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Fixed-Point Results with Applications in Generalized Neutrosophic Rectangular *b*-Metric Spaces

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Abstract: In this paper, we introduce several new concepts: generalized neutrosophic rectangular *b*-metric-like spaces (GNRBMLSs), generalized intuitionistic rectangular *b*-metric-like spaces (GIRBMLSs), and generalized fuzzy rectangular *b*-metric-like spaces (GFRBMLSs). These innovative spaces can expand various topological spaces, including neutrosophic rectangular extended *b*-metric-like spaces, intuitionistic fuzzy rectangular extended *b*-metric-like spaces, and fuzzy rectangular extended *b*-metric-like spaces. Moreover, we establish Banach's fixed point theorem and Ćirić's quasi-contraction theorem with respect to these spaces, and we explore an application regarding the existence and uniqueness of solutions for fuzzy fractional delay integro-differential equations, as derived from our main results.

Keywords: generalized neutrosophic rectangular *b*-metric-like space; generalized intuitionistic rectangular *b*-metric-like space; generalized fuzzy rectangular *b*-metric-like space; fuzzy fixed point; fuzzy fractional delay integro-differential equations

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1. Introduction and Preliminaries

Schweizer and Sklar [1] played a key role in the introduction of the concepts of continuous triangular norms (CtN) and continuous t norms (CtCN). Zadeh [2] later developed the theory of fuzzy sets (FSs). Extending Zadeh's foundation, Kramosil and Michalek introduced the concept of fuzzy metric spaces (FMSs) in their work [3]. Subsequently, George and Veeramani redefined FMS [4], and Grabiec, based on the work of Kramosil and Michalek, derived a well-known fixed point theorem known as the Banach contraction theorem (BCT) [5]. Gregori and Sapena [6] then generalized the fuzzy BCT to fuzzy metric spaces as defined by George and Veeramani.

While fuzzy sets are limited to membership functions, they leave a gap regarding non-membership functions. Atanassov [7] addressed this gap by introducing intuitionistic fuzzy sets (IFSs), which incorporate both membership and non-membership degrees. However, IFSs do not account for the concept of naturalness, which Smarandache [8] later tackled by introducing neutrosophic sets (NSs), a more general framework that extends IFSs.

Kirişci and Simsek [9] combined neutrosophic sets with metric spaces to form neutrosophic metric spaces (NMSs). Following this, Saleem et al. [10] introduced extended fuzzy rectangular b-metric spaces (EFRBMSs), while Saleem et al. [11] expanded upon this by presenting extended fuzzy rectangular metric-like spaces (EFRBMLSs). In later works, Hussain et al. [12] introduced fuzzy rectangular *b*-metric-like spaces (FRBMLSs), intuitionistic fuzzy rectangular *b*-metric-like spaces (IFRBMLSs), and neutrosophic rectangular *b*-metric-like spaces (NRBMLSs). Kattan et al. [13] proposed an extension to intuitionistic fuzzy rectangular b-metric spaces, defining extended intuitionistic rectangular *b*-metric spaces (EIRBMSs), thus creating a more generalized framework. Uddin et al. [14] introduced controlled neutrosophic b-metric-like spaces, while Saleem et al. [15] introduced neutrosophic extended b-metric spaces (ENRBMSs).

Despite these advances, the concepts of intuitionistic fuzzy rectangular extended b-metric-like spaces (IEFRBMLSs) and neutrosophic rectangular extended b-metric-like spaces (ENRBMLSs) remain relatively novel.

On another front, Ashraf et al. [16] introduced generalized FMSs by relaxing the triangular inequality in these systems. This relaxation allows for more flexibility in calculating distances between elements in a set, which is particularly advantageous when the standard triangle inequality cannot be applied, but a less strict version still provides a sufficient framework for practical applications, such as convergence analysis, fixed point theorems, and optimization issues. This generalized concept extends to various topological spaces, including FMSs, fuzzy b-MSs, and dislocated FMSs, with the BCT and Ćirić quasi-contraction theorem (CQT) proven in the context of generalized FMSs.

In this paper, inspired by the work of Ashraf et al. [16] and Hussain et al. [17], we define and expand the classes of generalized neutrosophic rectangular b-metric-like spaces (GNRBMLSs), generalized intuitionistic rectangular b-metric-like spaces (GIRBMLSs), and generalized fuzzy rectangular b-metric-like spaces (GFRBMLSs). We also extend and improve several fixed point (FP) theorems within the contexts of GNRBMLSs, GIRBMLSs, and GFRBMLSs. Our findings are applicable in the study of the existence and uniqueness of solutions for fuzzy fractional delay integro-differential equations (FFDIDEs). These new spaces offer a robust framework for addressing more complex problems in mathematical modeling, optimization, and decision making, particularly in situations where NMSs or fuzzy metric spaces are insufficient.

The structure of the remainder of the manuscript is as follows: Section 2 introduces GNRBMLSs, GIRBMLSs, and GFRBMLSs, explores the concept of Cauchy sequences and their convergence properties, and provides examples and propositions. Section 3 proves two major fixed point theorems and derives some corollaries, supported by non-trivial examples. Section 4 demonstrates the existence of a unique analytical solution for FFDIDEs. Finally, Section 5 discusses future work and presents two open problems.

Now, we will review several foundational concepts that are essential for understanding the subsequent sections.

Definition 1 ([15,18]). *Let $\mathcal{Q}$ be a non-empty set, $\Lambda : \mathcal{Q} \times \mathcal{Q} \to [1, +\infty)$ be a function, $*$ represent a CtN, and $\circ$ denote a CtCN. Furthermore, let $\mathcal{N}, \mathcal{O}, \mathcal{L} : \mathcal{Q} \times \mathcal{Q} \times (0, +\infty) \to [0, 1]$ be NSs. A six-tuple $(\mathcal{Q}, \mathcal{N}, \mathcal{O}, \mathcal{L}, *, \circ)$ is called an ENRBMS over $\mathcal{Q}$ if the following conditions hold for any $\xi, \eta \in \mathcal{Q}$, $\gamma, \lambda \in \mathcal{Q} \setminus \{\xi, \eta\}$, and $\tau, \iota, \sigma > 0$:*

(N1) $\mathcal{N}(\xi, \eta, \tau) + \mathcal{O}(\xi, \eta, \tau) + \mathcal{L}(\xi, \eta, \tau) \leq 3$;
(N2) $\mathcal{N}(\xi, \eta, \tau) > 0$;
(N3) $\mathcal{N}(\xi, \eta, \tau) = 1$ *if and only if* $\xi = \eta$;
(N4) $\mathcal{N}(\xi, \eta, \tau) = \mathcal{N}(\eta, \xi, \tau)$;
(N5) $\mathcal{N}(\xi, \eta, \Lambda(\xi, \eta)(\tau + \iota + \sigma)) \geq \mathcal{N}(\xi, \gamma, \tau) * \mathcal{N}(\gamma, \lambda, \iota) * \mathcal{N}(\lambda, \eta, \sigma)$;
(N6) $\mathcal{N}(\xi, \eta, .) : (0, +\infty) \to [0, 1]$ *is continuous and* $\lim_{\tau \to +\infty} \mathcal{N}(\xi, \eta, \tau) = 1$;
(N7) $\mathcal{O}(\xi, \eta, \tau) < 1$;
(N8) $\mathcal{O}(\xi, \eta, \tau) = 0$ *if and only if* $\xi = \eta$;
(N9) $\mathcal{O}(\xi, \eta, \tau) = \mathcal{O}(\eta, \xi, \tau)$;
(N10) $\mathcal{O}(\xi, \eta, \Lambda(\xi, \eta)(\tau + \iota + \sigma)) \leq \mathcal{O}(\xi, \gamma, \tau) \circ \mathcal{O}(\gamma, \lambda, \iota) \circ \mathcal{O}(\lambda, \eta, \sigma)$;
(N11) $\mathcal{O}(\xi, \eta, .) : (0, +\infty) \to [0, 1]$ *is continuous and* $\lim_{\tau \to +\infty} \mathcal{O}(\xi, \eta, \tau) = 0$;
(N12) $\mathcal{L}(\xi, \eta, \tau) < 1$;
(N13) $\mathcal{L}(\xi, \eta, \tau) = 0$ *if and only if* $\xi = \eta$;
(N14) $\mathcal{L}(\xi, \eta, \tau) = \mathcal{L}(\eta, \xi, \tau)$;
(N15) $\mathcal{L}(\xi, \eta, \Lambda(\xi, \eta)(\tau + \iota + \sigma)) \leq \mathcal{L}(\xi, \gamma, \tau) \circ \mathcal{L}(\gamma, \lambda, \iota) \circ \mathcal{L}(\lambda, \eta, \sigma)$;
(N16) $\mathcal{L}(\xi, \eta, .) : (0, +\infty) \to [0, 1]$ *is continuous and* $\lim_{\tau \to +\infty} \mathcal{L}(\xi, \eta, \tau) = 0$;

(N17) *If* $\tau \leq 0$ *then* $\mathcal{N}(\xi,\eta,\tau) = 0, \mathcal{O}(\xi,\eta,\tau) = 1, \mathcal{L}(\xi,\eta,\tau) = 1.$

Theorem 1 ([18])**.** *Let* $(\mathcal{Q},\mathcal{N},\mathcal{O},\mathcal{L},*,\circ)$ *be a complete ENRBMS in the company of* $\Lambda : \mathcal{Q} \times \mathcal{Q} \to [1,+\infty)$ *with* $\alpha \in (0,1)$. *Let* $\mathcal{T} : \mathcal{Q} \to \mathcal{Q}$ *be a mapping satisfying*

$$\mathcal{N}(\mathcal{T}\xi,\mathcal{T}\eta,\alpha\tau) \geq \mathcal{N}(\xi,\eta,\tau),$$
$$\mathcal{O}(\mathcal{T}\xi,\mathcal{T}\eta,\alpha\tau) \leq \mathcal{O}(\xi,\eta,\tau),$$
$$\mathcal{L}(\mathcal{T}\xi,\mathcal{T}\eta,\alpha\tau) \leq \mathcal{L}(\xi,\eta,\tau).$$

for all $\xi,\eta \in \mathcal{Q},\tau > 0$. *Furthermore, suppose that for arbitrary* $\xi_0 \in \mathcal{Q}$ *we have* $\Lambda(\xi_n,\xi_{n+m}) < \frac{1}{\alpha}$. *Then,* $\{\mathcal{T}^n\xi_0\}$ *will converge to a unique FP of* $\mathcal{T}$.

Remark 1. *According to Definition 1, we derive the following definitions:*

1. *Considering the following condition,*

   (L1) $\mathcal{N}(\xi,\eta,\tau) + \mathcal{O}(\xi,\eta,\tau) \leq 1$, *for all* $\xi,\eta \in \mathcal{Q},\tau > 0$,

   along with conditions (N2)–(N11), then $(\mathcal{Q},\mathcal{N},\mathcal{O},*,\circ)$ *characterizes an extended intuitionistic rectangular b-metric space (EIRBMS) on* $\mathcal{Q}$*; we refer the reader to [13].*
2. *Taking into account the following condition,*

   (L2) $0 < \mathcal{N}(\xi,\eta,\tau) \leq 1$, *for all* $\xi,\eta \in \mathcal{Q},\tau > 0$,

   along with conditions (N3)–(N6), then $(\mathcal{Q},\mathcal{N},*)$ *characterizes an extended fuzzy rectangular b-metric space (EFRBMS) on* $\mathcal{Q}$*; we direct the reader to [10]. Furthermore, as discussed in [11], EFRBMLSs can be derived by replacing (N3) in EFRBMSs with the axiom stated below:*

   (L3) $\mathcal{N}(\xi,\eta,\tau) = 1 \Rightarrow \xi = \eta$, *for every* $\xi,\eta \in \mathcal{Q},\tau > 0$.

In this paper, from Remark 1 and Definition 1, we are able to introduce the concepts of ENRBMLSs and EIRBMLSs, defined in $\mathcal{Q}$ as a generalization of ENRBMS and EIRBMS, if we replace (N8) and (N13) introduced in Definition 1 with the following axioms:

(L4) $\mathcal{O}(\xi,\eta,\tau) = 0 \Rightarrow \xi = \eta$, for all $\xi,\eta \in \mathcal{Q},\tau > 0$;
(L5) $\mathcal{L}(\xi,\eta,\tau) = 0 \Rightarrow \xi = \eta$, for all $\xi,\eta \in \mathcal{Q},\tau > 0$.

2. Generalized Neutrosophic Rectangular *b*-Metric Spaces

In this section, we present the concepts of GNRBMLSs, GIRBMLSs, and GFRBMLSs, and we demonstrate several FP theorems within these contexts. Let $\mathcal{Q}$ be a non-empty set and $\mathcal{N},\mathcal{O},\mathcal{L} : \mathcal{Q} \times \mathcal{Q} \times (0,+\infty) \to [0,1]$ be NSs. For $\xi \in \mathcal{Q}$, we define the sets below for any $\tau > 0$:

$$\mathcal{C}_1(\mathcal{Q},\mathcal{N},\xi) = \{\{\xi_n\} \subset \mathcal{Q} : \lim_{n\to+\infty} \mathcal{N}(\xi_n,\xi,\tau) = \mathcal{N}(\xi,\xi,\tau)\},$$
$$\mathcal{C}_2(\mathcal{Q},\mathcal{O},\xi) = \{\{\xi_n\} \subset \mathcal{Q} : \lim_{n\to+\infty} \mathcal{O}(\xi_n,\xi,\tau) = \mathcal{O}(\xi,\xi,\tau)\},$$
$$\mathcal{C}_3(\mathcal{Q},\mathcal{L},\xi) = \{\{\xi_n\} \subset \mathcal{Q} : \lim_{n\to+\infty} \mathcal{L}(\xi_n,\xi,\tau) = \mathcal{L}(\xi,\xi,\tau)\}.$$

Definition 2. *Let* $\mathcal{Q}$ *be a non-empty set;* $\Lambda : \mathcal{Q} \times \mathcal{Q} \to [1,+\infty)$ *be a given function;* $*$ *and* $\circ$ *be a CtN and CtCN, respectively; and let* $\mathcal{N},\mathcal{O},\mathcal{L} : \mathcal{Q} \times \mathcal{Q} \times (0,+\infty) \to [0,1]$ *be NSs. Then,* $(\mathcal{Q},\mathcal{N},\mathcal{O},\mathcal{L},*,\circ)$ *can be referred to as a GNRBMLS on* $\mathcal{Q}$. *If for any* $\xi,\eta \in \mathcal{Q}, \gamma \in \mathcal{Q} \setminus \{\xi,\eta\}$, *and all* $\tau,s,r > 0$ *then following conditions are satisfied:*

(S1) $\mathcal{N}(\xi,\eta,\tau) + \mathcal{O}(\xi,\eta,\tau) + \mathcal{L}(\xi,\eta,\tau) \leq 3$;
(S2) $\mathcal{N}(\xi,\eta,\tau) > 0$;
(S3) $\mathcal{N}(\xi,\eta,\tau) = 1$ *implies* $\xi = \eta$;
(S4) $\mathcal{N}(\xi,\eta,\tau) = \mathcal{N}(\eta,\xi,\tau)$;
(S5) $\mathcal{N}(\xi,\eta,\Lambda(\xi,\eta)(\tau+s+r)) \geq \limsup_{n\to+\infty}[\mathcal{N}(\xi,\gamma,\tau) * \mathcal{N}(\gamma,\eta_n,s) * \mathcal{N}(\eta,\eta,r)\}]$, *for all* $\{\eta_n\} \in \mathcal{C}_1(\mathcal{Q},\mathcal{N},\eta)$;

