pseudo-slant warped product submanifold. Then, if $\Sigma^\perp$ is totally geodesic in N and $\sigma(X_1, V_1) \in \nu$, then $\beta(X_1) = X_1(\ln f)$ for any $X \in \Gamma(\mathfrak{D}^\theta)$ and $V \in \Gamma(\mathfrak{D}^\perp)$.*

Proof. By virtue of Theorem 4 and the hypothesis of the theorem, we obtain the following equation.

$$cos^2\theta[g(\lambda, X_1) - X_1(\ln f)]g(V_1, V_2) = 0, \tag{19}$$

Since g is the Riemannian metric and N is proper pointwise pseudo-slant, the result follows from (19). □

To establish the characterization of pointwise pseudo-slant warped products, let us first recall Hiepko's Theorem.

Theorem 7 ([19])**.** *Let M be a Riemannian manifold and suppose that $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are two orthogonal distributions in M. Assume that $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are involutive such that $\mathfrak{D}_1$ is a totally geodesic foliation and $\mathfrak{D}_2$ is a spherical foliation. Then, N is locally isometric to a non-trivial warped product $N_1 \times_f N_2$, where N_1 and N_2 are integral manifolds of $\mathfrak{D}_1$ and $\mathfrak{D}_2$, respectively.*

We now present a characterization theorem that provides the necessary and sufficient conditions for a pointwise pseudo-slant submanifold to be a warped product of the form $\Sigma^\theta \times_f \Sigma^\perp$.

Theorem 8. *Let $\tilde{N}$ be an LCK-manifold. Then, a pointwise pseudo-slant submanifold N in $\tilde{N}$ is locally a non-trivial warped product manifold of the form $N = \Sigma^\theta \times_f \Sigma^\perp$ if and only if the Lee vector field λ tangent to N (orthogonal to $\mathfrak{D}^\perp$) and the shape operator A satisfies*

$$A_{FTX_1}V_1 - A_{JV_1}TX_1 = \cos^2\theta(\beta(X_1) - X_1(\mu))V_1, \ \forall\, V_1 \in \Gamma(\mathfrak{D}^\perp), \ X_1 \in \Gamma(\mathfrak{D}^\theta), \tag{20}$$

Here, Σ^θ is a pointwise slant submanifold and $\Sigma^\perp$ is an anti-invariant submanifold in $\tilde{N}$, while μ is a smooth function on N. Additionally, for any $W \in \Gamma(\mathfrak{D}^\perp)$, the condition $W(\mu) = 0$ holds.

Proof. If N is a warped product submanifold of an LCK-manifold $\tilde{N}$ of the form $\Sigma^\theta \times_f \Sigma^\perp$ such that Σ^θ is a proper pointwise slant submanifold and $\Sigma^\perp$ is an anti-invariant submanifold, then Theorem 3 (ii) directly implies Equation (20) with $\mu = \ln f$ and $\beta(X_1) = g(\lambda, X_1)$.

On the other hand, by applying Lemma 1 under the given conditions and noting that N is a pointwise pseudo-slant submanifold of an LCK-manifold $\tilde{N}$, we have the following result.

$$\cos^2\theta\, g(\nabla_{X_2}X_1, V_1) = 0 \ \forall\, V_1 \in \Gamma(\mathfrak{D}^\perp), \ X_1, X_2 \in \Gamma(\mathfrak{D}^\theta)$$

Given that N is a proper pointwise pseudo-slant submanifold of an LCK-manifold $\tilde{N}$, the above equality implies that the leaves of the distribution $\mathfrak{D}^\theta$ are totally geodesic in N. Moreover, we have the following by applying Lemma 2:

$$\cos^2\theta\, g(\nabla_{V2}V_1, X_1) = [g(A_{FTX_1}V_1 - A_{JV_1}TX_1, V_2) - g(V_1, V_2)g(\lambda, X_1)] \tag{21}$$

for any $X_1 \in \Gamma(\mathfrak{D}^\theta)$ and $V_1, V_2 \in \Gamma(\mathfrak{D}^\perp)$. Using the polarization identity, we have

$$\cos^2\theta\, g(\nabla_{V1}V_2, X_1) = [g(A_{FTX_1}V_2 - A_{JV_2}TX_1, V_1) - g(V_1, V_2)g(\lambda, X_1)] \tag{22}$$

