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Parametric Expansions of an Algebraic Variety near Its Singularities II

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Abstract: The paper is a continuation and completion of the paper Bruno, A.D.; Azimov, A.A. Parametric Expansions of an Algebraic Variety Near Its Singularities. *Axioms* **2023**, *5*, 469, where we calculated parametric expansions of the three-dimensional algebraic manifold Ω, which appeared in theoretical physics, near its 3 singular points and near its one line of singular points. For that we used algorithms of Nonlinear Analysis: extraction of truncated polynomials, using the Newton polyhedron, their power transformations and Formal Generalized Implicit Function Theorem. Here we calculate parametric expansions of the manifold Ω near its one more singular point, near two curves of singular points and near infinity. Here we use 3 new things: (1) computation in algebraic extension of the field of rational numbers, (2) expansions near a curve of singular points and (3) calculation of branches near infinity.

Keywords: algebraic variety; singular point; local parametrization; power geometry

MSC: 41A60

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1. Introduction

Here we continue and conclude the paper [1]. There, in Sections 1–5, we proposed a new method for solving the polynomial equation

$$f(x_1, \dots, x_n) = 0$$

near a singular point or curve of singular points of the polynomial f. In Sections 6–10 , this method was applied to compute the solutions to such a 12th degree equation with $n = 3$ that originated in theoretical physics. This new method is based on:

I. *Newton's polyhedron* to isolate the truncated equations,
II. *Power transformations* to simplify those equations, and
III. *Formal Generalized Implicit Function Theorem* to obtain solutions in the form of power expansions whose coefficients are rational functions of the parameters. Computer algebra is used in these calculations.

Newton's polyhedron is a multidimensional generalization of Newton's polygon (see [2–7]). Power transformations are a generalization of the sigma process used previously to resolve singularities of algebraic manifolds (see [8–10]). Algorithms for computing power transformations were proposed in [11]. The resolution of the singularity is done step-by-step until we come to the situation with a truncated equation containing a polynomial multiplier of degree one. If the roots of this multiplier are parameterized, the roots of the whole polynomial are obtained as a power expansion using a Generalized Implicit Function Theorem (Theorem 1 of [1]). All these are an application to algebraic equations of the general theory of Nonlinear Analysis [12], which is also suitable for differential equations. For its applications to systems of partial derivative equations, see [13].

According to [1] and [12] (Section 2) computational steps are the following:

Step 1. Introduction of local coordinates. For coordinates $X = (x_1, \dots, x_n)$ and singular point $X^0 = (x_1^0, \dots, x_n^0)$, they are $Y = X - X^0$, i.e., $y_i = x_i - x_i^0$, $i = 1, \dots, n$.

Step 2. Writing the initial polynomial $f(X)$ in local coordinates

$$g(Y) = f(X^0 + Y) = \sum g_Q Y^Q \text{ over } Q \in \mathbf{S}. \tag{1}$$

Here $Q = (q_1, \dots, q_n)$, $Y^Q = y_1^{q_1} \cdots y_n^{q_n}$, g_Q are real or complex coefficients, the sum has not similar terms, the set $\mathbf{S}(g) = \{Q : a_Q \not\equiv 0\}$, is called as *support* of the sum $g(Y)$. Here $0 \leqslant Q \in \mathbb{Z}^n$. Let the support $\mathbf{S}(g)$ consists of vectors $Q_1, \dots, Q_k$.

Step 3. The Newton polyhedron $\Gamma(g)$ is computating as the convex hull of the support $\mathbf{S}$:

$$\Gamma(g) = \left\{ Q = \sum_{i=1}^{k} \lambda_i Q_i,\ \lambda_i \geqslant 0,\ i = 1, \dots, k,\ \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

The boundary $\partial\Gamma$ of the polyhedron $\Gamma(g)$ consists from its generalized faces $\Gamma_j^{(d)}$, where d is dimension, $0 \leqslant d \leqslant n-1$, and j is a number of the face $\Gamma_j^{(d)}$. Each face $\Gamma_j^{(d)}$ corresponds to its truncated polynomial

$$\hat{g}_j^{(d)}(Y) = \sum g_Q Y^Q \text{ over } Q \in \mathbf{S} \cap \Gamma_j^{(d)}$$

and the normal cone $\mathbf{U}_j^{(d)}$, consisting of all normals to the face $\Gamma_j^{(d)}$, which are external to the polyhedron Γ. For their computation we use the `PolyhedralSets` package of the computer algebra system (CAS) `Maple`. In the steps below $n = 3$. Then $\Gamma_j^{(2)}$ is two dimensional face and normal cone $\mathbf{U}_j^{(2)}$ is a ray, spanned by external normal N_j to the face $\Gamma_j^{(2)}$.

Step 4. We select faces $\Gamma_j^{(2)}$ with normal $N_j \leqslant 0$ and corresponding truncated polynomials $\hat{g}_j^{(2)}(Y)$.

Step 5. For each selected truncated polynomial $\hat{g}_j^{(2)}(Y)$, we compute corresponding power transformation

$$(\ln y_1, \ln y_2, \ln y_3) = (\ln z_1, \ln z_2, \ln z_3)\alpha, \tag{2}$$

where α is an unimodular 3×3 matrix, such that

$$\hat{g}_j^{(2)}(Y) = h(z_1, z_2) z_3^l \tag{3}$$

with integral l.

Step 6. If the curve $h(z_1, z_2) = 0$ has parametrization

$$z_1 = b_1(t), \quad z_2 = b_2(t),$$

then it is obtained with the `algcurves` package from the CAS `Maple`. In that case we make the power transformation (2) in the full polynomial (1) and write it as

$$g(Y) = T(z_1, z_2, z_3) z_3^l = z_3^l \sum_{k=0}^{m} T_k(z_1, z_2) z_3^k,$$

with some natural m. Here polynomials $T_k(z_1, z_2)$ are computed by the command `coeff(T,z[k],m)` in CAS `Maple`. Here $T_0(z_1, z_2) = h(z_1, z_2)$ from (3).

Step 7. If $T_1(b_1(t), b_2(t)) \not\equiv 0$, we make the substitution

$$z_1 = b_1(t) + \varepsilon, \quad z_2 = b_2(t) + \varepsilon \tag{4}$$

into the polynomial $T(z_1, z_2, z_3)$, obtain function $u(\varepsilon, t, z_3) = T(z_1, z_2, z_3)$, apply to the equation $u(\varepsilon, t, z_3) = 0$ the Formal Generalized Implicit Function Theorem 1 [1] and get the parametric expansion

$$\varepsilon = \sum_{k=1}^{\infty} c_k(t) z_3^k. \tag{5}$$

Step 8. We compute several terms of expansion (5), substitute them into (4). The result is substituted in power transformation (2), and we obtain parametric expansion of Y in power series of z_3 with coefficients which are rational functions of t.
If $T_1(b_1(t), b_2(t)) \equiv 0$, we continue computation with new Newton polyhedron etc.

The method is new, with parts: the Newton polyhedron $\Gamma(g)$, polyhedron's faces $\Gamma_j^{(d)}$, polyhedron graph, normal cones $\mathbf{U}_j^{(d)}$ and power transformations (2) were proposed by the first author beginning 1962. Early such objects he calculated manually, but now there are programs for that.

In [1], this theory was applied to a problem arises in the study of Ricci flows (see [14–22]). The Ricci flows describe the evolution of Einstein's metrics on a variety. The equations of the normalized Ricci flow are reduced to a system of two differential equations with three parameters: a_1, a_2 and a_3:

$$\begin{aligned} \frac{dx_1}{dt} &= \tilde{f}_1(x_1, x_2, a_1, a_2, a_3), \\ \frac{dx_2}{dt} &= \tilde{f}_2(x_1, x_2, a_1, a_2, a_3), \end{aligned} \tag{6}$$

here, $\tilde{f}_1$ and $\tilde{f}_2$ are certain given functions.

The singular points of this system are associated with the invariant Einstein's metrics. At the singular (stationary) point x_1^0, x_2^0, system (6) has two eigenvalues, λ_1 and λ_2. If at least one of them is equal to zero, then the singular (stationary) point x_1^0, x_2^0 is said to be *degenerate*. It was proved in [14–22] that the set Ω of the values of the parameters a_1, a_2, a_3, in which system (6) has at least one degenerate singular point, is described by all solutions of the equation

$$\begin{aligned} Q(s_1, s_2, s_3) \stackrel{\text{def}}{\equiv} &(2s_1 + 4s_3 - 1)\Big(64s_1^5 - 64s_1^4 + 8s_1^3 + 240s_1^2 s_3 - 1536 s_1 s_3^2 - \\ &-4096 s_3^3 + 12 s_1^2 - 240 s_1 s_3 + 768 s_3^2 - 6s_1 + 60 s_3 + 1\Big) - \\ &- 8 s_1 s_2 (2s_1 + 4s_3 - 1)(2s_1 - 32 s_3 - 1)(10 s_1 + 32 s_3 - 5) - \\ &- 16 s_1^2 s_2^2 \Big(52 s_1^2 + 640 s_1 s_3 + 1024 s_3^2 - 52 s_1 - 320 s_3 + 13\Big) + \\ &+ 64(2s_1 - 1) s_2^3 (2s_1 - 32 s_3 - 1) + 2048 s_1 (2s_1 - 1) s_2^4 = 0, \end{aligned}$$

where s_1, s_2, s_3 are elementary symmetric polynomials, equal, respectively, to

$$s_1 = a_1 + a_2 + a_3, \quad s_2 = a_1 a_2 + a_1 a_3 + a_2 a_3, \quad s_3 = a_1 a_2 a_3.$$

Here Q is different from Q's in Steps 2 and 3, but the sign $\stackrel{\text{def}}{\equiv}$ means only a new notation.

Hence the polynomial $P(a_1, a_2, a_3) = Q(s_1, s_2, s_3)$ has degree 12. In [23], for symmetry reasons, the coordinates $\mathbf{a} = (a_1, a_2, a_3)$ were changed to the coordinates $\mathbf{A} = (A_1, A_2, A_3)$ by the linear transformation

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = M \cdot \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}, \quad M = \begin{pmatrix} (1+\sqrt{3})/6 & (1-\sqrt{3})/6 & 1/3 \\ (1-\sqrt{3})/6 & (1+\sqrt{3})/6 & 1/3 \\ -1/3 & -1/3 & 1/3 \end{pmatrix}$$

The resulting polynomial is

$$R(\mathbf{A}) = P(\mathbf{a}) \tag{7}$$

and has degree 12 again.

Definition 1. *Let $\varphi(X)$ be some polynomial, where $X = (x_1, \dots, x_n)$. A point $X = X^0$ of the set $\varphi(X) = 0$ is called the **singular point of the** k-**order**, if all partial derivatives of the polynomial $\varphi(X)$ with respect to $x_1, \dots, x_n$ turn into zero at this point, up to and including the k-th order derivatives, and at least one partial derivative of order $k+1$ is nonzero.*

In [23], all singular points of the variety Ω in coordinates $\mathbf{A} = (A_1, A_2, A_3)$ were found. The five points of the third order are:

Name	Coordinates A
$P_1^{(3)}$	$(0, 0, 3/4)$
$P_2^{(3)}$	$(0, 0, -3/2)$
$P_3^{(3)}$	$\left(-\frac{1+\sqrt{3}}{2}, \frac{\sqrt{3}-1}{2}, \frac{1}{2}\right)$
$P_4^{(3)}$	$\left(\frac{\sqrt{3}-1}{2}, -\frac{1+\sqrt{3}}{2}, \frac{1}{2}\right)$
$P_5^{(3)}$	$(1, 1, 1/2)$

three points of the second order

Name	Coordinates A
$P_1^{(2)}$	$\left(\frac{1+\sqrt{3}}{4}, \frac{1-\sqrt{3}}{4}, \frac{1}{2}\right)$
$P_2^{(2)}$	$\left(\frac{1-\sqrt{3}}{4}, \frac{1+\sqrt{3}}{4}, \frac{1}{2}\right)$
$P_3^{(2)}$	$(-1/2, -1/2, 1/2)$

and three more algebraic curves of singular points of the first order:

$$\mathcal{F} = \left\{ a_1 = a_2, \quad 16a_1^3 + 16a_1^2 a_3 - 4a_1 - 2a_3 + 1 = 0 \right\},$$

$$\mathcal{I} = \left\{ A_1 + A_2 + 1 = 0, \quad A_3 = \frac{1}{2} \right\},$$

$$\mathcal{K} = \left\{ A_1 = -\frac{9}{4} t h(t),\ A_2 = -\frac{9}{4} h(t),\ A_3 = \frac{3}{4},\ h(t) = \frac{t^2+1}{(t+1)(t^2-4t+1)} \right\}.$$

The points $P_3^{(3)}$, $P_4^{(3)}$ and $P_5^{(3)}$ are of the same type; they pass into each other when rotated in the plane A_1, A_2 by an angle $2\pi/3$, just as all points $P_1^{(2)}$, $P_2^{(2)}$, $P_3^{(2)}$. The curves $\mathcal{F}, \mathcal{I}, \mathcal{K}$ correspond to two more curves of the same type. Therefore, it is sufficient to study the variety Ω in the neighborhood of points $P_1^{(3)}$, $P_2^{(3)}$, $P_5^{(3)}$, $P_3^{(2)}$ and curves $\mathcal{F}$, $\mathcal{I}$ and $\mathcal{K}$. Moreover, in [23] there were computed sections of the variety Ω by planes $A_3 = \text{const}$, and was shown that in finite part of the space $\mathbb{R}^3 = \{A_1, A_2, A_3\}$ the variety Ω consists of two dimensional branches F_1, F_2, F_3, G_1, G_2, G_3 divided into parts $F_i^{\pm}$, $G_i^{\pm}$ with boundaries at the plane $A_3 = 1/2$.

In the paper [24], three variants of the global parametrization of the variety Ω were proposed. These parametrizations were computed using the parametric description of the discriminant set of a monic cubic polynomial [25] and can be written in radical form [26]. Such a global description of the variety Ω cannot provide an adequate picture of the Ω structure in the vicinity of its singular points.

In [1], parametric expansions of the variety Ω near the singular points $P_1^{(3)}$ (Section 7), $P_1^{(3)}$ (Section 8), $P_2^{(2)}$ (Section 8), $P_3^{(2)}$ (Section 10) and near the line of singular points $\mathfrak{I}$ (Section 9) were computed. Here these expansions are computed near the singular point $P_5^{(3)}$ (Section 2), near the curves of singular points $\mathcal{H}$ (Section 3) and $\mathcal{F}$ (Section 4), and near infinity (Section 5). Together they cover a wide range of cases. The following tactic has developed: if the truncated equation contains linear multipliers, they are used to do a linear transformation of the coordinates followed by the computation of Newton's polyhedron; and if they are nonlinear, a power transformation of the coordinates is done. To understand the present article it is necessary a knowledge with papers [1] (open access), [23] and the book [27].

2. The Structure of the Manifold Ω near the Singular Point $P_5^{(3)}$

2.1. Preliminary Computations

Near the point $P_5^{(3)}: (A_1, A_2, A_3) = (1, 1, 1/2)$ we introduce local coordinates x_1, x_2, x_3:

$$A_1 = x_1 + 1, \quad A_2 = x_2 + 1, \quad A_3 = \frac{1}{2} + x_3. \tag{8}$$

Then, from the polynomial $R(\mathbf{A})$ in (7) we get a polynomial of degree 12.

$$S_4(x_1, x_2, x_3) = R(\mathbf{A}) = Q(s_1, s_2, s_3).$$

We compute the support $\mathbf{S}$ of the polynomial S_4, the Newton polyhedron $\Gamma_4(S_4)$, its faces $\Gamma_j^{(2)}$ and their external normals using the PolyhedralSets package of the computer algebra system (CAS) Maple 2021 [27]. We obtain 5 faces $\Gamma_j^{(2)}$. The graph of the polyhedron Γ_4 is shown in Figure 1.

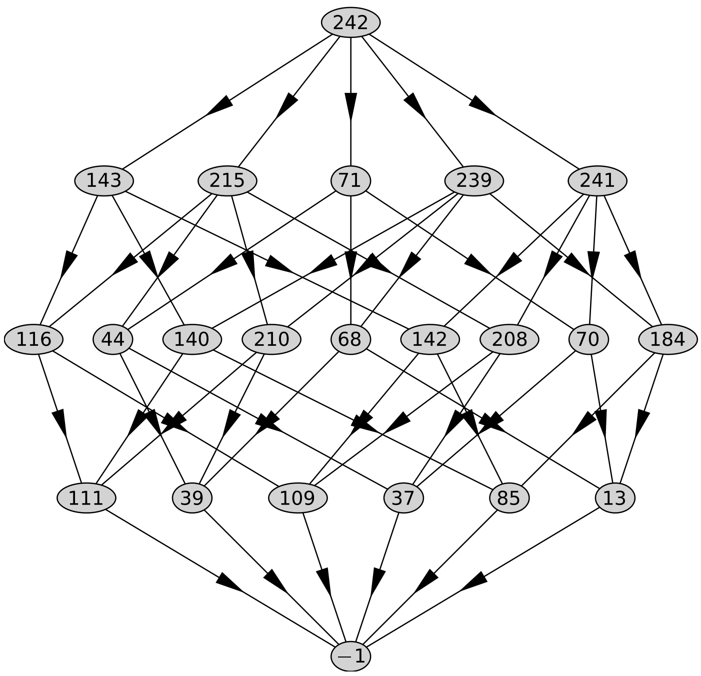

Figure 1. Graph of the polyhedron Γ_4.

In the top row—the whole polyhedron, in the next—all two-dimensional faces. Further down are the edges, then the vertices, and at the bottom is the empty set. The external normals to its two-dimensional faces are

$$N_{71} = (-1,-1,-1), N_{143} = (1,1,1), N_{215} = (-1,0,0), N_{239} = (0,-1,0), N_{241} = (0,0,-1).$$

Since $x_1,\ x_2,\ x_3 \longrightarrow 0$, we take the only normal that has all coordinates negative. This is $N_{71} = (-1,-1,-1)$. It corresponds to a truncated polynomial

$$\hat{f}_{71} = -16\left(2x_1^2 - 8x_1x_2 + 2x_1x_3 + 2x_2^2 + 2x_2x_3 - x_3^2\right)^2/81. \tag{9}$$

The quadratic polynomial bracketed in (9) does not factorize in the field of rational numbers, but it does factorize in the extension of this field with $\sqrt{3}$. We get

$$f = 2x_1^2 - 8x_1x_2 + 2x_1x_3 + 2x_2^2 + 2x_2x_3 - x_3^2.$$

We proceed according to [27]:

```
>alpha := RootOf(y^2-3);
>factor (f,alpha);
```

$$-\tfrac{1}{2}((2x_2 RootOf(Z^2-3) - x_3 RootOf(Z^2-3) + 2x_1 - 4x_2 + x_3)\cdot$$
$$\cdot(2x_2 RootOf(Z^2-3) - x_3 RootOf(Z^2-3) - 2x_1 + 4x_2 - x_3))$$

i.e.,

$$f = -\frac{1}{2}\left(2x_1 - \left(4-2\sqrt{3}\right)x_2 + \left(1-\sqrt{3}\right)x_3\right)\cdot\left(2x_1 - \left(4+2\sqrt{3}\right)x_2 + \left(1+\sqrt{3}\right)x_3\right).$$

Now we do a linear substitution of the coordinates

$$\begin{aligned} y_1 &= 2x_1 - \left(4-2\sqrt{3}\right)x_2 + \left(1-\sqrt{3}\right)x_3, \\ y_2 &= 2x_1 - \left(4+2\sqrt{3}\right)x_2 + \left(1+\sqrt{3}\right)x_3, \\ y_3 &= x_3. \end{aligned}$$

Its inverse substitution is

$$\begin{aligned} x_1 &= \frac{\left(2+\sqrt{3}\right)\sqrt{3}}{12}y_1 + \frac{\left(-2+\sqrt{3}\right)\sqrt{3}}{12}y_2 + \frac{1}{2}y_3, \\ x_2 &= \frac{\sqrt{3}}{12}y_1 - \frac{\sqrt{3}}{12}y_2 + \frac{1}{2}y_3, \\ x_3 &= y_3. \end{aligned} \tag{10}$$

We substitute it into the polynomial $S_4(\mathbf{x})$ and get the polynomial $S_5(\mathbf{y}) = S_4(\mathbf{x})$. For the polynomial $S_5(\mathbf{y})$, we compute Newton's polyhedron Γ_5. Its graph is shown in Figure 2. It has 11 two-dimensional faces with external normals

$$\begin{gathered} N_{14397} = (-1,-1,0),\ N_{15959} = (-1,0,-1),\ N_{19269} = (-2,-2,-1), \\ N_{39917} = (0,-1,-1),\ N_{111761} = (1,1,1),\ N_{131145} = (-1,0,0),\ N_{132735} = (0,0,1), \\ N_{135677} = (0,1,0),\ N_{137855} = (1,0,0),\ N_{159459} = (0,-1,0),\ N_{162019} = (0,0,-1). \end{gathered}$$

We parse the first 4 of them that have 2 or 3 coordinates negative, dedicating a subsection to each of them. Below we use notation from Maple [27].

2.2. The Normal $(-2,-2,-1)$

The corresponding truncated polynomial is

$$ftr19269 = -104976(5+3\alpha)(864\alpha y_1 y_3^6 - 3456\alpha y_2 y_3^6 + 117\alpha y_1^2 y_3^4 - 594\alpha y_1 y_2 y_3^4$$
$$+ 1287\alpha y_2^2 y_3^4 - 1728 y_1 y_3^6 + 6048 y_2 y_3^6 - 12\alpha y_1^2 y_2 y_3^2 + 48\alpha y_1 y_2^2 y_3^2 - 360\alpha y_2^3 y_3^2 - 117 y_1^2 y_3^4$$
$$+ 990 y_1 y_2 y_3^4 - 2223 y_2^2 y_3^4 + 6\alpha y_1^2 y_2^2 - 24 y_1^3 y_3^2 + 24 y_1^2 y_2 y_3^2 - 84 y_1 y_2^2 y_3^2 + 624 y_2^3 y_3^2 - 10 y_1^2 y_2^2),$$

where $\alpha = \sqrt{3}$. Here and below "ftr" number means truncated polynomial, corresponding to normal N_j with written number j. According to [11], we compute the matrix $\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{pmatrix}$ such that $(-2,-2,-1)\gamma = (0,0,-1)$. Since $\gamma^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$, then we do a power transformation

$$y_1 = z_1 z_3^2,\ y_2 = z_2 z_3^2,\ y_3 = z_3. \tag{11}$$

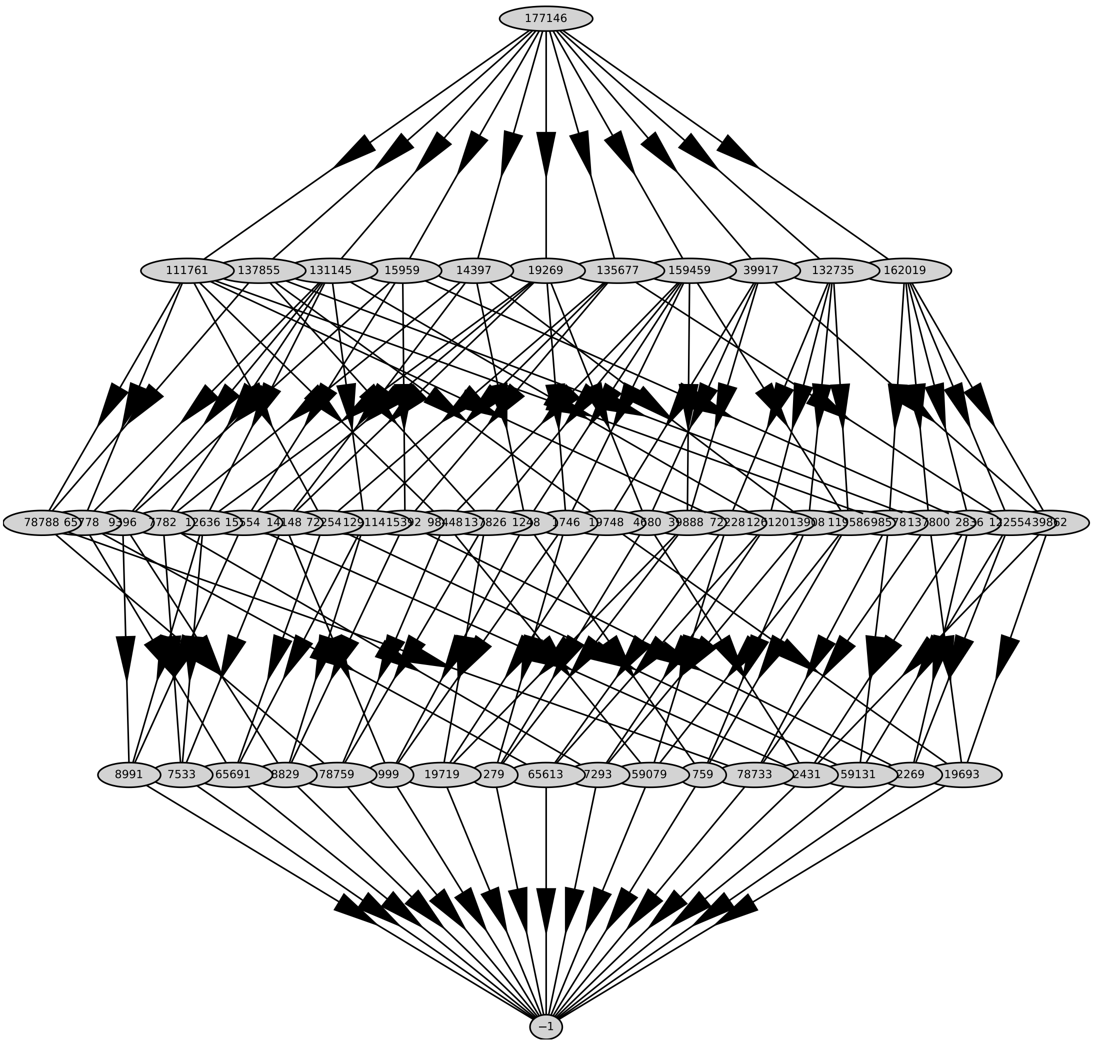

Figure 2. Graph of the polyhedron Γ_5.

We get after factorization

$$ftr19269 = -104976(5+3\alpha)(6\alpha z_1^2 z_2^2 - 12\alpha z_1^2 z_2 + 48\alpha z_1 z_2^2 - 360\alpha z_2^3 - 10 z_1^2 z_1^2 z_2^2$$
$$+117\alpha z_1^2 - 594\alpha z_1 z_2 + 1287 z_2^2 \alpha - 24 z_1^3 + 24 z_1^2 z_2 - 84 z_1 z_2^2 + 624 z_2^3 + 864\alpha z_1$$

$$-3456\alpha z_2 - 117z_1^2 + 990z_1z_2 - 2223z_2^2 - 1728z_1 + 6048z_2)z_3^8,$$

where $\alpha = \sqrt{3}$. The large polynomial in the parentheses is denoted by

$$f_1(z_1, z_2) = 6\alpha z_1^2 z_2^2 - 12\alpha z_1^2 z_2 + 48\alpha z_1 z_2^2 - 360\alpha z_2^3 - 10z_1^2 z_1^2 z_2^2$$
$$+117\alpha z_1^2 - 594\alpha z_1 z_2 + 1287 z_2^2 \alpha - 24z_1^3 + 24z_1^2 z_2 - 84z_1 z_2^2 + 624z_2^3 + 864\alpha z_1$$
$$-3456\alpha z_2 - 117z_1^2 + 990z_1z_2 - 2223z_2^2 - 1728z_1 + 6048z_2.$$

Consider the curve $f_1(z_1, z_2) = 0$. It has intersections with the axes: $z_1 = 0,\ z_2 = 0$. It is a curve of genus 0, with one singular point

$$(z_1, z_2) = \left(-6 + 6\sqrt{3}; -6 - 6\sqrt{3}\right) \approx (4.39230485;\ -16.39230485)$$

and parameterization

$$z_1 = b_1(t) = \frac{\left(5 + 3\sqrt{3}\right)\beta\left(2042820\sqrt{3}t - 521639194050t^2 + 64\sqrt{3} - 7660575t - 119\right)}{2(510705t + 2)^2},$$
$$z_2 = b_2(t) = \frac{\left(153\sqrt{3} + 265\right)\beta\left(2042820\sqrt{3}t - 260819597025t^2 + 120\sqrt{3} - 5617755t - 212\right)}{2(510705t + 2)},$$
$$\beta = 3\left(4\sqrt{3} + 510705t - 5\right).$$

We simplify these expressions by transforming $t = t_1/510705$ and obtaining

$$z_1 = b_1(t_1) = \frac{3\left(5 + 3\sqrt{3}\right)\left(4\sqrt{3}t_1 - 2t_1^2 + 64\sqrt{3} - 15t_1 - 119\right)\left(4\sqrt{3} + t_1 - 5\right)}{2(t + 2)^2},$$
$$z_2 = b_2(t_1) = \frac{3\left(265 + 153\sqrt{3}\right)\left(4\sqrt{3}t_1 - t_1^2 + 120\sqrt{3} - 11t_1 - 212\right)\left(4\sqrt{3} + t_1 - 5\right)}{2(t_1 + 2)}. \quad (12)$$

The curve is shown in Figure 3.