(S6) $\mathcal{N}(\xi,\eta,.):(0,+\infty)\to[0,1]$ *is continuous and* $\lim_{\tau\to+\infty}\mathcal{N}(\xi,\eta,\tau)=1$;
(S7) $\mathcal{O}(\xi,\eta,\tau)<1$;
(S8) $\mathcal{O}(\xi,\eta,\tau)=0$ *implies* $\xi=\eta$;
(S9) $\mathcal{O}(\xi,\eta,\tau)=\mathcal{O}(\eta,\xi,\tau)$;
(S10) $\mathcal{O}(\xi,\eta,\Lambda(\xi,\eta)(\tau+s+r))\leq\limsup_{n\to+\infty}[\mathcal{O}(\xi,\gamma,\tau)\circ\mathcal{O}(\gamma,\eta_n,s)\circ\mathcal{O}(\eta,\eta,r)\}]$ *for all* $\{\eta_n\}\in\mathcal{C}_2(\mathcal{Q},\mathcal{O},\eta)$;
(S11) $\mathcal{O}(\xi,\eta,.):(0,+\infty)\to[0,1]$ *is continuous and* $\lim_{\tau\to+\infty}\mathcal{O}(\xi,\eta,\tau)=0$;
(S12) $\mathcal{L}(\xi,\eta,\tau)<1$;
(S13) $\mathcal{L}(\xi,\eta,\tau)=0$ *implies* $\xi=\eta$;
(S14) $\mathcal{L}(\xi,\eta,\tau)=\mathcal{L}(\eta,\xi,\tau)$;
(S15) $\mathcal{L}(\xi,\eta,\Lambda(\xi,\eta)(\tau+s+r))\leq\limsup_{n\to+\infty}[\mathcal{L}(\xi,\gamma,\tau)\circ\mathcal{L}(\gamma,\eta_n,s)\circ\mathcal{L}(\eta,\eta,r)\}]$, *for all* $\{\eta_n\}\in\mathcal{C}_3(\mathcal{Q},\mathcal{L},\eta)$;
(S16) $\mathcal{L}(\xi,\eta,.):(0,+\infty)\to[0,1]$ *is continuous and* $\lim_{\tau\to+\infty}\mathcal{L}(\xi,\eta,\tau)=0$;
(S17) *If* $\tau\leq 0$, *then* $\mathcal{N}(\xi,\eta,\tau)=0,\mathcal{O}(\xi,\eta,\tau)=1,\mathcal{L}(\xi,\eta,\tau)=1$.

Remark 2. *From Definition 2, the following holds:*

1. *If the function* $\Lambda:\mathcal{Q}\times\mathcal{Q}\times\to[1,+\infty)$ *is given as* $\Lambda(\xi,\eta)=b, b\geq 1$, *then the structure* $(\mathcal{Q},\mathcal{N},\mathcal{O},\mathcal{L},*,\circ)$ *simplifies to a GNRBMLS.*
2. *If we consider only conditions (S2)–(S6) then* $(\mathcal{Q},\mathcal{N},*)$ *is a GFRBMLS on* $\mathcal{Q}$.
3. *Taking into account the condition (L1) along with conditions (S2)–(S11), then* $(\mathcal{Q},\mathcal{N},\mathcal{O},*,\circ)$ *characterizes a GIRBMS on* $\mathcal{Q}$.

Definition 3. *Let* $(\mathcal{Q},\mathcal{N},\mathcal{O},\mathcal{L},*,\circ)$ *be a GNRBMLS.*

(i) *A sequence* $\{\xi_n\}\subset\mathcal{Q}$ *is considered to converge to a point* ξ *if and only if* $\lim_{n\to+\infty}\mathcal{N}(\xi_n,\xi,\tau)=\mathcal{N}(\xi,\xi,\tau)$, $\lim_{n\to+\infty}\mathcal{O}(\xi_n,\xi,\tau)=\mathcal{O}(\xi,\xi,\tau)$, *and* $\lim_{n\to+\infty}\mathcal{L}(\xi_n,\xi,\tau)=\mathcal{L}(\xi,\xi,\tau)$ *for all* $\tau>0$.

(ii) $\{\xi_n\}$ *is called a Cauchy sequence if for all* $\tau>0,(m\geq 1),n,m\in\mathbb{N}$, $\lim_{n\to+\infty}\mathcal{N}(\xi_{n+m},\xi_n,\tau)$ *exists and is finite* , $\lim_{n\to+\infty}\mathcal{O}(\xi_{n+m},\xi_n,\tau)$ *exists and is finite* , *and* $\lim_{n\to+\infty}\mathcal{L}(\xi_{n+m},\xi_n,\tau)$ *exists and is finite.*

(iii) $(\mathcal{Q},\mathcal{N},\mathcal{O},\mathcal{L},*,\circ)$ *is called a complete GNRBMLS if every Cauchy sequence converges to some* $\xi\in\mathcal{Q}$, *such that*

$$\lim_{n\to\infty}\mathcal{N}(\xi_{n+m},\xi_n,\tau)=\mathcal{N}(\xi,\xi,\tau)=\lim_{n\to\infty}\mathcal{N}(\xi_n,\xi,\tau),$$
$$\lim_{n\to\infty}\mathcal{O}(\xi_{n+m},\xi_n,\tau)=\mathcal{O}(\xi,\xi,\tau)=\lim_{n\to\infty}\mathcal{O}(\xi_n,\xi,\tau),$$
$$\lim_{n\to\infty}\mathcal{L}(\xi_{n+m},\xi_n,\tau)=\mathcal{L}(\xi,\xi,\tau)=\lim_{n\to\infty}\mathcal{L}(\xi_n,\xi,\tau)\text{ for all }\tau>0,m\geq 1.$$

Example 1. *Let* $\mathcal{Q}=[0,2],k\in\mathbb{R}^+,m>0$ *and* $\Lambda:\mathcal{Q}\times\mathcal{Q}\to[1,+\infty)$ *be a function given by* $\Lambda(\xi,\eta)=1+\max\{\xi,\eta\}$. *Define* $\mathcal{N},\mathcal{O},\mathcal{L}:\mathcal{Q}\times\mathcal{Q}\times(0,+\infty)\to[0,1]$ *as follows:*

$$\mathcal{N}(\xi,\eta,\tau)=\frac{k\tau}{k\tau+m\max\{\xi,\eta\}},$$
$$\mathcal{O}(\xi,\eta,\tau)=\frac{\max\{\xi,\eta\}}{k\tau+m\max\{\xi,\eta\}},$$
$$\mathcal{L}(\xi,\eta,\tau)=\frac{m\max\{\xi,\eta\}^2}{k\tau+m\max\{\xi,\eta\}^2}.$$

Then, $(\mathcal{Q},\mathcal{N},\mathcal{O},\mathcal{L},*,\circ)$ *is a GNRBMLS, where "$*$" is taken as the product norm and "$\circ$" is taken as the maximum CtCN.*

Proof. We need to show that conditions (S5), (S10), and (S15) from Definition 2, the remaining hypotheses, are simpler to verify. Let $\xi, \eta \in \mathcal{Q}, \gamma \in \mathcal{Q} \setminus \{\xi, \eta\}$, and for all $\{\eta_n\}$, such that $\{\eta_n\} \in \mathcal{C}_1(\mathcal{Q}, \mathcal{N}, \eta)$, $\{\eta_n\} \in \mathcal{C}_2(\mathcal{Q}, \mathcal{O}, \eta)$, and $\{\eta_n\} \in \mathcal{C}_3(\mathcal{Q}, \mathcal{L}, \eta)$ for all $\tau > 0$, we obtain

$$\frac{k(1+\max\{\xi,\eta\})\tau}{k(1+\max\{\xi,\eta\})\tau + m\max\{\xi,\eta\}} \geq \limsup_{n\to+\infty}[\frac{\frac{k\tau}{3}}{\frac{k\tau}{3} + m\max\{\xi,\gamma\}} \cdot \frac{\frac{k\tau}{3}}{\frac{k\tau}{3} + m\max\{\gamma,\eta_n\}} \cdot \frac{\frac{k\tau}{3}}{\frac{k\tau}{3} + m\max\{\eta,\eta\}}],$$

implying that

$$\mathcal{N}(\xi,\eta,\Lambda(\xi,\eta)(\tau)) \geq \limsup_{n\to+\infty}[\mathcal{N}(\xi,\gamma,\frac{\tau}{3}) * \mathcal{N}(\gamma,\eta_n,\frac{\tau}{3}) * \mathcal{N}(\eta,\eta,\frac{\tau}{3})].$$

And,

$$\frac{m\max\{\xi,\eta\}}{k(1+\max\{\xi,\eta\})\tau + m\max\{\xi,\eta\}} \leq \max\left\{\frac{m\max\{\xi,\gamma\}}{\frac{k\tau}{3} + m\max\{\xi,\gamma\}}, \limsup_{n\to+\infty}\frac{m\max\{\gamma,\eta_n\}}{\frac{k\tau}{3} + m\max\{\gamma,\eta_n,\}}, \frac{m\max\{\eta,\eta\}}{\frac{k\tau}{3} + m\max\{\eta,\eta\}}\right\},$$

implies that

$$\mathcal{O}(\xi,\eta,\Lambda(\xi,\eta)\tau) \leq \limsup_{n\to+\infty}[\mathcal{O}(\xi,\gamma,\frac{\tau}{3}) \circ \mathcal{O}(\gamma,\eta_n,\frac{\tau}{3}) \circ \mathcal{O}(\eta,\eta,\frac{\tau}{3})].$$

Also,

$$\frac{m\min\{\xi,\eta\}^2}{k(1+\max\{\xi,\eta\})\tau + m\max\{\xi,\eta\}^2} \leq \max\left\{\frac{m\max\{\xi,\gamma\}^2}{\frac{k\tau}{3} + m\max\{\xi,\gamma\}^2}, \limsup_{n\to+\infty}\frac{m\max\{\gamma,\eta_n\}^2}{\frac{k\tau}{3} + m\max\{\gamma,\eta_n\}^2}, \frac{m\max\{\eta,\eta\}^2}{\frac{k\tau}{3} + m\max\{\eta,\eta\}^2}\right\},$$

implies that

$$\mathcal{L}(\xi,\eta,\Lambda(\xi,\eta)(\tau+s+r)) \leq \limsup_{n\to+\infty}[\mathcal{L}(\xi,\gamma,\frac{\tau}{3}) \circ \mathcal{L}(\gamma,\eta_n,\frac{\tau}{3}) \circ \mathcal{L}(\eta,\eta,\frac{\tau}{3})].$$

□

Remark 3. *In a GNRBMLS $(\mathcal{Q}, \mathcal{N}, \mathcal{O}, \mathcal{L}, *, \circ)$, the limit of a converging sequence might not be unique. Consider the GNRBMLS $(\mathcal{Q}, \mathcal{N}, \mathcal{O}, \mathcal{L}, *, \circ)$ from Example 1, where $m = k = 1$. Construct the sequence $\{\xi_n\}$ in $\mathcal{Q}$, such that $\xi_n = \frac{1}{n}$ for every $n \in \mathbb{N}$. If $\xi \geq 0$ then for any $\tau > 0$,*

$$\lim_{n\to+\infty}\mathcal{N}(\xi_n,\xi,\tau) = \lim_{n\to+\infty}\frac{\tau}{\tau+\max\{\xi_n,\xi\}} = \frac{\tau}{\tau+\xi} = \mathcal{N}(\xi,\xi,\tau),$$
$$\lim_{n\to+\infty}\mathcal{O}(\xi_n,\xi,\tau) = \lim_{n\to+\infty}\frac{\max\{\xi_n,\xi\}}{\tau+\max\{\xi_n,\xi\}} = \frac{\xi}{\tau+\xi} = \mathcal{O}(\xi,\xi,\tau),$$
$$\lim_{n\to+\infty}\mathcal{L}(\xi_n,\xi,\tau) = \lim_{n\to+\infty}\frac{\max\{\xi_n,\xi\}}{\tau+\max\{\xi_n,\xi\}} = \frac{\xi}{\tau+\xi} = \mathcal{L}(\xi,\xi,\tau).$$

Consequently, the sequence $\{\xi_n\}$ converges for any $\xi \in \mathcal{Q}$.

Remark 4. *In a GNRBMLS $(\mathcal{Q}, \mathcal{N}, \mathcal{O}, \mathcal{L}, *, \circ)$, a convergent sequence might not be a Cauchy sequence. Consider the GNRBMLS $(\mathcal{Q}, \mathcal{N}, \mathcal{O}, \mathcal{L}, *, \circ)$ from Example 1, where $m = k = 1$.*

Construct the sequence $\{\xi_n\}$ *in* $\mathcal{Q}$*, such that* $\xi_n = (-1)^n \frac{n}{2n+1} + \frac{n}{2n+1}$ *for every* $n \in \mathbb{N}$*. If* $\xi \geq 1$ *then for any* $\tau > 0$*,*

$$\lim_{n\to+\infty} \mathcal{N}(\xi_n, \xi, \tau) = \lim_{n\to+\infty} \frac{\tau}{\tau + \max\{\xi_n, \xi\}} = \frac{\tau}{\tau + \xi} = \mathcal{N}(\xi, \xi, \tau),$$
$$\lim_{n\to+\infty} \mathcal{O}(\xi_n, \xi, \tau) = \lim_{n\to+\infty} \frac{\max\{\xi_n, \xi\}}{\tau + \max\{\xi_n, \xi\}} = \frac{\xi}{\tau + \xi} = \mathcal{O}(\xi, \xi, \tau),$$
$$\lim_{n\to+\infty} \mathcal{L}(\xi_n, \xi, \tau) = \lim_{n\to+\infty} \frac{\max\{\xi_n, \xi\}}{\tau + \max\{\xi_n, \xi\}} = \frac{\xi}{\tau + \xi} = \mathcal{L}(\xi, \xi, \tau).$$

As a result, the sequence $\{\xi_n\}$ *converges for any* $\xi \geq 1$*; it fails to be a Cauchy sequence, since for every* $\tau > 0$ *and* $n, m \in \mathbb{N}$*, the limits* $\lim_{n\to+\infty} \mathcal{N}(\xi_{n+m}, \xi_n, \tau)$*,* $\lim_{n\to+\infty} \mathcal{O}(\xi_{n+m}, \xi_n, \tau)$*, and* $\lim_{n\to+\infty} \mathcal{L}(\xi_{n+m}, \xi_n, \tau)$ *do not exist.*

Proposition 1. *Any ENRBMLS* $(\mathcal{Q}, \mathcal{N}, \mathcal{O}, \mathcal{L}, *, \circ)$ *is a GNRBMLS.*

Proof. We confirm (S5), (S10), and (S15) of Definition 2 here, as the remaining conditions can be proven without difficulty.