Subtracting (22) from (21) and with the hypothesis of the theorem, we obtain

$$\cos^2\theta\, g([V_1, V_2], X_1) = 0.$$

Because $\cos^2\theta \neq 0$ due to N being a proper pointwise pseudo-slant submanifold, this implies that the distribution $\mathfrak{D}^\perp$ is integrable anti-invariant. Let $\sigma^\perp$ denote the second fundamental form of $N^\perp$ in N, where $N^\perp$ represents a leaf of $\mathfrak{D}^\perp$ on N. Then, we have for any $X_1 \in \Gamma(\mathfrak{D}^\theta)$ and $V_1, V_2 \in \Gamma(\mathfrak{D}^\perp)$

$$g(\sigma^\perp(V_1, V_2), X_1) = g(\nabla_{V_1}V_2, X_1) = g(\tilde{\nabla}_{V_1}V_2, X_1) = g(J\tilde{\nabla}_{V_1}V_2, JX_1).$$

Then, using Formula (2), we obtain the following equation.

$$\begin{aligned} g(\sigma^\perp(V_1, V_2), X_1) &= g(\tilde{\nabla}_{V_1}JV_2, JX_1) - g((\tilde{\nabla}_{V_1}J)V_2, JX_1) \\ &= -g(\tilde{\nabla}_{V_1}JX_1, JV_2) - g(V_1, V_2)g(\lambda, X_1). \end{aligned}$$

Applying Formula (6),

$$\begin{aligned} g(\sigma^\perp(V_1, V_2), X_1) &= -g(\tilde{\nabla}_{V_1}TX_1, JV_2) - g(\tilde{\nabla}_{V_1}FX_1, JV_2) - g(V_1, V_2)g(\lambda, X_1). \\ &= -g(\tilde{\nabla}_{V_1}TX_1, JV_2) + g(J\tilde{\nabla}_{V_1}FX_1, V_2) - g(V_1, V_2)g(\lambda, X_1). \end{aligned}$$

Hence,

$$\begin{aligned} g(\sigma^\perp(V_1, V_2), X_1) = &-g(A_{JV_2}TX_1, V_1) + g(\tilde{\nabla}_{V1}JFX_1, V_2) - g((\tilde{\nabla}_{V1}J)FX_1, V_2) \\ &- g(V_1, V_2)g(\lambda, X_1). \end{aligned}$$

By applying relation (11) under the theorem's hypotheses, the submanifold N is a proper pointwise pseudo-slant

$$g(\sigma^{\perp}(V_1, V_2), X_1) = -g(A_{JV_2}TX_1, V_1) + \sin 2\theta\, V_1(\theta) g(V_2, X_1) + \sin^2\theta\, g(\tilde{\nabla}_{V_1} V_2, X_1) + g(A_{FTX_1} V_2, V_1) - g(V_1, V_2) g(\lambda, X_1) \cos^2\theta.$$

Equivalently,

$$g(\sigma^{\perp}(V_1, V_2), X_1) = \sin^2\theta\, g(\tilde{\nabla}_{V_1} V_2, X_1) + g(A_{FTX_1} V_2 - A_{JV_2} TX_1, V_1) - g(V_1, V_2) g(\lambda, X_1) \cos^2\theta.$$

Then, from condition (20), we derive the following conclusion:

$$\cos^2\theta\, g(\sigma^{\perp}(V_1, V_2), X_1) = -\cos^2\theta\, X_1(\mu) g(V_1, V_2).$$