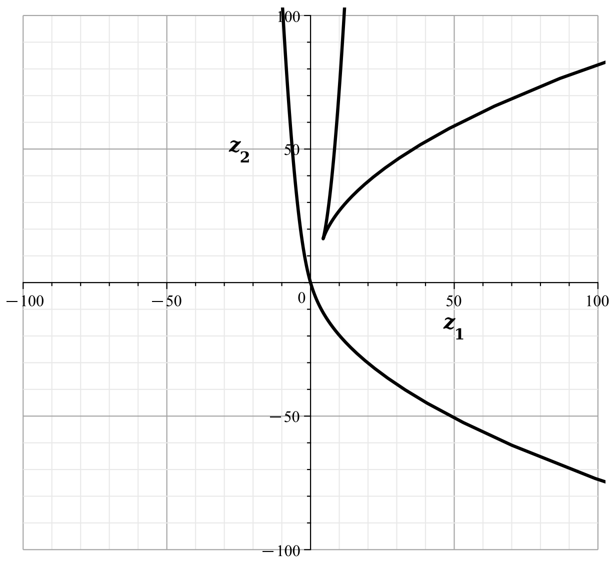

Figure 3. Curve $f_1(z_1, z_2) = 0$.

With z_3 fixed according to (8), (10) and (11) at the original coordinates **A**, we obtain a curve

$$A_1 = 1 + \frac{(2+\sqrt{3})\sqrt{3}}{12} z_1 z_3^2 + \frac{(-2+\sqrt{3})\sqrt{3}}{12} z_2 z_3^2 + \frac{1}{2} z_3,$$
$$A_2 = 1 + \frac{\sqrt{3}}{12} z_1 z_3^2 - \frac{\sqrt{3}}{12} z_2 z_3^2 + \frac{1}{2} z_3, \quad (13)$$

where $z_1 = b_1(t_1)$, $z_2 = b_2(t_1)$ according to (12).

If $z_3 = -0.05$, $z_1 = b_1(t_1)$, $z_2 = b_2(t_1)$ then according to (8), (10) and (13) $A_3 = 9/20 = 0.45$,

$$A_1 = \frac{(2+\sqrt{3})\sqrt{3}}{4800} z_1 + \frac{(-2+\sqrt{3})\sqrt{3}}{4800} z_2 + \frac{39}{40},$$
$$A_2 = \frac{\sqrt{3}}{4800} z_1 - \frac{\sqrt{3}}{4800} z_2 + \frac{39}{40}. \quad (14)$$

This curve is shown in Figure 4. This curve is similar to the curve of [23] (Figure 12) showing the cross-section of the variety Ω at $A_3 = 0.45$, with branches F_1^- and G_2^-.

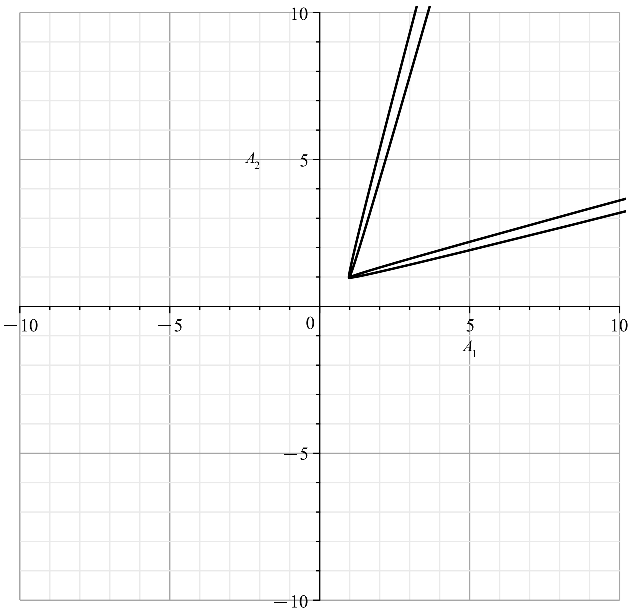

Figure 4. Curve (14).

If $z_3 = 0.005$, then $A_3 = 101/200 = 0.505$, hence according to (13)

$$A_1 = \frac{(2+\sqrt{3})\sqrt{3}}{480000} z_1 + \frac{(-2+\sqrt{3})\sqrt{3}}{480000} z_2 + \frac{401}{400},$$
$$A_2 = \frac{\sqrt{3}}{480000} z_1 - \frac{\sqrt{3}}{480000} z_2 + \frac{401}{400}. \quad (15)$$

When $z_1 = b_1(t_1)$, $z_2 = b_2(t_1)$, the curve (15) is shown in Figure 5. It is similar to Figure 15 in [23], which shows the section of the variety Ω at $A_3 = 0.505$, with coinsiding branches F_1^+ and G_2^+.

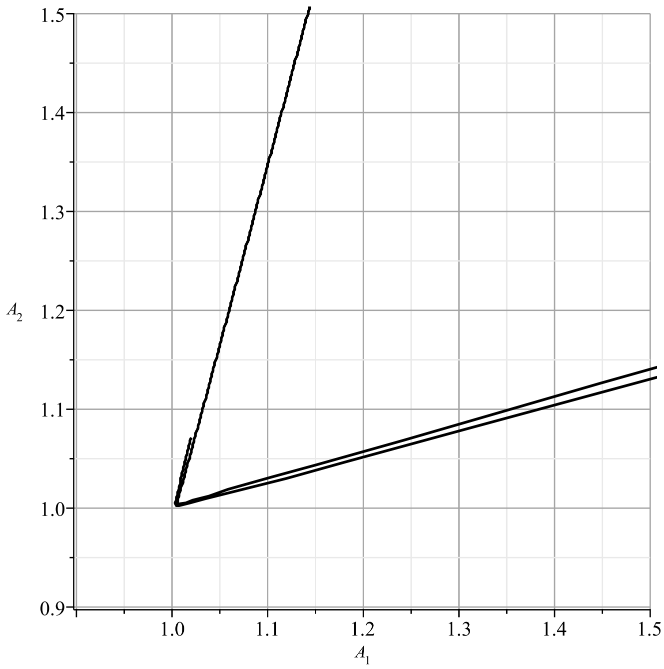

Figure 5. Curve (15).

In fact, the parametric expansion of the variety Ω can also be computed here. To do this, we substitute (11) into the polynomial $S_5(\mathbf{y})$ and get the polynomial of Step 6

$$T(\mathbf{z}) = z_3^8 \sum_{k=0}^{m} T_k(z_1, z_2) z_3^k, \tag{16}$$

where polynomials $T_k(z_1, z_2)$ are uniquely determined and can be computed by the command `coeff(T,z[k],m)`.

In it, according to (12), we make the substitution

$$z_1 = b_1(t) + \varepsilon, \quad z_2 = b_2(t) + \varepsilon. \tag{17}$$

We obtain that $T(\mathbf{z})/z_3^8 = u(\varepsilon, z_3)$ with coefficients depending on t via $b_1(t)$ and $b_2(t)$. We apply Theorem 1 in [1] to the equation $u(\varepsilon, z_3) = 0$ and obtain the expansion

$$\varepsilon = \sum_{k=1}^{\infty} c_k(t) z_3^k.$$

Returning to initial coordinates $\mathbf{A}$ via (8), (10)–(12) and (17), we obtain expansions

$$\begin{aligned} A_1 &= 1 + \frac{1}{2} z_3 + \frac{2\sqrt{3}+3}{12}\left(b_1(t_1) + \sum_{k=1}^{\infty} c_k(t_1) z_3^k\right) z_3^2 + \frac{3-2\sqrt{3}}{12}\left(b_2(t_1) + \sum_{k=1}^{\infty} c_k(t_1) z_3^k\right) z_3^2, \\ A_2 &= 1 + \frac{1}{2} z_3 + \frac{\sqrt{3}}{12}\left(b_1(t_1) + \sum_{k=1}^{\infty} c_k(t_1) z_3^k\right) z_3^2 - \frac{\sqrt{3}}{12}\left(b_2(t_1) + \sum_{k=1}^{\infty} c_k(t_1) z_3^k\right) z_3^2, \\ A_3 &= \frac{1}{2} + z_3 \end{aligned} \tag{18}$$

with parameters $t_1 \in \mathbb{R}$ and $z_3 \in \mathbb{R}$ for small $|z_3|$. Formula (13) contain first terms of expansions (18).

2.3. The Normal $(-1,-1,0)$

It corresponds to a truncated polynomial

$$ftr14397 = -11337408(1+\alpha)(\alpha y_2 + y_1 - 2y_2)(4y_3 - 1)(y_3+2)^3 y_3^6. \qquad (19)$$

Since it is linear on y_1, y_2, its root is the y_3-axis, i.e., $y_1 = y_2 = 0$, which we denote by N. This line N lies on the variety Ω and through N it passes one of its branches, which in the first approximation has the form $y_1 = (2-\alpha)y_2$.

According to (8) and (9) in the original coordinates $\mathbf{A}$, this line N has the following form

$$A_1 = \frac{1}{2}y_3 + 1,\ A_2 = \frac{1}{2}y_3 + 1,\ A_3 = \frac{1}{2} + y_3.$$

This is the straight line g_2 of [23] (Figure 3). In [23] (Figures 4–15), the points of the line N lie in the plane $M: A_1 = A_2$. Moreover, in Figure 6 $A_1 \in (-1,0)$, in Figures 8–11 $A_1 \in (0,1)$, in Figures 4, 13 and 14 $A_1 \in (1,2)$. According to (19), there are 3 singular points on the line N

$$y_3 = 0,\ y_3 = -2,\ y_3 = 1/4. \qquad (20)$$

In $\mathbf{A}$ coordinates, they look like this:

$$(1,1,1/2),\ (0,0,\ -3/2),\ (9/8,9/8,\ 3/4),$$

i.e., they are points $P_5^{(3)}$, $P_2^{(3)}$, and a point with $t=1$ on the curve $\mathcal{H}$ of singular points. The structure of the variety Ω near the point $P_5^{(3)}$ is dealt with in this Section, near the point $P_2^{(3)}$ was dealt with in Section 7 of [1], and in the next Section we will deal with the structure of the variety Ω near the curve $\mathcal{H}$.

We can obtain an expansion of the variety Ω near the line N. To do this, we substitute (10) in the polynomial $S_4(\mathbf{x})$ and obtain the polynomial $V(y_1,y_2,y_3) = S_4(\mathbf{x})$, which we write as

$$V(y_1,y_2,y_3) = \sum V_{q_1q_2}(y_3)y_1^{q_1}y_2^{q_2},$$

where $0 \le q_1,\ q_2 \in \mathbb{Z}$, $V_{q_1q_2}(y_3)-$ polynomials. Thus according to (19)

$$V_{00} = 0,$$

$$V_{10} = -11337408(1+\alpha)(4y_3-1)(y_3+2)^3 y_3^6,$$

$$V_{01} = -11337408(1+\alpha)(\alpha-2)(4y_3-1)(y_3+2)^3 y_3^6.$$

According to [1] (Theorem 1), the equation $V=0$ has a solution

$$y_1 = (2-\alpha)y_2 + \sum_{k=2}^{\infty} c_k(y_3)y_2^k.$$

Going to the original coordinates, we get the expansion for $\mathbf{A}$ with parameters y_2 and y_3. It is valid everywhere except the neighborhoods of the points (20).

2.4. The Normal $(-1,0,-1)$

It corresponds to a truncated polynomial

$$ftr15959 = -18(-26+15\alpha)(-y_2+6+6\alpha)^2(-y_2+3+3\alpha)^2 \cdot (-18y_2^2y_3^2\alpha + 6\alpha y_1^2 y_2$$
$$-108y_2y_3^2\alpha - y_1^2y_2^2 + 36y_2^2y_3^2 - 18\alpha y_1^2 + 6y_1^2y_2 + 108y_2y_3^2 - 36y_1^2) \cdot y_2^2.$$

Let's put

$$f_2 = -18y_2^2y_3^2\alpha + 6\alpha y_1^2y_2 - 108y_2y^2y_3^2\alpha - y_1^2y_2^2 + 36y_2^2y^2y_3^2 - 18\alpha y_1^2 + 6y_1^2y_2 + 108y_2y_3^2 - 36y_1^2.$$

According to [11], we compute the unimodular matrix $\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ such that $(-1,0,-1)\gamma = (0,0,-1)$. Since $\gamma^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, then we do a power transformation

$$y_1 = z_1z_3, \quad y_2 = z_2, \quad y_3 = z_3. \tag{21}$$

We get

$$f_2(z_1,z_2,z_3) = \left(6\alpha z_1^2z_2 - z_1^2z_2^2 - 18\alpha z_1^2 - 18\alpha z_2^2 + 6z_1^2z_2 - 108z_2\alpha - 36z_1^2 + 36z_2^2 + 108z_2\right)z_3^2$$

Let's denote

$$\begin{gathered} g(z_1,z_2) = 6\alpha z_1^2z_2 - z_1^2z_2^2 - 18\alpha z_1^2 - 18\alpha z_1^2 - 18\alpha z_2^2 + 6z_1^2z_2 - 108z_2\alpha - 36z_1^2 \\ +36z_2^2 + 108z_2 = -z_1^2z_2^2 + 6(\alpha+1)z_1^2z_2 - 18(\alpha+2)z_1^2 + 18(-\alpha+2)z_2^2 + 108(-\alpha+1)z_2. \end{gathered} \tag{22}$$

The curve $g(z_1,z_2) = 0$ has genus 0, intersections with axes

$$(z_1,z_2) = (0,0),\ (z_1,z_2) = \left(0, -\frac{6\left(\sqrt{3}-1\right)}{\sqrt{3}-2}\right) = (0,\ 16.39230485), \tag{23}$$

parameterization

$$\begin{aligned} z_1 = b_1(t) = &\\ &\frac{3(173465063\sqrt{3} - 1091281895)\beta_2(-176651t + 52563 + 26043\sqrt{3})}{70130447(2376102210\sqrt{3}t - 7846989697t^2 - 814835898\sqrt{3} + 4779544314t - 1425469788)}, \\ z_2 = b_2(t) = &-\frac{\left(2662513 + 1729681\sqrt{3}\right)\beta_2^2}{1891308\left(-45683t + 17823 + 9267\sqrt{3}\right)\left(-131t + 25 + 11\sqrt{3}\right)}, \\ \beta_2 = &(717 + 381\sqrt{3} - 397t). \end{aligned} \tag{24}$$

The singular points are reached at

$$t_1 = \frac{717 + 381\sqrt{3}}{397} \approx 3.468290574,\ t_2 = \frac{52563 + 26043\sqrt{3}}{176651} \approx 0.5529026114,$$

The plot of the curve is shown in Figure 6:

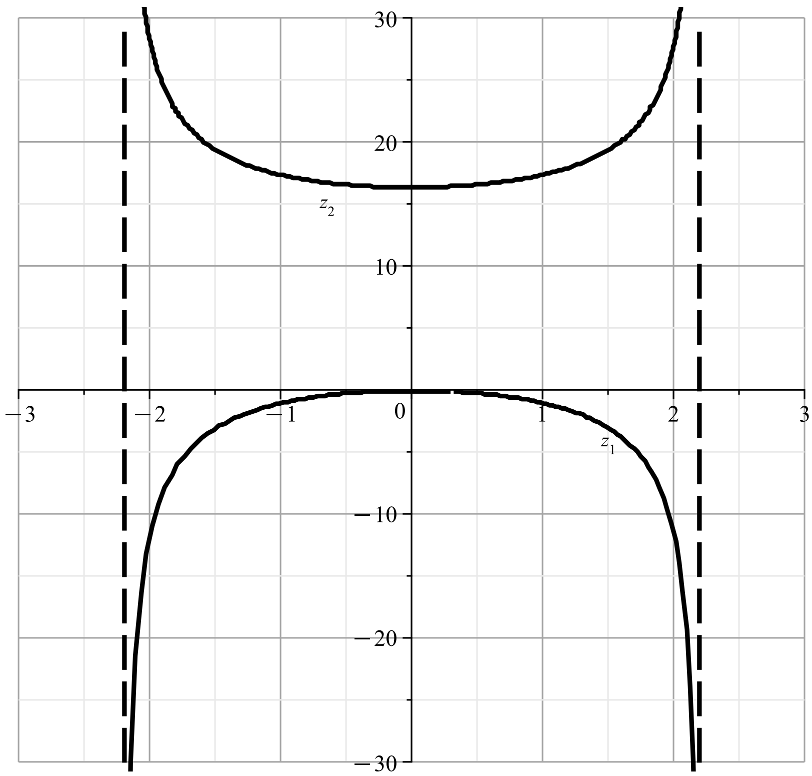

Figure 6. Plot of the curve $g(z_1, z_2) = 0$.

According to (8), (10) and (11) in coordinates A_1, A_2, A_3, we obtain

$$\begin{aligned} A_1 &= 1 + \frac{(2+\sqrt{3})\sqrt{3}}{12} z_1 z_3 + \frac{(-2+\sqrt{3})\sqrt{3}}{12} z_2 + \frac{1}{2} z_3, \\ A_2 &= 1 + \frac{\sqrt{3}}{12} z_1 z_3 - \frac{\sqrt{3}}{12} z_2 + \frac{1}{2} z_3, A_3 = \frac{1}{2} + z_3. \end{aligned} \tag{25}$$

If $z_3 = 0$ then,

$$A_1 = 1 + \frac{\left(-2+\sqrt{3}\right)\sqrt{3}}{12} z_2,\ A_2 = 1 - \frac{\sqrt{3}}{12} z_2,\ A_3 = \frac{1}{2}. \tag{26}$$

This is the singular line of [23] (Figure 5), which is obtained from the singular line $\mathfrak{I}$ by rotating by an angle $2\pi/3$. If $z_3 = -0.05$, then according to (25) $A_3 = 9/20 = 0.45$ and

$$\begin{aligned} A_1 &= -\frac{\left(2+\sqrt{3}\right)\sqrt{3}}{240} z_1 + \frac{\left(-2+\sqrt{3}\right)\sqrt{3}}{12} z_2 + \frac{39}{40}, \\ A_2 &= -\frac{\sqrt{3}}{240} z_1 - \frac{\sqrt{3}}{12} z_2 + \frac{39}{40}. \end{aligned} \tag{27}$$

When parameterized by (24), this curve is shown in Figure 7. It is similar to the part of [23] (Figure 12) corresponding to $A_3 = 0.45$, with parts of branches F_1^- and G_2^-.

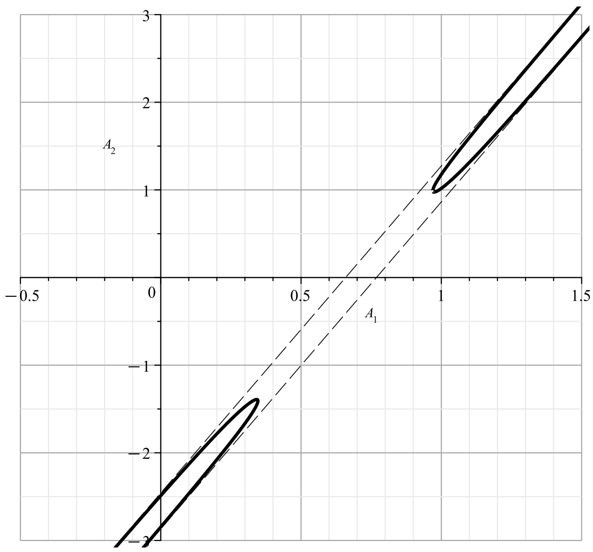

Figure 7. Curve (27).

If $z_3 = 0.005$, then $A_3 = \frac{101}{200} = 0.505$ and

$$\begin{aligned} A_1 &= -\frac{\left(2+\sqrt{3}\right)\sqrt{3}}{2400} z_1 + \frac{\left(-2+\sqrt{3}\right)\sqrt{3}}{12} z_2 + \frac{401}{400}, \\ A_2 &= \frac{\sqrt{3}}{2400} z_1 - \frac{\sqrt{3}}{12} z_2 + \frac{401}{400}. \end{aligned} \tag{28}$$

When parameterized by (24), this curve is shown in Figure 8. According to (23) in this figure, when $z_2 < 0$ and $z_2 > \frac{6(\sqrt{3}-1)}{2-\sqrt{3}}$, 2 branches each merge into one line. These results are similar to those of Section 9 of [1], differing from them by the angle $2\pi/3$ rotation.

Figure 8 is similar to the part of Figure 15 in [23]corresponding to $A_3 = 0.505$ with parts of branches F_1^+ and G_2^+.

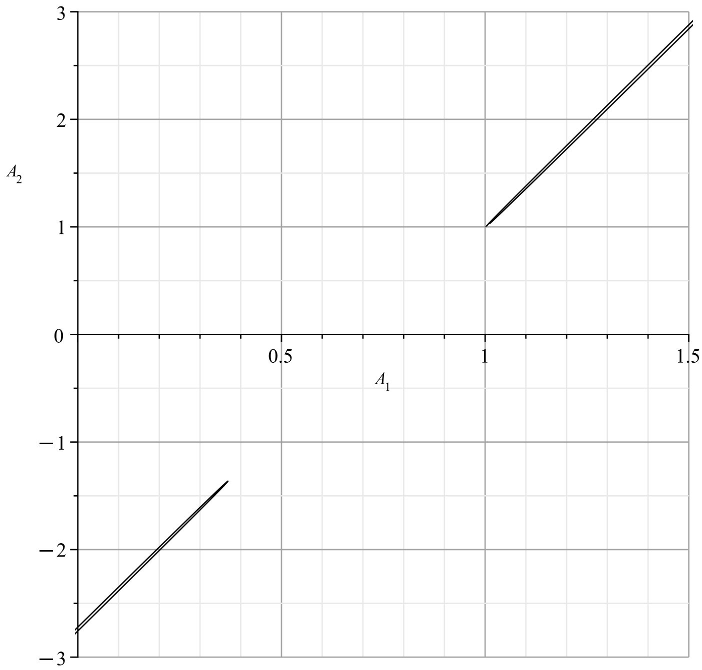

Figure 8. Curve (28).

Here the branches are very close and a different scale is needed. We can draw each branch separately, as in Figure 9.

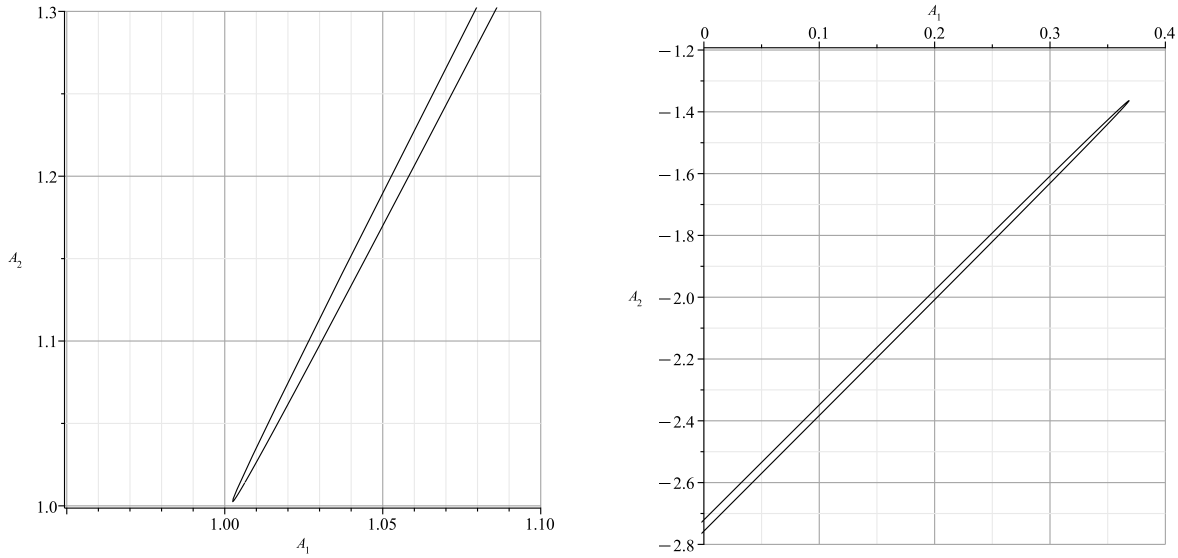

Figure 9. Curve (28) in more detail.

In that case also there exists a parametric expansions. In the polynomial $V(\mathbf{y}) = S(\mathbf{x})$ we make the power transformation (21) and obtain

$$V(\mathbf{y}) = W(\mathbf{z}) = z_3^2 \sum_{k=0}^{10} w_k(z_1, z_2) z_3^k.$$

Here $w_0(z_1, z_2) = g(z_1, z_2)$ from (22). According to (24) we substitute (17) into polynomials $w_k(z_1, z_2)$. We obtain $W(\mathbf{z})/z_3^2 = u_1(\varepsilon, z_3)$ with coefficients depending on t via $b_1(t)$ and $b_2(t)$. We apply Theorem 1 in [1] to the equation $u_1(\varepsilon, z_3) = 0$ and obtain the expansion

$$\varepsilon = \sum_{k=1}^{\infty} \tilde{c}_k(t) z_3^k.$$

So according to (25)

$$\begin{aligned} A_1 &= 1 + \frac{(2\sqrt{3}+3)}{12}(b_1(t)+\varepsilon)z_3 + \frac{(3-2\sqrt{3})}{12}(b_2(t)+\varepsilon) + \frac{1}{2}z_3, \\ A_2 &= 1 + \frac{\sqrt{3}}{12}(b_1(t)+\varepsilon)z_3 - \frac{\sqrt{3}}{12}(b_2(t)+\varepsilon) + \frac{1}{2}z_3, A_3 = \frac{1}{2} + z_3. \end{aligned} \tag{29}$$

Formulas (26) and (27) give the initial terms of them.