Let $\xi, \eta \in \mathcal{Q} \setminus \{\gamma\}$, $\{\eta_n\} \in \mathcal{C}_1(\mathcal{Q}, \mathcal{N}, \eta)$, $\{\eta_n\} \in \mathcal{C}_2(\mathcal{Q}, \mathcal{O}, \eta)$, and $\{\eta_n\} \in \mathcal{C}_3(\mathcal{Q}, \mathcal{L}, \eta)$. We then obtain the following:

$$\mathcal{N}(\xi, \eta, \tau) \geq \mathcal{N}(\xi, \gamma, \frac{\tau}{3\Lambda(\xi, \eta)}) * \mathcal{N}(\gamma, \eta_n, \frac{\tau}{3\Lambda(\xi, \eta)}) * \mathcal{N}(\eta_n, \eta, \frac{\tau}{3\Lambda(\xi, \eta)}), \tag{1}$$
$$\mathcal{O}(\xi, \eta, \tau) \leq \mathcal{O}(\xi, \gamma, \frac{\tau}{3\Lambda(\xi, \eta)}) \circ \mathcal{O}(\gamma, \eta_n, \frac{\tau}{3\Lambda(\xi, \eta)}) \circ \mathcal{O}(\eta_n, \eta, \frac{\tau}{3\Lambda(\xi, \eta)}), \tag{2}$$
$$\mathcal{L}(\xi, \eta, \tau) \leq \mathcal{L}(\xi, \gamma, \frac{\tau}{3\Lambda(\xi, \eta)}) \circ \mathcal{L}(\gamma, \eta_n, \frac{\tau}{3\Lambda(\xi, \eta)}) \circ \mathcal{L}(\eta_n, \eta, \frac{\tau}{3\Lambda(\xi, \eta)}). \tag{3}$$

Taking $n \to +\infty$ in (1)–(3), it can be seen that (S5), (S10), and (S15) in Definition 2 are satisfied. □

Following the same reasoning as in Proposition 1, we obtain the subsequent propositions.

Proposition 2. *Any EIRBMLS* $(\mathcal{Q}, \mathcal{N}, \mathcal{O}, *, \circ)$ *is a GIRBMLS.*

Proposition 3. *Any EFRBMLS* $(\mathcal{Q}, \mathcal{N}, *)$ *is a GFRBMLS.*

Remark 5. *A GNRBMLS may not always satisfy the conditions of being an ENRBMLS; the following example supports our contention. Consequently, a GIRBMLS and a GFRBMLS may not always satisfy the conditions of being an EIRBMLS and an EFRBMLS, respectively.*

Example 2. *Let* $\mathcal{Q} = [0, 1]$*,* $\Lambda : \mathcal{Q} \times \mathcal{Q} \to [1, +\infty)$ *be a function given by* $\Lambda(\xi, \eta) = 1 + \max\{\xi, \eta\}$*, and define* $\mathcal{N}, \mathcal{O}, \mathcal{L} : \mathcal{Q} \times \mathcal{Q} \times (0, +\infty) \to [0, 1]$ *as follows:*

$$\begin{cases} \mathcal{N}(\xi, \eta, \tau) = \exp \frac{-(\xi+\eta)}{\tau}, & \text{if } \xi, \eta \in \mathcal{Q} \setminus \{0\}, \\ \mathcal{N}(\xi, 0, \tau) = \mathcal{N}(0, \xi, \tau) = \exp \frac{-\xi}{3\tau}, & \xi \in \mathcal{Q}, \end{cases}$$
$$\begin{cases} \mathcal{O}(\xi, \eta, \tau) = 1 - \exp \frac{-(\xi+\eta)^2}{\tau}, & \text{if } \xi, \eta \in \mathcal{Q} \setminus \{0\}, \\ \mathcal{O}(\xi, 0, \tau) = \mathcal{O}(0, \xi, \tau) = 1 - \exp \frac{-\xi}{3\tau}, & \xi \in \mathcal{Q}, \end{cases}$$
$$\begin{cases} \mathcal{L}(\xi, \eta, \tau) = 1 - \exp \frac{-(\xi+\eta)}{\tau}, & \text{if } \xi, \eta \in \mathcal{Q} \setminus \{0\}, \\ \mathcal{L}(\xi, 0, \tau) = \mathcal{L}(0, \xi, \tau) = 1 - \exp \frac{-\xi^2}{3\tau}, & \xi \in \mathcal{Q}. \end{cases}$$

For every $\xi, \eta \in \mathcal{Q}$ *and* τ*. Then,* $(\mathcal{Q}, \mathcal{N}, \mathcal{O}, \mathcal{L}, *, \circ)$ *is GNRBMLS but does not meet the criteria for ENRBMLS, where "*$*$*" and "*$\circ$*" are taken as the minimum CtN and maximum CtCN,* *respectively. We now demonstrate that conditions (S5), (S10), and (S15) in Definition 2 are satisfied, as the remaining conditions are evident.*

Case 1: *Let $\xi,\eta,\in\mathcal{Q}\setminus\{0\},\gamma\in\mathcal{Q}\setminus\{\xi,\eta\}$; for all $\tau>0$, the sets $\mathcal{C}_1(\mathcal{Q},\mathcal{N},\eta),\mathcal{C}_2(\mathcal{Q},\mathcal{O},\eta)$, and $\mathcal{C}_3(\mathcal{Q},\mathcal{L},\eta)$ contain only the eventually constant sequences $\{\eta_n=\eta\}_{n\geq N}, n\in\mathbb{N}$. We obtain the following two subcases:*

- *Subcase 1: If $\gamma\neq 0$ then the following inequality holds:*

$$e^{\frac{-(\xi+\eta)}{(1+\max\{\xi,\eta\})\tau}}\geq\min\{e^{\frac{-3(\xi+\gamma)}{\tau}},e^{\frac{-3(\gamma+\eta)}{\tau}},e^{\frac{-6\eta}{\tau}}\}$$
$$=e^{\frac{-(\xi+\gamma)}{\frac{\tau}{3}}}*e^{\frac{-(\gamma+\eta)}{\frac{\tau}{3}}}*e^{\frac{-2\eta}{\frac{\tau}{3}}},$$

  which implies that

$$\mathcal{N}(\xi,\eta,\Lambda(\xi,\eta)(\tau))\geq\limsup_{n\to+\infty}[\mathcal{N}(\xi,\gamma,\frac{\tau}{3})*\mathcal{N}(\gamma,\eta_n,\frac{\tau}{3})*\mathcal{N}(\eta,\eta,\frac{\tau}{3})\}].$$

  Also, we observe that the following inequality is valid:

$$1-e^{\frac{-(\xi+\eta)^2}{(1+\max\{\xi,\eta\})\tau}}\leq\max\{1-e^{\frac{-3(\xi+\gamma)^2}{\tau}},1-e^{\frac{-3(\gamma+\eta)^2}{\tau}},1-e^{\frac{-12\eta^2}{\tau}}\}$$
$$=(1-e^{\frac{-(\xi+\gamma)^2}{\frac{\tau}{3}}})\circ(1-e^{\frac{-(\gamma+\eta)^2}{\frac{\tau}{3}}})\circ(1-e^{\frac{-4\eta^2}{\frac{\tau}{3}}}),$$

  which implies that

$$\mathcal{O}(\xi,\eta,\Lambda(\xi,\eta)(\tau))\leq\limsup_{n\to+\infty}[\mathcal{O}(\xi,\gamma,\frac{\tau}{3})\circ\mathcal{O}(\gamma,\eta_n,\frac{\tau}{3})\circ\mathcal{O}(\eta,\eta,\frac{\tau}{3})\}].$$

  Additionally, the following inequality holds:

$$1-e^{\frac{-(\xi+\eta)}{(1+\max\{\xi,\eta\})\tau}}\leq\max\{1-e^{\frac{-3(\xi+\gamma)}{\tau}},1-e^{\frac{-3(\gamma+\eta)}{\tau}},1-e^{\frac{-6\eta}{\tau}}\}$$
$$=(1-e^{\frac{-(\xi+\gamma)}{\frac{\tau}{3}}})\circ(1-e^{\frac{-(\gamma+\eta)}{\frac{\tau}{3}}})\circ(1-e^{\frac{-2\eta}{\frac{\tau}{3}}}),$$

  which implies that

$$\mathcal{L}(\xi,\eta,\Lambda(\xi,\eta)(\tau))\leq\limsup_{n\to+\infty}[\mathcal{L}(\xi,\gamma,\frac{\tau}{3})\circ\mathcal{L}(\gamma,\eta_n,\frac{\tau}{3})\circ\mathcal{L}(\eta,\eta,\frac{\tau}{3})\}].$$

- *Subcase 2: If $\gamma=0$, we obtain*

$$e^{\frac{-(\xi+\eta)}{(1+\max\{\xi,\eta\})\tau}}\geq\min\{e^{\frac{-\xi}{\tau}},e^{\frac{-\eta}{\tau}},e^{\frac{-6\eta}{\tau}}\}$$
$$=e^{\frac{-\xi}{\tau}}*e^{\frac{-\eta}{\tau}}*e^{\frac{-2\eta}{\frac{\tau}{3}}},$$

  which implies that

$$\mathcal{N}(\xi,\eta,\Lambda(\xi,\eta)(\tau))\geq\limsup_{n\to+\infty}[\mathcal{N}(\xi,\gamma,\frac{\tau}{3})*\mathcal{N}(\gamma,\eta_n,\frac{\tau}{3})*\mathcal{N}(\eta,\eta,\frac{\tau}{3})\}]$$

  Also, we find

$$1-e^{\frac{-(\xi+\eta)^2}{(1+\max\{\xi,\eta\})\tau}}\leq\max\{1-e^{\frac{-\xi}{\tau}},1-e^{\frac{-\eta}{\tau}},1-e^{\frac{-12\eta^2}{\tau}}\}$$
$$=(1-e^{\frac{-\xi}{\tau}})\circ(1-e^{\frac{-\eta}{\tau}})\circ(1-e^{\frac{-4\eta^2}{\frac{\tau}{3}}}),$$

which implies that

$$\mathcal{O}(\xi,\eta,\Lambda(\xi,\eta)(\tau)) \leq \limsup_{n\to+\infty}[\mathcal{O}(\xi,\gamma,\frac{\tau}{3})\circ\mathcal{O}(\gamma,\eta_n,\frac{\tau}{3})\circ\mathcal{O}(\eta,\eta,\frac{\tau}{3})\}].$$

Finally, we also find the following inequality:

$$\begin{aligned} 1-e^{\frac{-(\xi+\eta)}{(1+\max\{\xi,\eta\})\tau}} &\leq \max\{(1-e^{\frac{-\xi^2}{\tau}}),(1-e^{\frac{-\eta^2}{\tau}}),(1-e^{\frac{-6\eta}{\tau}})\} \\ &= (1-e^{\frac{-\xi^2}{\tau}})\circ(1-e^{\frac{-\eta^2}{\tau}})\circ(1-e^{\frac{-2\eta}{\frac{\tau}{3}}}), \end{aligned}$$

which implies that

$$\mathcal{L}(\xi,\eta,\Lambda(\xi,\eta)(\tau)) \leq \limsup_{n\to+\infty}[\mathcal{L}(\xi,\gamma,\frac{\tau}{3})\circ\mathcal{L}(\gamma,\eta_n,\frac{\tau}{3})\circ\mathcal{L}(\eta,\eta,\frac{\tau}{3})\}$$

Case 2: *If ξ or $\eta = 0$, let $\eta = 0, \gamma \neq 0$, we have*

$$\begin{aligned} e^{\frac{-\xi}{(1+\max\{\xi,\eta\})3\tau}} &\geq \min\{e^{\frac{-3(\xi+\gamma)}{\tau}},e^{\frac{-3(\gamma+\eta n)}{\tau}},1\} \\ &= e^{\frac{-(\xi+\gamma)}{\frac{\tau}{3}}} * e^{\frac{-(\gamma+\eta n)}{\frac{\tau}{3}}} * 1, \end{aligned}$$

which implies that

$$\mathcal{N}(\xi,\eta,\Lambda(\xi,\eta)(\tau)) \geq \limsup_{n\to+\infty}[\mathcal{N}(\xi,\gamma,\frac{\tau}{3})*\mathcal{N}(\gamma,\eta_n,\frac{\tau}{3})*\mathcal{N}(\eta,\eta,\frac{\tau}{3})\}].$$

And the subsequent inequality is valid:

$$\begin{aligned} 1-e^{\frac{-\xi}{(1+\max\{\xi,\eta\})3\tau}} &\leq \max\{1-e^{\frac{-3(\xi+\gamma)^2}{\tau}},1-e^{\frac{-3(\gamma+\eta n)^2}{\tau}},0\} \\ &= (1-e^{\frac{-(\xi+\gamma)^2}{\frac{\tau}{3}}})\circ(1-e^{\frac{-(\gamma+\eta n)^2}{\frac{\tau}{3}}})\circ 0, \end{aligned}$$

which implies that

$$\mathcal{O}(\xi,\eta,\Lambda(\xi,\eta)(\tau)) \leq \limsup_{n\to+\infty}[\mathcal{O}(\xi,\gamma,\frac{\tau}{3})\circ\mathcal{O}(\gamma,\eta_n,\frac{\tau}{3})\circ\mathcal{O}(\eta,\eta,\frac{\tau}{3})\}].$$

Also, the subsequent inequality is valid:

$$\begin{aligned} 1-e^{\frac{-\xi^2}{(1+\max\{\xi,\eta\})3\tau}} &\leq \max\{1-e^{\frac{-3(\xi+\gamma)}{\tau}},1-e^{\frac{-3(\gamma+\eta n)}{\tau}},0\} \\ &= (1-e^{\frac{-(\xi+\gamma)}{\frac{\tau}{3}}})\circ(1-e^{\frac{-(\gamma+\eta n)}{\frac{\tau}{3}}})\circ 0, \end{aligned}$$

which implies that

$$\mathcal{L}(\xi,\eta,\Lambda(\xi,\eta)(\tau)) \leq \limsup_{n\to+\infty}[\mathcal{L}(\xi,\gamma,\frac{\tau}{3})\circ\mathcal{L}(\gamma,\eta_n,\frac{\tau}{3})\circ\mathcal{L}(\eta,\eta,\frac{\tau}{3})\}],$$

Therefore, it follows that $(\mathcal{Q},\mathcal{N},\mathcal{O},\mathcal{L},*,\circ)$ is a GNRBMLS. However, it is not an ENRBMLS. By selecting $\xi = \frac{1}{2}, \gamma = \frac{8}{10}, \zeta = \frac{2}{10}$, and $\eta = \frac{3}{10}$, we obtain

$$\begin{aligned} &\mathcal{N}(\xi,\gamma,\frac{\tau}{3})*\mathcal{N}(\gamma,\zeta,\frac{\tau}{3})*\mathcal{N}(\zeta,\eta,\frac{\tau}{3}) \\ &= \mathcal{N}(\frac{1}{2},\frac{8}{10},\frac{\tau}{10})*\mathcal{N}(\frac{8}{10},\frac{2}{2},\frac{\tau}{3})*\mathcal{N}(\frac{2}{10},\frac{3}{10},\frac{\tau}{3}) \\ &= \min\{e^{\frac{-39}{10\tau}},e^{\frac{-\tau}{2}},e^{\frac{-3}{2\tau}}\} \end{aligned}$$

$$
\begin{aligned}
&= e^{\frac{-39}{10\tau}} \\
&\geq e^{\frac{-\frac{8}{10}}{(1+\max\{\frac{1}{2},\frac{3}{10}\})\tau}} \\
&= e^{\frac{-8}{15\tau)}} \\
&= \mathcal{N}(\xi,\eta,\Lambda(\xi,\eta)\tau).
\end{aligned}
$$

3. Fixed-Point Results

3.1. Main Results

In this subsection, we introduce Banach's fixed point theorem and Ćirić's quasi-contraction theorem within the context of GNRBMLSs, and we offer two illustrative examples.