Hence, by the gradient of the function μ, we conclude that

$$g(\sigma^{\perp}(V_1, V_2), X_1) = -\vec{\nabla}\mu g(V_1, V_2)$$

Therefore, $\Sigma^{\perp}$ is totally umbilical in N and $H^{\perp} = -\vec{\nabla}\mu$ represents the mean curvature vector. Further, we can prove that the mean curvature vector $H^{\perp}$ is parallel to the normal connection of $\Sigma^{\perp}$ in N (see [19] for more details). Hence, it is an extrinsic sphere. Thus, according to a result by Theorem 7 , we conclude that N is a warped product submanifold $\Sigma^{\theta} \times_f \Sigma^{\perp}$, where μ is the warping function. □

5. Consequences of the Main Results

We provide a variety of special situations in this section derived from our previous results; some of them are obtained from important theorems that were proven in earlier research. This means that the conclusions presented in this study are extensions and generalizations of basic theories. We present the following applications:

Theorem 4 extends the following theories in particular special cases.

For the pseudo-slant warped product submanifold of an LCK-manifold $\tilde{N}$, the slant function θ of Σ^{θ} in Theorem 4 is replaced by a constant value rather than a function, and as a result, we obtain the following:

Theorem 9. *If $N = \Sigma^{\theta} \times_f \Sigma^{\perp}$ is a mixed totally geodesic pseudo-slant warped product submanifold of an LCK-manifold $\tilde{N}$, then for any $X \in \Gamma(\mathfrak{D}^{\theta}), V \in \Gamma(\mathfrak{D}^{\perp})$*

$$A_{FTX_1} V_1 = \cos^2\theta [\beta(X_1) - X_1(\ln f)] V_1.$$

Currently, by applying Theorem 4 and setting $\theta = 0$ in the equation, it yields result (3.3) in [9]. Consequently, Theorem 4 is valid for CR-warped product submanifolds of the form $N = \Sigma^{T} \times_f \Sigma^{\perp}$, demonstrated as follows:

Theorem 10. *If $N = \Sigma^{T} \times_f \Sigma^{\perp}$ is a mixed totally geodesic CR-warped product submanifold of an LCK-manifold $\tilde{N}$, we have for any $X_1 \in \Gamma(\mathfrak{D})$*

$$X_1(\ln f) = \beta(X_1)$$

Therefore, Proposition 3.3 in [9] can be regarded as a special case of Theorem 4.

Moreover, a Kähler manifold can be considered an LCK-manifold with $\lambda = 0$. Thus, Theorem 5 leads to the following result.

Theorem 11. *Let $N = \Sigma^\theta \times_f \Sigma^\perp$ be a mixed totally geodesic pointwise pseudo-slant warped product of Kähler manifold $\tilde{N}$. Then, for any $X_1 \in \Gamma(\mathfrak{D}^\theta)$ and $V_1 \in \Gamma(\mathfrak{D}^\perp)$,*

$$A_{FTX_1}V_1 = -\cos^2\theta[X_1(\ln f)]V_1.$$

Additionally, Theorem 5 implies the following.

Theorem 12 ([20])**.** *A pointwise pseudo-slant warped product submanifold $N = \Sigma^\theta \times_f \Sigma^\perp$ in a Kähler manifold $\tilde{N}$ is simply a locally Riemannian product $X_1(\ln f) = 0$ if and only if $A_{FTX_1}V_1 = 0$ for any $X_1 \in \Gamma(\mathfrak{D}^\theta)$ and $V_1 \in \Gamma(\mathfrak{D}^\perp)$.*

The previous theorem is merely Theorem (3.3), which was proven in [20].

Now, we may deduce the following outcome specifically for the case in which $\lambda = 0$ and $\theta = 0$ in Theorem 5, i.e., N is a CR-warped product submanifold in Kähler manifold $\tilde{N}$.

Theorem 13. *On a Kähler manifold $\tilde{N}$, no mixed geodesic CR-warped product submanifold of the form $N = \Sigma^T \times_f \Sigma^\perp$ exists in $\tilde{N}$.*

Next, we present the following consequences of Theorem 8.