2.5. *The Normal* $(0, -1, -1)$

It corresponds to a truncated polynomial

$$ftr39917 = 18(26+15\alpha)(y_1 - 3 + 3\alpha)^2 (y_1 - 6 + 6\alpha)^2 y_1^2 \cdot$$
$$\left(18y_1^2y_3^2\alpha - 6\alpha y_1 y_2^2 + 108y_1y_3^2\alpha - y_1^2y_2^2 + 36y_1^2y_3^2 + 18y_2^2\alpha + 6y_1y_2^2 + 108y_1y_3^2 - 36y_2^2\right).$$

Let's denote

$$f_3 = 18y_1^2y_3^2\alpha - 6\alpha y_1 y_2^2 + 108y_1y_3^2\alpha - y_1^2y_2^2 + 36y_1^2y_3^2 + 18y_2^2\alpha + 6y_1y_2^2 + 108y_1y_3^2 - 36y_2^2.$$

According to [11], we compute the matrix $\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ such that $(0,-1,-1)\gamma = (0,0,-1)$. Since $\gamma^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, then we do a power transformation.

$$y_1 = z_1,\ y_2 = z_2 z_3,\ y_3 = z_3$$

and we get

$$f_3 = \left(-6\alpha z_1 z_2^2 - z_1^2 z_2^2 + 18\alpha z_1^2 + 18\alpha z_2^2 + 6 z_1 z_2^2 + 108\alpha z_1 + 36 z_1^2 - 36 z_2^2 + 108 z_1\right) z_3^2.$$

Let us denote

$$g_1(z_1, z_2) = -z_1^2 z_2^2 + (18\alpha + 36) z_1^2 + (-6\alpha + 6) z_1 z_2^2 + (108\alpha + 108) z_1 + (18\alpha - 36) z_2^2 =$$
$$= -z_1^2 z_2^2 + 6(-\alpha + 1) z_1 z_2^2 + 18(\alpha + 2) z_1^2 + 18(\alpha - 2) z_2^2 + 108(\alpha + 1) z_1.$$

According to (22) the polynomial $g_1(z_1, z_2)$ is obtained from the polynomial $g(z_1, z_2)$ by the transposition

$$(z_1, z_2, \alpha) \to (z_2, z_1, -\alpha). \tag{30}$$

So here everything is similar to Section 2.4, but in the plane A_1, A_2 rotated by an angle of $2\pi/3$. Also here there are expansions symmetric to (29) by transpositions (30) and

$$((b_1(t), b_2(t)) \to ((b_2(t), b_1(t)) \tag{31}$$

- Result of Section 2

Theorem 1. *Near the singular point $P_5^{(3)}$ the variety Ω has 3 local singular parametric expansions (18) and (29) and symmetric to (29) by transpositions (30) and (31). The expansions (18) describe parts of branches F_1 and G_2 very near the point $P_5^{(3)}$. Expansions (29) and its symmetric describes branches F_1 and G_2 near line (26) and symmetry line to it. For $\varepsilon \to 0$ branches F_1 and G_2 coincide at parts of these lines.*

3. The Structure of the Manifold Ω near the Curve $\mathcal{H}$ of Singular Points

Recall that the curve $\mathcal{H}$ is given by equations

$$\begin{aligned} A_1 &= b_1(t) = -\frac{9t(t^2+1)}{4(t+1)(t^2-4t+1)}, \\ A_2 &= b_2(t) = -\frac{9(t^2+1)}{4(t+1)(t^2-4t+1)}, \\ A_3 &= \frac{3}{4}. \end{aligned} \tag{32}$$

In the polynomial $R(\mathbf{A}) = Q(\mathbf{s})$, substitute

$$A_3 = \frac{3}{4} + \mu$$

and write the result as

$$\tilde{R}(A_1, A_2, \mu) = \sum_{k=0}^{m} R_k(A_1, A_2)\mu^k. \tag{33}$$

The polynomials R_0, R_1, R_2, and R_3 are computed using the command coeff (R, mu, k) [27]. After factorization, the polynomials R_0, R_1, R_2 and R_3 have the form:

$$R_0(A_1,A_2) = -(256A_1^6 - 1536A_1^5A_2 + 768A_1^4A_2^2 + 5120A_1^3A_2^3 + 768A_1^2A_2^4 - 1536A_1A_2^5 + 256A_2^6 - 5184A_1^4 - 10368A_1^2A_2^2 - 5184A_2^4 + 17496A_1^2 + 17496A_2^2 - 19683)(4A_1^3 - 12A_1^2A_2 - 12A_1A_2^2 + 4A_2^3 + 9A_1^2 + 9A_2^2)^2/2125764,$$

$$R_1(A_1,A_2) = (-1/177147)(8(4A_1^3 - 12A_1^2A_2 - 12A_1A_2^2 + 4A_2^3 + 9A_1^2 + 9A_2^2)(64A_1^8 - 384A_1^7A_2 + 256A_1^6A_2^2 + 896A_1^5A_2^3 + 384A_1^4A_2^4 + 896A_1^3A_2^5 + 256A_1^2A_2^6 - 384A_1A_2^7 + 64A_2^8 - 576A_1^7 + 1728A_1^6A_2 + 576A_1^5A_2^2 + 2880A_1^4A_2^3 + 2880A_1^3A_2^4 + 576A_1^2A_2^5 + 1728A_1A_2^6 - 576A_2^7 - 2592A_1^6 - 7776A_2^2A_1^4 - 7776A_2^4A_1^2 - 2592A_2^6 + 3888A_1^5 - 11664A_1^4A_2 - 7776A_1^3A_2^2 - 7776A_1^2A_2^3 - 11664A_1A_2^4 + 3888A_2^5 + 13122A_1^4 + 26244A_1^2A_1^2 + 13122A_2^4 - 6561A_1^3 + 19683A_1^2A_2 + 19683A_1A_2^2 - 6561A_2^3 - 19683A_1^2 - 19368A_2^2)),$$

$$R_2(A_1,A_2) = -\frac{544}{81}A_2^6 - \frac{16}{3}A_1^3 - \frac{544}{81}A_1^6 - \frac{16}{3}A_2^3 - \frac{512}{27}A_2^5A_1 + 16A_1^2A_2 + 16A_1A_2^2 - \frac{512}{27}A_2A_1^5 - \frac{544}{27}A_2^2A_1^4 + \frac{5120}{81}A_2^3A_1^3 - \frac{544}{27}A_2^4A_1^2 + \frac{5120}{19683}A_1^2A_2^8 + \frac{10240}{19683}A_1^4A_2^6 + \frac{8192}{19683}A_1^3A_2^7 + \frac{5120}{19683}A_1^8A_2^2 + \frac{8192}{19683}A_1^7A_2^3 + \frac{10240}{19683}A_1^6A_2^4 + \frac{28672}{19683}A_1^5A_2^5 - \frac{2048}{6561}A_1^9A_2 - \frac{7168}{2187}A_1^2A_2^6 + \frac{256}{27}A_1^2A_2^5 + \frac{3584}{729}A_1^7A_2 - \frac{7168}{2187}A_1^6A_2^2 + \frac{256}{9}A_1^6A_2 - \frac{25088}{2187}A_1^5A_2^3 + \frac{256}{27}A_1^5A_2^2 - \frac{3584}{729}A_1^4A_2^4 + \frac{1280}{27}A_1^4A_2^3 - \frac{688}{9}A_1^4A_2 - \frac{25088}{2187}A_1^3A_2^5 + \frac{1280}{27}A_1^3A_2^4 - \frac{1376}{27}A_1^3A_2^2 + \frac{160}{3}A_1^2A_2^2 - \frac{2048}{6561}A_1A_2^9 + \frac{3584}{729}A_1A_2^7 + \frac{256}{9}A_1A_2^6 - \frac{688}{9}A_1A_2^4 - \frac{1376}{27}A_1^2A_2^3 - \frac{16384}{6561}A_1^8A_2 + \frac{14336}{6561}A_1^7A_2^2 - \frac{2048}{19683}A_1^6A_2^3 - \frac{75776}{6561}A_1^5A_2^4 - \frac{75776}{6561}A_1^4A_2^5 - \frac{2048}{19683}A_1^3A_2^6 + \frac{14336}{6561}A_1^2A_2^7 - \frac{16384}{6561}A_2^8A_1 - \frac{256}{27}A_2^7 + \frac{688}{27}A_2^5 + \frac{80}{3}A_2^4 + \frac{10240}{19683}A_1^{10} - \frac{1792}{2187}A_2^8 + \frac{80}{3}A_1^4 + \frac{688}{27}A_1^5 - \frac{256}{27}A_1^7 - \frac{1792}{2187}A_1^8 + \frac{1024}{19683}A_2^{10} + \frac{34816}{59049}A_2^9 + \frac{34816}{59049}A_1^9,$$

$$R_3(A_1,A_2) = \frac{5120}{729}A_2^6 - \frac{1216}{27}A_1^3 + \frac{5120}{729}A_1^6 - \frac{1216}{27}A_2^3 - \frac{5120}{243}A_2^5A_1 + \frac{1216}{9}A_1^2A_2 + \frac{1216}{9}A_2^2A_1 - \frac{5120}{243}A_1^5A_2 + \frac{5120}{243}A_2^2A_1^4 + \frac{51200}{729}A_1^3A_2^3 + \frac{5120}{243}A_2^4A_1^2 - \frac{163840}{19683}A_1^2A_2^6 + \frac{23552}{2187}A_1^2A_2^5 + \frac{8192}{6561}A_1^7A_2 - \frac{163840}{19683}A_1^6A_2^2 + \frac{23552}{729}A_1^6A_2 - \frac{57344}{19683}A_1^5A_2^3 + \frac{23552}{2187}A_1^5A_2^2 - \frac{81920}{6561}A_1^4A_2^4 + \frac{117760}{2187}A_1^4A_2^3 - \frac{14272}{81}A_1^4A_2 - \frac{57344}{19683}A_1^3A_2^5 + \frac{117760}{2187}A_1^3A_2^4 - \frac{28544}{243}A_1^3A_2^2 + \frac{1280}{27}A_1^2A_2^2 + \frac{8192}{6561}A_1A_2^7 + \frac{23552}{729}A_1A_2^6 - \frac{14272}{81}A_1A_2^4 - \frac{28544}{243}A_1^2A_2^3 - \frac{65536}{59049}A_1^8A_2 + \frac{57344}{59049}A_1^7A_2^2 - \frac{8192}{177147}A_1^6A_2^3 - \frac{303104}{59049}A_1^5A_2^4 - \frac{303104}{59049}A_1^4A_2^5 - \frac{8192}{177147}A_1^3A_2^6 + \frac{57344}{59049}A_1^2A_2^7 - \frac{65536}{59049}A_1A_2^8 - \frac{23552}{2187}A_2^7 + \frac{14272}{243}A_2^5 + \frac{640}{27}A_2^4 - \frac{40960}{19683}A_2^8 + \frac{640}{27}A_1^4 + \frac{14272}{243}A_1^5 - \frac{23552}{2187}A_1^7 - \frac{40960}{19683}A_1^8 + \frac{139264}{531441}A_2^9 + \frac{139264}{531441}A_1^9.$$

Let's denote

$$\Phi(A_1,A_2) = 4\left(A_1^3 + A_2^3\right) - 12\left(A_1^2A_2 + A_1A_2^2\right) + \left(A_1^2 + A_2^2\right).$$

Then R_0 is divided by Φ^2, R_1 is divided by Φ, and R_2 and R_3 are not divided by Φ. The curve $\Phi(A_1, A_2) = 0$ has genus 0, its parameterization is

$$\{A_1, A_2\} = \{-\frac{9t(t^2+1)}{4(t+1)(t^2-4t+1)}, -\frac{9(t^2+1)}{4(t+1)(t^2-4t+1)}\} \tag{34}$$

and is shown in Figure 10.

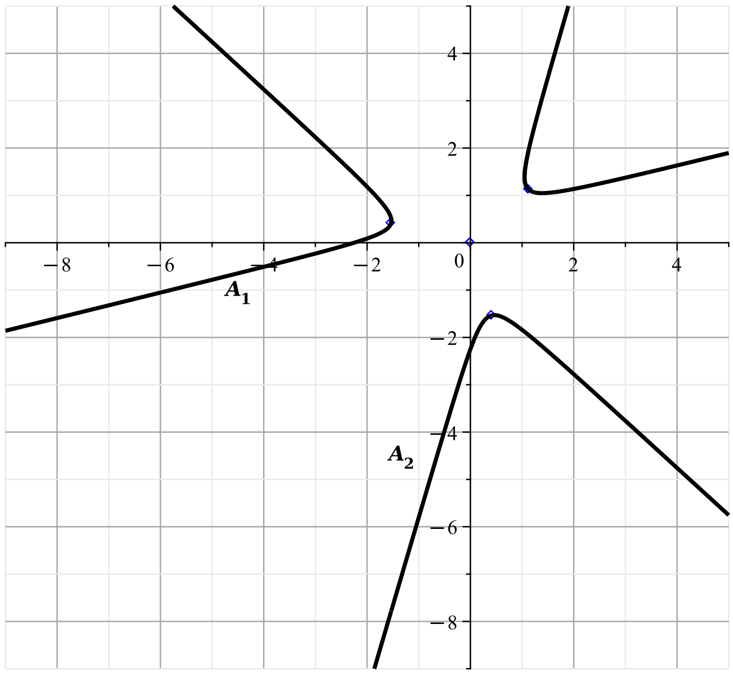

Figure 10. Curve $\Phi(A_1, A_2) = 0$.

The parameterization (34) is the same as the formulas of (32). The curve $\mathcal{H}$ goes to infinity at

$$t_1 = -1, t_2 = 2 + \sqrt{3} \approx 3.732050808, t_3 = 2 - \sqrt{3} \approx 0.267949192.$$

According to (32) we substitute

$$A_1 = b_1(t) + \varepsilon, A_2 = b_2(t). \tag{35}$$

into the polynomials $R_k(A_1, A_2)$. Then the polynomial (33) will become a polynomial

$$\tilde{R}(A_1, A_2, \mu) = u(\varepsilon, \mu) = \sum u_{pq}(t)\varepsilon^p \mu^q, \tag{36}$$

whereby

$$u_{pq} = \frac{1}{p!} \cdot \frac{\partial^p R_q}{\partial A_1^p},$$

where $A_1 = b_1(t)$, $A_2 = b_2(t)$ according to (32). In particular, we obtain

$$u_{00} = u_{10} = u_{01} \equiv 0, \tag{37}$$

$$
\begin{aligned}
u_{20}(t) = \frac{1}{2}\cdot\frac{\partial^2 R_0}{\partial A_1^2} = & -\frac{28160}{59049}A_1^9 - \frac{9728}{59049}A_2^9 + \frac{320}{243}A_1^8 + \frac{512}{81}A_1^7 + \frac{3}{2}A_2^2 - \frac{112}{9}A_1^5 - \frac{14}{9}A_2^4 - \frac{70}{9}A_1^4 + \frac{16}{27}A_2^5 + \frac{9}{2}A_1^2 \\
& + \frac{320}{2187}A_2^8 - \frac{4}{3}A_2^3 + \frac{64}{243}A_2^6 + \frac{20}{3}A_1^3 + \frac{448}{243}A_1^6 + \frac{80}{3}A_1^4A_2 - \frac{5120}{243}A_1^4A_2^3 + \frac{3200}{729}A_1^4A_2^4 + \frac{3584}{729}A_1^5A_2^3 \\
& - \frac{28}{3}A_1^2A_2^2 + \frac{160}{9}A_1^2A_2^3 - \frac{512}{81}A_1^2A_2^5 + \frac{1280}{729}A_1^2A_2^6 + \frac{160}{27}A_1^3A_2^2 - \frac{2560}{243}A_1^3A_2^4 + \frac{17920}{2187}A_1^3A_2^5 - \frac{3584}{243}A_1^6A_2 \\
& + \frac{8960}{2187}A_1^6A_2^2 + \frac{716800}{59049}A_1^6A_2^4 - \frac{512}{81}A_1^7A_2 - \frac{16384}{59049}A_1^7A_2^3 - \frac{71680}{19683}A_1^8A_2^2 + \frac{80}{9}A_1A_2^4 - \frac{81920}{59049}A_1^3A_2^7 \\
& - \frac{839680}{59049}A_1^4A_2^6 + \frac{51200}{19683}A_1^2A_2^8 - \frac{57344}{19683}A_1^5A_2^5 - 4A_1A_2^2 - \frac{448}{81}A_1A_2^5 - 12A_1^2A_2 + \frac{64}{27}A_1^2A_2^4 \\
& - \frac{4480}{243}A_1^3A_2^3 + \frac{448}{27}A_1^5A_2 + \frac{320}{81}A_1^4A_2^2 + \frac{71680}{19683}A_1^3A_2^6 + \frac{2048}{729}A_1^2A_2^7 - \frac{3584}{6561}A_1A_2^8 + \frac{35840}{6561}A_1^4A_2^5 - \frac{38912}{6561}A_1^7A_2^2 \\
& - \frac{100352}{19683}A_1^6A_2^3 + \frac{7168}{729}A_1^5A_2^4 + \frac{225280}{177147}A_1^9A_2 + \frac{2560}{729}A_1^8A_2 + \frac{512}{729}A_1A_2^7 - \frac{1024}{243}A_1A_2^6 - \frac{80}{9}A_1^3A_2 \\
& + \frac{80}{9}A_1A_2^3 - \frac{4096}{177147}A_1A_2^9 - \frac{22528}{177147}A_1^{10} - \frac{14336}{177147}A_2^{10} \\
= & \frac{243\left(t^2+1\right)^2\left(t^4+6t^2-8t-3\right)^2\left(5t^6+24t^5+15t^4-80t^3+15t^2+24t+5\right)^2}{1024(t+1)^8\left(t^2-4t+1\right)^8},
\end{aligned}
$$

$$
\begin{aligned}
u_{11}(t) = \frac{\partial R_1}{\partial A_1} = & \, 32A_1A_2^2 - \frac{224}{9}A_1^5 + \frac{71680}{6561}A_1^4A_2^5 + \frac{7168}{243}A_1^6A_2 + 32A_1^3 - \frac{512}{243}A_2^8 \\
& - \frac{22528}{177147}A_1^{10} - \frac{56}{3}A_2^4 - \frac{64}{9}A_2^5 + \frac{512}{81}A_1^8 + \frac{1024}{243}A_2^7 + \frac{2048}{729}A_1^7 - \frac{320}{9}A_1^4A_2 - \frac{35840}{729}A_1^4A_2^3 \\
& - \frac{5120}{243}A_1^4A_2^4 + \frac{2048}{243}A_1^5A_2^2 + \frac{2048}{19683}A_2^{10} + \frac{280}{9}A_1^4 + \frac{640}{9}A_1^2A_2^3 - \frac{8192}{243}A_1^5A_2^3 - \frac{112}{3}A_1^2A_2^2 \\
& + \frac{2048}{243}A_1^3A_2^4 - \frac{4096}{243}A_1^3A_2^5 + \frac{28672}{6561}A_1^6A_2^4 - \frac{38912}{19683}A_1^8A_2^2 - \frac{4096}{243}A_1^7A_2 - \frac{224}{9}A_1A_2^4 \\
& - \frac{114688}{59049}A_1^7A_2^3 + \frac{143360}{59049}A_1^4A_2^6 - \frac{14336}{19683}A_1^2A_2^8 + \frac{57344}{19683}A_1^5A_2^5 + \frac{16384}{6561}A_1^3A_2^7 + \frac{6400}{81}A_1^3A_2^3 \\
& + \frac{1600}{27}A_1^2A_2^4 + \frac{640}{81}A_1A_2^5 + \frac{4096}{2187}A_1^2A_2^7 + \frac{1600}{81}A_1^4A_2^2 + \frac{10240}{2187}A_1^5A_2^4 + \frac{28672}{6561}A_1^6A_2^3 \\
& + \frac{20480}{6561}A_1^7A_2^2 + \frac{20480}{6561}A_1^3A_2^6 - \frac{1024}{243}A_1^8A_2 - \frac{77824}{177147}A_1A_2^9 + \frac{20480}{19683}A_1^9A_2 + \frac{2048}{729}A_1A_2^6 \\
& + \frac{5120}{6561}A_1A_2^8 + \frac{640}{9}A_1^5A_2 - \frac{224}{3}A_1^3A_2 - \frac{224}{9}A_1A_2^3 + \frac{5120}{6561}A_1^9 - \frac{1024}{2187}A_2^9 - \frac{2240}{81}A_1^6 \\
& + \frac{320}{27}A_2^6 - \frac{4096}{243}A_1^2A_2^6 - \frac{448}{9}A_1^3A_2^2 - \frac{7168}{243}A_1^2A_2^5 = \\
= & \frac{729\left(t^2+1\right)^4\left(t^4+6t^2-8t-3\right)\left(5t^6+24t^5+15t^4-80t^3+15t^2+24t+5\right)^2}{512(t+1)^8\left(t^2-4t+1\right)^8},
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
& u_{02}(t) = R_2(b_1(t), b_2(t)) = \\
& = -\frac{243\left(t^2+1\right)^3\left(t^2+8t+1\right)\left(11t^4+8t^3-42t^2+8t+11\right)\left(5t^6+24t^5+15t^4-80t^3+15t^2+24t+5\right)^2}{1024(t+1)^8\left(t^2-4t+1\right)^8}.
\end{aligned}
$$

From the Formulas (37) and (38), we can see that the Newton's polygon Γ of the polynomial $u(\varepsilon, \mu)$ (36) in the plane p, q has an edge $\Gamma_1^{(1)}$ containing the points $(2, 0)$, $(1, 1)$, $(0, 2)$ (Figure 11) with external normal $N_1 = (-1, -1)$.

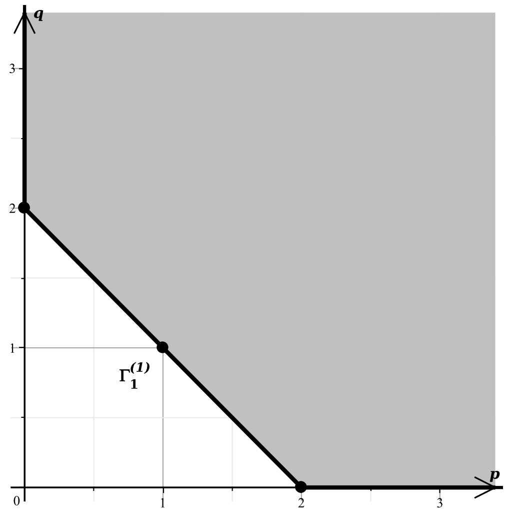

Figure 11. The lower left side of the polygon Γ.

So we're doing a power transformation

$$\varepsilon = \mu\delta. \tag{39}$$

Then the polynomial $u(\varepsilon, \mu)$ becomes a polynomial.

$$\mu^2 V(\delta, \mu) = \mu^2 \sum V_{pr}(t)\delta^p \mu^r = u(\varepsilon, \mu), \tag{40}$$

where

$$u_{p,q}(t) = V_{p,p+q-2}(t).$$

The support and Newton's polygon for the polynomial $V(\delta, \mu)$ are shown in Figure 12. The truncated equation corresponding to the edge $\widetilde{\Gamma}_1^{(1)}$ is

$$u_{20}(t)\delta^2 + u_{11}(t)\delta + u_{02}(t) = 0. \tag{41}$$

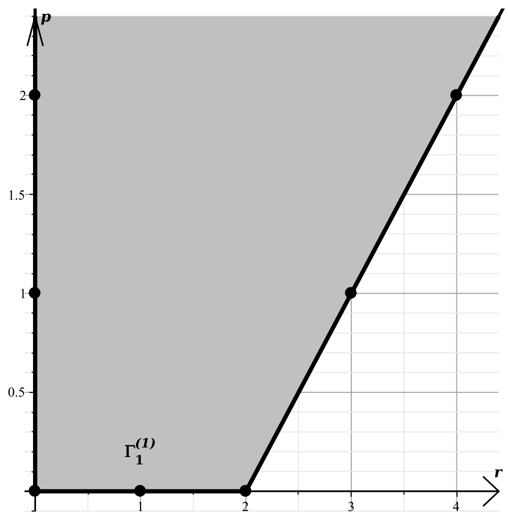

Figure 12. The support and polygon of the polynomial $V(\delta, \mu)$.

It has two roots

$$\begin{aligned} \delta_1(t) &= \frac{-3\left(t^2+1\right)^2 + 2\sqrt{5t^8+24t^7+20t^6-56t^5+30t^4-56t^3+20t^2+24t+5}}{t^4+6t^2-8t-3}, \\ \delta_2(t) &= \frac{-3\left(t^2+1\right)^2 - 2\sqrt{5t^8+24t^7+20t^6-56t^5+30t^4-56t^3+20t^2+24t+5}}{t^4+6t^2-8t-3}. \end{aligned} \tag{42}$$

Their denominator goes to zero at two real points

$$t_4 \approx 1.324070992, t_5 \approx -0.3044219351. \tag{43}$$

These points are indicated in Figure 10.

Next, we consider the expansions of the Ω manifold for the cases of two roots (42).

In the polynomial $V(\delta,\mu)$ of (40), we make the substitutions

$$\delta = \delta_i + \varkappa_i,\ i = 1,2. \tag{44}$$

where δ_i are given by the formula (42). We get

$$W_i(\varkappa_i,\mu) = V(\delta,\mu) = \sum W_{isr}(t)\varkappa_i^s\mu^r, i = 1,2,$$

where integers $s, r \geq 0, r+s \geq 1$. In this case.

$$W_{isr}(t) = \sum_{p\geq s} V_{ps}(t) C_p^s \delta_i^{p-s}, i = 1,2,$$

where $C_p^s = \dfrac{p!}{s!(p-s)!}$ are binomial coefficients. In particular, according to (38), (41) and (42), we have

$$W_{i00} \equiv 0,\ W_{i10} \equiv 2u_{20}(t)\cdot\delta_i(t) + u_{11}(t), i = 1,2.$$

More specifically,

$$\begin{aligned} W_{110} &= 2u_{20}(t)\delta_1(t) + u_{11}(t) \\ &= \frac{243\left(t^4+6t^2-8t-3\right)\left(t^2+1\right)^2\left(5t^6+24t^5+15t^4-80t^3+15t^2+24t+5\right)^2}{256(t+1)^8} \\ &\times \frac{\sqrt{(t^2+1)(5t^6+24t^5+15t^4-80t^3+15t^2+24t+5)}}{\left(t^2-4t+1\right)^8}, \end{aligned} \tag{45}$$

$$\begin{aligned} W_{210} &= 2u_{20}(t)\delta_2(t) + u_{11}(t) \\ &= -\frac{243\left(t^4+6t^2-8t-3\right)\left(t^2+1\right)^2\left(5t^6+24t^5+15t^4-80t^3+15t^2+24t+5\right)^2}{256(t+1)^8} \\ &\times \frac{\sqrt{(t^2+1)(5t^6+24t^5+15t^4-80t^3+15t^2+24t+5)}}{(t^2-4t+1)^8}, \end{aligned} \tag{46}$$

i.e., $W_{210} = -W_{110} \not\equiv 0$. Hence Theorem 1 in [1] is applicable, which for solutions of equations $W_i(\varkappa_i,\mu) = 0$ according to (45) and (46) gives the expansions

$$\varkappa_i(t) = \sum_{k=1}^{\infty} c_{ik}(t)\mu^k,\ i = 1,2.$$

Let's go from coordinates $(\varkappa_i,\mu)$ to coordinates (δ,μ) by (44), with decompositions

$$\delta^{(1)} = \delta_1(t) + \varkappa_1(t) = \delta_1 + \sum_{k=1}^{\infty} c_{1k}\mu^k$$

and

$$\delta^{(2)} = \delta_2(t) + \varkappa_2(t) = \delta_2 + \sum_{k=1}^{\infty} c_{2k}\mu^k.$$

Then we go to the coordinates $\varepsilon, \mu : \varepsilon^{(i)} = \mu\delta^{(i)},\ i = 1, 2$ by (39) to the coordinates $A_1,\ \mu$ by (35):

$$A_1^{(1)} = b_1(t) + \varepsilon^{(1)} = b_1(t) + \mu(\delta_1 + \varkappa_1) = b_1(t) + \delta_1(t)\mu + \mu\sum_{k=1}^{\infty} c_{1k}(t)\mu^k, \quad (47)$$

$$A_1^{(2)} = b_1(t) + \varepsilon^{(2)} = b_1(t) + \mu(\delta_2 + \varkappa_2) = b_1(t) + \delta_2(t)\mu + \mu\sum_{k=1}^{\infty} c_{2k}(t)\mu^k, \quad (48)$$

$$A_2 = b_2(t) = -\frac{9(t^2+1)}{4(t+1)(t^2-4t+1)}. \quad (49)$$

With μ fixed, the Formulas (47) and (49) are defined in the plane A_1, A_2 with $A_3 = 3/4 + \mu$ the first curve $K^{(1)}$, and the Formulas (48) and (49) define there the second curve $K^{(2)}$. We obtain four curves. Restricting ourselves to the initial terms of the expansions, we draw them. When $\mu = \frac{1}{8}$, the curve $K^{(1)}$ is

$$K^{(1)} = \left\{A_1^{(1)} = b_1(t) + \delta_1(t)\mu,\ \ A_2 = b_2(t)\right\}$$
$$= \left\{-\frac{9t(t^2+1)}{4(t+1)(t^2-4t+1)} + \frac{-3\left(t^2+1\right)^2 + 2\sqrt{(t^2+1)(5t^6+24t^5+15t^4-80t^3+15t^2+24t+5)}}{8\cdot(t^4+6t^2-8t-3)}, b_2(t)\right\},$$

and it is shown in Figure 13.