Definition 4. *Let $(\mathcal{Q},\mathcal{N},\mathcal{O},\mathcal{L},*,\circ)$ be a GNRBMLS. A mapping $\mathcal{T}:\mathcal{Q}\to\mathcal{Q}$ is said to be an α-contraction if for all $\xi,\eta\in\mathcal{Q},\tau>0$ and for some $\alpha\in(0,1)$ it holds that*

$$
\begin{aligned}
&\mathcal{N}(\mathcal{T}\xi,\mathcal{T}\eta,\alpha\tau)\geq\mathcal{N}(\xi,\eta,\tau),\\
&\mathcal{O}(\mathcal{T}\xi,\mathcal{T}\eta,\alpha\tau)\leq\mathcal{O}(\xi,\eta,\tau),\\
&\mathcal{L}(\mathcal{T}\xi,\mathcal{T}\eta,\alpha\tau)\leq\mathcal{L}(\xi,\eta,\tau).
\end{aligned} \tag{4}
$$

Theorem 2. *Let $(\mathcal{Q},\mathcal{N},\mathcal{O},\mathcal{L},*,\circ)$ be a complete GNRBMLS and $\mathcal{T}$ be an α contraction. If there exists $\xi_0\in\mathcal{Q}$, such that for all $\tau>0$ the subsequent expressions are satisfied*

$$
\begin{aligned}
\delta_1(\mathcal{N},\mathcal{T},\xi_0,\tau)&=\inf_{i,j\in\mathbb{N}}\{\mathcal{N}(\mathcal{T}^i\xi_0,\mathcal{T}^j\xi_0,\tau)\}>0,\\
\delta_2(\mathcal{O},\mathcal{T},\xi_0,\tau)&=\sup_{i,j\in\mathbb{N}}\{\mathcal{O}(\mathcal{T}^i\xi_0,\mathcal{T}^j\xi_0,\tau)\}<1,\\
\delta_3(\mathcal{L},\mathcal{T},\xi_0,\tau)&=\sup_{i,j\in\mathbb{N}}\{\mathcal{L}(\mathcal{T}^i\xi_0,\mathcal{T}^j\xi_0,\tau)\}<1,
\end{aligned}
$$

then $\mathcal{T}$ has a unique FP $\xi\in\mathcal{Q}$, with $\mathcal{N}(\xi,\xi,\tau)=1,\mathcal{O}(\xi,\xi,\tau)=0,\mathcal{L}(\xi,\xi,\tau)=0$ for all $\tau>0$. Moreover, $\{\mathcal{T}^n\xi_0\}$ will converge to a unique FP of $\mathcal{T}$.

Proof. Let $\xi_0\in\mathcal{Q}$, such that for all $\tau>0$ we have

$$
\begin{aligned}
\delta_1(\mathcal{N},\mathcal{T},\xi_0,\tau)&=\inf_{i,j\in\mathbb{N}}\{\mathcal{N}(\mathcal{T}^i\xi_0,\mathcal{T}^j\xi_0,\tau)\}>0,\\
\delta_2(\mathcal{O},\mathcal{T},\xi_0,\tau)&=\sup_{i,j\in\mathbb{N}}\{\mathcal{O}(\mathcal{T}^i\xi_0,\mathcal{T}^j\xi_0,\tau)\}<1,\\
\delta_3(\mathcal{L},\mathcal{T},\xi_0,\tau)&=\sup_{i,j\in\mathbb{N}}\{\mathcal{L}(\mathcal{T}^i\xi_0,\mathcal{T}^j\xi_0,\tau)\}<1.
\end{aligned}
$$

It therefore holds that $\xi_n=\mathcal{T}^n\xi_0$ for all $\tau>0$ and all fixed $p=0,1,2\ldots,$

$$
\begin{aligned}
\delta_1(\mathcal{N},\mathcal{T}^{p+1},\xi_0,\tau)&=\inf_{i,j\in\mathbb{N}}\{\mathcal{N}(\mathcal{T}^{p+i}\xi_0,\mathcal{T}^{p+j}\xi_0,\tau)\},\\
\delta_2(\mathcal{O},\mathcal{T}^{p+1},\xi_0,\tau)&=\sup_{i,j\in\mathbb{N}}\{\mathcal{O}(\mathcal{T}^{p+i}\xi_0,\mathcal{T}^{p+j}\xi_0,\tau)\},\\
\delta_3(\mathcal{L},\mathcal{T}^{p+1},\xi_0,\tau)&=\sup_{i,j\in\mathbb{N}}\{\mathcal{L}(\mathcal{T}^{p+i}\xi_0,\mathcal{T}^{p+j}\xi_0,\tau)\}.
\end{aligned}
$$

For all $\tau>0$, $p=0,1,2\ldots$ and $i,j\in\mathbb{N}$, we obtain

$$
\begin{aligned}
&\{\mathcal{N}(\mathcal{T}^{p+i}\xi_0,\mathcal{T}^{p+j}\xi_0,\tau)\}\subseteq\{\mathcal{N}(\mathcal{T}^i\xi_0,\mathcal{T}^j\xi_0,\tau)\},\\
&\{\mathcal{O}(\mathcal{T}^{p+i}\xi_0,\mathcal{T}^{p+j}\xi_0,\tau)\}\subseteq\{\mathcal{O}(\mathcal{T}^i\xi_0,\mathcal{T}^j\xi_0,\tau)\},\\
&\{\mathcal{L}(\mathcal{T}^{p+i}\xi_0,\mathcal{T}^{p+j}\xi_0,\tau)\}\subseteq\{\mathcal{L}(\mathcal{T}^i\xi_0,\mathcal{T}^j\xi_0,\tau)\},
\end{aligned}
$$

which implies that

$$\delta_1(\mathcal{N},\mathcal{T}^{p+1},\xi_0,\tau) \geq \delta_1(\mathcal{N},\mathcal{T},\xi_0,\tau) > 0, \tag{5}$$

$$\delta_2(\mathcal{O},\mathcal{T}^{p+1},\xi_0,\tau) \leq \delta_2(\mathcal{O},\mathcal{T},\xi_0,\tau) < 1, \tag{6}$$

$$\delta_3(\mathcal{L},\mathcal{T}^{p+1},\xi_0,\tau) \leq \delta_3(\mathcal{L},\mathcal{T},\xi_0,\tau) < 1. \tag{7}$$

Next, for all $i, j \in \mathbb{N}$ and $n \geq 1, \tau > 0$, we can use (5) to obtain

$$\begin{aligned}
\mathcal{N}(\mathcal{T}^{n+i}\xi_0,\mathcal{T}^{n+j}\xi_0,\tau) &\geq \mathcal{N}(\mathcal{T}^{n+i-1}\xi_0,\mathcal{T}^{n+j-1}\xi_0,\frac{\tau}{\alpha}) \\
&\geq \delta_1(\mathcal{N},\mathcal{T}^n,\xi_0,\frac{\tau}{\alpha}) \\
&\geq \mathcal{N}(\mathcal{T}^{n+i}\xi_0,\mathcal{T}^{n+j}\xi_0,\tau) \\
&\geq \delta_1(\mathcal{N},\mathcal{T}^{n+1},\xi_0,\tau) \\
&\geq \delta_1(\mathcal{N},\mathcal{T}^n,\xi_0,\frac{\tau}{\alpha}) \\
&\geq \delta_1(\mathcal{N},\mathcal{T}^{n-1},\xi_0,\frac{\tau}{\alpha^2}) \\
&\vdots \\
&\geq \delta_1(\mathcal{N},\mathcal{T},\xi_0,\frac{\tau}{\alpha^n}).
\end{aligned}$$

Therefore, for every $m \geq 1$, we attain

$$\mathcal{N}(\xi_{n+m},\eta_n,\tau) = \mathcal{N}(\mathcal{T}^{n+m}\xi_0,\mathcal{T}^n\xi_0,\tau) \geq \delta_1(\mathcal{N},\mathcal{T}^n,\xi_0,\tau) \to 1 \text{ as } n \to +\infty,$$

as $\delta_1(\mathcal{N},\mathcal{T}^n,\xi_0,\tau) > 0$ for all $\tau > 0$ and $\alpha \in (0,1)$, according to (S6) in Definition 2. Therefore, we have

$$\lim_{n\to+\infty} \mathcal{N}(\xi_{n+m},\xi_n,\tau) = 1. \tag{8}$$

Then, for all $i, j \in \mathbb{N}$, $n \geq 1$, and $\tau > 0$, it holds that

$$\begin{aligned}
\mathcal{O}(\mathcal{T}^{n+i}\xi_0,\mathcal{T}^{n+j}\xi_0,\tau) &\leq \mathcal{O}(\mathcal{T}^{n+i-1}\xi_0,\mathcal{T}^{n+j-1}\xi_0,\frac{\tau}{\alpha}) \\
&\leq \delta_2(\mathcal{O},\mathcal{T}^n,\xi_0,\frac{\tau}{\alpha}) \\
&\leq \mathcal{O}(\mathcal{T}^{n+i}\xi_0,\mathcal{T}^{n+j}\xi_0,\tau) \\
&\leq \delta_2(\mathcal{O},\mathcal{T}^{n+1},\xi_0,\tau) \\
&\leq \delta_2(\mathcal{O},\mathcal{T}^n,\xi_0,\frac{\tau}{\alpha}) \\
&\leq \delta_2(\mathcal{O},\mathcal{T}^{n-1},\xi_0,\frac{\tau}{\alpha^2}) \\
&\vdots \\
&\leq \delta_2(\mathcal{O},\mathcal{T},\xi_0,\frac{\tau}{\alpha^n}).
\end{aligned}$$

Therefore, for every $m \geq 1$, we can use (6) to obtain

$$\mathcal{O}(\xi_{n+m},\xi_n,\tau) = \mathcal{O}(\mathcal{T}^{n+m}\xi_0,\mathcal{T}^n\xi_0,\tau) \leq \delta_2(\mathcal{O},\mathcal{T}^n,\xi_0,\tau) \to 0 \text{ as } n \to +\infty,$$

as $\delta_2(\mathcal{O},\mathcal{T}^n,\xi_0,\tau) < 1$ for all $\tau > 0$ and $\alpha \in (0,1)$, according to (S11) in Definition 2. Therefore, we have

$$\lim_{n\to+\infty} \mathcal{O}(\xi_{n+m},\xi_m,\tau) = 0. \tag{9}$$

Now, for all $i, j \in \mathbb{N}$, $n \geq 1$, and $\tau > 0$,

$$\begin{aligned}
\mathcal{L}(\mathcal{T}^{n+i}\xi_0, \mathcal{T}^{n+j}\xi_0, \tau) &\leq \mathcal{L}(\mathcal{T}^{n+i-1}\xi_0, \mathcal{T}^{n+j-1}\xi_0, \frac{\tau}{\alpha}) \\
&\leq \delta_3(\mathcal{L}, \mathcal{T}^n, \xi_0, \frac{\tau}{\alpha}) \\
&\leq \mathcal{L}(\mathcal{T}^{n+i}\xi_0, \mathcal{T}^{n+j}\xi_0, \tau) \\
&\leq \delta_3(\mathcal{L}, \mathcal{T}^{n+1}, \xi_0, \tau) \\
&\leq \delta_3(\mathcal{L}, \mathcal{T}^n, \xi_0, \frac{\tau}{\alpha}) \\
&\leq \delta_3(\mathcal{L}, \mathcal{T}^{n-1}, \xi_0, \frac{\tau}{\alpha^2}) \\
&\vdots \\
&\leq \delta_3(\mathcal{L}, \mathcal{T}, \xi_0, \frac{\tau}{\alpha^n}).
\end{aligned}$$

Therefore, for every $m \geq 1$, we can use (7) to obtain

$$\mathcal{L}(\xi_{n+m}, \xi_n, \tau) = \mathcal{L}(\mathcal{T}^{n+m}\xi_0, \mathcal{T}^m\xi_0, \tau) \leq \delta_3(\mathcal{L}, \mathcal{T}^n, \xi_0, \tau) \to 0 \text{ as } n \to +\infty,$$

as $\delta_3(\mathcal{L}, \mathcal{T}^n, \xi_0, \tau) < 1$ for all $\tau > 0$ and $\alpha \in (0, 1)$, according to (S16) in Definition 2. Therefore, we have

$$\lim_{n\to+\infty} \mathcal{L}(\xi_{n+m}, \xi_n, \tau) = 0. \tag{10}$$

Thus, $\{\xi_n\}$ is a Cauchy sequence. By completeness of $(\mathcal{Q}, \mathcal{N}, \mathcal{O}, \mathcal{L}, *, \circ)$, this sequence converges to some $\xi \in \mathcal{Q}$, such that

$$\begin{aligned}
\lim_{n\to+\infty} \mathcal{N}(\xi_n, \xi, \tau) &= \mathcal{N}(\xi, \xi, \tau) = \lim_{n\to+\infty} \mathcal{N}(\xi_{n+m}, \xi_n, \tau) = 1, \\
\lim_{n\to+\infty} \mathcal{O}(\xi_n, \xi, \tau) &= \mathcal{O}(\xi, \xi, \tau) = \lim_{n\to+\infty} \mathcal{O}(\xi_{n+m}, \xi_n, \tau) = 0, \\
\lim_{n\to+\infty} \mathcal{L}(\xi_n, \xi, \tau) &= \mathcal{L}(\xi, \xi, \tau) = \lim_{n\to+\infty} \mathcal{L}(\xi_{n+m}, \xi_n, \tau) = 0.
\end{aligned}$$

Now, we have the following for all $\tau > 0$:

$$\begin{aligned}
\mathcal{N}(\mathcal{T}\xi, \xi, \Lambda(\xi, \mathcal{T}\xi)\tau) &\geq \limsup_{n\to+\infty}[\mathcal{N}(\mathcal{T}\xi, \xi_{n+1}, \frac{\tau}{3}) * \mathcal{N}(\xi_{n+1}, \xi_n, \frac{\tau}{3}) * \mathcal{N}(\xi, \xi, \frac{\tau}{3})] \\
&\geq \limsup_{n\to+\infty}[\mathcal{N}(\xi, \xi_n, \frac{\tau}{3}) * \mathcal{N}(\xi_{n+1}, \xi_n, \frac{\tau}{3}) * \mathcal{N}(\xi, \xi, \frac{\tau}{3})] \\
&\geq 1,
\end{aligned}$$

$$\begin{aligned}
\mathcal{O}(\mathcal{T}\xi, \xi, \Lambda(\xi, \mathcal{T}\xi)\tau) &\leq \limsup_{n\to+\infty}[\mathcal{O}(\mathcal{T}\xi, \xi_{n+1}, \frac{\tau}{3}) \circ \mathcal{O}(\xi_{n+1}, \xi_n, \frac{\tau}{3}) \circ \mathcal{O}(\xi, \xi, \frac{\tau}{3})] \\
&\leq \limsup_{n\to+\infty}[\mathcal{O}(\xi, \xi_n, \frac{\tau}{3}) \circ \mathcal{O}(\xi_{n+1}, \xi_n, \frac{\tau}{3}) \circ \mathcal{O}(\xi, \xi, \frac{\tau}{3})] \\
&\leq 0,
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}(\mathcal{T}\xi, \xi, \Lambda(\xi, \mathcal{T}\xi)\tau) &\leq \limsup_{n\to+\infty}[\mathcal{L}(\mathcal{T}\xi, \xi_{n+1}, \frac{\tau}{3}) \circ \mathcal{L}(\xi_{n+1}, \xi_n, \frac{\tau}{3}) \circ \mathcal{L}(\xi, \xi, \frac{\tau}{3})] \\
&\leq \limsup_{n\to+\infty}[\mathcal{L}(\xi, \xi_n, \frac{\tau}{3}) \circ \mathcal{L}(\xi_{n+1}, \xi_n, \frac{\tau}{3}) \circ \mathcal{L}(\xi, \xi, \frac{\tau}{3})] \\
&\leq 0.
\end{aligned}$$

Then, we achieve $\mathcal{T}\xi = \xi$, where ξ is a FP of $\mathcal{T}$. Now, let $v \in \mathcal{Q}$ be another FP of $\mathcal{T}$, such that $\mathcal{N}(\xi, v, \tau) > 0$. Then, according to (4), for all $\tau > 0, n \in \mathbb{N}$ we find that

$$\begin{aligned}
\mathcal{N}(v,\xi,\tau) \geq& \mathcal{N}(\mathcal{T}v,\mathcal{T}\xi,\frac{\tau}{\alpha}) \\
=& \mathcal{N}(v,\xi,\frac{\tau}{\alpha}) \\
\geq& \mathcal{N}(\mathcal{T}v,\mathcal{T}\xi,\frac{\tau}{\alpha^2}) \\
&\vdots \\
\geq& \mathcal{N}(\mathcal{T}v,\mathcal{T}\xi,\frac{\tau}{\alpha^n}) \to 1 \text{ as } n \to +\infty,
\end{aligned} \tag{11}$$

$$\begin{aligned}
\mathcal{O}(v,\xi,\tau) \leq& \mathcal{O}(\mathcal{T}v,\mathcal{T}\xi,\frac{\tau}{\alpha}) \\
=& \mathcal{O}(v,\xi,\frac{\tau}{\alpha}) \\
\leq& \mathcal{O}(\mathcal{T}v,\mathcal{T}\xi,\frac{\tau}{\alpha^2}) \\
&\vdots \\
\leq& \mathcal{O}(\mathcal{T}v,\mathcal{T}\xi,\frac{\tau}{\alpha^n}) \to 0 \text{ as } n \to +\infty,
\end{aligned} \tag{12}$$

$$\begin{aligned}
\mathcal{L}(v,\xi,\tau) \leq& \mathcal{L}(\mathcal{T}v,\mathcal{T}\xi,\frac{\tau}{\alpha}) \\
=& \mathcal{L}(v,\xi,\frac{\tau}{\alpha}) \\
\leq& \mathcal{L}(\mathcal{T}v,\mathcal{T}\xi,\frac{\tau}{\alpha^2}) \\
&\vdots \\
\leq& \mathcal{L}(\mathcal{T}v,\mathcal{T}\xi,\frac{\tau}{\alpha^n}) \to 0 \text{ as } n \to +\infty.
\end{aligned} \tag{13}$$

From (11)–(13), we can conclude that $v = \xi$. □

The subsequent result is derived from Theorem 2 and Proposition 1.