If we consider that the slant function θ is a constant in Theorem 8, then the submanifold N is a pseudo-slant submanifold of an LCK-manifold. Accordingly, Theorem 8 indicates the following for this case.

Theorem 14. *Let $\tilde{N}$ be an LCK-manifold. Then, a pseudo-slant submanifold N in $\tilde{N}$ is locally a non-trivial warped product manifold of the form $N = \Sigma^\theta \times_f \Sigma^\perp$ if and only if the Lee vector field λ tangent to N and the shape operator A satisfies*

$$A_{FTX_1}V_1 - A_{JV_1}TX_1 = \cos^2\theta(\beta(X_1) - X_1(\mu))V_1,\ \ \forall\ V_1 \in \Gamma(\mathfrak{D}^\perp),\ X_1 \in \Gamma(\mathfrak{D}^\theta), \tag{23}$$

Here, Σ^θ is a slant submanifold, and $\Sigma^\perp$ is an anti-invariant submanifold in $\tilde{N}$, while μ is a smooth function on N. Additionally, for any $W \in \Gamma(\mathfrak{D}^\perp)$, the condition $W(\mu) = 0$ holds.

In Theorem 8, if $\theta = 0$, the submanifold N is a CR-submanifold of an LCK-manifold. Hence, in this instance, Theorem 8 states the following.

Theorem 15 ([21])**.** *Let $\tilde{N}$ be an LCK-manifold. Then, a proper CR-submanifold N in $\tilde{N}$ is locally a CR-warped product if and only if the Lee vector field λ tangent to N (orthogonal to $\mathfrak{D}^\perp$) and the shape operator A satisfies*

$$A_{JV_1}X_1 = -(JX_1(\mu) + g(J\lambda, X_1))V_1\ X_1 \in \Gamma(\mathfrak{D}),\ V_1 \in \Gamma(\mathfrak{D}^\perp), \tag{24}$$

Here, $\mathfrak{D}$ is an invariant distribution, and $\mathfrak{D}^\perp$ is an anti-invariant distribution of N, while μ is a smooth function on N. Additionally, for any $W \in \Gamma(\mathfrak{D}^\perp)$, the condition $W(\mu) = 0$ holds.

On the other hand, when $\lambda = 0$ is considered, we arrive at a particular case of Theorem 8 where the submanifold N becomes a submanifold within a Kähler manifold. This particular case was studied in [20]. In this context, Theorem 8 corresponds to the characterization theorem (Theorem 4.2) presented in [20].

Theorem 16. *Let $\tilde{N}$ be a Kähler manifold. Then, a pointwise pseudo-slant submanifold N in $\tilde{N}$ is locally a non-trivial warped product manifold of the form $N = \Sigma^\theta \times_f \Sigma^\perp$ if and only if the shape operator A satisfies*

$$A_{FTX_1}V_1 - A_{JV_1}X_1 = -(\cos^2\theta)X_1(\mu)V_1, \forall\ X_1 \in \Gamma(\mathfrak{D}^\theta),\ V_1 \in \Gamma(\mathfrak{D}^\perp), \tag{25}$$

Here, Σ^θ is a pointwise slant submanifold, and $\Sigma^\perp$ is an anti-invariant submanifold in $\tilde{N}$, while μ is a smooth function on N. Additionally, for any $W \in \Gamma(\mathfrak{D}^\perp)$, the condition $W(\mu) = 0$ holds.

Consequently, the main result of Theorem in [20] represents a specific instance of Theorem 8.

However, if $\lambda = 0$ and θ is a slant function in Theorem 8, the pointwise pseudo-slant warped product submanifold simplifies to the pseudo-slant warped product submanifold. Then, the characterization theorem (Theorem 5.1) of [22] is derived from Theorem 8.