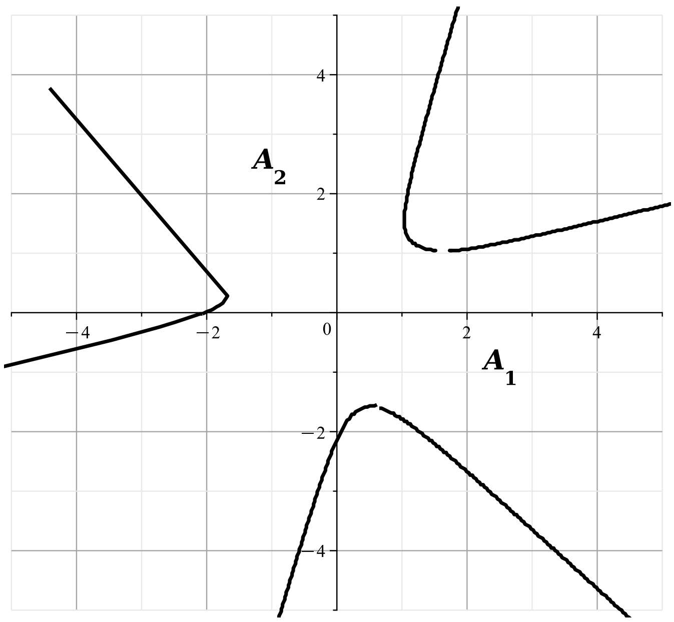

Figure 13. The curve $K^{(1)}$ at $\mu = 1/8$.

The curve $K^{(2)}$ is

$$K^{(2)} = \left\{A_1^{(2)} = b_1(t) + \delta_2(t)\mu, A_2 = b_2(t)\right\}$$
$$= \left\{-\frac{9t(t^2+1)}{4(t+1)(t^2-4t+1)} + \frac{-3\left(t^2+1\right)^2 - 2\sqrt{(t^2+1)(5t^6+24t^5+15t^4-80t^3+15t^2+24t+5)}}{8\cdot(t^4+6t^2-8t-3)}, b_2(t)\right\},$$

It's shown in Figure 14.

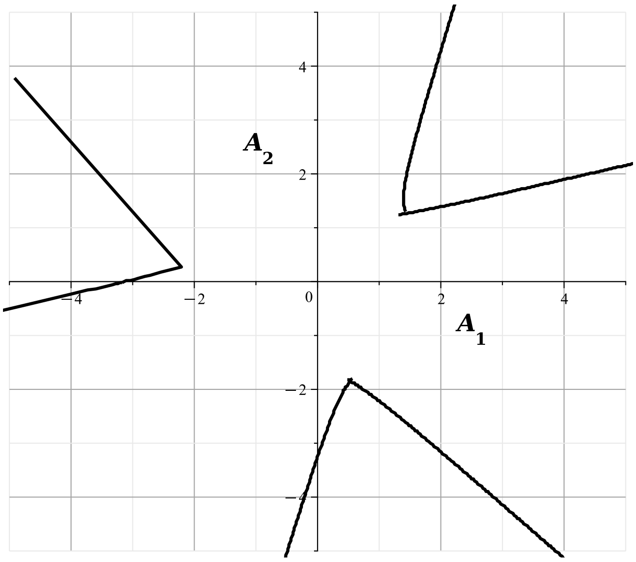

Figure 14. The curve $K^{(2)}$ at $\mu = 1/8$.

When $\mu = -1/8$, the curve

$$K^{(1)} = \left\{ A_1^{(1)} = b_1(t) + \delta_1(t)\mu, A_2 = b_2(t) \right\}$$

$$= \left\{ -\frac{9t\left(t^2+1\right)}{4(t+1)(t^2-4t+1)} - \frac{-3\left(t^2+1\right)^2 + 2\sqrt{(t^2+1)(5t^6+24t^5+15t^4-80t^3+15t^2+24t+5)}}{8\cdot(t^4+6t^2-8t-3)}, b_2(t) \right\}$$

is shown in Figure 15.

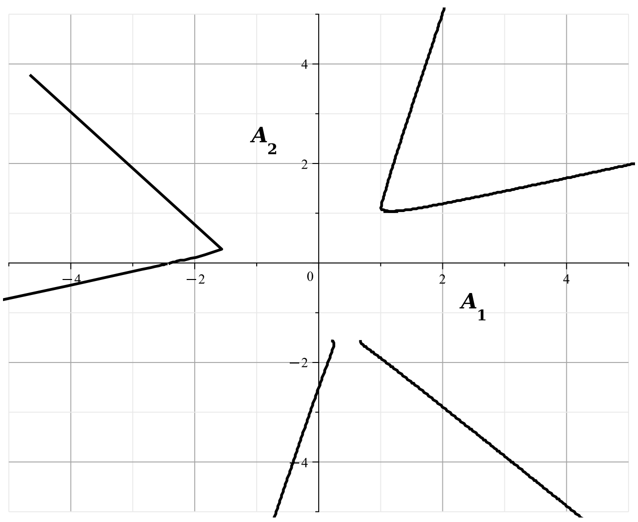

Figure 15. The curve $K^{(1)}$ at $\mu = -1/8$.

and the curve

$$K^{(2)} = \left\{ A_1^{(2)} = b_1(t) + \delta_2(t)\mu, A_2 = b_2(t) \right\}$$

$$= \left\{ -\frac{9t\left(t^2+1\right)}{4(t+1)(t^2-4t+1)} + \frac{3\left(t^2+1\right)^2 + 2\sqrt{(t^2+1)(5t^6+24t^5+15t^4-80t^3+15t^2+24t+5)}}{8\cdot(t^4+6t^2-8t-3)}, b_2(t) \right\}$$

is shown in Figure 16.

The curves of Figures 13 and 14 correspond to $A_3 = 7/8$ and are similar to the curves of Figure 13 of [23], showing the section of the variety Ω by the plane $A_3 = 1$. The curves of Figures 15 and 16 correspond to $A_3 = 5/8$ and they are similar to the curves of Figure 14 of [23], showing the cross section of Ω by the plane $A_3 = 5/8$. This confirms the correctness of the calculated expansions.

In Figures 13, 15 and 16, there are discontinuities in the curves at the places of the roots (43) of the denominators $\delta_1(t)$ and $\delta_2(t)$ in (42). They can be eliminated by substituting

$$A_1 = b_1(t) + \varepsilon,\ A_2 = b_2(t) + \varepsilon$$

instead of substitution (35) and calculate the corresponding expansion.

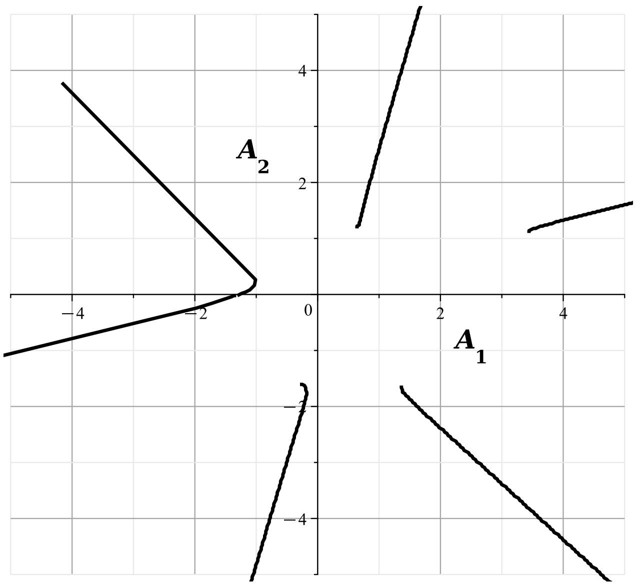

Figure 16. The curve $K^{(2)}$ at $\mu = -1/8$.

- Result of Section 3

Theorem 2. *Near the curve $\mathcal{H}$ of singular points the variety Ω has two singular parametric expansions* (47), (49) *and* (48), (49)*. They represent parts of branches G_2^+ and G_3^- correspondingly. At $A_3 = 3/4$ they coincide with curve $\mathcal{H}$.*

4. The Structure of the Variety Ω near the Curve $\mathcal{F}$ of Singular Points

We take the polynomial $Q(\mathbf{s}) = Q(s_1, s_2, s_3)$, where $s_1 = a_1 + a_2 + a_3$, $s_2 = a_1 \cdot a_2 + a_1 \cdot a_3 + a_1 \cdot a_3$, $s_3 = a_1 \cdot a_2 \cdot a_3$ are elementary symmetric polynomials, and we substitute $a_1 = a_2$. Then the polynomial $Q(\mathbf{s})$ takes the form

$$\tilde{Q}(a_1, a_3) = -(1 + 2a_3)\left(8a_1a_3 + 8a_3^2 - 4a_1 - 4a_3 + 1\right)\left(16a_1^3 + 16a_1^2a_3 - 4a_1 - 2a_3 + 1\right)^3.$$

Let's write the polynomial $16a_1^3 + 16a_1^2a_3 - 4a_1 - 2a_3 + 1$ in $\mathbf{A}$ coordinates, substituting

$$a_1 = \frac{1+\sqrt{3}}{6}A_1 + \frac{1-\sqrt{3}}{6}A_1 + \frac{1}{3}A_3, a_3 = -\frac{1}{3}A_1 - \frac{1}{3}A_1 + \frac{1}{3}A_3$$

with $A_1 = A_2$.

Then we get a polynomial $-\frac{1}{27}\left(16A_1^3 - 48A_1A_3^2 - 32A_3^3 + 54A_3 - 27\right)$. We put

$$\mathcal{F}(A_1, A_3) = 16A_1^3 - 48A_1A_3^2 - 32A_3^3 + 54A_3 - 27.$$

The curve $\mathcal{F} = 0$ consists of singular points, has genus 0, and parameterization

$$[A_1, A_3] = \left[b_1(t) = -\frac{(5t+2)(t+4)^2}{6t(t^2-16t-8)}, \quad b_2(t) = \frac{11t^3 - 48t^2 - 48t - 16}{6t(t^2-16t-8)} \right]. \tag{50}$$

The curve is shown in Figure 17.

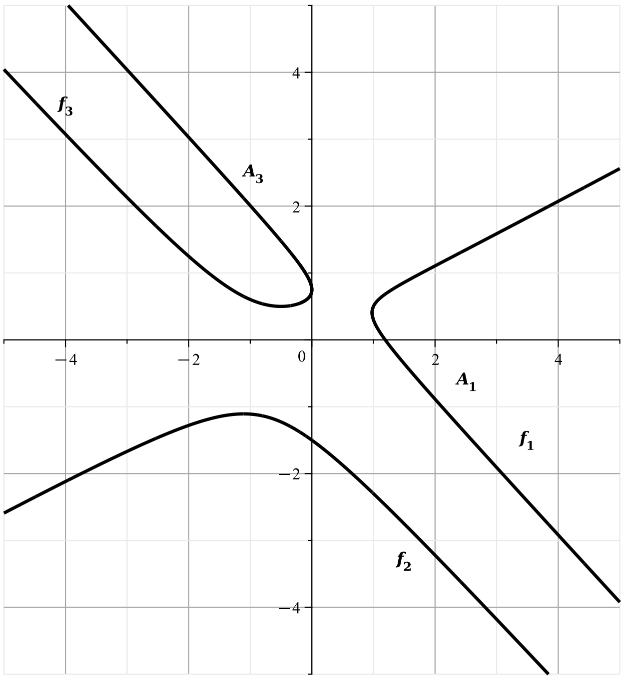

Figure 17. The curve $\mathcal{F}(A_1, A_3) = 0$.

In [23] (Figure 3), the components $f_1^{\pm}$, f_2, $f_3^{\pm}$ of this curve are shown in gray. The scales on the axes are different there. In the polynomial $R(\mathbf{A}) = Q(\mathbf{s})$ we substitute

$$A_1 = B_1,\ A_2 = B_1 + B_2,\ A_3 = B_3, \tag{51}$$

and we get a polynomial depending on three variables,

$$K(B_1,\ B_2, B_3) = \sum_{l=0}^{12} K_l(B_1,\ B_3) B_2^l. \tag{52}$$

We factorize K_l for $l = 0, 1, 2, 3$ because they are need for our computation and get

$$\begin{aligned} -531441 K_0(B_1, B_3) &= (-2B_3 - 3 + 4B_1)\left(16B_1^2 - 40B_1B_3 + 16B_3^2 + 12B_1 - 24B_3 + 9\right) \\ &\times \left(16B_1^3 - 48B_1B_3^2 - 32B_3^3 + 54B_3 - 27\right)^3 = \\ &= (-2B_3 - 3 + 4B_1)\left(16B_1^2 - 40B_1B_3 + 16B_3^2 + 12B_1 - 24B_3 + 9\right)\mathcal{F}^3(B_1, B_3). \end{aligned}$$

and

$$\begin{aligned} -177147 K_1(B_1, B_3) &= 8B_1\left(16B_1^3 - 48B_1B_3^2 - 32B_3^3 + 54B_3 - 27\right)^2 \\ &\times \left(256B_1^4 - 704B_1^3B_3 + 96B_1^2B_3^2 + 736B_1B_3^3 - 320B_3^4 + 378B_1B_3 - 432B_3^2 - 189B_1 + 216B_3\right) \\ &= 8B_1\left(256B_1^4 - 704B_1^3B_3 + 96B_1^2B_3^2 + 736B_1B_3^3 - 320B_3^4 + 378B_1B_3 - 432B_3^2 - 189B_1 + 216B_3\right)\mathcal{F}^2(B_1, B_3). \end{aligned}$$

Then

$$K_2(B_1,B_3) = -\frac{45056}{729}B_1^7B_3 + \frac{1856}{27}B_1^4B_3 - \frac{131072}{19683}B_1^{10} - \frac{467}{27}B_1^4 + \frac{22528}{729}B_1^7 + \frac{1310720}{177147}B_3^{10}$$
$$-\frac{32768}{2187}B_3^8 + \frac{16384}{2187}B_3^7 - \frac{128}{3}B_3^3 + \frac{2048}{81}B_3^4 + \frac{1024}{81}B_3^5 - \frac{1024}{81}B_3^6 + \frac{64}{3}B_3^2 - \frac{32}{9}B_3 - \frac{65536}{729}B_1^2B_3^5$$
$$+\frac{106496}{2187}B_1^3B_3^4 + \frac{136192}{729}B_1^4B_3^3 + \frac{512}{27}B_1^2B_3^2 + \frac{2048}{81}B_1^3B_3 - \frac{212992}{2187}B_1^3B_3^5 - \frac{40960}{729}B_1^5B_3^2$$
$$-\frac{177152}{2187}B_1^6B_3 + \frac{131072}{729}B_1^2B_3^6 + \frac{81920}{729}B_1^5B_3^3 - \frac{272384}{729}B_1^4B_3^4 + \frac{354304}{2187}B_1^6B_3^2 + \frac{3080192}{177147}B_1^9B_3$$
$$-\frac{1638400}{19683}B_1^7B_3^3 - \frac{5472256}{59049}B_1^6B_3^4 + \frac{2621440}{19683}B_1^5B_3^5 + \frac{2768896}{19683}B_1^4B_3^6 + \frac{671744}{19683}B_1^8B_3^2 - \frac{1441792}{19683}B_1^2B_3^8$$
$$-\frac{3407872}{59049}B_1^3B_3^7 + \frac{2048}{27}B_1^2B_3^4 + \frac{8192}{81}B_1^3B_3^3 - \frac{1856}{27}B_1^4B_3^2 - \frac{8192}{81}B_1^3B_3^2 - \frac{2048}{27}B_1^2B_3^3,$$
$$K_3(B_1,B_3) = \frac{1024}{27}B_1B_3^4 + \frac{65536}{729}B_1B_3^6 - \frac{1024}{27}B_1B_3^3 - \frac{32768}{729}B_1B_3^5 + \frac{256}{27}B_1B_3^2 - \frac{1952}{243}B_1^3$$
$$+\frac{336896}{6561}B_1^6 - \frac{3276800}{531441}B_1^9 - \frac{917504}{531441}B_3^9 - \frac{90112}{6561}B_3^7 + \frac{3424}{243}B_3^3 - \frac{5632}{243}B_3^4 + \frac{5632}{243}B_3^5 + \frac{45056}{6561}B_3^6$$
$$-\frac{112}{9}B_3^2 + \frac{56}{3}B_3 - \frac{720896}{19683}B_1B_3^8 - \frac{28}{27} - \frac{851968}{59049}B_1^2B_3^7 + \frac{23953408}{177147}B_1^3B_3^6 + \frac{655360}{6561}B_1^4B_3^5$$
$$-\frac{6324224}{59049}B_1^5B_3^4 - \frac{13991936}{177147}B_1^6B_3^3 + \frac{1015808}{59049}B_1^8B_3 - \frac{53248}{2187}B_1^2B_3^5 - \frac{2772992}{6561}B_1^3B_3^4 + \frac{20480}{243}B_1^4B_3^3$$
$$-\frac{2048}{81}B_1^2B_3^2 + \frac{7808}{243}B_1^3B_3 + \frac{604160}{2187}B_1^5B_3^2 - \frac{673792}{6561}B_1^6B_3 + \frac{720896}{19683}B_1^7B_3^2 + \frac{512}{81}B_1^2B_3 + \frac{26624}{2187}B_1^2B_3^4$$
$$+\frac{1386496}{6561}B_1^3B_3^3 - \frac{10240}{243}B_1^4B_3^2 - \frac{302080}{2187}B_1^5B_3 - \frac{7808}{243}B_1^3B_3^2 + \frac{2048}{81}B_1^2B_3^3.$$

The multiplier $\mathcal{F}(B_1,B_3)$ enters in $K_0(B_1,B_3)$ in the third degree, in $K_1(B_1,B_3)$ in the second degree, in $K_2(B_1,B_3)$, and in $K_3(B_1,B_3)$ it does not enter. Then K_0 is divisible by $\mathcal{F}^3$, K_1 is divisible by $\mathcal{F}^2$, and K_2 and K_3 are not divisible by $\mathcal{F}$. The curve $\mathcal{F}(B_1,B_3)=0$ has genus 0, parameterization (50).

The curve $\mathcal{F}=0$ goes to infinity at

$$t_1 = 0, t_2 = 2\left(4+3\sqrt{2}\right) \approx 16.48528137,\quad t_3 = 2\left(4-3\sqrt{2}\right) \approx -0.485281372.$$

Into the polynomials $K_l(B_1,B_3)$ we substitute

$$B_1 = b_1(t) + \varepsilon, B_3 = b_2(t) \tag{53}$$

according to (50). Then the polynomial (52) will become a polynomial

$$K(B_1,B_3,B_2) = u(\varepsilon,B_2) = \sum_{p,q\geq 0} u_{pq}(t)\varepsilon^p B_2^q, \text{ whereby } u_{pq} = \frac{1}{p!}\cdot\frac{\partial^p K_q}{\partial B_1^p}(b_1(t),b_2(t)), \tag{54}$$

where $B_1 = b_1(t)$, $B_3 = b_2(t)$ according to (50). In particular, we obtain

$$\begin{aligned} u_{00} =& u_{10} = u_{01} \equiv 0, & u_{20}(t) =& \frac{1}{2}\cdot\left(\frac{\partial^2 K_0}{\partial B_1^2}\right)(b_1(t),b_2(t)) = 0, \\ u_{11}(t) =& \left(\frac{\partial K_1}{\partial B_1}\right)(b_1(t),b_2(t)) = 0, & u_{02}(t) =& K_2(b_1(t),b_2(t)) = \frac{64(5t+2)^2(t+4)^4(t-2)^4(t+1)^4}{t^2(t^2-16t-8)^6}. \end{aligned} \tag{55}$$

$$
\begin{aligned}
u_{30} = \frac{1}{6}\cdot\frac{\partial^3 K_0}{\partial B_1^3} == & \frac{40960}{81}B_1^2B_3^3 + \frac{14417920}{59049}B_1^8B_3 + \frac{10240}{81}B_1^2B_3 - \frac{286720}{729}B_1^4B_3^2 - \frac{57671680}{531441}B_1^9 \\
& + \frac{360448}{6561}B_3^7 + \frac{4259840}{6561}B_1^3B_3^3 + \frac{3670016}{531441}B_3^9 + \frac{22528}{243}B_3^4 - \frac{22528}{243}B_3^5 - \frac{180224}{6561}B_3^6 \\
& + \frac{1835008}{2187}B_1^5B_3^2 + \frac{102400}{243}B_1^3B_3 + \frac{2621440}{6561}B_1^7B_3^2 - \frac{2981888}{6561}B_1^6B_3 - \frac{150470656}{177147}B_1^6B_3^3 \\
& - \frac{2883584}{19683}B_1B_3^8 + \frac{4096}{27}B_1B_3^4 - \frac{4096}{27}B_1B_3^3 - \frac{131072}{729}B_1B_3^5 + \frac{262144}{729}B_1B_3^6 + \frac{573440}{729}B_1^4B_3^3 \\
& + \frac{1490944}{6561}B_1^6 - \frac{224}{9}B_3 - \frac{13696}{243}B_3^3 + \frac{448}{9}B_3^2 - \frac{25600}{243}B_1^3 + \frac{112}{27} - \frac{917504}{2187}B_1^5B_3 \\
& - \frac{102400}{243}B_1^3B_3^2 - \frac{17039360}{59049}B_1^2B_3^7 + \frac{103546880}{177147}B_1^3B_3^6 + \frac{1024}{27}B_1B_3^2 - \frac{40370176}{59049}B_1^5B_3^4 \\
& - \frac{1064960}{2187}B_1^2B_3^5 + \frac{18350080}{19683}B_1^4B_3^5 + \frac{532480}{2187}B_1^2B_3^4 - \frac{40960}{81}B_1^2B_3^2 - \frac{8519680}{6561}B_1^3B_3^4 \\
& = -\frac{8192(5t+2)(t+4)^2(t-2)^2(t+1)^3\left(8t^3-3t^2+24t+8\right)^3}{6561t^5\left(t^2-16t-8\right)^6}.
\end{aligned}
\tag{56}
$$

From the Formulas (55) and (57), we can see that the Newton's polygon Γ of the polynomial $u(\varepsilon, B_2)$ given by (54) in the plane p, q has an edge $\Gamma_1^{(1)}$, containing the points $(3,0)$, $(0,2)$ (Figure 18) with external normal $N_1 = (-2,-3)$.

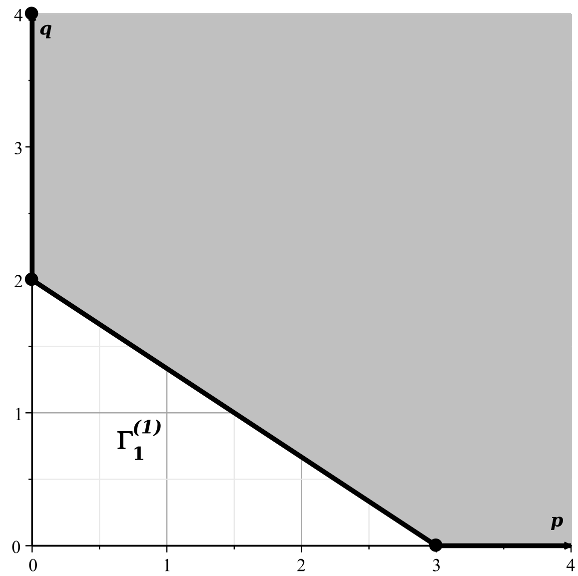

Figure 18. Bottom left of the polygon Γ.

A truncated polynomial corresponds to this edge

$$\varepsilon^3 u_{30}(t) + B_2^2 u_{02}(t) = 0 \tag{57}$$

According to [11] we find the unimodular matrix $\alpha = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$ for N_1 such that $N_1\alpha = (0,-1)$. Therefore, we need to do a power transformation

$$(\ln\delta, \ln D) = (\ln\varepsilon, \ln B_2)\cdot\alpha,$$

where δ and D are new variables, i.e.,

$$(\ln\varepsilon, \ln B_2) = (\ln\delta, \ln D)\cdot\alpha^{-1}.$$

Since $\alpha^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$, then $\varepsilon = \delta D^2,\ B_2 = \delta D^3$. Hence we can write

$$K(B_1, B_3, B_2) \ = \ u(\varepsilon, B_2) \ = \ \sum u_{pq}(t)\varepsilon^p B_2^q \ = \ \sum u_{pq}(t)\delta^{p+q} D^{2p+3q} \ = \ \delta^2 D^6 V(\delta, D).$$

Then the polynomial $u(\varepsilon, B_2)$ becomes a polynomial

$$\delta^2 D^6 V(\delta, D) = \delta^2 D^6 \sum V_{rs}(t)\delta^r D^s = u(\varepsilon, B_2),$$

where $V_{r,s}(t) = V_{p+q,2p+3q}(t) = u_{p,q}(t)$. Thus the polygon Γ of Figure 18 takes the form shown in Figure 19. For the polynomial $V(\delta, D)$ the polygon is shown in Figure 20. The truncated Equation (57) takes the form of

$$\delta^2 D^6 (u_{30}(t)\delta + u_{02}(t)) = 0.$$

From where

$$\delta_0(t) = c_0(t) = -\frac{u_{02}(t)}{u_{30}(t)} = \frac{6561(5t+2)\cdot(t+4)^2 \cdot (t-2)^2 (t+1) t^3}{128(8t^3 - 3t^2 + 24t + 8)^3}.$$

The only real root of the denominator is

$$t_4 = -\frac{3\left(13 + 16\sqrt{2}\right)^{\frac{1}{3}}}{8} + \frac{21}{8\left(13 + 16\sqrt{2}\right)^{\frac{1}{3}}} + \frac{1}{8} \approx -0.3111842957. \qquad (58)$$

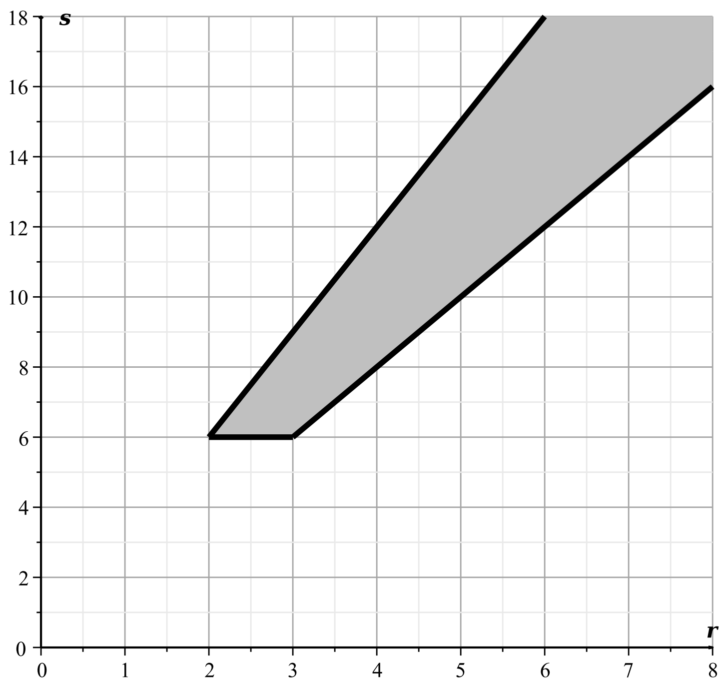

Figure 19. The Newton's polygon of the polynomial $\delta^2 D^6 V(\delta, D)$.

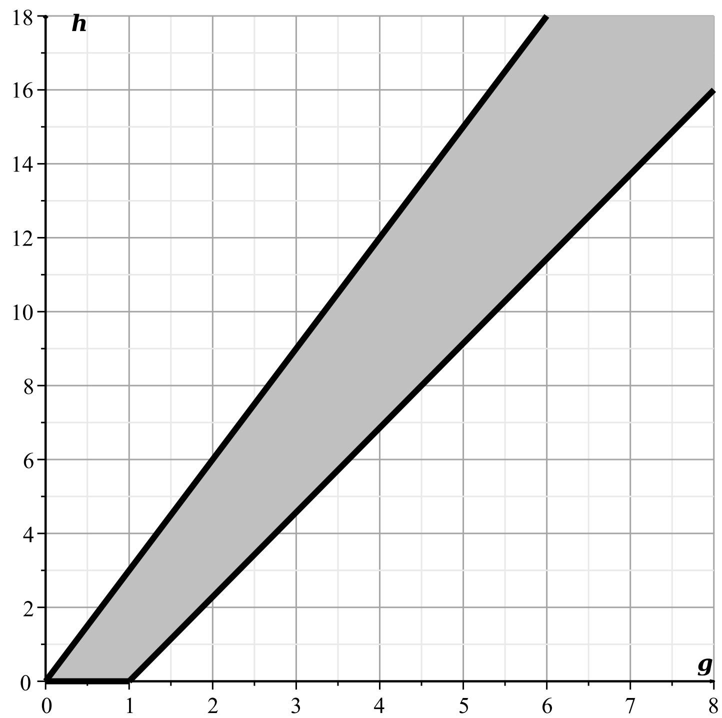

Figure 20. Newton's polygon of the polynomial $V(\delta, D)$.