Example 3. *In Example 1, let $k = 2, m = 3$, and $\mathcal{T} : \mathcal{Q} \to \mathcal{Q}$ be defined by $\mathcal{T}(\xi) = \frac{\xi}{2}$ for all $\xi \in [0,2]$. Then, by Theorem 2, we ascertain that $\mathcal{T}$ possesses a unique FP at $\xi = 0$. It is clear that $\mathcal{T}$ is an α-contraction, with $\alpha = \frac{1}{2} \in (0,1)$. It is observable that for any $\xi_0 \in [0,2], \mathcal{T}^n(\xi_0) = \frac{\xi_0}{2^n}$ converges to the fixed point 0 as $n \to +\infty$ and $\delta_1(\mathcal{N},\mathcal{T},\xi_0,\tau) = 1, \delta_2(\mathcal{O},\mathcal{T},\xi_0,\tau) = 0, \delta_3(\mathcal{L},\mathcal{T},\xi_0,\tau) = 0$, and $\mathcal{N}(0,0,\tau) = 1, \mathcal{O}(0,0,\tau) = 0, \mathcal{L}(0,0,\tau) = 0$ for all $\tau > 0$.*

Definition 5. *Let $(\mathcal{Q},\mathcal{N},\mathcal{O},\mathcal{L},*,\circ)$ be a GNRBMLS. A mapping $\mathcal{T} : \mathcal{Q} \to \mathcal{Q}$ is an α-quasi contraction (α-QC) for every $\xi, \eta \in \mathcal{Q}$, $\tau > 0$ and a certain $\alpha \in (0,1)$ if it holds that*

$$\mathcal{N}(\mathcal{T}\xi,\mathcal{T}\eta,\alpha\tau) \geq \min\{\mathcal{N}(\xi,\eta,\tau),\mathcal{N}(\xi,\mathcal{T}\eta,\tau),\mathcal{N}(\eta,\mathcal{T}\xi,\tau)\}, \tag{14}$$
$$\mathcal{O}(\mathcal{T}\xi,\mathcal{T}\eta,\alpha\tau) \leq \max\{\mathcal{O}(\xi,\eta,\tau),\mathcal{O}(\xi,\mathcal{T}\eta,\tau),\mathcal{O}(\eta,\mathcal{T}\xi,\tau)\}, \tag{15}$$
$$\mathcal{L}(\mathcal{T}\xi,\mathcal{T}\eta,\alpha\tau) \leq \max\{\mathcal{L}(\xi,\eta,\tau),\mathcal{L}(\xi,\mathcal{T}\eta,\tau),\mathcal{L}(\eta,\mathcal{T}\xi,\tau)\}. \tag{16}$$

Theorem 3. *Let $(\mathcal{Q}, \mathcal{N}, \mathcal{O}, \mathcal{L}, *, \circ)$ be a complete GNRBMLS and $\mathcal{T}$ be an α-quasi contraction. If there exists $\xi_0 \in \mathcal{Q}$, and for all $\tau > 0$*

$$\delta_1(\mathcal{N}, \mathcal{T}, \xi_0, \tau) = \inf_{i,j\in\mathbb{N}}\{\mathcal{N}(\mathcal{T}^i\xi_0, \mathcal{T}^j\xi_0, \tau)\} > 0,$$
$$\delta_2(\mathcal{O}, \mathcal{T}, \xi_0, \tau) = \sup_{i,j\in\mathbb{N}}\{\mathcal{O}(\mathcal{T}^i\xi_0, \mathcal{T}^j\xi_0, \tau)\} < 1,$$
$$\delta_3(\mathcal{L}, \mathcal{T}, \xi_0, \tau) = \sup_{i,j\in\mathbb{N}}\{\mathcal{L}(\mathcal{T}^i\xi_0, \mathcal{T}^j\xi_0, \tau)\} < 1,$$

then $\mathcal{T}$ has a unique FP $\xi \in \mathcal{Q}$, with $\mathcal{N}(\xi, \xi, \tau) = 1, \mathcal{O}(\xi, \xi, \tau) = 0, \mathcal{L}(\xi, \xi, \tau) = 0$ for all $\tau > 0$. Moreover, $\{\mathcal{T}^n\xi_0\}$ will converge to a unique FP of $\mathcal{T}$.

Proof. Let $\xi_0 \in \mathcal{Q}$ be arbitrary, such that the following conditions are satisfied for all $\tau > 0$:

$$\delta_1(\mathcal{N}, \mathcal{T}, \xi_0, \tau) = \inf_{i,j\in\mathbb{N}}\{\mathcal{N}(\mathcal{T}^i\xi_0, \mathcal{T}^j\xi_0, \tau)\} > 0,$$
$$\delta_2(\mathcal{O}, \mathcal{T}, \xi_0, \tau) = \sup_{i,j\in\mathbb{N}}\{\mathcal{O}(\mathcal{T}^i\xi_0, \mathcal{T}^j\xi_0, \tau)\} < 1,$$
$$\delta_3(\mathcal{L}, \mathcal{T}, \xi_0, \tau) = \sup_{i,j\in\mathbb{N}}\{\mathcal{L}(\mathcal{T}^i\xi_0, \mathcal{T}^j\xi_0, \tau)\} < 1,$$

where $\xi_n = \mathcal{T}^n\xi_0$ and

$$\delta_1(\mathcal{N}, \mathcal{T}^p, \xi_0, \tau) = \inf_{i,j\in\mathbb{N}}\{\mathcal{N}(\mathcal{T}^{p+i}\xi_0, \mathcal{T}^{p+j}\xi_0, \tau)\} > 0,$$
$$\delta_2(\mathcal{O}, \mathcal{T}^p, \xi_0, \tau) = \sup_{i,j\in\mathbb{N}}\{\mathcal{O}(\mathcal{T}^{p+i}\xi_0, \mathcal{T}^{p+j}\xi_0, \tau)\} < 1,$$
$$\delta_3(\mathcal{L}, \mathcal{T}^p, \xi_0, \tau) = \sup_{i,j\in\mathbb{N}}\{\mathcal{L}(\mathcal{T}^{p+i}\xi_0, \mathcal{T}^{p+j}\xi_0, \tau)\} < 1.$$

Observe that for every $i, j \in \mathbb{N}, \tau > 0$,

$$\{\mathcal{N}(\mathcal{T}^{p+i}\xi_0, \mathcal{T}^{p+j}\xi_0, \tau)\} \subseteq \{\mathcal{N}(\mathcal{T}^i\xi_0, \mathcal{T}^j\xi_0, \tau)\},$$
$$\{\mathcal{O}(\mathcal{T}^{p+i}\xi_0, \mathcal{T}^{p+j}\xi_0, \tau)\} \subseteq \{\mathcal{O}(\mathcal{T}^i\xi_0, \mathcal{T}^j\xi_0, \tau)\},$$
$$\{\mathcal{L}(\mathcal{T}^{p+i}\xi_0, \mathcal{T}^{p+j}\xi_0, \tau)\} \subseteq \{\mathcal{L}(\mathcal{T}^i\xi_0, \mathcal{T}^j\xi_0, \tau)\},$$

which implies that

$$\delta_1(\mathcal{N}, \mathcal{T}^{p+1}, \xi_0, \tau) \geq \delta_1(\mathcal{N}, \mathcal{T}, \xi_0, \tau), \tag{17}$$
$$\delta_2(\mathcal{O}, \mathcal{T}^{p+1}, \xi_0, \tau) \leq \delta_2(\mathcal{O}, \mathcal{T}, \xi_0, \tau), \tag{18}$$
$$\delta_3(\mathcal{L}, \mathcal{T}^{p+1}, \xi_0, \tau) \leq \delta_3(\mathcal{L}, \mathcal{T}, \xi_0, \tau). \tag{19}$$

Now, for all $i, j \in \mathbb{N}$,

$$\begin{aligned}\mathcal{N}(\mathcal{T}^{n+i}\xi_0, \mathcal{T}^{n+j}\xi_0, \tau) \geq \min\{&\mathcal{N}(\mathcal{T}^{n+i-1}\xi_0, \mathcal{T}^{n+j-1}\xi_0, \frac{\tau}{\alpha}),\\ &\mathcal{N}(\mathcal{T}^{n+i-1}\xi_0, \mathcal{T}^{n+j}\xi_0, \frac{\tau}{\alpha}), \mathcal{N}(\mathcal{T}^{n+i}\xi_0, \mathcal{T}^{n+j-1}\xi_0, \frac{\tau}{\alpha})\}.\end{aligned} \tag{20}$$

Then, for all $i, j \in \mathbb{N}, \tau > 0$, it holds that

$$\begin{aligned}\min\{\mathcal{N}(\mathcal{T}^{n+i}\xi_0, \mathcal{T}^{n+j}\xi_0, \tau)\} \geq \min\{\min\{&\mathcal{N}(\mathcal{T}^{n+i-1}\xi_0, \mathcal{T}^{n+j-1}\xi_0, \frac{\tau}{\alpha}),\\ &\mathcal{N}(\mathcal{T}^{n+i-1}\xi_0, \mathcal{T}^{n+j}\xi_0, \frac{\tau}{\alpha}), \mathcal{N}(\mathcal{T}^{n+i}\xi_0, \mathcal{T}^{n+j-1}\xi_0, \frac{\tau}{\alpha})\}\},\end{aligned}$$

which implies that

$$\delta_1(\mathcal{N},\mathcal{T}^{n+1},\xi_0,\tau) \geq \min\{\delta_1(\mathcal{N},\mathcal{T}^n,\xi_0,\frac{\tau}{\alpha}),\delta_1(\mathcal{N},\mathcal{T}^n,\xi_0,\frac{\tau}{\alpha}),\delta_1(\mathcal{N},\mathcal{T}^n,\xi_0,\frac{\tau}{\alpha})\}. \tag{21}$$

It follows that, for all $n \geq 1, \tau > 0$,

$$\begin{aligned}\mathcal{N}(\mathcal{T}^{n+i}\xi_0,\mathcal{T}^{n+j}\xi_0,\tau) &\geq \delta_1(\mathcal{N},\mathcal{T}^{n+1},\xi_0,\tau)\\ &\geq \delta_1(\mathcal{N},\mathcal{T}^n,\xi_0,\frac{\tau}{\alpha})\\ &\geq \delta_1(\mathcal{N},\mathcal{T}^{n-1},\xi_0,\frac{\tau}{\alpha^2})\\ &\vdots\\ &\geq \delta_1(\mathcal{N},\mathcal{T},\xi_0,\frac{\tau}{\alpha^n}).\end{aligned} \tag{22}$$

Again, for all $i,j \in \mathbb{N}$,

$$\begin{aligned}\mathcal{O}(\mathcal{T}^{n+i}\xi_0,\mathcal{T}^{n+j}\xi_0,\tau) \leq \max\{&\mathcal{O}(\mathcal{T}^{n+i-1}\xi_0,\mathcal{T}^{n+j-1}\xi_0,\frac{\tau}{\alpha}),\\ &\mathcal{O}(\mathcal{T}^{n+i-1}\xi_0,\mathcal{T}^{n+j}\xi_0,\frac{\tau}{\alpha}),\\ &\mathcal{O}(\mathcal{T}^{n+i}\xi_0,\mathcal{T}^{n+j-1}\xi_0,\frac{\tau}{\alpha})\}.\end{aligned} \tag{23}$$

Then, for all $i,j \geq 1$, it holds that

$$\begin{aligned}\max\{\mathcal{O}(\mathcal{T}^{n+i}\xi_0,\mathcal{T}^{n+j}\xi_0,\tau)\} \leq \max\{\max\{&\mathcal{O}(\mathcal{T}^{n+i-1}\xi_0,\mathcal{T}^{n+j-1}\xi_0,\frac{\tau}{\alpha}),\\ &\mathcal{O}(\mathcal{T}^{n+i-1}\xi_0,\mathcal{T}^{n+j}\xi_0,\frac{\tau}{\alpha}),\\ &\mathcal{O}(\mathcal{T}^{n+i}\xi_0,\mathcal{T}^{n+j-1}\xi_0,\frac{\tau}{\alpha})\}\},\end{aligned} \tag{24}$$

which implies that

$$\begin{aligned}\delta_2(\mathcal{O},\mathcal{T}^{n+1},\xi_0,\tau) \leq \max\{&\delta_2(\mathcal{O},\mathcal{T}^n,\xi_0,\frac{\tau}{\alpha}),\delta_2(\mathcal{O},\mathcal{T}^n,\xi_0,\frac{\tau}{\alpha}),\\ &\delta_2(\mathcal{O},\mathcal{T}^n,\xi_0,\frac{\tau}{\alpha})\}.\end{aligned} \tag{25}$$

It follows that, for all $n \geq 1, \tau > 0$,

$$\begin{aligned}\mathcal{O}(\mathcal{T}^{n+i}\xi_0,\mathcal{T}^{n+j}\xi_0,\tau) &\leq \delta_2(\mathcal{O},\mathcal{T}^{n+1},\xi_0,\tau)\\ &\leq \delta_2(\mathcal{O},\mathcal{T}^n,\xi_0,\frac{\tau}{\alpha})\\ &\leq \delta_2(\mathcal{O},\mathcal{T}^{n-1},\xi_0,\frac{\tau}{\alpha^2})\\ &\vdots\\ &\leq \delta_2(\mathcal{O},\mathcal{T},\xi_0,\frac{\tau}{\alpha^n}).\end{aligned} \tag{26}$$