Theorem 17. *Let $\tilde{N}$ be the Kähler manifold. Then, a pseudo-slant submanifold N in $\tilde{N}$ is locally a non-trivial warped product manifold of the form $N = \Sigma^\theta \times_f \Sigma^\perp$ if and only if the shape operator A satisfies*

$$A_{FTX_1}V_1 - A_{JV_1}TX_1 = -(\cos^2\theta)X_1(\mu)V_1, \ \forall\ X_1 \in \Gamma(\mathfrak{D}^\theta),\ V_1 \in \Gamma(\mathfrak{D}^\perp), \tag{26}$$

Here, Σ^θ is a slant submanifold, and $\Sigma^\perp$ is an anti-invariant submanifold in $\tilde{N}$, while μ is a smooth function on N. Additionally, for any $W \in \Gamma(\mathfrak{D}^\perp)$, the condition $W(\mu) = 0$ holds.

Furthermore, this theorem is proven in [22] (Theorem 5.1). We can prove Theorem 5.1 without using the mixed condition. Consequently, Theorem 8 extends Theorem 5.1 of [22].

By setting $\lambda = 0$ and $\theta = 0$ in Theorem 8, the submanifold N becomes a CR-submanifold in a Kähler manifold, a case that has been examined in [3]. A characterization theorem for such submanifolds is provided below. Therefore, Theorem 8 simplifies to Theorem 4.2 from [3].

Theorem 18. *Let $\tilde{N}$ be the Kähler manifold. Then, a proper CR-submanifold N in $\tilde{N}$ is locally a CR-warped product if and only if*

$$A_{JV_1}X_1 = -JX_1(\mu)V_1,\ X_1 \in \Gamma(\mathfrak{D}),\ V_1 \in \Gamma(\mathfrak{D}^\perp), \tag{27}$$

Here, $\mathfrak{D}$ is an invariant submanifold, and $\mathfrak{D}^\perp$ is an anti-invariant submanifold in $\tilde{N}$, while μ is a smooth function on N. Additionally, for any $W \in \Gamma(\mathfrak{D}^\perp)$, the condition $W(\mu) = 0$ holds.

This result corresponds to Theorem 4.2 from [3]. Therefore, the main result of [3] is a particular case of Theorem 8.

6. Examples

To conclude this study, we provide the following examples of a non-trivial warped product pointwise pseudo-slant submanifold of an LCK-manifold.

Now, let $(\mathbb{R}^{2n}, J, g)$ be a usual Kähler manifold with the Cartesian coordinates

$$(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

of a Euclidean $2n$-space $\mathbb{R}^{2n}$ equipped with the Euclidean metric g_0, and the standard complex structure J is defined:

$$J(\frac{\partial}{\partial x_i}) = -\frac{\partial}{\partial y_i}, \quad J(\frac{\partial}{\partial y_j}) = \frac{\partial}{\partial x_j}\ , 1 \leq i, j \leq n. \tag{28}$$

Remark 3. *By applying the same method as Proposition 2.2 of [18], we may prove the same result for warped product pointwise pseudo-slant submanifolds N in LCK-manifolds.*

Proposition 2. *Let $N = \Sigma^\theta \times_f \Sigma^\perp$ be a warped product pointwise pseudo-slant submanifold of a Kähler manifold $\tilde{N}$. Then, N is also a warped product pointwise pseudo-slant submanifold with the* *same slant function in an LCK-manifold* $(\tilde{N}, J, \tilde{g})$ *with* $\tilde{g} = e^{-f} g$, *where* f *is any smooth function on* $\tilde{N}$.