In this case.

$$V_{g,h}(t) = V_{p+q-2,2p+3q-6}(t) = u_{p,q}(t).$$

After substitution $\delta = \delta_0(t) + \xi$, into the polynomial $V(\delta, D)$, we obtain

$$W(\xi, D) = V(\delta_0(t) + \xi, D).$$

When $\xi = 0$, the polynomial $W(0, D)$ is calculated using the command $coeff\ (M(0,D),\ 0)$ [27]. The quotient at D of degree zero is zero. The coefficient on the first degree of D is obtained by

$$a(t) = coeff\ (M(0,D),\ D, 1) = \frac{1594323t^6(5t+2)^2(t+4)^4(t-2)^4(t+1)^4}{2}.$$

Therefore, Theorem 1 in [1] is applied to equation $W(\xi, D) = 0$, and according to it a solution is

$$\xi = \sum_{k=1}^{\infty} c_k(t) D^k. \tag{59}$$

When we get the truncated equation $u_{30}(t) \cdot \xi + a(t) \cdot D = 0$, then it follows

$$\xi = -\frac{a(t)}{u_{30}(t)} \cdot D = \frac{10460353203t^{11}(t+1)(5t+2)(t+4)^2(t-2)^2(t^2-16t-8)^6}{16384(8t^3-3t^2+24t+8)^3} \cdot D = c_1(t)D.$$

Now let's go back and get an approximation

$$\varepsilon = \delta D^2 = (\delta_0(t) + \xi)D^2 \approx \delta_0(t)D^2 + c_1(t)D^3, \tag{60}$$

$$B_2 = (\delta_0(t) + \xi)D^3 \approx \delta_0(t)D^3 + c_1(t)D^4. \tag{61}$$

Therefore, from the formula (51) we get

$$A_1 = B_1 = b_1(t) + \delta_0(t)D^2 + c_1(t)D^3, \tag{62}$$

$$A_2 = B_1 + B_2 = b_1(t) + \delta_0(t)D^2 + (c_1(t) + \delta_0(t))D^3 + c_1(t)D^4,$$
$$A_3 = B_3 = b_2(t) = \frac{11t^3 - 48t^2 - 48t - 16}{6t(t^2 - 16t - 8)}. \tag{63}$$

The curves (62) and (63) at $D = \pm 0.1$ are shown in Figures 21 and 22, respectively. The gaps in these curves are the neighborhoods of the point t_4 of (58). They can be filled in if instead of substituting (51) we do

$$B_1 = b_1(t) + \varepsilon, \ B_3 = b_2(t) + \varepsilon.$$

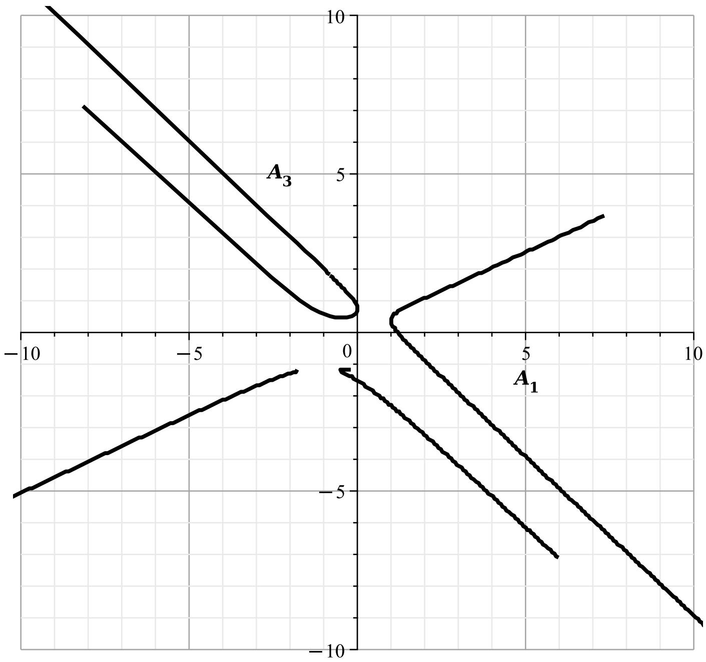

Figure 21. Curve (62) and (63) at $D = 1/10$.

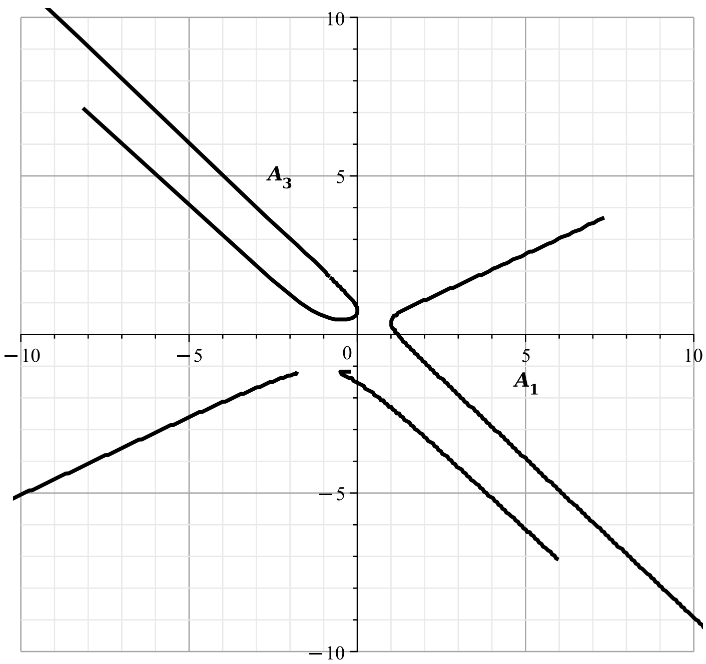

Figure 22. Curve (62) and (63) at $D = -1/10$.

The closeness of these curves to the curve of Figure 17 confirms the correctness of the found parametric expansion of the (59) of the variety Ω near the curve of singular points.

According to (61) and (62) branches F_i intersect curve $\mathcal{F}$ with singularity of type

$$\sqrt{\frac{A_1 - b_1(t)}{\delta_0(t)}} \approx \left(\frac{B_2}{\delta_0(t)}\right)^{1/3}.$$

- Result of Section 4

Theorem 3. *Near the curve $\mathcal{F}$ of singular points the variety Ω has one singular parametric expansions (59)–(61) and (63). They represent parts of branches $F_1^{\pm}$, F_2, $F_3^{\pm}$. At $A_1 = A$ they coincide with curve $\mathcal{F}$, having points of curve $\mathcal{F}$ as singular points.*

5. The Variety Ω at Infinity

The number of branches of the variety Ω at infinity exceeds their number near its singularities. Their complete study would exceed 7 sections on branches in the finite domain (4 Section in [1] and 3 Section here). Therefore, we study here only those branches corresponding to the first nonlinear polynomial multiplier included in the truncated polynomial in degree one.

5.1. Reducing the Study at Infinity to the Study in the Finite Domain

In the polynomial $S(\mathbf{A}) = Q(\mathbf{s})$, we do a power transformation

$$A_1 = B_1 B_3, \quad A_2 = B_2 B_3, \quad A_3 = B_3. \tag{64}$$

The resulting polynomial is divided by B_3^{12} and factorized, we get

$$S(B_1B_3, B_2B_3, B_3)/B_3^{12} = -\frac{1}{531441}\widetilde{T}(B_1, B_2, B_3).$$

In the sum $\widetilde{T}(B_1,\ B_2, B_3)$, which is not a polynomial, we substitute

$$B_1 = C_1,\ B_2 = C_2,\ B_3 = C_3^{-1}. \tag{65}$$

The resulting polynomial is $J(C_1,\ C_2, C_3) = \widetilde{T}(B_1,\ B_2, B_3)$. Let us explain the meaning of these transformations for the two-dimensional case, restricting ourselves to coordinates A_2 and A_3. The polyhedron of the original polynomial $S(\mathbf{A})$ has the form shown in Figure 23.

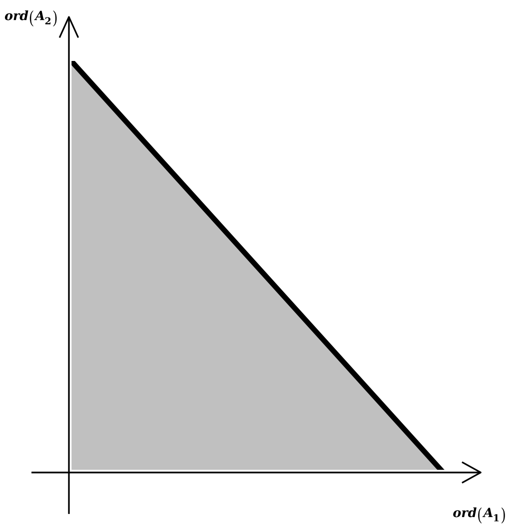

Figure 23. Projection of the polyhedron of the polynomial $S(\mathbf{A})$ onto the plane $ordA_2$, $ordA_3$.

After replacing (64), it takes the form shown in Figure 24.

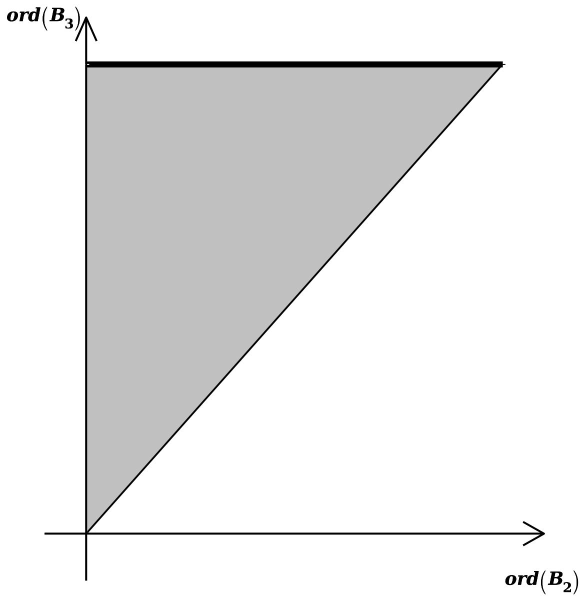

Figure 24. Projection of the polyhedron of the polynomial $S(B_1 B_3, B_2 B_3, B_3)$ in coordinates $ordB_2$, $ordB_3$.

The polyhedron of sum $T(\mathbf{B})$ is shown in Figure 25.

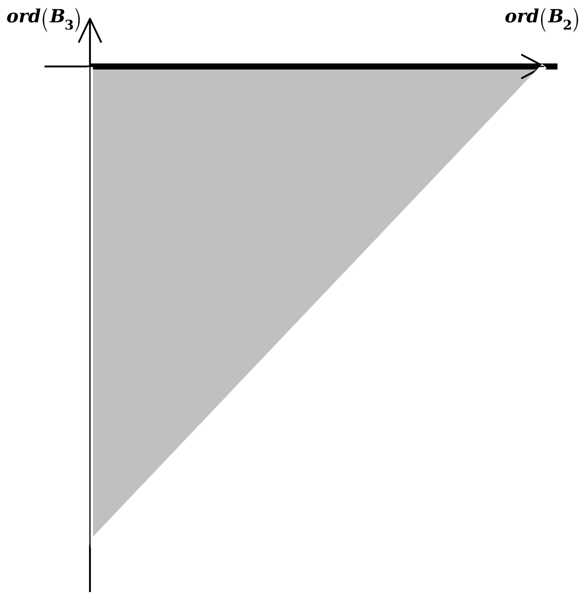

Figure 25. Projection of the polyhedron of sum $T(\mathbf{B})$.

After substituting (65), we get the polynomial $J(\mathbf{C})$, whose polyhedron is shown in Figure 26.

In Figures 23–26, the edge that corresponds to infinity in coordinates **A** is bolded.

Now we need to study the polynomial $J(\mathbf{C})$ at $C_1, C_2 \longrightarrow const$, $C_3 \to 0$. For this purpose, we compute the Newton's polyhedron Γ_8 of the polynomial $J(\mathbf{C})$. Its graph is shown in Figure 27.

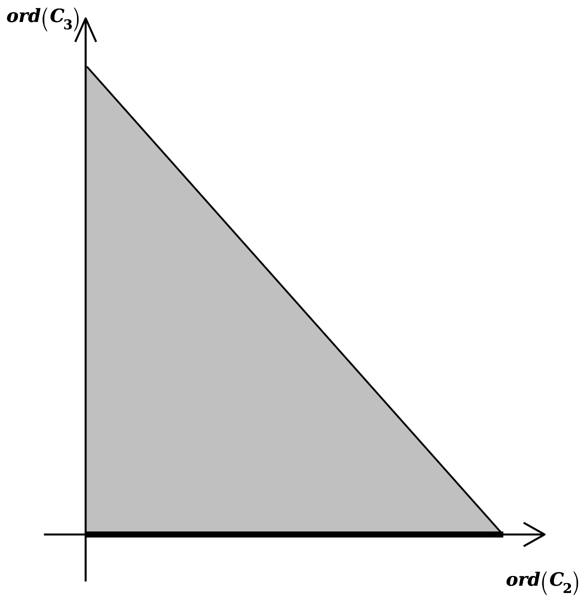

Figure 26. Projection of the polyhedron of the polynomial $J(\mathbf{C})$.

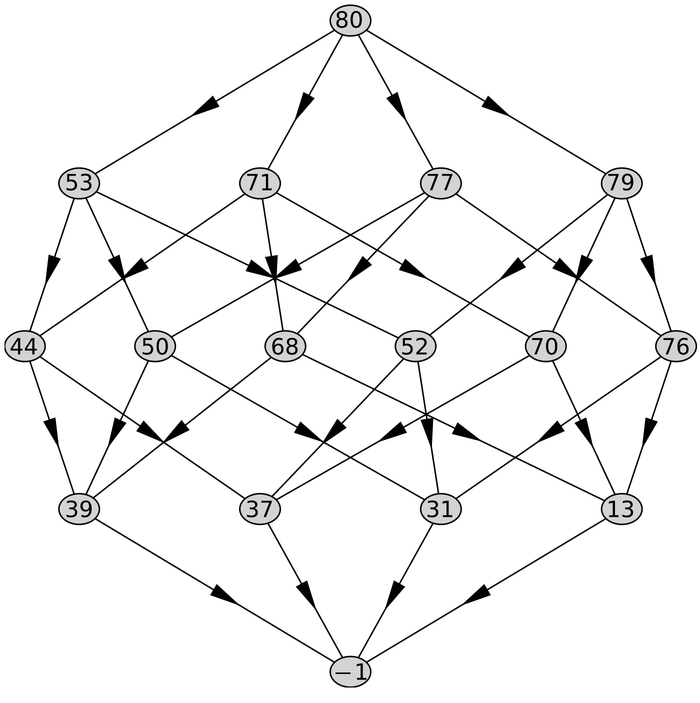

Figure 27. Graph of polyhedron Γ_8.

It has 4 two-dimensional faces with external normals

$$N_{53} = (1,1,1),\ N_{71} = (-1,0,0),\ N_{77} = (0,-1,0),\ N_{79} = (0,0,-1).$$

Since C_1 and $C_2 \to$ const, and $C_3 \to 0$, we select the only normal $N_{79} = (0,0,-1)$, which has only the third coordinate negative. After factorization, the corresponding truncated polynomial $ftr79$ has the form:

$$ftr79 = 1024\Big(C_1^2 - 4C_1C_2 + C_2^2 - 2C_1 - 2C_2 - 2\Big)^2 \times \Big(C_1^2 - 4C_1C_2 + C_2^2 + 4C_1 + 4C_2 - 8\Big)^2 (C_1 + C_2 + 2)^2 (C_1 + C_2 - 1)^2. \quad (66)$$

We will devote a separate subsection to each of its multipliers.

5.2. The First Multiplier in (66)

Multiplier

$$f_1 = C_1^2 - 4C_1C_2 + C_2^2 - 2C_1 - 2C_2 - 2$$

does not factorize in the field of rational numbers, but does factorize in the extension of that field with $\alpha = \sqrt{3}$. It is the product of two linear forms $f_1 = D_1D_2$, and we will consider the whole thing as a coordinate substitution, where

$$D_1 = C_1 - C_2\left(\sqrt{3}+2\right) - \left(\sqrt{3}+1\right),$$

$$D_2 = C_1 + C_2\left(\sqrt{3}-2\right) + \left(\sqrt{3}-1\right),$$

$$D_3 = C_3.$$

and put $\mathbf{D} = (D_1, D_2, D_3)$. Its inverse substitution is

$$C_1 = \frac{(-2+\sqrt{3})\sqrt{3}}{6}D_1 + \frac{(2+\sqrt{3})\sqrt{3}}{6}D_2 - 1,$$

$$C_2 = -\frac{\sqrt{3}}{6}D_1 + \frac{\sqrt{3}}{6}D_2 - 1, \;\; C_3 = D_3. \tag{67}$$

We substitute it into the polynomial $J(\mathbf{C})$ and get the polynomial $S_1(\mathbf{D}) = J(\mathbf{C})$. For the polynomial $S_1(\mathbf{D})$, we compute its Newton's polyhedron Γ_9.

Its graph is shown in Figure 28. It has 11 two-dimensional faces with external normals

$$N_{5645} = (-1,0,-1), N_{6101} = (-1,-1,-1), N_{13631} = (0,-1,-1), N_{116147} = (1,1,1),$$
$$N_{122463} = (-1,0,0), N_{122607} = (0,3,2), N_{124121} = (0,1,0), N_{133225} = (0,0,-1),$$
$$N_{150921} = (3,0,2),\; N_{164051} = (1,0,0), N_{175461} = (0,-1,0).$$

Since D_1, D_2 and $D_3 \to 0$, we select the only normal that has all coordinates negative. This is $N_{6101} = (-1,-1,-1)$. It corresponds to the truncated polynomial

$$\begin{aligned} ftr6101 = {} & -23328(-2+\alpha)(288\alpha D_1^4D_3^2 - 128\alpha D_1^3D_2^3 + 256\alpha D_1^2D_2^4 - 288\alpha D_1^2D_2^2D_3^2 \\ & +1728\alpha D_1D_2^3D_3^2 - 1782\alpha D_1D_2D_3^4 - 4320\alpha D_2^4D_3^2 + 4212\alpha D_2^2D_3^4 - 2916\alpha D_3^6 + 64D_1^4D_2^2 \\ & -576D_1^4D_3^2 - 256D_1^3D_2^3 + 432D_1^3D_2D_3^2 + 448D_1^2D_2^4 - 576D_1^2D_2^2D_3^2 + 1053D_1^2D_3^4 \\ & +3024D_1D_2^3D_3^2 - 3564D_1D_2D_3^4 - 7488D_2^4D_3^2 + 7371D_2^2D_3^4 - 5832D_3^6). \end{aligned}$$

After the power transformation

$$D_1 = M_1M_3, D_2 = M_2M_3, D_3 = M_3, \tag{68}$$

we get $ftr6101 = F_{10}(M_1, M_2) \cdot M_3^6$, where

$$\begin{aligned} F_{10}(M_1, M_2) = {} & (-128\alpha M_1^3M_2^3 + 256\alpha M_1^2M_2^4 + 64M_1^4M_2^2 - 256M_1^3M_2^3 + 448M_1^2M_2^4 + 288\alpha M_1^4 \\ & -288\alpha M_1^2M_2^2 + 1728\alpha M_1M_2^3 - 4320\alpha M_2^4 - 576M_1^4 + 432M_1^3M_2 - 576M_1^2M_2^2 + 3024M_1M_2^3 \\ & -7488M_2^4 - 1782\alpha M_1M_2 + 4212\alpha M_2^2 + 1053M_1^2 - 3564M_1M_2 + 7371M_2^2 - 2916\alpha - 5832)M_3^6. \end{aligned}$$

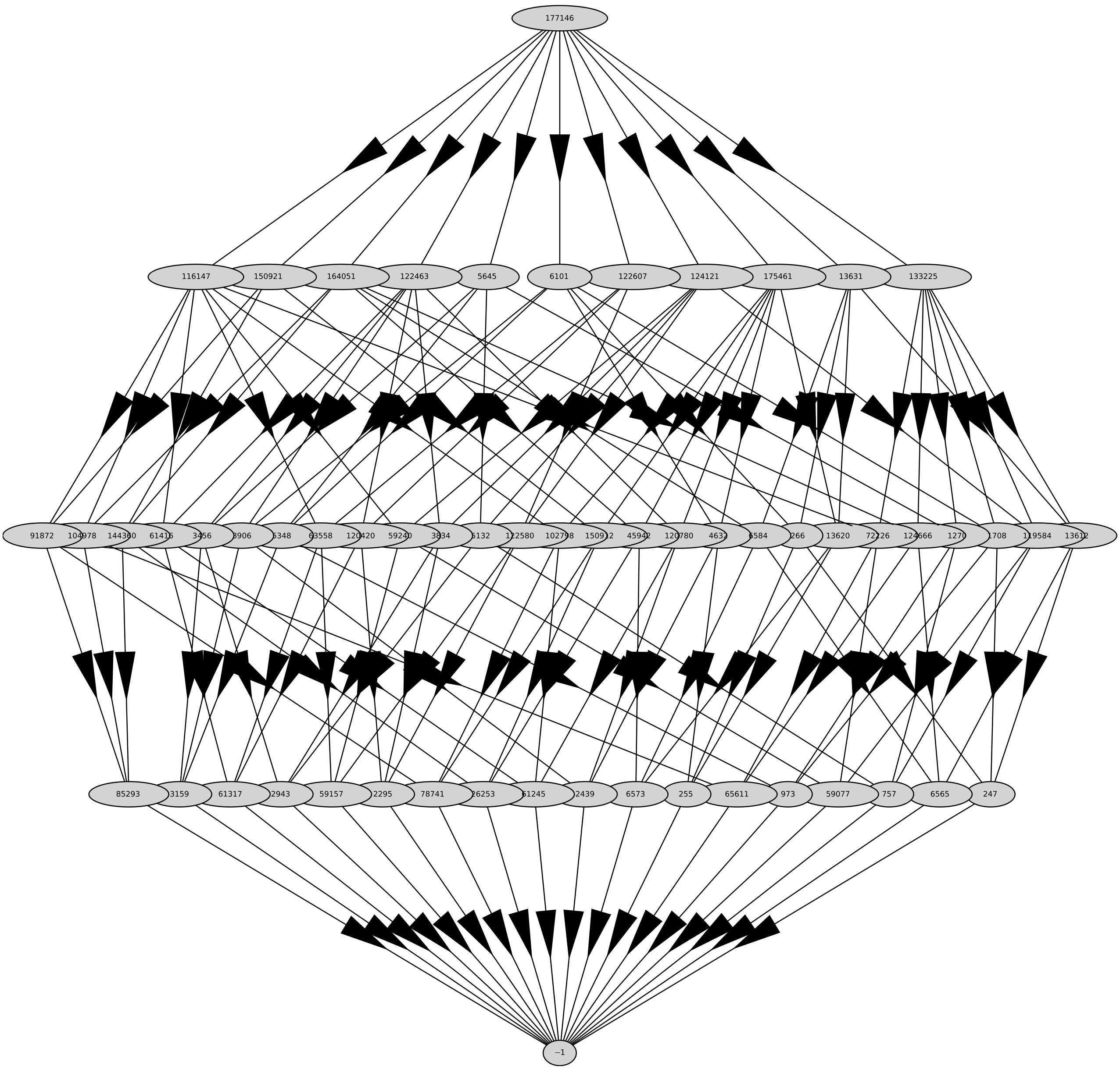

Figure 28. Graph of polyhedron Γ_9.

The curve $F_{10} = 0$ has genus 0, and parameterization

$$\begin{aligned} M_1 = b_1(t) = -(3(35677231 + 53951067\alpha)(2260415192296864844349440000\alpha t^3 \\ + 312865218289492809482240000t^4 + 2969245022145483876802560 0\alpha t^2 + \\ 2752150186688110972108800t^3 + 194709074673587078659795 20\alpha t \\ + 205018459636432060173312000t^2 + 431078468622082108802982\alpha \\ + 552278881165356491537664t + 297648199997558112581626 5))/ \\ (812429853035062962611 20t\times \\ \times (2346560t + 357378 + 757301\alpha)(56960\alpha t + 204800t^2 - 26880t + 25527)), \quad (69) \end{aligned}$$

$$
\begin{aligned}
M_2 = b_2(t) = -((&-327921327 + 145854781\alpha)(845687115552141577420800000\alpha t^3 \\
&+ 1527829243442567000883232000 0t^4 - 582100343712291111321600\alpha t^2 \\
&- 837947563507193362513139200 0t^3 + 225971250125139024049161 60\alpha t \\
&+ 129684691757394325122969600t^2 + 86215693724416421760 5964\alpha \\
&+ 7644517497617158949374080t + 429493664258516213443937 7))/ \\
&(12398051292851259813 8\left(56960\alpha t + 204800t^2 - 26880t + 25527\right) \\
&(56960t + 8509\alpha)(-8509 + 8960t)),
\end{aligned} \tag{70}
$$

and is shown in Figure 29.

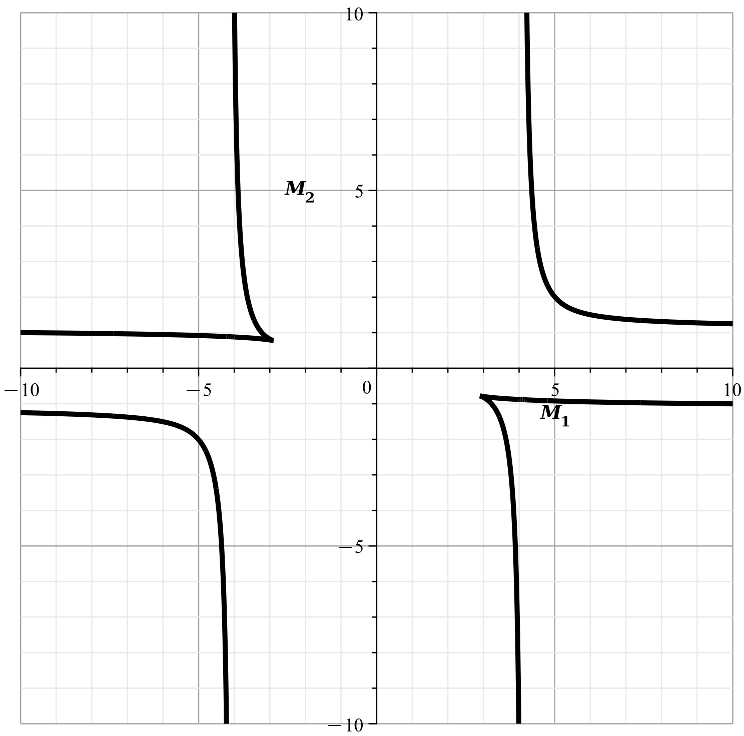

Figure 29. Curve $F_{10} = 0$.

In (70), the denominator has 2 real roots

$$t_1 = 0, t_2 \approx -0.7112802609. \tag{71}$$

In (70), the denominator has 2 real roots

$$t_3 = \frac{8509}{8960} \approx 0.9496651786, t_4 = -\frac{8509\sqrt{3}}{56960} \approx -0.2587433343 \tag{72}$$

In fact, the parametric expansion of the variety Ω can also be calculated here. To do this, we perform a power transformation (68) to the polynomial $S_1(\mathbf{D})$ and similarly to (16) get the polynomial

$$-\frac{T(\mathbf{M})}{M_3^6} = \sum_{k=0}^{6} T_k(M_1, M_2)M_3^k,$$

where $\mathbf{M} = (M_1, M_2, M_3)$. We substitute into the polynomials $T_k(M_1, M_2)$ according to (69) and (70)

$$M_1 = b_1(t) + \varepsilon, \quad M_2 = b_2(t) + \varepsilon. \tag{73}$$

We obtain the polynomial $u(\varepsilon,\ M_3) = -T(M_1, M_2, M_3)/M_3^6$ with coefficients depending on t via $b_1(t)$ and $b_2(t)$. In this polynomial.

$$u(\varepsilon, M_3) = \sum_{k=0}^{6} T_k(b_1+\varepsilon, b_2+\varepsilon)M_3^k = \sum_{p,q\geq 0} u_{pq}\varepsilon^p M_3^q,$$

where

$$u_{pq} = \sum_{p_1+p_2=p\geq 1} \frac{1}{p_1!p_2!}\cdot\frac{\partial^p T_q}{\partial M_1^{p_1}\partial M_2^{p_2}}, \tag{74}$$

when $M_i = b_i(t)$, $i = 1, 2$, $p_1, p_2 \geq 0$, $p \geq 1$,

Specifically

$$\begin{aligned}
u_{00} \equiv& 0,\\
u_{10} =& \frac{\partial T_0(M_1, M_2)}{\partial M_1} + \frac{\partial T_0(M_1, M_2)}{\partial M_2} =\\
& 2985984\alpha M_1^4 M_2 + 5971968\alpha M_1^3 M_2^2 - 5971968\alpha M_1^2 M_2^3 - 2985984\alpha M_1 M_2^4\\
& - 5971968 M_1^4 M_2 - 2985984 M_1^3 M_2^2 - 2985984 M_1^2 M_2^3 - 5971968 M_1 M_2^4\\
& - 97417728\alpha M_1^3 + 30233088\alpha M_1^2 M_2 - 30233088\alpha M_1 M_2^2 + 97417728\alpha M_2^3\\
& + 167961600 M_1^3 - 47029248 M_1^2 M_2 - 47029248 M_1 M_2^2 + 167961600 M_2^3\\
& + 49128768\alpha M_1 - 49128768\alpha M_2 - 56687040 M_1 - 56687040 M_2 \stackrel{\text{def}}{=} H(b_1(t), b_2(t)),\\
u_{01} =& T_1(b_1(t), b_2(t)) \stackrel{\text{def}}{=} G(b_1(t), b_2(t)).
\end{aligned}$$

Here the sign $\stackrel{\text{def}}{=}$ means new notation.