Similarly, we find that for all $n \geq 1, \tau > 0$,

$$\mathcal{L}(\mathcal{T}^{n+i}\xi_0,\mathcal{T}^{n+j}\xi_0,\tau) \leq \delta_2(\mathcal{L},\mathcal{T},\xi_0,\frac{\tau}{\alpha^n}). \tag{27}$$

Therefore, for every $m \geq 1$, we can use (22), (26), and (27) to obtain

$$\mathcal{N}(\xi_{n+m},\xi_n,\tau) = \mathcal{N}(\mathcal{T}^{n+m}\xi_0,\mathcal{T}^n\xi_0,\tau) \geq \delta_1(\mathcal{N},\mathcal{T}^n,\xi_0,\tau) \geq \delta_1(\mathcal{N},\mathcal{T},\xi_0,\frac{\tau}{\alpha^n}),$$

$$\mathcal{O}(\xi_{n+m}, \xi_n, \tau) = \mathcal{O}(\mathcal{T}^{n+m}\xi_0, \mathcal{T}^n\xi_0, \tau) \leq \delta_2(\mathcal{O}, \mathcal{T}^n, \xi_0, \tau) \leq \delta_2(\mathcal{O}, \mathcal{T}, \xi_0, \frac{\tau}{\alpha^n}),$$
$$\mathcal{L}(\xi_{n+m}, \xi_n, \tau) = \mathcal{L}(\mathcal{T}^{n+m}\xi_0, \mathcal{T}^n\xi_0, \tau) \leq \delta_3(\mathcal{L}, \mathcal{T}^n, \xi_0, \tau) \leq \delta_3(\mathcal{L}, \mathcal{T}, \xi_0, \frac{\tau}{\alpha^n}).$$

As $\delta_1(\mathcal{N}, \mathcal{T}^n, \xi_0, \tau) > 0, \delta_2(\mathcal{O}, \mathcal{T}^n, \xi_0, \tau) < 1$, and $\delta_3(\mathcal{L}, \mathcal{T}^n, \xi_0, \tau) < 1$ for all $\tau > 0, \alpha \in (0,1)$, according to (S6), (S11), and (S16) in Definition 2, we can then obtain

$$\lim_{n\to+\infty} \mathcal{N}(\xi_{n+m}, \xi_n, \tau) = 1,$$
$$\lim_{n\to+\infty} \mathcal{O}(\xi_{n+m}, \xi_n, \tau) = 0,$$
$$\lim_{n\to+\infty} \mathcal{L}(\xi_{n+m}, \xi_n, \tau) = 0.$$

Hence, the sequence $\{\xi_n\}$ is a Cauchy sequence. Given that $(\mathcal{Q}, \mathcal{N}, \mathcal{O}, \mathcal{L}, *, \circ)$ is complete, this sequence converges to a certain $\zeta \in \mathcal{Q}$. Now,

$$\begin{aligned}
\mathcal{N}(\xi_{n+1}, \mathcal{T}\zeta, \tau) &\geq \min\{\mathcal{N}(\xi_n, \zeta, \frac{\tau}{\alpha}), \mathcal{N}(\xi_n, \mathcal{T}\zeta, \frac{\tau}{\alpha}), \mathcal{N}(\zeta, \mathcal{T}\xi_n, \frac{\tau}{\alpha})\} \\
&= \min\{\mathcal{N}(\xi_n, \zeta, \frac{\tau}{\alpha}), \mathcal{N}(\xi_n, \mathcal{T}\zeta, \frac{\tau}{\alpha}), \mathcal{N}(\zeta, \xi_{n+1}, \frac{\tau}{\alpha})\} \\
&= \mathcal{N}(\xi_n, \mathcal{T}\zeta, \frac{\tau}{\alpha}) \\
&\geq \\
&\vdots \\
&\geq \min\{\mathcal{N}(\xi_0, \zeta, \frac{\tau}{\alpha^{n+1}}), \mathcal{N}(\xi_0, \mathcal{T}\zeta, \frac{\tau}{\alpha^{n+1}}), \mathcal{N}(\zeta, \mathcal{T}\xi_0, \frac{\tau}{\alpha^{n+1}})\} \\
&\to 1 \text{ as } n \to +\infty.
\end{aligned}$$

Furthermore, according to (S5) and (S6) in Definition 2, we have

$$\mathcal{N}(\zeta, \mathcal{T}\zeta, \Lambda(\zeta, \mathcal{T}\zeta)\tau) \geq \limsup_{n\to+\infty}[\mathcal{N}(\zeta, \xi_n, \frac{\tau}{3}) * \mathcal{N}(\xi_n, \xi_{n+1}, \frac{\tau}{3}) * \mathcal{N}(\zeta, \zeta, \frac{\tau}{3})]. \tag{28}$$

We obtain

$$\mathcal{N}(\zeta, \mathcal{T}\zeta, \tau) \geq 1 * 1 * 1 = 1. \tag{29}$$

Also,

$$\begin{aligned}
\mathcal{O}(\xi_{n+1}, \mathcal{T}\zeta, \tau) &\leq \max\{\mathcal{O}(\xi_n, \zeta, \frac{\tau}{\alpha}), \mathcal{O}(\xi_n, \mathcal{T}\zeta, \frac{\tau}{\alpha}), \mathcal{O}(\zeta, \mathcal{T}\xi_n, \frac{\tau}{\alpha})\} \\
&= \max\{\mathcal{O}(\xi_n, \zeta, \frac{\tau}{\alpha}), \mathcal{O}(\xi_n, \mathcal{T}\zeta, \frac{\tau}{\alpha}), \mathcal{O}(\zeta, \xi_{n+1}, \frac{\tau}{\alpha})\} \\
&= \mathcal{O}(\xi_n, \mathcal{T}\zeta, \frac{\tau}{\alpha}) \\
&\leq \\
&\vdots \\
&\leq \max\{\mathcal{O}(\xi_0, \zeta, \frac{\tau}{\alpha^{n+1}}), \mathcal{O}(\xi_0, \mathcal{T}\zeta, \frac{\tau}{\alpha^{n+1}}), \mathcal{O}(\zeta, \mathcal{T}\xi_0, \frac{\tau}{\alpha^{n+1}})\} \\
&\to 0 \text{ as } n \to +\infty.
\end{aligned}$$

Furthermore, according to (S10) of Definition 2, we gain

$$\mathcal{O}(\mathcal{T}\zeta, \zeta, \Lambda(\zeta, \mathcal{T}\zeta)\tau) \leq \limsup_{n\to+\infty}[\mathcal{O}(\mathcal{T}\zeta, \zeta_{m+1}, \frac{\tau}{3}) \circ \mathcal{O}(\xi_{m+1}, \zeta, \frac{\tau}{3}) \circ \mathcal{O}(\zeta, \zeta, \frac{\tau}{3})]. \tag{30}$$

We can then obtain

$$\mathcal{O}(\zeta, \mathcal{T}\zeta, \tau) \leq 0 \circ 0 \circ 0 = 0. \tag{31}$$

In a similar fashion, we can show that

$$\mathcal{L}(\zeta, \mathcal{T}\zeta, \tau) = 0. \tag{32}$$

Therefore, according to (S3), (S8), and (S13) of Definition 2, it holds that $\zeta = \mathcal{T}\zeta$.

Consider $v \in \mathcal{Q}$ to be a different FP of $\mathcal{T}$, such that $\mathcal{N}(\zeta, v, \tau) > 0$. Then, due to the α-quasi contractions (14)–(16), it is evident that

$$\begin{aligned}
\mathcal{N}(v, \zeta, \tau) &\geq \min\{\mathcal{N}(v, \zeta, \frac{\tau}{\alpha}), \mathcal{N}(\mathcal{T}v, \zeta, \frac{\tau}{\alpha}), \mathcal{N}(v, \mathcal{T}\zeta, \frac{\tau}{\alpha})\} \\
&= \mathcal{N}(v, \zeta, \frac{\tau}{\alpha}) \\
&\geq \mathcal{N}(v, \zeta, \frac{\tau}{\alpha^2}) \\
&\vdots \\
&\geq \mathcal{N}(v, \zeta, \frac{\tau}{\alpha^n}) \to 1 \text{ as } n \to +\infty, \\
\mathcal{O}(v, \zeta, \tau) &\leq \max\{\mathcal{O}(v, \zeta, \frac{\tau}{\alpha}), \mathcal{O}(\mathcal{T}v, \zeta, \frac{\tau}{\alpha}), \mathcal{O}(v, \mathcal{T}\zeta, \frac{\tau}{\alpha})\} \\
&= \mathcal{O}(v, \zeta, \frac{\tau}{\alpha}) \\
&\leq \mathcal{O}(v, \zeta, \frac{\tau}{\alpha^2}) \\
&\vdots \\
&\leq \mathcal{O}(v, \zeta, \frac{\tau}{\alpha^n}) \to 0 \text{ as } n \to +\infty, \\
\mathcal{L}(v, \zeta, \tau) &\leq \max\{\mathcal{L}(v, \zeta, \frac{\tau}{\alpha}), \mathcal{L}(\mathcal{T}v, \zeta, \frac{\tau}{\alpha}), \mathcal{L}(v, \mathcal{T}\zeta, \frac{\tau}{\alpha})\} \\
&= \mathcal{L}(v, \zeta, \frac{\tau}{\alpha}) \\
&\leq \mathcal{L}(v, \zeta, \frac{\tau}{\alpha^2}) \\
&\vdots \\
&\leq \mathcal{L}(v, \zeta, \frac{\tau}{\alpha^n}) \to 0 \text{ as } n \to +\infty;
\end{aligned}$$

that is, $v = \zeta$. □

Example 4. *Let $\mathcal{Q} = \mathbb{M}_n(\mathbb{R})$ be the space of all upper triangular matrices of order n and $\Lambda : \mathcal{Q} \times \mathcal{Q} \to [1, +\infty)$ be a function given by*

$$\Lambda(\Xi, \Pi) = \begin{cases} 1, & \text{if } \Xi = \Pi, \\ 1 + det(\Xi) + det(\Pi), & \text{if } \Xi \neq \Pi. \end{cases}$$

Let $\mathcal{N}, \mathcal{O}, \mathcal{L} : \mathcal{Q} \times \mathcal{Q} \times (0, +\infty) \to [0,1]$ be defined by

$$\mathcal{N}(\Xi, \Pi, \tau) = \exp \frac{-\sum_{s,r=1}^{n} (|u_{s,r}| + |v_{s,r}|)^2}{\tau},$$
$$\mathcal{O}(\Xi, \Pi, \tau) = 1 - \exp \frac{-\sum_{s,r=1}^{n} (|u_{s,r}| + |v_{s,r}|)^2}{\tau},$$
$$\mathcal{L}(\Xi, \Pi, \tau) = 1 - \exp \frac{-\sum_{s,r=1}^{n} \max\{|u_{s,r}|, |v_{s,r}|\}}{\tau},$$

for all $\Xi = (u_{s,r}), \Pi = (v_{s,r}) \in \mathcal{Q}$. Define $\mathcal{T} : \mathcal{Q} \to \mathcal{Q}$ by

$$\mathcal{T}(\Xi) = (\frac{u_{s,r}}{2})$$

for all $\Xi \in \mathcal{Q}$. $(\mathcal{Q}, \mathcal{N}, \mathcal{O}, \mathcal{L}, *, \circ)$ is a complete GNRBMLS, where $*$ is a product continuous CtN and $\circ$ is a maximum CtCN.

We check that T is an α-QC on $(\mathcal{Q}, \mathcal{N}, \mathcal{O}, \mathcal{L}, *, \circ)$ with $\alpha = \frac{1}{4}$. Indeed, let $\Xi = (u_{i,j})$ and $\Pi = (v_{i,j}) \in \mathcal{Q}$. Then,

$$\begin{aligned}
\mathcal{N}(\mathcal{T}\Xi, \mathcal{T}\Pi, \alpha\tau) &= \exp \frac{-\sum_{s,r=1}^{n} (|\frac{u_{s,r}}{2}| + |\frac{v_{s,r}}{2}|)^2}{\alpha\tau} \\
&= \exp \frac{-\sum_{s,r=1}^{n} (|u_{s,r}| + |v_{s,r}|)^2}{4\alpha\tau}, \alpha = \frac{1}{4} \\
&\geq \min\left\{ \exp \frac{-\sum_{s,r=1}^{n} (|u_{s,r}| + |v_{s,r}|)^2}{\tau}, \exp \frac{-\sum_{s,r=1}^{n} (|u_{s,r}| + |\frac{v_{s,r}}{2}|)^2}{\tau}, \right. \\
&\left. \exp \frac{-\sum_{s,r=1}^{n} (|\frac{u_{s,r}}{2}| + |v_{s,r}|)^2}{\tau} \right\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{O}\mathcal{T}\Xi, \mathcal{T}\Pi, \alpha\tau) &= 1 - \exp \frac{-\sum_{s,r=1}^{n} (|\frac{u_{s,r}}{2}| + |\frac{v_{s,r}}{2}|)^2}{\alpha\tau} \\
&= 1 - \exp \frac{-\sum_{s,r=1}^{n} (|u_{s,r}| + |v_{s,r}|)^2}{4\alpha\tau}, \alpha = \frac{1}{4} \\
&\leq \max\left\{ 1 - \exp \frac{-\sum_{s,r=1}^{n} (|u_{s,r}| + |v_{s,r}|)^2}{\tau}, 1 - \exp \frac{-\sum_{s,r=1}^{n} (|u_{s,r}| + |\frac{v_{s,r}}{2}|)^2}{\tau}, \right. \\
&\left. 1 - \exp \frac{-\sum_{s,r=1}^{n} (|\frac{u_{s,r}}{2}| + |v_{s,r}|)^2}{\tau} \right\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}(\mathcal{T}\Xi, \mathcal{T}\Pi, \alpha\tau) &= 1 - \exp \frac{-\sum_{s,r=1}^{n} \max\{|\frac{u_{s,r}}{2}|, |\frac{v_{s,r}}{2}|\}}{\alpha\tau} \\
&= 1 - \exp \frac{-\sum_{s,r=1}^{n} \max\{|u_{s,r}|, |v_{s,r}|\}}{4\alpha\tau}, \alpha = \frac{1}{4} \\
&\leq \max\{1 - \exp \frac{-\sum_{s,r=1}^{n} \max\{|u_{s,r}|, |v_{s,r}|\}}{\tau}, \\
&1 - \exp \frac{-\sum_{s,r=1}^{n} \max\{|u_{s,r}|, |\frac{v_{s,r}}{2}|\}}{\tau}, \\
&1 - \exp \frac{-\sum_{s,r=1}^{n} \max\{|\frac{u_{s,r}}{2}|, |v_{s,r}|\}}{\tau}\}.
\end{aligned}$$

Now, we can construct a sequence $\xi_{n+1} = \mathcal{T}\xi_n$ for all $n \in \mathbb{N} \cup \{0\}$ by taking some $0 \neq A = (a_{s,r}) \in \mathcal{Q}$. We can obtain a non-trivial sequence as follows:

$$\{\xi_n\} = \{(a_{s,r}), (\frac{a_{s,r}}{2}), (\frac{a_{s,r}}{2^2}), (\frac{a_{s,r}}{2^3}), \ldots\},$$

which implies that for any fixed $\tau > 0$,

$$\delta_1(\mathcal{N},\mathcal{T},A,\tau) = \inf_{i,j\in\mathbb{N}}\{\mathcal{N}(\mathcal{T}^iA,\mathcal{T}^jA,\tau)\}$$
$$= \inf_{i,j\in\mathbb{N}}\{\exp\frac{-\sum_{l,m=1}^{n}[|\frac{a_{s,r}}{2^{i-1}}|+|\frac{|a_{s,r}}{2^{j-1}}|]^2}{\tau}\} > 0,$$
$$\delta_2(\mathcal{O},\mathcal{T},A,\tau) = \sup_{i,j\in\mathbb{N}}\{\mathcal{O}(\mathcal{T}^iA,\mathcal{T}^jA,\tau)\}$$
$$= \sup_{i,j\in\mathbb{N}}\{1-\exp\frac{-\sum_{l,m=1}^{n}[|\frac{a_{s,r}}{2^{i-1}}|+|\frac{|a_{s,r}}{2^{j-1}}|]^2}{\tau}\} < 1,$$
$$\delta_3(\mathcal{L},\mathcal{T},A,\tau) = \sup_{i,j\in\mathbb{N}}\{\mathcal{L}(\mathcal{T}^iA,\mathcal{T}^jA,\tau)\}$$
$$= \sup_{i,j\in\mathbb{N}}\{1-\exp\frac{-\sum_{l,m=1}^{n}\max\{|\frac{a_{s,r}}{2^{i-1}}|,|\frac{|a_{s,r}}{2^{j-1}}|\}}{\tau}\} < 1.$$

Thus, $\mathcal{T}$ *fulfills all the requirements stated in Theorem 3, and the null matrix* $O_{n\times n}$ *is the unique FP of* $\mathcal{T}$*, satisfying* $\mathcal{N}(O_{n\times n},O_{n\times n},\tau) = 1, \mathcal{O}(O_{n\times n},O_{n\times n},\tau) = 0, \mathcal{L}(O_{n\times n},O_{n\times n},\tau) = 0$ *for all* $\tau > 0$.