Example 1. Consider a submanifold N of a Kähler manifold $(\mathbb{R}^6, J, g_0)$ defined in $\mathbb{R}^6$. Let N be given by the following immersion:

$$\begin{aligned} x_1 &= a_1 \cos a_2, \quad x_2 = a_1 + a_3, \quad x_3 = a_1 \sin a_2, \\ y_1 &= a_3 \cos a_2, \quad y_2 = a_1 - a_3, \quad y_3 = a_3 \sin a_2, \end{aligned} \tag{29}$$

for non-vanishing functions a_1, a_2 and a_3 on N. Then, the TN of N is spanned by

$$\begin{aligned} E_1 &= \cos a_2 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2} + \sin a_2 \frac{\partial}{\partial x_3}, \\ E_2 &= \cos a_2 \frac{\partial}{\partial y_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_2} + \sin a_2 \frac{\partial}{\partial y_3}, \\ E_3 &= -a_1 \sin a_2 \frac{\partial}{\partial x_1} - a_3 \sin a_2 \frac{\partial}{\partial y_1} + a_1 \cos a_2 \frac{\partial}{\partial x_3} + a_1 \cos a_2 \frac{\partial}{\partial y_3}. \end{aligned}$$

Clearly, we see that JE_3 is orthogonal to TN. Then, $\mathfrak{D}^\perp = \mathrm{Span}\{E_3\}$ and $\mathfrak{D}^\theta = \mathrm{Span}\{E_1, E_2\}$ if we consider $\mathfrak{D}^\perp$ is the anti-invariant distribution and $\mathfrak{D}^\theta$ is the slant distribution. Furthermore, N is a proper pseudo-slant submanifold, and the slant angle $\theta = \cos^{-1}(1/3)$.

Consider the integral manifolds of $\mathfrak{D}^\theta$ and $\mathfrak{D}^\perp$ represented by $\Sigma^\perp$ and Σ^θ, respectively. Consequently, the induced metric tensor $\hat{g}$ of $N = \Sigma^\theta \times \Sigma^\perp$ is

$$\hat{g} = g_\theta + g_{\Sigma^\perp}, \quad g_\theta = d{a_1}^2 + da_3^2, \quad g_{\Sigma^\perp} = (a_1^2 + a_3^2) da_2^2. \tag{30}$$

Let f be a smooth, non-constant function on $\mathbb{R}^6$ depending on a_1, a_3. Consider the Riemannian metric $\tilde{g} = e^{-f} g_0$ on $\mathbb{R}^6$, which is conformal to the standard metric g_0. Under this construction, $\tilde{N} = (\mathbb{R}^6, J, \tilde{g})$ forms a GCK-manifold. Consequently, the metric induced on N from this GCK-manifold is the warped product metric.

$$g_N = g_{\Sigma^\theta} + e^{-f} g_{\Sigma^\perp}, \quad g_{\Sigma^\theta} = e^{-f} g_\theta. \tag{31}$$

Moreover, (N, g_N) is a proper warped product pseudo-slant submanifold by applying Proposition 2. Furthermore, since f is a smooth function on $\mathbb{R}^6$ depending only on a_1, a_3, it follows from (29) that, when restricted to the submanifold N, the corresponding Lee form is expressed as

$$\beta = df = \frac{\partial f}{\partial a_1} da_1 + \frac{\partial f}{\partial a_3} da_3. \tag{32}$$

As a result, the Lee vector field λ is tangent to Σ^θ and, by extension, tangent to N, as shown by (31) and (37).

Example 2. Let us consider $\mathbb{R}^6$, the Kähler manifold with the standard Kähler structure. For non-vanishing functions b_1, b_2 and b_3, where $b_1 b_2 \neq 1$, $b_3 \neq 0$ and $(b_1 - b_2) \in \left(0, \frac{\pi}{2}\right)$, we define a submanifold N in $\mathbb{R}^6$ as follows:

$$\begin{aligned} x_1 &= b_3, \quad x_2 = b_1 \cos b_2, \quad x_3 = b_1 \sin b_2, \\ y_1 &= b_3, \quad y_2 = b_2 \cos b_1, \quad y_3 = b_2 \sin b_1, \end{aligned} \tag{33}$$