Indeed functions $u_{10}(t)$ and $u_{01}(t)$ have very complicated forms. So we omit them and give only some their properties.

The function $u_{10}(t)$ has two multiple roots

$$\begin{aligned}
t_5 &= -\frac{5553288233\sqrt{3}}{11245663040} - \frac{1415395569}{2811415760} + \frac{6443874209\sqrt{6}}{22491326080} + \frac{828223515\sqrt{2}}{2249132608} \approx -0.1361976710,\\
t_6 &= -\frac{5553288233\sqrt{3}}{11245663040} - \frac{1415395569}{2811415760} - \frac{6443874209\sqrt{6}}{22491326080} - \frac{828223515\sqrt{2}}{2249132608} \approx -2.581322779
\end{aligned} \tag{75}$$

of multiplicity 6, and the function $u_{01}(t)$ has the same roots of multiplicity 8. The denominators of the functions $u_{10}(t)$ and $u_{01}(t)$ each have four multiple roots of (71) and (72). By the implicit function theorem [1] (Theorem 1), the equation $u(\varepsilon, M_3) = 0$ has a solution as a power series on M_3

$$\varepsilon = \sum_{k=1}^{\infty} c_k(t) M_3^k, \tag{76}$$

where $c_k(t)$ are the rational functions of t, which are expressed through the coefficients $u_{pq}(t)$, which in turn are expressed through $b_1(t)$ and $b_2(t)$ according to (74). This expansion is valid for all values of t, except maybe the neighborhood of the roots of (75). In particular,

$$
\begin{aligned}
c_1(t) = -\frac{u_{01}}{u_{10}} = -\frac{G}{H} = \Big(3\Big(5000596138840425\sqrt{3} - 6061042284824999\Big)\Big(7108208938240\sqrt{3}t + 15394617958400t^2 + \\
1623856300668\sqrt{3} - 1613861867520t + 6311014555365)\Big(7108208938240\sqrt{3}t + 7197224345600t^2 \\
+541285433556\sqrt{3} + 7246825313280t + 1592795378919)\Big)/(42266280808032016\times \\
\times \Big(6937422113089418874060800 0\sqrt{3}t^2 + 30391107347952695246848000 0t^4 - 9161742964858347934924800\sqrt{3}t^2 \\
- 16842682857716693965209600 0t^3 + 2122351603209593191744512 0\sqrt{3}t \\
+ 8051169050590661262581760 0t^2 + 2536997119105720789608561\sqrt{3} \\
+2142879268179561462016128 0t + 5952963999951162251632530))
\end{aligned}
$$

where the denominator has no real roots. According to (76) approximately.

$$
\varepsilon \approx c_1(t) M_3.
$$

By the sequence of substitutions (64), (65), (67), (68) and (73), we return to the original coordinates, which at small $|M_3|$ by Ω are approximated with

$$
D_1 = (b_1(t) + c_1(t)M_3)M_3, \;\; D_2 = (b_2(t) + c_1(t)M_3)M_3, \;\; D_3 = M_3;
$$

$$
\begin{aligned}
C_1 &= -\frac{\left(2\sqrt{3} - 3\right)}{6} \cdot \left(-2c_1(t)\left(2\sqrt{3} + 3\right)M_3^2 + \left(b_1(t) - 4\sqrt{3}b_2(t) - 7b_2(t)\right)M_3 + 4\sqrt{3} + 6\right), \\
C_2 &= -\frac{\sqrt{3}}{6} \cdot \left(b_1(t)M_3 - b_2(t)M_3 + 2\sqrt{3}\right), \\
C_3 &= M_3;
\end{aligned}
$$

According to (65)

$$
B_1 = C_1, \;\; B_2 = C_2, \;\; B_3 = \frac{1}{C_3} = \frac{1}{M_3}.
$$

According to (64)

$$
A_1 = B_1 B_3 = b_1(t)\left(\frac{1}{2} - \frac{\sqrt{3}}{3}\right) + b_2(t)\left(\frac{1}{2} + \frac{\sqrt{3}}{3}\right) + c_1(t)M_3 - \frac{1}{M_3}, \tag{77}
$$

$$
A_2 = B_2 B_3 = -\frac{\sqrt{3}}{6} \cdot b_1(t) + \frac{\sqrt{3}}{6} \cdot b_2(t) - \frac{1}{M_3}, \tag{78}
$$

$$
A_3 = \frac{1}{M_3}.
$$

When $M_3 = -0.1$, the curve (77) and (78) is shown in Figure 30.
At $M_3 = 0.1$, the curve (77) and (78) is shown in Figure 31.

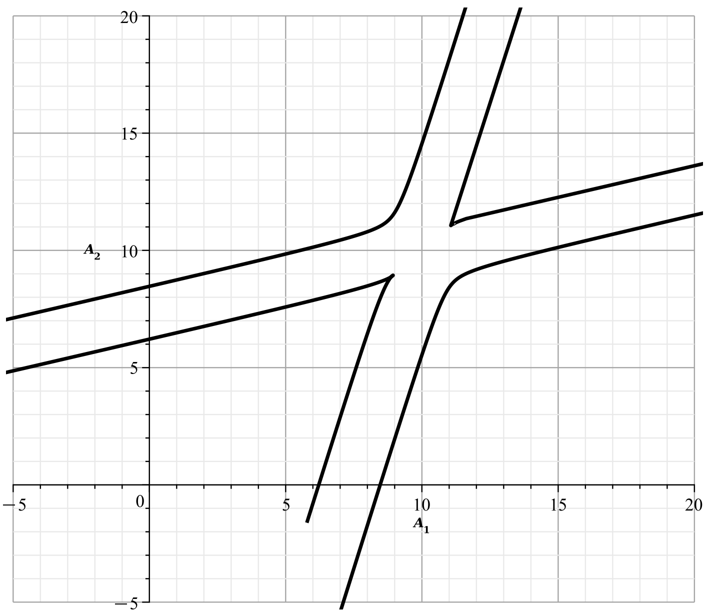

Figure 30. Curve (77) and (78) at $M_3 = -0.1$.

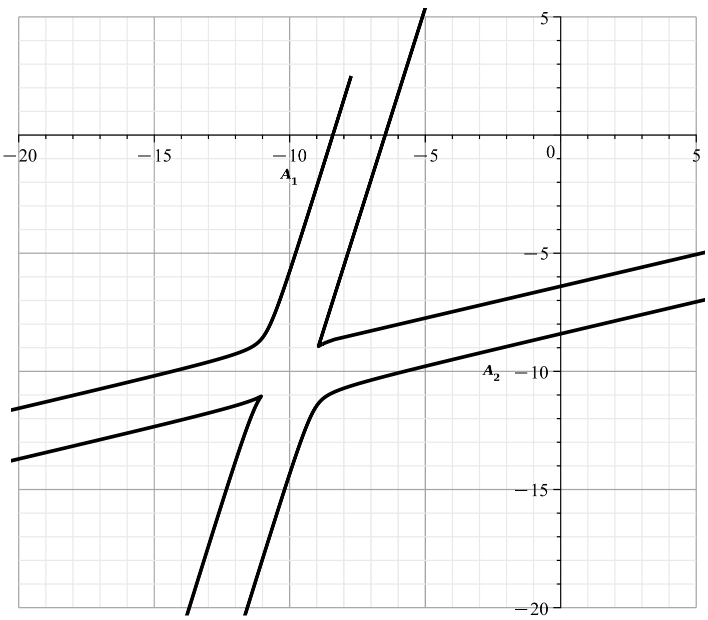

Figure 31. Curve (77), (78) at $M_3 = 0.1$.

5.3. Second Multiplier in (66)

Polynomial

$$f_2 = C_1^2 - 4C_1C_2 + C_2^2 + 4C_1 + 4C_2 - 8$$

does not factorize in the field of rational numbers, but does factorize in the extension of that field with $\alpha = \sqrt{3}$. It is the product of two linear forms $f_2 = D_1 \cdot D_2$, which we treat as coordinate substitutions

$$D_1 = C_1 - C_2\left(\sqrt{3}+2\right) + \left(2\sqrt{3}+2\right), D_2 = C_1 + C_2\left(\sqrt{3}-2\right) - \left(2\sqrt{3}-2\right), D_3 = C_3.$$

Its inverse substitution is

$$\begin{aligned} C_1 =& \frac{(-2+\sqrt{3})\sqrt{3}}{6}D_1 + \frac{(2+\sqrt{3})\sqrt{3}}{6}D_2 + 2, \\ C_2 =& -\frac{\sqrt{3}}{6}D_1 + \frac{\sqrt{3}}{6}D_2 + 2, \\ C_3 =& D_3. \end{aligned} \tag{79}$$

We substitute it into the polynomial $J(\mathbf{C})$ and get the polynomial $S_2(\mathbf{D}) = J(\mathbf{C})$. For the polynomial $S_2(\mathbf{D})$, we calculate Newton's polyhedron Γ_{10}.

Its graph is shown in Figure 32. It has 11 two-dimensional faces with external normals

$$N_{2049} = (-2,-2,-1), N_{6293} = (-2,0,-1),\ N_{52673} = (0,-2,-1),$$
$$N_{117443} = (1,1,1),\ N_{122283} = (-2,1,0),\ N_{122987} = (0,1,0),\ N_{123867} = (-1,0,0),$$
$$N_{132855} = (0,-1,0),\ N_{150911} = (1,0,0),\ N_{158011} = (0,0,-1),\ N_{164049} = (1,-2,0).$$

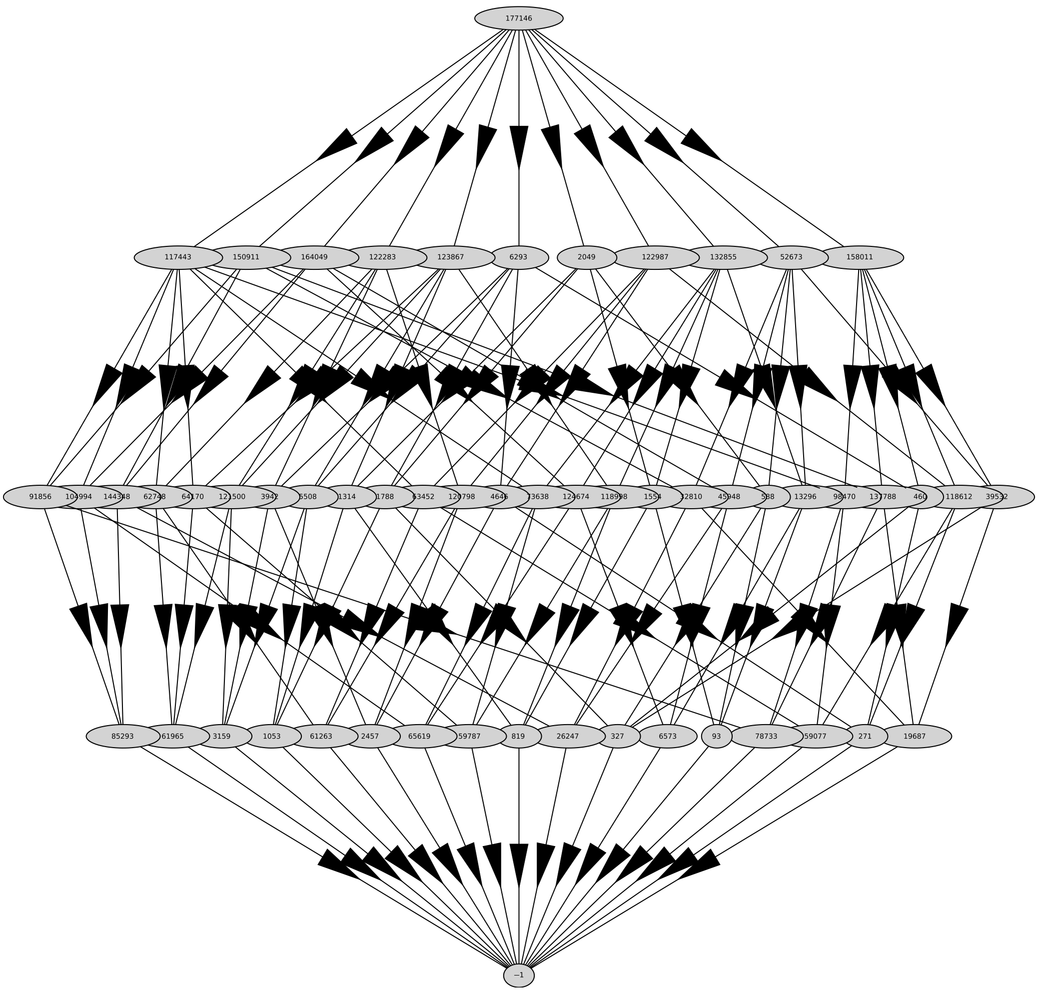

Figure 32. Graph of Polyhedron Γ_{10}.

Since D_1, D_2 and $D_3 \to 0$, we select the only normal that has all coordinates negative. This is the normal $N_{2049} = (-2, -2, -1)$. It corresponds to the truncated polynomial

$$ftr2049 = 26244(3\alpha D_3^2 + 3D_3^2 - 8D_1)^2(3\alpha D_3^2 - 3D_3^2 + 8D_2)^2.$$

Assume

$$x_1 = -8D_1 + (3\alpha + 3)D_3^2, \;\; x_2 = 8D_2 + (3\alpha - 3)D_3^2, \;\; x_3 = D_3.$$

The inverse transformation is

$$D_1 = -\frac{1}{8}x_1 + \frac{(3+3\alpha)}{8}x_3^2, D_2 = \frac{1}{8}x_2 + \frac{(3-3\alpha)}{8}x_3^2, D_3 = x_3. \tag{80}$$

We substitute it into the polynomial $S_2(\mathbf{D}) = J(\mathbf{C})$ and get the polynomial $S_3(\mathbf{x}) = J(\mathbf{C})$. For the polynomial $S_3(\mathbf{x})$, we compute Newton's polyhedron Γ_{11}.

Its graph is shown in Figure 33. It has 9 two-dimensional faces with external normals

$$N_{1919} = (-2, 0, -1), \; N_{2049} = (-2, -2, -1), \; N_{4559} = (0, -2, -1),$$
$$N_{12467} = (2, 2, 1), \; N_{14571} = (-1, 0, 0), \; N_{15095} = (0, 1, 0),$$
$$N_{15335} = (1, 0, 0), N_{18043} = (0, 0, -1), \; N_{19131} = (0, -1, 0).$$

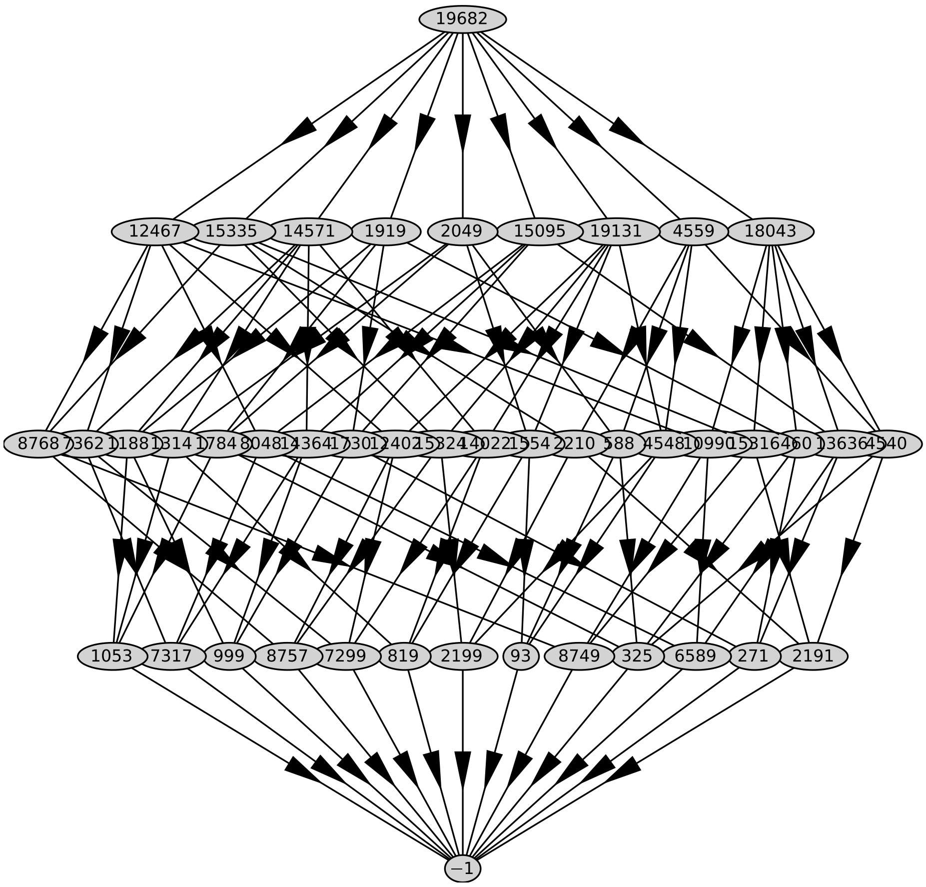

Figure 33. Graph of polyhedron Γ_{11}.

Since x_1, x_2, and $x_3 \to 0$, we select the only normal that has all coordinates negative. This is $N_{2049} = (-2, -2, -1)$. According to results of our program, it corresponds to the truncated polynomial

$$\begin{aligned} ftr2049 = &-13695130288521216\alpha^2x_3^8 - 126806761930752\alpha^2x_1^2x_3^4 - 507227047723008\alpha^2x_1x_2x_3^4 \\ &-126806761930752\alpha^2x_2^2x_3^4 + 41085390865563648x_3^8 + 380420285792256x_1^2x_3^4 \\ &+1521681143169024x_3^4x_1x_2 + 380420285792256x_2^2x_3^4 + 7044820107264x_1^2x_2^2. \end{aligned}$$

Doing the power transformation

$$x_1 = y_1y_3^2, x_2 = y_2y_3^2, x_3 = y_3 \tag{81}$$

and factorize we get

$$ftr2049 = -7044820107264\Big(18\alpha^2y_1^2 + 72\alpha^2y_1y_2 + 18\alpha^2y_2^2 - y_1^2y_2^2 + 1944\alpha^2 - 54y_1^2 - 216y_1y_2 - 54y_2^2 - 5832\Big)y_3^8.$$

If we substitute $\alpha = \sqrt{3}$, into the polynomial in parentheses, it is equal to $-y_1^2y_2^2$. Therefore, the power transformation (81) is substituted into the large polynomial $S_3(\mathbf{x})$ and divided by $(-7+4\sqrt{3})y_3^8$ we get the polynomial $S_4(\mathbf{y})$. We compute its Newton polyhedron Γ_{12}.

Its graph is shown in Figure 34. It has 11 two-dimensional faces with external normals

$$N_{4367} = (-1,0,0),\ N_{5643} = (-1,0,-1),\ \ N_{6099} = (-1,-1,-1),$$

$$N_{13629} = (0,-1,-1), N_{57365} = (0,-1,0), N_{111755} = (2,2,-1), N_{122609} = (0,1,0),$$

$$N_{124119} = (0,2,-1),\ N_{150923} = (1,0,0), N_{162019} = (0,0,1), N_{164049} = (2,0,-1).$$

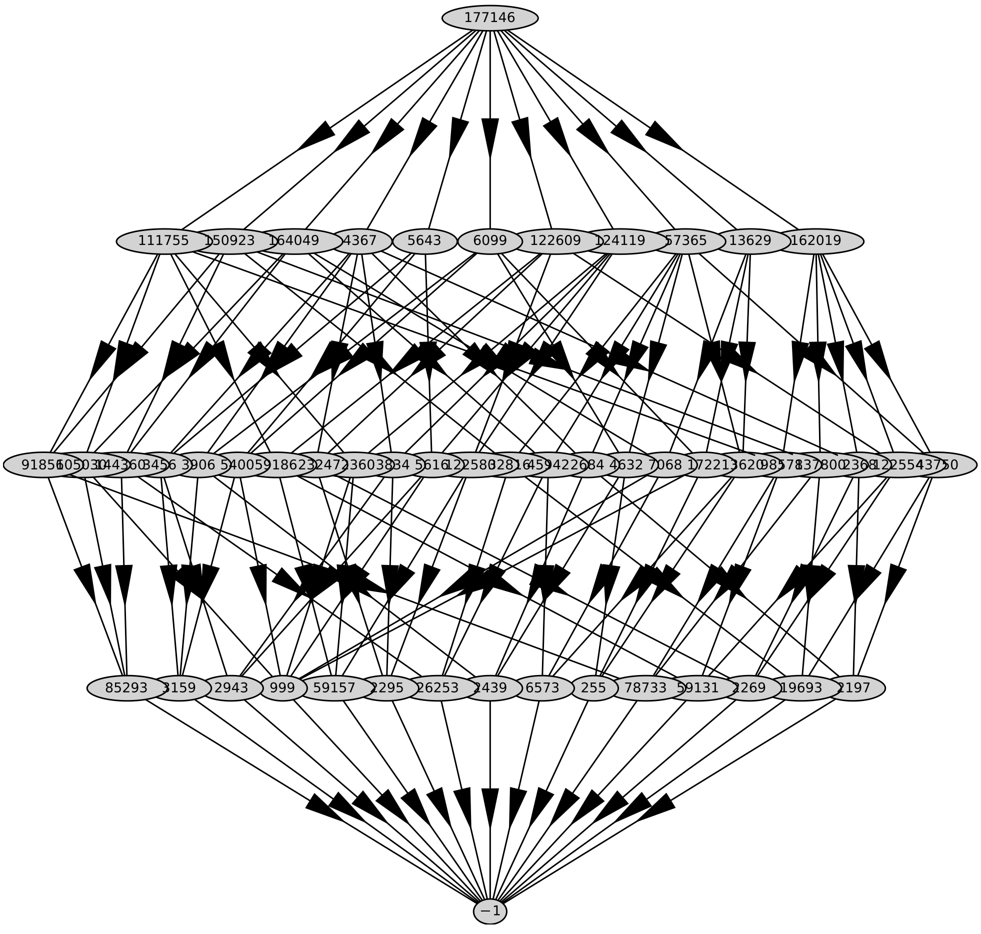

Figure 34. Graph of polyhedron Γ_{12}.

Since y_1, y_2, and $y_3 \to 0$, we select the only normal that has all coordinates negative. This is $N_{6099} = (-1,-1,-1)$. The corresponding shortening

$$\begin{gathered} ftr6099 = -28179280429056y_1^2y_2^2\alpha + 63403380965376y_1^2y_2y_3\alpha - 31701690482688y_1^2y_3^2\alpha \\ +232479063539712y_1y_2^2y_3\alpha - 507227047723008y_1y_2y_3^2\alpha + 285315214344192y_1y_3^3\alpha \\ -475525357240320y_2^2y_3^2\alpha + 1046155785928704y_2y_3^3\alpha - 570630428688384y_3^4\alpha \\ -49313740750848y_1^2y_2^2 + 105672301608960y_1^2y_2y_3 - 63403380965376y_1^2y_3^2 \\ +401554746114048y_1y_2^2y_3 - 887647333515264y_1y_2y_3^2 + 475525357240320y_1y_3^3 \\ -824243952549888y_2^2y_3^2 + 1806996357513216y_2y_3^3 - 998603250204672y_3^4 \end{gathered}$$

Doing the power transformation.

$$y_1 = z_1z_3, y_2 = z_2z_3, y_3 = z_3. \tag{82}$$

and we get

$$\begin{aligned} ftr6099pow = -1761205026816z_3^4(16\alpha z_1^2z_2^2 - 36\alpha z_1^2z_2 - 132\alpha z_1z_2^2 + 28z_1^2z_2^2 + 18\alpha z_1^2 \\ +288\alpha z_1z_2 + 270\alpha z_2^2 - 60z_1^2z_2 - 288z_1z_2^2 - 162\alpha z_1 - 594\alpha z_2 + 36z_1^2 + 504z_1z_2 + 468z_2^2 + 324\alpha \\ -270z_1 - 1026z_2 + 567) = -1761205026816z_3^4 \cdot F_{20}(z_1,z_2), \end{aligned}$$

where addition "*pow*" indicate that it is after power transformation.

If we substitute $\alpha = \sqrt{3}$ inside the brackets, we get the factorization

$$F_{20}(z_1,z_2) = \frac{(4\sqrt{3}+7)(3\sqrt{3}-2z_1+3)^2(-2z_2+3\sqrt{3}-3)^2}{4}.$$

The power transformation (82) we do in the polynomial $S_4(\mathbf{y})$, we get the polynomial $z_3^4S_5(\mathbf{z}) = S_4(\mathbf{y})$.

In $S_5(\mathbf{z})$, we substitute, introducing new variables L_i,

$$z_1 = L_1 + \frac{3(\sqrt{3}+1)}{2}, \quad z_2 = L_2 + \frac{3(\sqrt{3}-1)}{2}, \quad z_3 = L_3. \tag{83}$$

We get the polynomial $S_6(\mathbf{L}) = S_5(\mathbf{z})$ and for it we calculate Newton's polyhedron Γ_{13}. Its graph is shown in Figure 35. The computer computed the polyhedron Γ_{13} in 87 h and 23 min. It has 9 two-dimensional faces with exterior normals

$$\begin{gathered} N_{2165} = (0,-1,0),\ N_{5047} = (0,0,1),\ N_{5649} = (3,1,-1), \\ N_{6083} = (1,0,0),\ N_{11169} = (1,3,-1),\ N_{13283} = (-1,0,0), \\ N_{14775} = (-1,-1,-1), N_{17273} = (3,3,-1),\ N_{17669} = (0,1,0). \end{gathered}$$

Since L_1, L_2, and $L_3 \to 0$, we select the only normal that has all coordinates negative. This is $N_{14775} = (-1,-1,-1)$. The corresponding truncation

$$\begin{aligned} ftr14775 = -440301256704(\alpha+1)(384\alpha L_1^2L_2^2 + 672\alpha L_1^2L_2L_3 - 315\alpha L_1^2L_3^2 + 2688\alpha L_1L_2^2L_3 \\ + 5778\alpha L_1L_2L_3^2 + 756\alpha L_1L_3^3 - 2880\alpha L_2^3L_3 - 3465\alpha L_2^2L_3^2 + 3024\alpha L_2L_3^3 + 972\alpha L_3^4 - 192L_1^3L_3 \\ + 640L_1^2L_2^2 + 1344L_1^2L_2L_3 - 315L_1^2L_3^2 + 4704L_1L_2^2L_3 + 9630L_1L_2L_3^2 + 1512L_1L_3^3 \\ - 4992L_2^3L_3 - 5985L_2^2L_3^2 + 5292L_2L_3^3 + 1620L_3^4) \end{aligned}$$

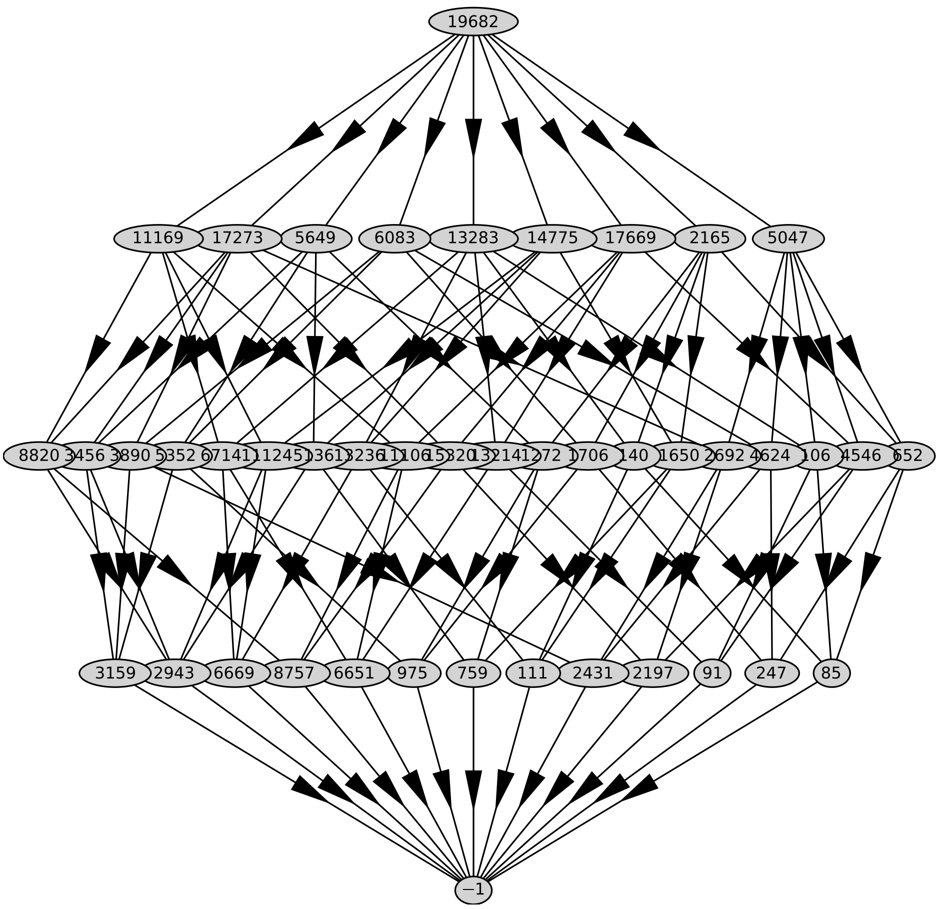

Figure 35. Graph of polyhedron Γ_{13}.