3.2. Consequences

In this subsection, we present the findings from Section 3.1 in the framework of ERBMLSs, GIRBMLS, EIRBMLS, EFRBMLS, and GFRBMLS.

Corollary 1. *Let* $(\mathcal{Q},\mathcal{N},\mathcal{O},\mathcal{L},*,\circ)$ *be a complete ENRBMLS and* $\mathcal{T}$ *be an* α*-contraction. If for some* $\xi_0 \in \mathcal{Q}$ *and all* $\tau > 0$ *the subsequent expressions are satisfied,*

$$\delta_1(\mathcal{N},\mathcal{T},\xi_0,\tau) = \inf_{i,j\in\mathbb{N}}\{\mathcal{N}(\mathcal{T}^i\xi_0,\mathcal{T}^j\xi_0,\tau)\} > 0,$$
$$\delta_2(\mathcal{O},\mathcal{T},\xi_0,\tau) = \sup_{i,j\in\mathbb{N}}\{\mathcal{O}(\mathcal{T}^i\xi_0,\mathcal{T}^j\xi_0,\tau)\} < 1,$$
$$\delta_3(\mathcal{L},\mathcal{T},\xi_0,\tau) = \sup_{i,j\in\mathbb{N}}\{\mathcal{L}(\mathcal{T}^i\xi_0,\mathcal{T}^j\xi_0,\tau)\} < 1,$$

then $\{\mathcal{T}^n\xi_0\}$ *converges to a unique FP of* $\mathcal{T}$.

Definition 6. *Let* $(\mathcal{Q},\mathcal{N},\mathcal{O},*,\circ)$ *be a GIRBMLS. A mapping* $\mathcal{T}:\mathcal{Q}\to\mathcal{Q}$ *is said to be an* α*-contraction. If for all* $\xi,\eta\in\mathcal{Q}$ *some* $\alpha\in(0,1)$ *and all* $\tau > 0$*, it holds that*

$$\begin{aligned}&\mathcal{N}(\mathcal{T}\xi,\mathcal{T}\eta,\alpha\tau) \geq \mathcal{N}(\xi,\eta,\tau),\\&\mathcal{O}(\mathcal{T}\xi,\mathcal{T}\eta,\alpha\tau) \leq \mathcal{O}(\xi,\eta,\tau).\end{aligned} \tag{33}$$

Corollary 2. *Let* $(\mathcal{Q},\mathcal{N},\mathcal{O},*,\circ)$ *be a complete GIRBMLS and* $\mathcal{T}$ *be an* α*-contraction. If there exists* $\xi_0\in\mathcal{Q}$*, such that subsequent expressions are held to be*

$$\delta_1(\mathcal{N},\mathcal{T},\xi_0,\tau) = \inf_{i,j\in\mathbb{N}}\{\mathcal{N}(\mathcal{T}^i\xi_0,\mathcal{T}^j\xi_0,\tau)\} > 0,$$
$$\delta_2(\mathcal{O},\mathcal{T},\xi_0,\tau) = \sup_{i,j\in\mathbb{N}}\{\mathcal{O}(\mathcal{T}^i\xi_0,\mathcal{T}^j\xi_0,\tau)\} < 1,$$

then $\{\mathcal{T}^n\xi_0\}$ *converges to a unique FP of* $\mathcal{T}$.

Proof. This proof follows similarly to the proof of Theorem 2, but without considering an NS $\mathcal{L}$. □

The subsequent corollary is derived from Corollary 2 and Proposition 2.

Corollary 3. *Let $(\mathcal{Q}, \mathcal{N}, \mathcal{O}, *, \circ)$ be a complete EIRBMLS and $\mathcal{T}$ be an α-contraction. If for some $\xi_0 \in \mathcal{Q}$ and all $\tau > 0$ the subsequent expressions hold,*

$$\delta_1(\mathcal{N}, \mathcal{T}, \xi_0, \tau) = \inf_{i,j \in \mathbb{N}} \{\mathcal{N}(\mathcal{T}^i \xi_0, \mathcal{T}^j \xi_0, \tau)\} > 0,$$
$$\delta_2(\mathcal{O}, \mathcal{T}, \xi_0, \tau) = \sup_{i,j \in \mathbb{N}} \{\mathcal{O}(\mathcal{T}^i \xi_0, \mathcal{T}^j \xi_0, \tau)\} < 1,$$

then $\{\mathcal{T}^n \xi_0\}$ converges to a unique FP of $\mathcal{T}$.

Definition 7. *Let $(\mathcal{Q}, \mathcal{N}, *)$ be a GFRBMLS. A mapping $\mathcal{T} : \mathcal{Q} \to \mathcal{Q}$ is said to be an α-contraction if for all $\xi, \eta \in \mathcal{Q}$, some $\alpha \in (0, 1)$, and all $\tau > 0$, it holds that*

$$\mathcal{N}(\mathcal{T}\xi, \mathcal{T}\eta, \alpha\tau) \geq \mathcal{N}(\xi, \eta, \tau). \tag{34}$$

Corollary 4. *Let $(\mathcal{Q}, \mathcal{N}, *)$ be a complete GFRBMLS and $\mathcal{T}$ be an α-contraction. If for some $\xi_0 \in \mathcal{Q}$ and all $\tau > 0$ the subsequent expression holds*

$$\delta_1(\mathcal{N}, \mathcal{T}, \xi_0, \tau) = \inf_{i,j \in \mathbb{N}} \{\mathcal{N}(\mathcal{T}^i \xi_0, \mathcal{T}^j \xi_0\} > 0,$$

then $\{\mathcal{T}^n \xi_0\}$ converges to a unique FP of $\mathcal{T}$.

Proof. This proof follows similarly to the proof of Corollary 2, but without considering the NSs $\mathcal{O}$ and $\mathcal{L}$. □

The subsequent result is a direct consequence of Corollary 4 and Proposition 3.

Corollary 5. *Let $(\mathcal{Q}, \mathcal{N}, *)$ be a complete EFRBMLS and $\mathcal{T}$ be a α contraction. If for some $\xi_0 \in \mathcal{Q}$ and all $\tau > 0$ the subsequent expression holds,*

$$\delta_1(\mathcal{N}, \mathcal{T}, \xi_0, \tau) = \inf_{i,j \in \mathbb{N}} \{\mathcal{N}(\mathcal{T}^i \xi_0, \mathcal{T}^j \xi_0, \tau)\} > 0,$$

then $\{\mathcal{T}^n \xi_0\}$ converges to a unique FP of $\mathcal{T}$.

Definition 8. *Let $(\mathcal{Q}, \mathcal{N}, \mathcal{O}, *, \circ)$ be a GIRBMLS. A mapping $\mathcal{T} : \mathcal{Q} \to \mathcal{Q}$ is an α-QC for every $\xi, \eta \in \mathcal{Q}$, $\tau > 0$ and a certain $\alpha \in (0, 1)$ if it holds that*

$$\mathcal{N}(\mathcal{T}\xi, \mathcal{T}\eta, \alpha\tau) \geq \min\{\mathcal{N}(\xi, \eta, \tau), \mathcal{N}(\xi, \mathcal{T}\eta, \tau), \mathcal{N}(\eta, \mathcal{T}\xi, \tau)\}, \tag{35}$$
$$\mathcal{O}(\mathcal{T}\xi, \mathcal{T}\eta, \alpha\tau) \leq \max\{\mathcal{O}(\xi, \eta, \tau), \mathcal{O}(\xi, \mathcal{T}\eta, \tau), \mathcal{O}(\eta, \mathcal{T}\xi, \tau)\}. \tag{36}$$

Corollary 6. *Let $(\mathcal{Q}, \mathcal{N}, \mathcal{O}, *, \circ)$ be a complete GIRBMLS and $\mathcal{T}$ be an α-quasi contraction. If for some $\xi_0 \in \mathcal{Q}$ and all $\tau > 0$ the subsequent expressions hold*

$$\delta_1(\mathcal{N}, \mathcal{T}, \xi_0, \tau) = \inf_{i,j \in \mathbb{N}} \{\mathcal{N}(\mathcal{T}^i \xi_0, \mathcal{T}^j \xi_0, \tau)\} > 0,$$
$$\delta_2(\mathcal{O}, \mathcal{T}, \xi_0, \tau) = \sup_{i,j \in \mathbb{N}} \{\mathcal{O}(\mathcal{T}^i \xi_0, \mathcal{T}^j \xi_0, \tau)\} < 1.$$

then $\{\mathcal{T}^n \xi_0\}$ converges to a unique FP of $\mathcal{T}$.

Proof. This demonstration follows a similar approach to Theorem 3, except for the inclusion of the neutrosophic set $\mathcal{L}$. □

Definition 9. *Let $(\mathcal{Q}, \mathcal{N}, *)$ be a GFRBMLS. A mapping $\mathcal{T} : \mathcal{Q} \to \mathcal{Q}$ is an α-QC for every $\xi, \eta \in \mathcal{Q}$, any $\tau > 0$ and a certain $\alpha \in (0,1)$ if it holds that*

$$\mathcal{N}(\mathcal{T}\xi, \mathcal{T}\eta, \alpha\tau) \geq \min\{\mathcal{N}(\xi, \eta, \tau), \mathcal{N}(\xi, \mathcal{T}\eta, \tau), \mathcal{N}(\eta, \mathcal{T}\xi, \tau)\}. \tag{37}$$

Corollary 7. *Let $(\mathcal{Q}, \mathcal{N}, *)$ be a complete GFRBMLS and $\mathcal{T}$ be an α-quasi contraction. If for some $\xi_0 \in \mathcal{Q}$ and all $\tau > 0$ the subsequent expression holds,*

$$\delta_1(\mathcal{N}, \mathcal{T}, \xi_0, \tau) = \inf_{i,j \in \mathbb{N}} \{\mathcal{N}(\mathcal{T}^i \xi_0, \mathcal{T}^j \xi_0, \tau)\} > 0,$$

then $\{\mathcal{T}^n \xi_0\}$ converges to a unique FP of $\mathcal{T}$.

Proof. This proof proceeds in a manner analogous to the proof of Theorem 3, except that it does not take into account NSs $\mathcal{O}$ and $\mathcal{L}$. □

Remark 6. *Similar to Propositions 1–3, one can derive FP theorems for CQT within the context of ENRBMLSs, EIRBMLSs, and EFRBMLSs as a result of Theorem 3 and Corollarys 6 and 7.*

4. Application to FFDIDE with the OBCFFD

In 2024, Dwivedi et al. [19] refined and introduced the Odibat–Baleanu–Caputo fuzzy fractional derivative (OBCFFD), a generalized version of the Caputo-type fractional derivative in a fuzzy setting. To the best of our knowledge, FFDIDEs with the OBCFFD have not yet been investigated in the existing literature. Motivated by these results, we aim to explore and establish the existence and uniqueness of the solution to the following FFDIDE:

$$\begin{cases} {}^{OBC}_{t_0}\mathcal{D}^{\varkappa,\varrho}_t P(t) = \mathcal{F}(t, P(t), P_t) + \int_{t_0}^{\theta} \mathcal{H}(t, s, P(s), P_s) ds, t \in [t_0, \theta], \\ P(t) = \psi(t - t_0), t \in [t_0 - a, t_0], a > 0, \end{cases} \tag{38}$$

where $\varkappa \in (0,1)$, $\tilde{\mathcal{M}} = \{(t,s); t_0 < s \leq t \leq \theta\}$, $\psi \in C_\beta(C([-\beta, 0], \mathbb{F}_\mathbb{R}))$, $P_t = P(t+s)$, $t \in [t_0, \theta]$, $s \in [-\beta, 0]$, and $\mathcal{F} : [t_0, \theta] \times \mathbb{F}_\mathbb{R} \times C_\beta \to \mathbb{F}_\mathbb{R}$ and $\mathcal{H} : \tilde{\mathcal{M}} \times \mathbb{F}_\mathbb{R} \times C_v \to \mathbb{F}_\mathbb{R}$ are continuous. For more information about FFDIDEs, please refer to [20].

Definition 10 ([21])**.** *The Hausdorff distance $D_H : \mathbb{F}_\mathbb{R} \times \mathbb{F}_\mathbb{R}$ between P_1 and $P_2 \in \mathbb{F}_\mathbb{R}$ is given by*

$$D_H(P_1, P_2) = \sup_{\varsigma \in [0,1]} \max\{|P_{1_l}(\varsigma) - P_{2_l}(\varsigma)|, |P_{1_u}(\varsigma) - P_{2_u}(\varsigma)|\}.$$

Definition 11 ([21])**.** *The generalized Hukuhara difference of two fuzzy numbers $P_1, P_2 \in \mathbb{F}_\mathbb{R}$, if they exist, is defined as follows:*

$$\begin{cases} P_1 \ominus_{gH} P_2 = P_3 \text{ if and only if } \begin{cases} (i) P_1 = P_2 + P_3, & \text{if } diam([P_1]^r) \geq diam([P_2]^r) \\ (ii) P_2 = P_1 + (-1)P_3, & \text{if } diam([P_1]^r) \leq diam([P_2]^r), \end{cases} \end{cases}$$

where $diam([P]^r) = P_u(r) - P_l(r)$, $r \in (0,1)$.