Next, the tangent bundle of N is spanned by

$$\psi_{b3} = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}\right)$$
$$\psi_{b1} = \frac{1}{\sqrt{1+b_2^2}}\left(\cos b_2 \frac{\partial}{\partial x_2} + \sin b_2 \frac{\partial}{\partial x_3} - b_2 \sin b_1 \frac{\partial}{\partial y_2} + b_2 \cos b_1 \frac{\partial}{\partial y_3}\right),$$
$$\psi_{b2} = \frac{1}{\sqrt{1+b_1^2}}\left(-b_1 \sin b_2 \frac{\partial}{\partial x_2} + b_1 \cos b_2 \frac{\partial}{\partial x_3} + \cos b_1 \frac{\partial}{\partial y_2} + \sin b_1 \frac{\partial}{\partial y_3}\right).$$

It is clear that N is a proper pointwise pseudo-slant submanifold with pointwise slant distribution $\mathfrak{D}^\theta = \mathrm{Span}\{\psi_{b1}, \psi_{b2}\}$ and anti-invariant distribution $\mathfrak{D}^\perp = \mathrm{Span}\ \{\psi_{b3}\}$. Also, the slant function θ of the pointwise slant distribution satisfies

$$\cos^2\theta = \frac{(b_1 b_2 - 1)^2 \cos^2(b_1 - b_2)}{(1+b_1^2)(1+b_2^2)}.$$

It is easy to verify that $\mathfrak{D}^\theta$ and $\mathfrak{D}^\perp$ are integrable and totally geodesic in N. Thus, the metric $\hat{g}$ on $N = \Sigma^\theta \times \Sigma^\perp$ such that Σ^θ and $\Sigma^\perp$ are integral submanifolds to $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$ is given by

$$\hat{g} = g_\theta + g_{\Sigma^\perp}, \tag{34}$$

where

$$g_\theta = (1+b_2^2)db_1^2 + (1+b_1^2)db_2^2, \quad g_{\Sigma^\perp} = 2db_3^2. \tag{35}$$

Now, consider the Riemannian metric $\tilde{g} = e^{-f} g_0$, in which f is a smooth function on $\mathbb{R}^6$ and $\tilde{g}$ is conformal to the standard metric g_0 on $\mathbb{R}^6$ as in Example 1.In this case, the warped product metric is

$$g_N = g_{\Sigma^\theta} + e^{-f} g_{\Sigma^\perp}, \quad g_{\Sigma^\theta} = e^{-f} g_\theta. \tag{36}$$

Furthermore, Proposition 2 implies that (N, g_N) is a non-trivial warped product pointwise pseudo-slant submanifold. Moreover, the corresponding Lee form is expressed as

$$\beta = df = \frac{\partial f}{\partial b_1} db_1 + \frac{\partial f}{\partial b_2} db_2. \tag{37}$$

As a consequence, the Lee vector field λ is tangent to Σ^θ and, by extension, tangent to N, as shown by (31) and (37).

7. Conclusions

This paper introduces the concept of pointwise pseudo-slant warped products in locally conformal Kähler manifolds (LCK-manifolds), building upon and generalizing the idea of CR-warped products. We studied warped products of the form $\Sigma^\theta \times_f \Sigma^\perp$, where Σ^θ denotes proper pointwise slant submanifolds and $\Sigma^\perp$ represents anti-invariant submanifolds. Through this framework, we established necessary and sufficient conditions for these submanifolds to be classified as warped product submanifolds.

Additionally, we derived several key results that expand and generalize existing findings in [3,9,20–22]. Non-trivial examples were provided to illustrate the properties and validate the theoretical results presented. This research contributes to the understanding of these submanifolds and their warped products, paving the way for future research in differential geometry and related fields.

Funding: This work was funded by the University of Jeddah, Jeddah, Saudi Arabia, under grant No. (UJ-24-DR-3029-1).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The author thanks the University of Jeddah for its technical and financial support. The author is also grateful to the unknown referees for their many constructive suggestions, which improved the presentation of the paper.

Conflicts of Interest: The author declares no conflicts of interest.

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