We do a power transformation.

$$L_1 = M_1 M_3,\ L_2 = M_2 M_3,\ L_3 = M_3. \tag{84}$$

Then

$$\begin{aligned} ftr14775pow = {} & -440301256704(\alpha+1)(384\alpha M_1^2 M_2^2 + 672\alpha M_1^2 M_2 + 2688\alpha M_1 M_2^2 - 2880\alpha M_2^3 \\ & + 640 M_1^2 M_2^2 - 315\alpha M_1^2 + 5778\alpha M_1 M_2 - 3465\alpha M_2^2 - 192 M_1^3 + 1344 M_1^2 M_2 + 4704 M_1 M_2^2 \\ & - 4992 M_2^3 + 756\alpha M_1 + 3024\alpha M_2 - 315 M_1^2 + 9630 M_1 M_2 - 5985 M_2^2 + 972\alpha + 1512 M_1 + 5292 M_2 + 1620) M_3^4 \\ & = -440301256704(\alpha+1) M_3^4 \cdot F_{40}. \end{aligned}$$

The curve $F_{40} = 0$ has genus 0, and parameterization

$$\begin{aligned} [M_1, M_2] = [b_1(t), b_2(t)] = {} & \\ = \Big\{ & (5 - 3\sqrt{3})(1344\sqrt{3}t^2 + t^3 + 2352t^2 - 345047040\sqrt{3} - 597639168)/(256t^2), \\ & (265 - 153\sqrt{3})(-t^3 + 3612672\sqrt{3}t + 1380188160\sqrt{3} + 6257664t + 2390556672)/(49152t) \Big\} \end{aligned} \tag{85}$$

and is shown in Figure 36.

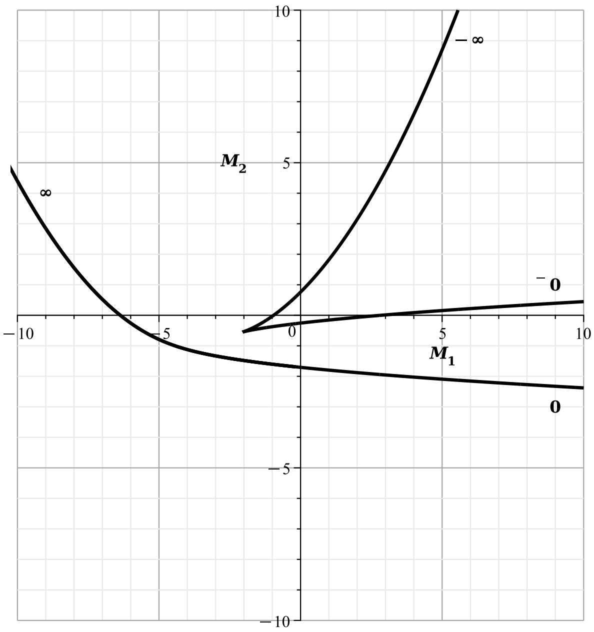

Figure 36. Curve $F_{40}(M_1, M_2) = 0$.

Figure 36 shows the limiting values of $t = -\infty, -0, +0, +\infty$, when the branch goes to infinity. The approximate values of the zeros of the numerators in (85) are. $t_1 \approx 481,241,\ t_2 \approx -4623,972,\ t_3 \approx -537,144$ for $b_1(t)$ and $t_4 \approx 3715,095,\ t_5 \approx -3328,446,\ t_6 \approx -386,649$ for $b_2(t)$.

We do a power transformation (84) to $S_6(\mathbf{L})$, and we get

$$-M_3^4 E(\mathbf{M}) = S_6(\mathbf{L}) = -M_3^4 \sum_{k=0} E_k(M_1, M_2) M_3^k,$$

where

$$\begin{aligned}
E_0 = {} & 440301256704(1+\alpha)(384\alpha M_1^2 M_2^2 + 672\alpha M_1^2 M_2 + 2688\alpha M_1 M_2^2 - 2880\alpha M_2^3 + 640 M_1^2 M_2^2 \\
& -315\alpha M_1^2 + 5778\alpha M_1 M_2 - 3456\alpha M_2^2 - 192 M_1^3 + 1344 M_1^2 M_2 + 4704 M_1 M_2^2 - 4992 M_2^3 + 756\alpha M_1 \\
& +3024\alpha M_2 - 315 M_1^2 + 9630 M_1 M_2 - 5985 M_2^2 + 972\alpha + 1512 M_1 + 5292 M_2 + 1620), \\
E_1 = {} & -2641807540224(1+\alpha)(112\alpha M_1^2 M_2 + 448\alpha M_1 M_2^2 - 480\alpha M_2^3 - 105\alpha M_1^2 + 1926\alpha M_1 M_2 \\
& -1155\alpha M_2^2 - 32 M_1^3 + 224 M_1^2 M_2 + 784 M_1 M_2^2 - 832 M_2^3 + 378\alpha M_1 + 1512\alpha M_2 - 105 M_1^2 + 3210 M_1 M_2 \\
& -1995 M_2^2 + 648\alpha + 756 M_1 + 2646 M_2 + 1080), \\
E_2 = {} & -55037657088(1+\alpha)(7680\alpha M_1^2 M_2^2 + 7728\alpha M_1^2 M_2 + 30912\alpha M_1 M_2^2 - 21600\alpha M_2^3 + 12800 M_1^2 M_2^2 \\
& +4599\alpha M_1^2 - 32778\alpha M_1 M_2 + 50589\alpha M_2^2 - 1440 M_1^3 + 15456 M_1^2 M_2 + 54096 M_1 M_2^2 - 37440 M_2^3 \\
& -17766\alpha M_1 - 71064\alpha M_2 + 4599 M_1^2 - 54630 M_1 M_2 + 87381 M_2^2 - 52002\alpha - 35532 M_1 - 124362 M_2 - 86670).
\end{aligned} \tag{86}$$

Into the polynomial $E(\mathbf{M})$ we substitute

$$M_1 = b_1(t), M_2 = b_2(t) + \varepsilon \tag{87}$$

according to (85). Then the polynomial $E(\mathbf{M})$ becomes a polynomial

$$u(\varepsilon, M_3) = \sum_{p,q \geq 0} u_{pq}(t) \varepsilon^p M_3^q,$$

whereby

$$u_{pq} = \frac{1}{p!} \cdot \frac{\partial^p E_q}{\partial M_2^p}(b_1(t), b_2(t)), \tag{88}$$

where $M_1 = b_1(t)$, $M_2 = b_2(t)$ according to (85). In particular, from (86)–(88) we obtain

$$u_{00} = E(b_1(t), b_2(t)) \equiv 0,$$

$$u_{10}(M_1, M_2) = \frac{\partial E_0}{\partial M_2} =$$

$$u_{10}(M_1, M_2) = \frac{\partial E_0}{\partial M_2} = 1761205026816(7+4\alpha)\Big(168\alpha M_1^2 + 336\alpha M_1 M_2 - 432\alpha M_2^2 + 128 M_1^2 M_2$$
$$-315\alpha M_2 - 168 M_1^2 + 336 M_1 M_2 - 720 M_2^2 + 189\alpha + 963 M_1 - 630 M_2 + 189)\Big)$$
$$= \frac{69984(571\alpha - 989)\left(384\alpha t - t^2 - 516096\alpha + 672t - 893952\right)^3 (t + 672 + 384\alpha)^3}{t^5},$$

$$u_{01} = E_1(b_1(t), b_2(t)) = ((729(151316\alpha - 262087)\Big(2304\alpha t + t^2 + 516096\alpha + 4032t + 893952\Big)$$
$$(384\alpha t - t^2 - 516096\alpha + 672t - 893952)^3 (t + 672 + 384\alpha)^4)/(256 t^6)),$$

$$u_{11}(M_1, M_2) = \frac{\partial E_1}{\partial M_2} = -5283615080448(5+3\alpha)(112\alpha M_1 M_2 - 192\alpha M_2^2 + 321\alpha M_1 - 315\alpha M_2$$
$$+56 M_1^2 + 224 M_1 M_2 - 336 M_2^2 + 189\alpha + 321 M_1 - 525 M_2 + 378) = \frac{1}{t^4}(209952(571\alpha - 989)\cdot$$
$$\cdot(-3333609787974746112 - 18912215826432 t^2 + 14958900893712384 t + t^6 + 1792 t^5 + 2681856 t^4$$
$$+1024\alpha t^5 + 1548288\alpha t^4 - 4600627200\alpha t^3 - 1924660508460318720\alpha - 10918972882944\alpha t^2$$
$$+8636525457702912\alpha t - 7968522240 t^3)(t + 672 + 384\alpha)^2),$$

$$u_{20}(M_1, M_2) = \frac{1}{2}\left(\frac{\partial^2 E_0}{\partial M_2^2}\right) = 880602513408(7+4\alpha)\Big(336\alpha M_1 - 864\alpha M_2 + 128 M_1^2 - 315\alpha$$
$$+336 M_1 - 1440 M_2 - 630) = (6879707136(\alpha - 2)(144\alpha t - t^2 - 69120\alpha + 240t - 119808)\times$$
$$\times\Big(528\alpha t + t^2 + 963072\alpha + 912t + 1668096\Big)(t + 240 + 144\alpha)(-t + 912 + 528\alpha)/t^4).$$

According to [1] (Theorem 1), the solution to the equation $u(\varepsilon, M_3) = 0$ has the form, $\varepsilon = \sum_{k=1}^{\infty} c_k(t) M_3^k$, where

$$c_1(t) = -\frac{u_{01}(t)}{u_{10}(t)} = \frac{(-265 + 153\alpha)\left(2304\alpha t + t^2 + 516096\alpha + 4032t + 893952\right)(t + 672 + 384\alpha)}{49152t}.$$

The denominators in $c_1(t)$ and in (85) have root $t = 0$.

Now let's go back and approximately from (84) obtain

$$L_1 = b_1(t) M_3, \quad L_2 = (b_2(t) + c_1(t) M_3) M_3 = b_2(t) M_3 = b_2(t) M_3 + c_1(t) M_3^2, \quad L_3 = M_3.$$

Substitute that into (83).

$$z_1 = b_1(t) M_3 + \frac{3(\sqrt{3}+1)}{2}, \; z_2 = b_2(t) M_3 + c_1(t) M_3^2 + \frac{3(\sqrt{3}-1)}{2}, \; z_3 = M_3. \tag{89}$$

We substitute the values of (89) into (82) and obtain

$$\begin{aligned} y_1 &= b_1(t) M_3^2 + \frac{3(\sqrt{3}+1)}{2} M_3, \\ y_2 &= c_1(t) M_3^3 + b_2(t) M_3^2 + \frac{3(\sqrt{3}-1)}{2} M_3, \\ y_3 &= M_3. \end{aligned} \tag{90}$$

We substitute the values of (90) into (81) and obtain

$$\begin{aligned} x_1 =& y_1 y_3^2 = b_1(t) M_3^4 + \frac{3(\sqrt{3}+1)}{2} M_3^3, \\ x_2 =& y_2 y_3^2 = c_1(t) M_3^5 + b_2(t) M_3^4 + \frac{3(\sqrt{3}-1)}{2} M_3^3, \\ x_3 =& y_3 = M_3. \end{aligned} \tag{91}$$

We substitute the values of (91) into (80) and obtain

$$\begin{aligned} D_1 =& -\frac{1}{8} x_1 + \frac{(3+3\alpha)}{8} x_3^2 = -\frac{1}{8} b_1(t) M_3^4 - \frac{3\left(\sqrt{3}+1\right)}{16} M_3^3 + \frac{\left(3+3\sqrt{3}\right)}{8} M_3^2, \\ D_2 =& \frac{1}{8} x_2 + \frac{(3-3\alpha)}{8} x_3^2 = \frac{1}{8} c_1(t) M_3^5 + b_2(t) M_3^4 + \frac{3(\sqrt{3}-1)}{16} M_3^3 + \frac{(3-3\sqrt{3})}{8} M_3^2, \\ D_3 =& M_3. \end{aligned} \tag{92}$$

We substitute the values of (92) into (79) and obtain

$$\begin{aligned} C_1 =& \frac{\left(-2+\sqrt{3}\right)\sqrt{3}}{6} D_1 + \frac{\left(2+\sqrt{3}\right)\sqrt{3}}{6} D_2 + 2 = \frac{\left(2\sqrt{3}+3\right)}{48}\Big(c_1(t) M_3^5 - 4\sqrt{3} b_1(t) M_3^4 \\ & + 7 b_1(t) M_3^4 + 8 b_2(t) M_3^4 - 9 M_3^3 + 6\sqrt{3} M_3^3 + 18 M_3^2 - 12\sqrt{3} M_3^2 - 96 + 64\sqrt{3}\Big), \\ C_2 =& -\frac{\sqrt{3}}{6} D_1 + \frac{\sqrt{3}}{6} D_2 + 2 = \frac{\sqrt{3}}{48}\Big(c_1(t) M_3^5 + b_1(t) M_3^4 + 8 b_2(t) M_3^4 + 3\sqrt{3} M_3^3 - 6\sqrt{3} M_3^2 + 32\sqrt{3}\Big), \\ C_3 =& M_3. \end{aligned} \tag{93}$$

We substitute the values of (93) into (65) and obtain

$$\begin{aligned} B_1 =& C_1 = \frac{\left(2\sqrt{3}+3\right)}{48}(c_1(t) M_3^5 - 4\sqrt{3} b_1(t) M_3^4 + 7 b_1(t) M_3^4 + 8 b_2(t) M_3^4 - 9 M_3^3 \\ & + 6\sqrt{3} M_3^3 + 18 M_3^2 - 12\sqrt{3} M_3^2 - 96 + 64\sqrt{3}), \\ B_2 =& C_2 = \frac{\sqrt{3}}{48}\Big(c_1(t) M_3^5 + b_1(t) M_3^4 + 8 b_2(t) M_3^4 + 3\sqrt{3} M_3^3 - 6\sqrt{3} M_3^2 + 32\sqrt{3}\Big), \\ B_3 =& C_3^{-1} = \frac{1}{M_3}. \end{aligned} \tag{94}$$

Finally, we substitute the values of (94) into (64) and obtain values A_1, A_2, A_3:

$$\begin{aligned} A_1 =& B_1 B_3 = \frac{\left(2\sqrt{3}+3\right)}{48}(c_1(t) M_3^4 - 4\sqrt{3} b_1(t) M_3^3 + 7 b_1(t) M_3^3 + 8 b_2(t) M_3^3 \\ & - 9 M_3^2 + 6\sqrt{3} M_3^2 + 18 M_3 - 12\sqrt{3} M_3 - \frac{96}{M_3} + \frac{64\sqrt{3}}{M_3}), \end{aligned} \tag{95}$$

$$A_2 = B_2 B_3 = \frac{\sqrt{3}}{48}(c_1(t) M_3^4 + b_1(t) M_3^3 + 8 b_2(t) M_3^3 + 3\sqrt{3} M_3^2 - 6\sqrt{3} M_3 + \frac{32\sqrt{3}}{M_3}, \tag{96}$$

$$A_3 = B_3 = \frac{1}{M_3}.$$

We need them to drive figures. Figures 37 and 38 show the curves of (95) and (96) at $M_3 = 0.1$ and $M_3 = -0.1$, respectively.

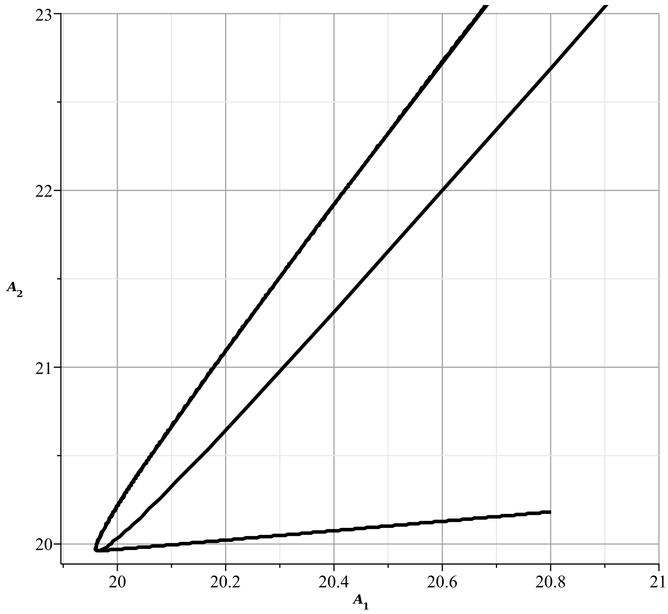

Figure 37. Curve (95) and (96) for $M_3 = 0.1$.

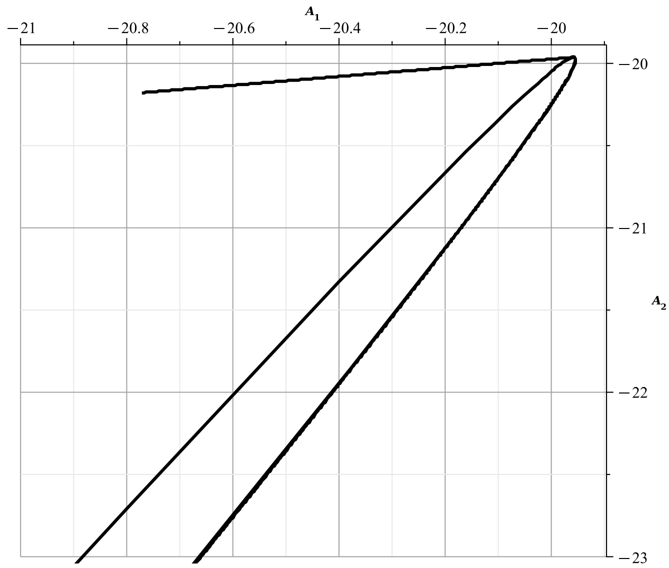

Figure 38. Curve (95) and (96) for $M_3 = -0.1$.

5.4. Third Multiplier in (66)

It is a linear multiplier defined with

$$f_3 = C_1 + C_2 + 2.$$

The substitution is

$$D_1 = C_1 + C_2 + 2, \;\; D_2 = C_2, \;\; D_3 = C_3.$$

Its inverse substitution is

$$C_1 = D_1 - D_2 - 2, \; C_2 = D_2, \;\; C_3 = D_3. \tag{97}$$

Let's consider all of this as a coordinate substitution in the polynomial $J(\mathbf{C})$. We substitute it into the polynomial $J(\mathbf{C})$ and get the polynomial $S_7(\mathbf{D}) = J(\mathbf{C})$. For the polynomial $S_7(\mathbf{D})$, we compute Newton's polyhedron Γ_{14}.

Its graph is shown in Figure 39. It has 7 two-dimensional faces with external normals

$$N_{701} = (-2, 0, -1),\ N_{1403} = (1, 1, 1),\ \ N_{1537} = (0, 0, -1),$$

$$N_{1679} = (0, -1, 0),\ N_{1935} = (-2, 1, 0),\ N_{2009} = (0, 1, 0),\ N_{2085} = (-1, 0, 0).$$

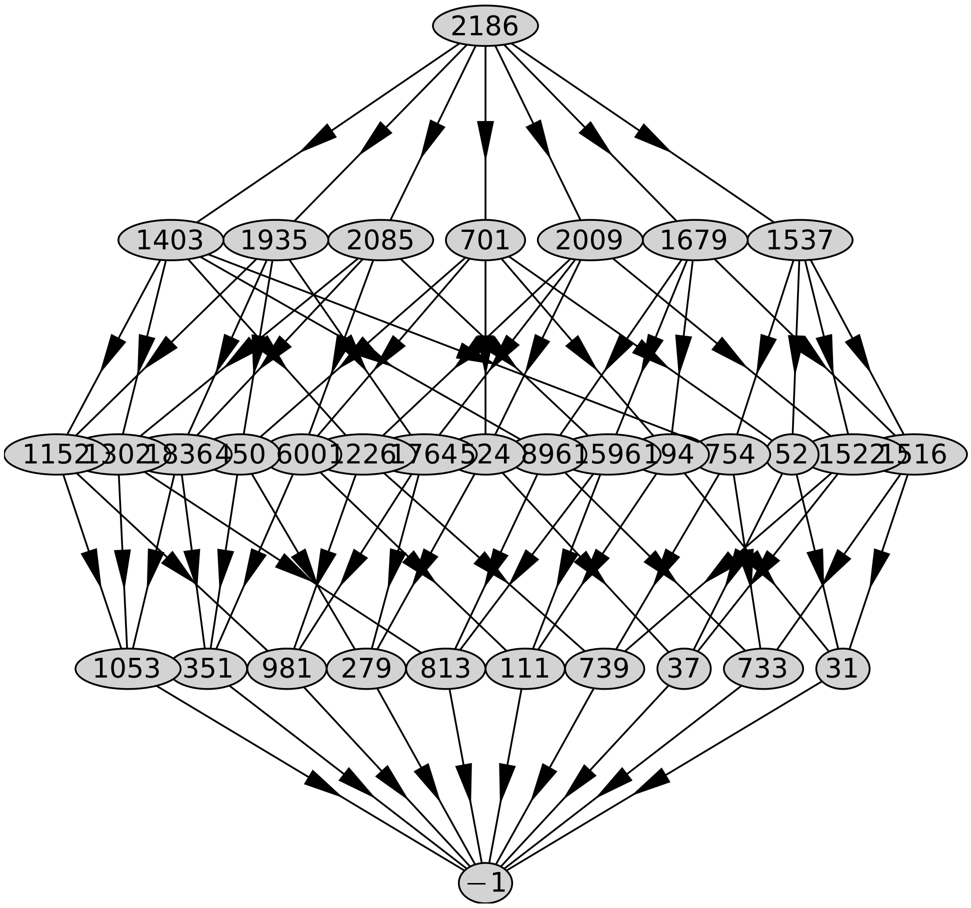

Figure 39. Graph of polyhedron Γ_{14}.

Since $D_1 \to 0$, $D_2 \to const$, and $D_3 \to 0$, we select the only normal whose first and third coordinates are negative. This is the normal $N_{701} = (-2, 0, -1)$. It corresponds to a truncated polynomial

$$ftr701 = 186624(D_2+1)^2\left(D_2^2+2D_2-2\right)^2 \times$$
$$\times\left(-48D_1D_2^2D_3^2+64D_1^2D_2^2-96D_1D_2D_3^2+27D_3^4+128D_1^2D_2-48D_1D_3^2+64D_1^2\right).$$

Do a power transformation.

$$D_1 = x_1x_3^2, D_2 = x_2, D_3 = x_3. \tag{98}$$

Then ftr701 after the power transformation (98) is

$$ftr701pow = 186624(x_2+1)^2\Big(x_2^2+2x_2-2\Big)^2 x_3^4\Big(64x_1^2x_2^2+128x_1^2x_2-48x_1x_2^2+64x_1^2-96x_1x_2-48x_1+27\Big)$$
$$= 186624(x_2+1)^2\Big(x_2^2+2x_2-2\Big)^2 x_3^4 \cdot F_{30}(x_1,x_2),$$

where $F_{30}(x_1,x_2) = 64x_1^2x_2^2+128x_1^2x_2-48x_1x_2^2+64x_1^2-96x_1x_2-48x_1+27$. The curve $F_{30}=0$ has genus 0, and parameterization

$$\{x_1,x_2\} = \{b_1(t),b_2(t)\} = \left\{\frac{81(29t+3)^2}{4(23283t^2+5466t+499)}, -\frac{27459t^2+8682t+787}{48(3t+2)(29t+3)}\right\} \quad (99)$$

and is shown in the Figure 40.

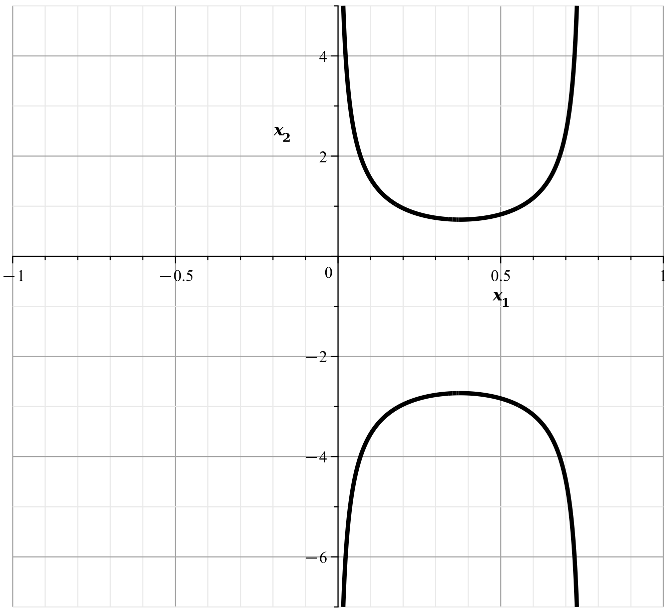

Figure 40. Curve $F_{30}(x_1,x_2)=0$.