Definition 12 ([19])**.** *The generalized fractional integral of order $\varkappa > 0$ of $P \in L_{1,loc}([t_0, t], \mathbb{F}_\mathbb{R})$ is expressed as follows:*

$${}^{K}_{t_0}\mathfrak{I}^{\varkappa,\varrho}_t P(t) = \frac{\varrho^{1-\varkappa}}{\Gamma(\varkappa)} \odot \int_{t_0}^{t} \varsigma^{\varrho-1}(t^\varrho - \varsigma^\varrho)^{\varkappa-1} \odot P(\varsigma) d\varsigma. \tag{39}$$

Definition 13 ([19])**.** *The OBCFFD of order $m - 1 < \varkappa < m$ of $P \in L_{1,loc}([t_0, t], \mathbb{F}_\mathbb{R})$ is given by*

$${}^{OBC}_{t_0}\mathcal{D}^{\varkappa,\varrho}_t P = \frac{\varrho^{\varkappa-m+1}}{\Gamma(m-\varkappa)} \odot \int_{t_0}^{t} \varsigma^{\varrho-1}(t^\varrho - \varsigma^\varrho)^{m-\varkappa-1} \left(\varsigma^{1-\varrho} \frac{d}{d\varsigma}\right)^m \odot P(\varsigma) d\varsigma, \tag{40}$$

where the gH-derivative of P^m is defined as

$$P^m(t) = \lim_{h \to 0} \frac{P^{m-1}(t_0 + h) \ominus_g P^{m-1}(t_0)}{h}.$$

Lemma 1. *The generalized OBC FFDIDE corresponding to (38) has two distinct forms in the context of fuzzy logic.*

Proof. By applying the generalized fractional integral of order $\varkappa > 0$ to both sides of FFDIDE (38), we obtain

$$\begin{aligned} {}^{K}_{t_0}\mathfrak{I}^{\varkappa,\varrho}_{t}\,{}^{OBC}_{t_0}\mathcal{D}^{\varkappa,\varrho}_{t} P(t) = P(t) \ominus P(t_0) &= {}^{K}_{t_0}\mathfrak{I}^{\varkappa,\varrho}_{t} \left[\mathcal{F}(t, P(t), P_t) + \int_{t_0}^{\theta} \mathcal{H}(t, s, P(s), P_s) ds \right] \\ &= \frac{\varrho^{1-\varkappa}}{\Gamma(\varkappa)} \odot \int_{t_0}^{t} \varsigma^{\varrho-1}(t^{\varrho} - \varsigma^{\varrho})^{\varkappa-1} \odot \big[\mathcal{F}(t, P(t), P_t) \\ &\quad + \int_{t_0}^{\theta} \mathcal{H}(t, s, P(s), P_s) ds \big] d\varsigma. \end{aligned}$$

Applying this in IVP of (38), we obtain

$$\begin{aligned} &P(t) \ominus \psi(0) \\ &= \frac{\varrho^{1-\varkappa}}{\Gamma(\varkappa)} \odot \int_{t_0}^{t} \varsigma^{\varrho-1}(t^{\varrho} - \varsigma^{\varrho})^{\varkappa-1} \odot \left[\mathcal{F}(t, P(t), P_t) + \int_{t_0}^{\theta} \mathcal{H}(t, s, P(s), P_s) ds \right] d\varsigma. \end{aligned} \tag{41}$$

Equation (41) branches into two distinct integral equations, contingent upon whether $P(t)$ is differentiable in the i-gH sense or the ii-gH sense.

1. When $P(t)$ is i-gH,

$$\begin{aligned} P(t) = &\psi(0) \oplus \frac{\varrho^{1-\varkappa}}{\Gamma(\varkappa)} \odot \int_{t_0}^{t} \varsigma^{\varrho-1}(t^{\varrho} - \varsigma^{\varrho})^{\varkappa-1} \odot \Bigg[\mathcal{F}(t, P(t), P_t) \\ &+ \int_{t_0}^{\theta} \mathcal{H}(t, s, P(s), P_s) ds \Bigg] d\varsigma. \end{aligned} \tag{42}$$

2. When $P(t)$ is ii-gH,

$$\begin{aligned} P(t) = &\psi(0) \ominus_g (-1) \frac{\varrho^{1-\varkappa}}{\Gamma(\varkappa)} \odot \int_{t_0}^{t} \varsigma^{\varrho-1}(t^{\varrho} - \varsigma^{\varrho})^{\varkappa-1} \odot \Bigg[\mathcal{F}(t, P(t), P_t) \\ &+ \int_{t_0}^{\theta} \mathcal{H}(t, s, P(s), P_s)\, ds \Bigg] d\varsigma. \end{aligned} \tag{43}$$

□

Theorem 4. *Let $\mathcal{F}$ be continuous, and let there exist $\mathcal{A}, \mathcal{B} > 0$, such that*

$$\begin{aligned} &D_H(\mathcal{F}(t, U(t), U_t(t)), \mathcal{F}(t, V(t), V_t(t))) \leq \mathcal{A} D_H(U(t), V(t)), \\ &D_H(\mathcal{H}(t, s, U(s), U_s(s)), \mathcal{H}(t, s, V(s)), V_s(s))) \leq \mathcal{B} D_H(U(s), V(s)), \end{aligned}$$

for all $U, V \in \mathcal{Q}$. Then, (41) has a unique solution for each case, provided that $\frac{\varrho^{1-\varkappa}}{\Gamma(\varkappa)}(\frac{\theta^{\varrho} - t_0^{\varrho})^{\varkappa}}{\varkappa})(\mathcal{A} + (\theta - t_0)\mathcal{B}) < 1$.

Proof. Considering the first case (42), as mentioned in Lemma 1,

$$P(t) = \psi(0) \oplus \frac{\varrho^{1-\varkappa}}{\Gamma(\varkappa)} \odot \int_{t_0}^{t} \varsigma^{\varrho-1}(t^{\varrho} - \varsigma^{\varrho})^{\varkappa-1} \odot \left[\mathcal{F}(t, P(t), P_t) + \int_{t_0}^{\theta} \mathcal{H}(t, s, P(s), P_s)ds\right]d\varsigma.$$

With this consideration, we introduce the subsequent operator on $\mathcal{Q} = C([t_0, \theta], \mathbb{F}_{\mathbb{R}})$:

$$(\mathcal{T}P)(t) = \psi(0) \oplus \frac{\varrho^{1-\varkappa}}{\Gamma(\varkappa)} \odot \int_{t_0}^{t} \varsigma^{\varrho-1}(t^{\varrho} - \varsigma^{\varrho})^{\varkappa-1} \odot G_U(\varsigma)d\varsigma,$$

where $G_U(\varsigma) = \left[\mathcal{F}(t, P(t), P_t) + \int_{t_0}^{\theta} \mathcal{H}(t, s, P(s), P_s)ds\right]$. This operator is well defined, as the expression on the right-hand side is valid for any t. Let $(\mathcal{Q}, \mathcal{N}, *)$ be a complete GFRBMLS, where $\mathcal{N} : \mathcal{Q} \times \mathcal{Q} \times [0, +\infty) \to [0, 1]$ is defined by

$$\mathcal{N}(U, V, \tau) = \exp \sup_{t \in [t_0, \theta]} \frac{-D_H(U(t), V(t))}{\tau},$$

$\Lambda : \mathcal{Q} \times \mathcal{Q} \to [1, +\infty)$ is given by $\Lambda(U, V) = 1 + \max\{|U|, |V|\}$, and $*$ denotes the product CtN.

Note that, for some $P_0(t) \in \mathcal{Q}, \tau > 0$, it holds that

$$\begin{aligned}\delta_1(\mathcal{N}, \mathcal{T}, P_0, t) &= \inf_{i,j \in \mathbb{N}} \{\mathcal{N}(\mathcal{T}^i P_0(t), \mathcal{T}^j P_0(t), \tau)\} \\ &= \inf_{i,j \in \mathbb{N}} \{\exp \sup_{t \in [t_0, \theta]} \frac{-D_H(\mathcal{T}^i P_0(t), \mathcal{T}^j P_0(t))}{\tau}\} > 0.\end{aligned}$$

We also have

$$\begin{aligned}-D_H(\mathcal{T}U(t), \mathcal{T}V(t)) = &- D_H(\psi(0) \oplus \frac{\varrho^{1-\varkappa}}{\Gamma(\varkappa)} \odot \int_{t_0}^{t} \varsigma^{\varrho-1}(t^{\varrho} - \varsigma^{\varrho})^{\varkappa-1} \odot G_U(\varsigma)d\varsigma, \\ &\psi(0) \oplus \frac{\varrho^{1-\varkappa}}{\Gamma(\varkappa)} \odot \int_{t_0}^{t} \varsigma^{\varrho-1}(t^{\varrho} - \varsigma^{\varrho})^{\varkappa-1} \odot G_V(\varsigma)d\varsigma) \\ &\geq -\frac{\varrho^{1-\varkappa}}{\Gamma(\varkappa)} (\int_{t_0}^{t} \varsigma^{\varrho-1}(t^{\varrho} - \varsigma^{\varrho})^{\varkappa-1} D_H(G_U(\varsigma), G_V(\varsigma))d\varsigma).\end{aligned}$$

We can also show that $\mathcal{T}$ is an α-contraction, as follows:

$$\begin{aligned}&\mathcal{N}(\mathcal{T}U(t), \mathcal{T}V(t), \tau) \\ &= \exp \sup_{t \in [t_0, \theta]} \frac{-D_H(\mathcal{T}U(t), \mathcal{T}V(t))}{\tau} \\ &\geq \exp \sup_{t \in [t_0, \theta]} \frac{-\frac{\varrho^{1-\varkappa}}{\Gamma(\varkappa)} (\int_{t_0}^{t} \varsigma^{\varrho-1}(t^{\varrho} - \varsigma^{\varrho})^{\varkappa-1} D_H(G_U(\varsigma), G_V(\varsigma))d\varsigma)}{\tau} \\ &\geq \exp \sup_{t \in [t_0, \theta]} \frac{-\frac{\varrho^{1-\varkappa}}{\Gamma(\varkappa)} (\int_{t_0}^{t} \varsigma^{\varrho-1}(t^{\varrho} - \varsigma^{\varrho})^{\varkappa-1} D_H(G_U(\varsigma), G_V(\varsigma))d\varsigma)}{\tau} \\ &\geq \exp \sup_{t \in [t_0, \theta]} \frac{-\frac{\varrho^{1-\varkappa}}{\Gamma(\varkappa)} (\int_{t_0}^{t} \varsigma^{\varrho-1}(t^{\varrho} - \varsigma^{\varrho})^{\varkappa-1} [\mathcal{A}D_H(U(\varsigma), V(\varsigma)) + \int_{t_0}^{\theta} \mathcal{B}D_H(U(\varsigma), V(\varsigma))d\varsigma])}{\tau} \\ &\geq \exp \sup_{t \in [t_0, \theta]} \frac{-\frac{\varrho^{1-\varkappa}}{\Gamma(\varkappa)} (\int_{t_0}^{t} \varsigma^{\varrho-1}(t^{\varrho} - \varsigma^{\varrho})^{\varkappa-1} (\mathcal{A} + (t - t_0)\mathcal{B}) D_H(U(t), V(t)))}{\tau}\end{aligned}$$

$$\geq \exp \sup_{t\in[t_0,\theta]} \frac{-\frac{\varrho^{1-\varkappa}}{\Gamma(\varkappa)}(\frac{t^{\varrho}-t_0^{\varrho})^{\varkappa}}{\varkappa})(\mathcal{A}+(t-t_0)\mathcal{B})D_H(U(t),V(t))}{\tau}$$

$$\geq \exp \sup_{t\in[t_0,\theta]} \frac{-D_H(U(t),V(t))}{\frac{\tau}{\alpha}}, \alpha = \frac{\varrho^{1-\varkappa}}{\Gamma(\varkappa)}(\frac{T^{\varrho}-t_0^{\varrho})^{\varkappa}}{\varkappa})(\mathcal{A}+(\theta-t_0)\mathcal{B}) < 1.$$

Following a similar approach, we can prove the second case stated in Lemma 1 for Equation (42). Thus, based on Corollary 4, the proof is concluded. □

5. Conclusions and Future Works

This paper presented the definitions of generalized neutrosophic rectangular b-metric-like spaces, generalized intuitionistic rectangular b-metric- spaces, and generalized fuzzy rectangular b-metric-like spaces. Various fixed point theorems were established with respect to these frameworks, accompanied by examples to substantiate the results. Our results extend and enrich the existing knowledge that is present in the literature. Furthermore, we used our findings to show that FFDIDEs possess a unique solution. Additionally, these findings could unlock new opportunities and introduce novel methods for their application across different domains, including mathematical modeling, decision making, pattern recognition, image processing, and data analysis, which are evolving. This allows researchers to refer to papers [22,23], develop more advanced predictive models, and participate in discussions about their findings. This research proposes two unsolved issues:

Problem 1. *Excluding the axioms* $\lim_{\tau\to+\infty} \mathcal{N}(\xi,\eta,\tau) = 1$, $\lim_{\tau\to+\infty} \mathcal{O}(\xi,\eta,\tau) = 0$, *and* $\lim_{\tau\to+\infty} \mathcal{L}(\xi,\eta,\tau) = 0$ *from Definition 2, we question whether we can ensure the validity of Theorems 2 and 3 by substituting the α-contraction condition with*

$$\begin{aligned}
&\frac{1}{\mathcal{N}(\mathcal{T}\xi,\mathcal{T}\eta,\tau)} - 1 \leq \alpha\Big[\frac{1}{\mathcal{N}(\xi,\eta,\tau)}\Big],\\
&\mathcal{O}(\mathcal{T}\xi,\mathcal{T}\eta,\tau) \leq \alpha\mathcal{O}(\xi,\eta,\tau),\\
&\mathcal{L}(\mathcal{T}\xi,\mathcal{T}\eta,\tau) \leq \alpha\mathcal{L}(\xi,\eta,\tau),
\end{aligned}$$

for any $\xi,\eta \in \mathcal{Q}, \tau > 0$, *and* $\alpha \in (0,1)$. *Moreover, consider replacing the α-quasi contraction condition with*

$$\begin{aligned}
&\frac{1}{\mathcal{N}(\mathcal{T}\xi,\mathcal{T}\eta,\tau)} - 1 \leq \alpha \max\{\frac{1}{\mathcal{N}(\xi,\eta,\tau)} - 1, \frac{1}{\mathcal{N}(\xi,\mathcal{T}\eta,\tau)} - 1, \frac{1}{\mathcal{N}(\eta,\mathcal{T}\xi,\tau)} - 1\},\\
&\mathcal{O}(\mathcal{T}\xi,\mathcal{T}\eta,\tau) \leq \alpha \max\{\mathcal{O}(\xi,\eta,\tau), \mathcal{O}(\xi,\mathcal{T}\eta,\tau), \mathcal{O}(\eta,\mathcal{T}\xi,\tau)\},\\
&\mathcal{L}(\mathcal{T}\xi,\mathcal{T}\eta,\tau) \leq \alpha \max\{\mathcal{L}(\xi,\eta,\tau), \mathcal{L}(\xi,\mathcal{T}\eta,\tau), \mathcal{L}(\eta,\mathcal{T}\xi,\tau)\},
\end{aligned}$$

for any $\xi,\eta \in \mathcal{Q}, \tau > 0$, *and* $\alpha \in (0,1)$.

Problem 2. *How can the results in this paper be generalized in the context of extended rectangular graphical neutrosophic b-metric spaces?*

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Abbreviations

The following abbreviations are used in this manuscript:

GNRBMLS	generalized neutrosophic rectangular b-metric-like space
GIRBMLS	generalized intuitionistic rectangular b-metric-like space
GFRBMLS	generalized fuzzy rectangular b-metric-like space
OBCFFD	Odibat–Baleanu–Caputo fuzzy fractional derivative
CtN	continuous triangular norm
CtCN	continuous t-conorm
FS	fuzzy set
FP	fixed point
BCT	Banach contraction theorem
IFS	intuitionistic fuzzy set
NMS	neutrosophic metric space
FRBMLS	fuzzy rectangular b-metric-like space
IFRBMLS	intuitionistic fuzzy rectangular b-metric-like space
NRBMLS	neutrosophic rectangular b-metric-like space
ENRBMLS	neutrosophic rectangular extended b- metric-like space
CQT	Ćirić quasi-contraction theorem
FFDIDE	fuzzy fractional delay integro-differential equation

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