The curve $F_{30}=0$ goes to infinity $x_2=\pm\infty$ at

$$t_1 = -\frac{3}{29} \approx -0.1034482759, \quad t_2 = -\frac{2}{3} \approx -0.6666666667. \quad (100)$$

We do a power transformation of (98) to the polynomial $S_7(\mathbf{D})$ and get the polynomial

$$x_3^4 W(\mathbf{x}) = S_7(\mathbf{D}) = x_3^4 \sum_{k=0} W_k(x_1,x_2)x_3^k.$$

where

$$\begin{aligned}
W_0 =&186624(x_2+1)^2\Big(x_2^2+2x_2-2\Big)^2\Big(64x_1^2x_2^2+128x_1^2x_2-48x_1x_2^2+64x_1^2-96x_1x_2-48x_1+27\Big),\\
W_1 =&559872(x_2+1)^2\Big(x_2^2+2x_2-2\Big)^2\Big(8x_1x_2^2+16x_1x_2+8x_1-9\Big),\\
W_2 =&-15552(x_2^2+2x_2-2)(512x_1^3x_2^6+6144x_1^3x_2^5+19968x_1^3x_2^4-2304x_1^2x_2^5+24064x_1^3x_2^3\\
&-10080x_1^2x_2^4+6144x_1^3x_2^2-13824x_1^2x_2^3-7680x_1^3x_2-8352x_1^2x_2^2+972x_1x_2^3-81x_2^4-4096x_1^3\\
&-4032x_1^2x_2+4860x_1x_2^2-324x_2^3-1728x_1^2+5832x_1x_2-243x_2^2+1944x_1+162x_2-1296).
\end{aligned} \quad (101)$$

Into the polynomial $W(\mathbf{x})$ we substitute

$$x_1 = b_1(t), \quad x_2 = b_2(t) + \varepsilon \tag{102}$$

according to (99). Then the polynomial $W(\mathbf{x})$ becomes a polynomial

$$u(\varepsilon, x_3) = \sum_{p,q \geq 0} u_{pq}(t)\varepsilon^p x_3^q,$$

whereby

$$u_{pq} = \frac{1}{p!} \cdot \frac{\partial^p W_q}{\partial x_2^p}(x_1, x_2), \tag{103}$$

where $x_1 = b_1(t)$, $x_2 = b_2(t)$ according to (99). In particular, from (101)–(103) we obtain

$$\begin{aligned} u_{00} &= W(b_1(t), b_2(t)) \equiv 0, \\ u_{10}(x_1, x_2) = \frac{\partial W_0}{\partial x_2} &= 373248(x_2+1)\left(x_2^2 + 2x_2 - 2\right) \\ &\times \Big(256x_1^2x_2^4 + 1024x_1^2x_2^3 - 192x_1x_2^4 + 1152x_1^2x_2^2 - \\ -768x_1x_2^3 + 256x_1^2x_2 - 864x_1x_2^2 &- 128x_1^2 - 192x_1x_2 + 81x_2^2 + 96x_1 + 162x_2\Big)\Big) \\ &= \frac{81(23283t^2 + 5466t + 499)\left(22131t^2 + 3930t - 13\right)^4}{2048(29t+3)^5(3t+2)^5}, \\ u_{01} = W_1(b_1(t), b_2(t)) &= \frac{27\left(23283t^2 + 5466t + 499\right)^2\left(22131t^2 + 3930t - 13\right)^5}{8388608(29t+3)^6(3t+2)^8}. \end{aligned}$$

According to Theorem 1 of [1], the solution of equation $u(\varepsilon,\ x_3) = 0$ is $\varepsilon = \sum_{k=1}^{\infty} c_k(t)x_3^k$, where

$$c_1(t) = -\frac{u_{01}(t)}{u_{10}(t)} = -\frac{\left(23283t^2 + 5466t + 499\right)\left(22131t^2 + 3930t - 13\right)}{12288(29t+3)(3t+2)^3}.$$

The denominator in $c_1(t)$ has roots (100).
Now let's go back and approximate from (102) obtain

$$x_1 = b_1(t), \;\; x_2 = b_2(t) + c_1(t)x_3.$$

Substitute that into (98) and we get

$$D_1 = b_1(t)x_3^2, D_2 = b_2(t) + c_1(t)x_3, D_3 = x_3. \tag{104}$$

We substitute the expression (104) into (97) and obtain

$$\begin{aligned} C_1 =& D_1 - D_2 - 2 = b_1(t)x_3^2 - b_2(t) - c_1(t)x_3 - 2, \\ C_2 =& b_2(t) + c_1(t)x_3, \\ C_3 =& x_3. \end{aligned} \tag{105}$$

We substitute (105) into (65) and we get

$$B_1 = C_1 = b_1(t)x_3^2 - b_2(t) - c_1(t)x_3 - 2, B_2 = C_2 = b_2(t) + c_1(t)x_3, B_3 = C_3^{-1} = \frac{1}{x_3}.$$

Finally, according to (64)

$$A_1 = B_1B_3 = \Big(b_1(t)x_3^2 - b_2(t) - c_1(t)x_3 - 2\Big)/x_3 = b_1(t)x_3 - c_1(t) - (b_2(t)+2)/x_3, \tag{106}$$
$$A_2 = B_2B_3 = c_1(t) + b_2(t)/x_3, \tag{107}$$
$$A_3 = B_3 = 1/x_3.$$

In Figures 41–44, are shown the curves of (106) and (107) for values $x_3 = -1, -0.1, 1, 0.1$, respectively.

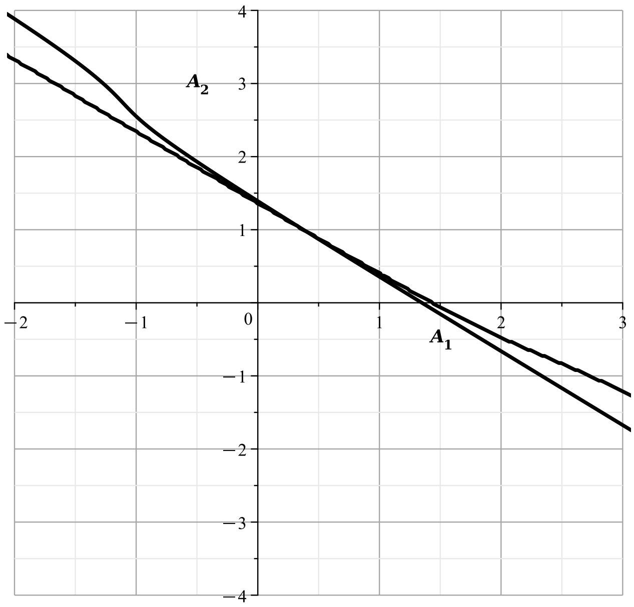

Figure 41. Curve (106) and (107) at $x_3 = -1$.

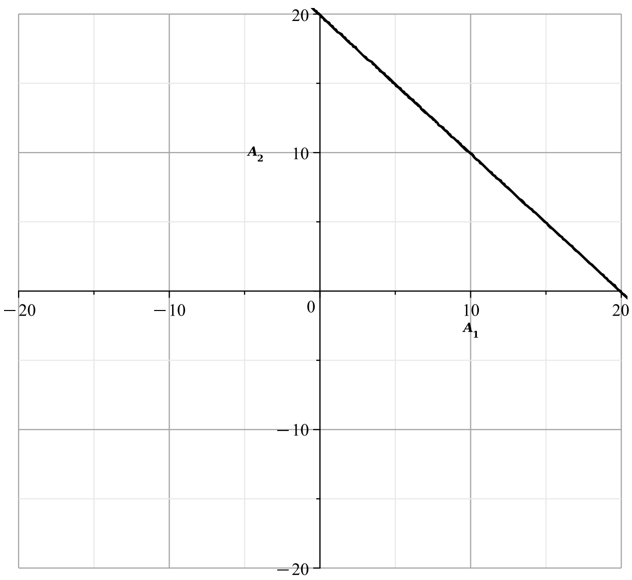

Figure 42. Curve (106) and (107) at $x_3 = -0.1$.

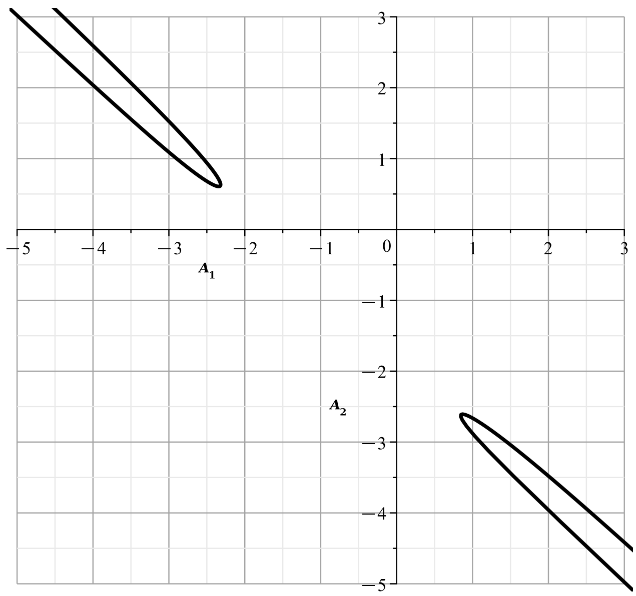

Figure 43. Curve (106) and (107) when $x_3 = 1$.

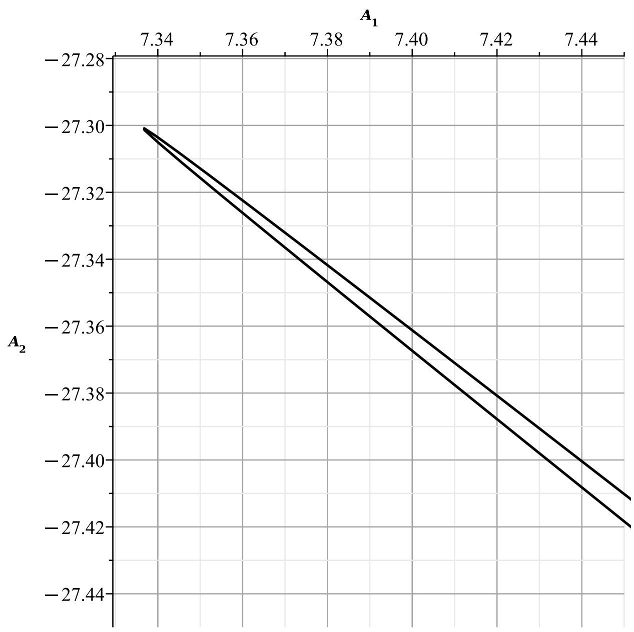

Figure 44. Curve of (106) and (107) at $x_3 = 0.1$.

The another branch is symmetric to this one with respect to the line $A_1 = A_2$.

5.5. Fourth Multiplier in (66)

5.5.1. Preliminary Calculations

The 4th multiplier is a linear multiplier $f_4 = C_1 + C_2 - 1$. Let's do the substitution $D_1 = C_1 + C_2 - 1$, $D_2 = C_2$, $D_3 = C_3$. Its inverse substitution is

$$C_1 = D_1 - D_2 + 1, C_2 = D_2, C_3 = D_3. \tag{108}$$

We treat it all as a coordinate transformation in the polynomial $J(\mathbf{C})$. We substitute it into the polynomial $J(\mathbf{C})$ and get the polynomial $S_8(\mathbf{D}) = J(\mathbf{C})$. For the polynomial

$S_8(\mathbf{D})$, we compute Newton's polyhedron Γ_{15}, with graph given in Figure 45. It has 7 two-dimensional faces with external normals

$$\begin{aligned} N_{593} =& (-1,0,-1), N_{1295} = (1,1,1), N_{1645} = (0,0,-1), \\ N_{1941} =& (-1,0,0), N_{1989} = (0,3,2), N_{2111} = (0,-1,0), N_{2171} = (0,1,0). \end{aligned}$$

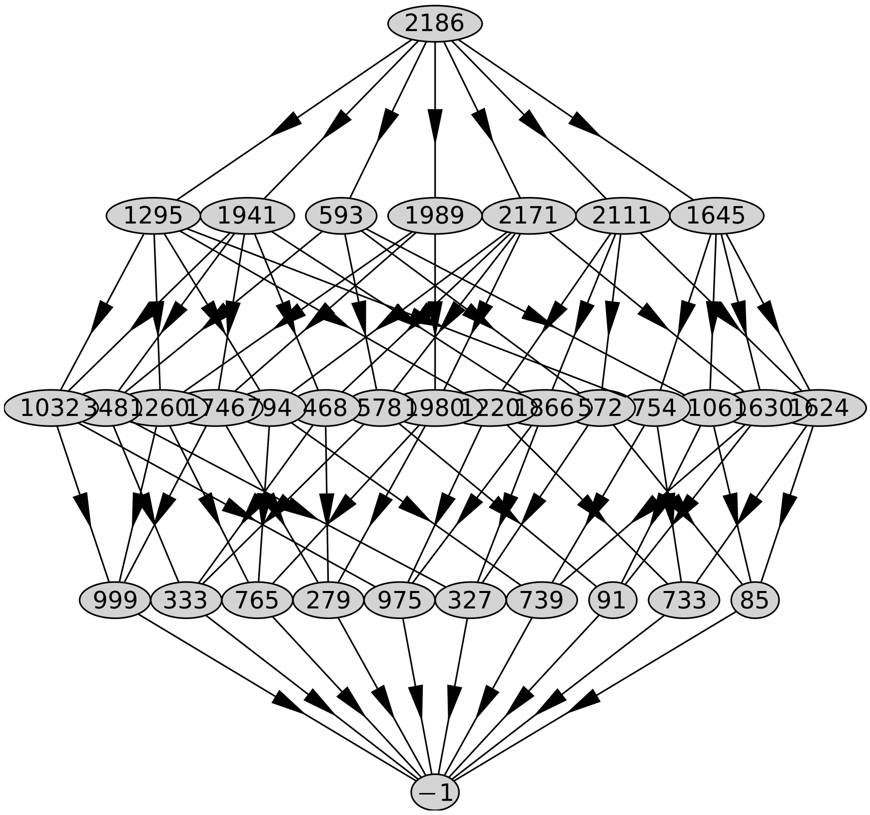

Figure 45. Graph of Polyhedron Γ_{15}.

Since $D_1 \to 0$, $D_2 \to$ const, and $D_3 \to 0$, we select the only normal that has negative first and third coordinates. This is $N_{593} = (-1,0,-1)$ and it corresponds to the truncated polynomial

$$ftr593 = 186624\left(2D_2^2 - 2D_2 - 1\right)^4 (2D_1 - 3D_3)(2D_1 + 3D_3).$$

Let's do the substitution $x_1 = 2D_1 - 3D_3$, $x_2 = D_2$, $x_3 = 2D_1 + 3D_3$. Its inverse substitution is

$$D_1 = \frac{1}{4}x_1 + \frac{1}{4}x_3,\ D_2 = x_2,\ D_3 = -\frac{1}{6}x_1 + \frac{1}{6}x_3, \tag{109}$$

and treat it all as a coordinate change in the polynomial $S_8(\mathbf{D})$. Substitute it into the polynomial $S_8(\mathbf{D})$ and get the polynomial $S_9(\mathbf{x}) = S_8(\mathbf{D})$. For the polynomial $S_9(\mathbf{x})$, we calculate Newton's polyhedron Γ_{17}.

Its graph is shown in Figure 46. It has 8 two-dimensional faces with external normals

$$\begin{aligned} N_{593} = (-1,0,-2),\ N_{1923} = (-2,0,-1),\ N_{3725} = (1,1,1),\ N_{4561} = (0,0,-1), \\ N_{4601} = (0,1,-2), N_{5649} = (-1,0,0),\ N_{6363} = (0,1,0), N_{6485} = (0,-1,0). \end{aligned} \tag{110}$$

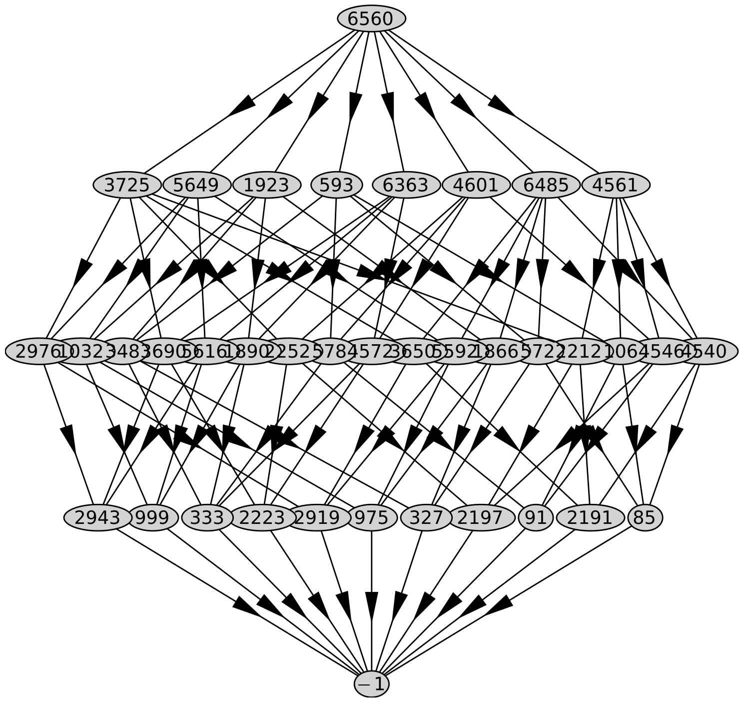

Figure 46. Graph of polyhedron Γ_{17}.

Since $x_1 \to 0$, $x_2 \to const$, and $x_3 \to 0$, we select two normals whose first and third coordinates are negative. These are $N_{593} = (-1, 0, -2)$ and $N_{1923} = (-2, 0, -1)$. We will deal with them in separate subsubsections.

5.5.2. The Normal $N_{593} = (-1, 0, -2)$

According to result of our program it corresponds to a truncated polynomial

$$ftr593 = 764411904 x_1 \left(2x_2^2 - 2x_2 - 1\right)^3 (32x_2^2 x_3 + x_1^2 - 32x_2 x_3 - 16x_3).$$

Making a power transformation

$$x_1 = y_1,\ x_2 = y_2,\ x_3 = y_1^2 y_3, \tag{111}$$

We get a polynomial

$$ftr593pow = 764411904 y_1^3 \left(2y_2^2 - 2y_2 - 1\right)^3 \left(32y_2^2 y_3 - 32y_2 y_3 - 16y_3 + 1\right) = \\ = 764411904 y_1^3 \left(2y_2^2 - 2y_2 - 1\right)^3 \cdot F_{50}(y_2, y_3),$$

where $F_{50} = 32y_2^2 y_3 - 32y_2 y_3 - 16y_3 + 1$. The curve $F_{50} = 0$ has genus 0, parameterization

$$\{y_2, y_3\} = \{b_2(t), b_3(t)\} = \left\{t, -\frac{1}{16(2t^2 - 2t - 1)}\right\} \tag{112}$$

and is shown in Figure 47.

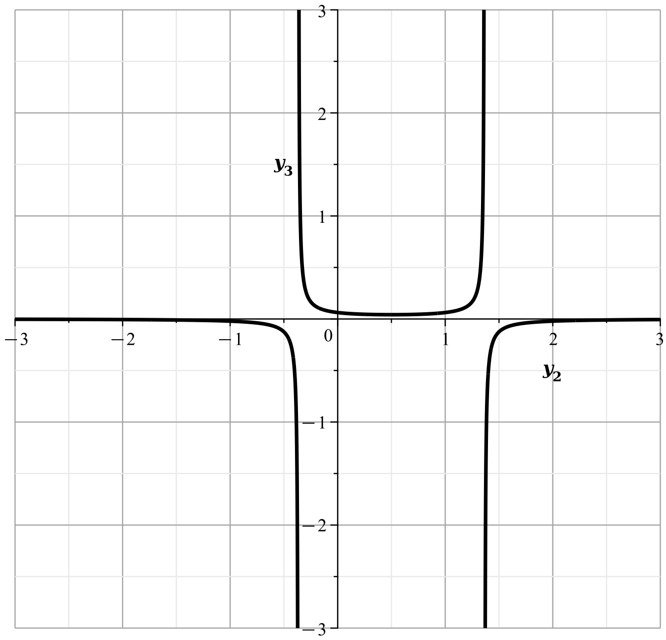

Figure 47. Curve $F_{50}(y_2, y_3) = 0$.

In (112), the denominator in $b_2(t)$ has 2 real roots

$$t_1 = \frac{1}{2} + \frac{\sqrt{3}}{2} \approx 1.366025404, t_2 = \frac{1}{2} - \frac{\sqrt{3}}{2} \approx -0.3660254040. \quad (113)$$

In fact, here we can also compute the parametric expansion of the Ω manifold. To do this, we do the power transformation (110) in the polynomial $S_9(\mathbf{x})$ and get the polynomial

$$y_1^3 U(\mathbf{y}) = S_9(\mathbf{x}) = y_1^3 \sum_{k=0} U_k(y_2, y_3) y_1^k.$$

In the polynomials $U_k(y_2, y_3)$ according to (112) we substitute

$$y_2 = b_1(t) + \varepsilon, \;\; y_3 = b_2(t) + \varepsilon.$$

We obtain the polynomial $u(\varepsilon,\, y_1) = U(y_1, y_2, y_3)$ with coefficients depending on t through $b_1(t)$ and $b_2(t)$. In this polynomial

$$u(\varepsilon, y_1) = \sum_{k=0}^{m} U_k(b_1 + \varepsilon, b_2 + \varepsilon) y_1^k = \sum_{p,q \geq 0} u_{pq} \varepsilon^p y_1^q,$$

where $u_{pq} = \sum_{p_1+p_2=p\geq 1} \frac{1}{p_1! p_2!} \cdot \frac{\partial^p U_q}{\partial y_2^{p_1} \partial y_3^{p_2}}$ when $y_i = b_i(t)$, $i = 2, 3$, p_1, $p_2 \geq 0$, $p \geq 1$. Specifically,

$$u_{00} \equiv 0, u_{10} = \frac{\partial U_0(y_2, y_3)}{\partial y_2} + \frac{\partial U_0(y_2, y_3)}{\partial y_3}$$

$$= 1528823808\left(2y_2^2 - 2y_2 - 1\right)^2 \left(32y_2^4 + 256y_2^3 y_3 - 64y_2^3 - 384y_2^2 y_3 + 38y_2 + 64y_3 + 5\right)$$

$$= 1528823808(2t-3)\left(16t^3 - 8t^2 - 12t - 3\right)\left(2t^2 - 2t - 1\right)^2 \overset{\text{def}}{=} H(b_1(t), b_2(t)),$$

$$u_{01} = U_1(b_1, b_2) = 31850496(2t+5)(2t-1)\left(2t^2 - 2t - 1\right)^2 \overset{\text{def}}{=} G(b_1(t), b_2(t)).$$

The function $u_{10}(t)$ has real roots

$$\begin{aligned} t_1 =& \frac{3}{2} = 1.5, \\ t_2 =& \frac{1}{2} + \frac{\sqrt{3}}{2} \approx 1.366025404 \text{ 2-multiple}, \\ t_3 =& \frac{1}{2} - \frac{\sqrt{3}}{2} \approx -0.3660254040, \text{ 2-multiple}, \\ t_4 \approx& 1.232176060, \end{aligned} \tag{114}$$

and the function $u_{01}(t)$ has 2-multiple roots t_2, t_3 and

$$t_5 = -\frac{5}{2} = -2.5, t_6 = \frac{1}{2}.$$

By the Implicit Function Theorem [1] (Theorem 1), the equation $u(\varepsilon, y_1) = 0$ has a solution as a power series on y_1

$$\varepsilon = \sum_{k=1}^{\infty} c_k(t) y_1^k,$$

where $c_k(t)$ are rational functions of t, which are expressed through the coefficients $u_{pq}(t)$, which in turn are expressed through $b_1(t)$ and $b_2(t)$ according to (110). This expansion is valid for all values of t, except maybe the neighborhood of the roots of (114). In particular,

$$c_1(t) = -\left(\frac{u_{01}}{u_{10}}\right) = -\frac{G}{H} = -\frac{(2t+5)(2t-1)}{48(2t-3)(16t^3 - 8t^2 - 12t - 3)},$$

where the denominator has 2 real roots t_1 and t_4 of (114). Approximately we get

$$\varepsilon \approx c_1(t) y_1.$$

Let's return to the previous coordinates, which are approximated to be equal for small $|y_3|$ on the manifold Ω

$$y_2 = b_1(t) + c_1(t) y_1, \;\; y_3 = b_2(t) + c_1(t) y_1.$$

$$x_1 = y_1, x_2 = y_2 = b_1(t) + c_1(t) y_1, x_3 = y_1^2 y_3 = b_2(t) y_1^2 + c_1(t) y_1^3. \tag{115}$$

We substitute the expressions (115) into the transformation (109) and get

$$\begin{aligned} D_1 =& \frac{1}{4} x_1 + \frac{1}{4} x_3 = \frac{1}{4} y_1 + \frac{1}{4} b_2(t) y_1^2 + \frac{1}{4} c_1(t) y_1^3, \\ D_2 =& x_2 = b_1(t) + c_1(t) y_1, \\ D_3 =& -\frac{1}{6} x_1 + \frac{1}{6} x_3 = -\frac{1}{6} y_1 + \frac{1}{6} b_2(t) y_1^2 + \frac{1}{6} c_1(t) y_1^3. \end{aligned} \tag{116}$$

We substitute the expressions (116) into the transformation (109) and obtain variables defind in (108)

$$\begin{aligned} C_1 =& D_1 = b_1(t) + c_1(t) y_1, \\ C_2 =& D_1 - D_2 + 1 = \frac{1}{4} y_1 + \frac{1}{4} b_2(t) y_1^2 + \frac{1}{4} c_1(t) y_1^3 - b_1(t) - c_1(t) y_1 + 1, \\ C_3 =& D_3 = -\frac{1}{6} y_1 + \frac{1}{6} b_2(t) y_1^2 + \frac{1}{6} c_1(t) y_1^3. \end{aligned} \tag{117}$$

Substitute (117) into (65) and obtain

$$\begin{aligned} B_1 =& C_1 = b_1(t) + c_1(t)y_1, \\ B_2 =& C_2 = \frac{1}{4}c_1(t)y_1^3 + \frac{1}{4}b_2(t)y_1^2 + \frac{1}{4}y_1 - c_1(t)y_1 - b_1(t) + 1, \\ B_3 =& C_3^{-1} = \frac{1}{C_3} = \frac{1}{-\frac{1}{6}y_1 + \frac{1}{6}b_2(t)y_1^2 + \frac{1}{6}c_1(t)y_1^3}. \end{aligned} \tag{118}$$

Finally, we substitute the expressions (118) into the transformation (64) and obtain

$$A_1 = B_1 B_3 = \frac{b_1(t) + c_1(t)y_1}{-\frac{1}{6}y_1 + \frac{1}{6}b_2(t)y_1^2 + \frac{1}{6}c_1(t)y_1^3} = -\frac{6(b_1(t) + c_1(t)y_1)}{y_1 - b_2(t)y_1^2 - c_1(t)y_1^3}, \tag{119}$$

$$\begin{aligned} A_2 = B_2 B_3 =& \frac{\frac{1}{4}c_1(t)y_1^3 + \frac{1}{4}b_2(t)y_1^2 + \frac{1}{4}y_1 - c_1(t)y_1 - b_1(t) + 1}{-\frac{1}{6}y_1 + \frac{1}{6}b_2(t)y_1^2 + \frac{1}{6}c_1(t)y_1^3} \\ =& -\frac{\frac{3}{2}c_1(t)y_1^3 + \frac{3}{2}b_2(t)y_1^2 + \frac{3}{2}y_1 - 6c_1(t)y_1 - 6b_1(t) + 6}{y_1 - b_2(t)y_1^2 - c_1(t)y_1^3}, \end{aligned} \tag{120}$$

$$A_3 = B_3 = -\frac{6}{y_1 - b_2(t)y_1^2 - c_1(t)y_1^3}.$$

Figures 48 and 49, show the curves (119) and (120) for values $y_3 = 1$ and -1, respectively.

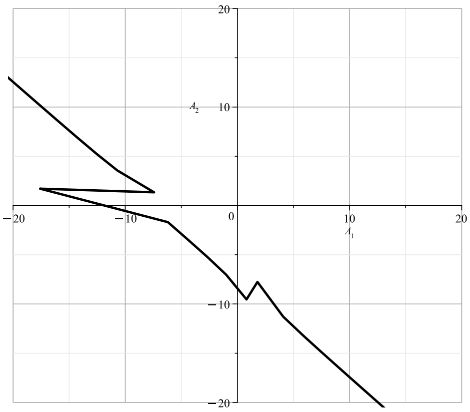

Figure 48. Curve (119) and (120) at $y_1 = 1$.

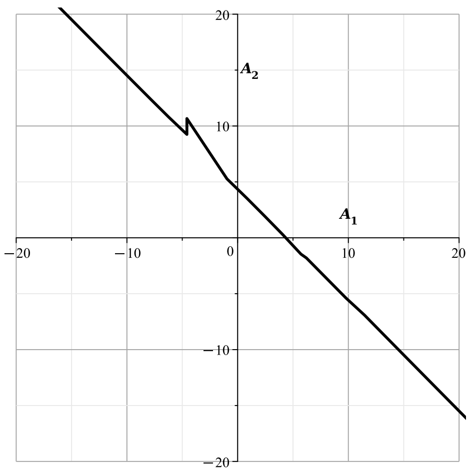

Figure 49. Curve (119) and (120) at $y_1 = -1$.

5.5.3. The Normal $N_{1923} = (-2, 0, -1)$ from (110)

It corresponds to a truncated polynomial

$$ftr1923 = 254803968x_3\left(2x_2^2 - 2x_2 - 1\right)^3\left(4x_2^2x_3^2 + 96x_1x_2^2 - 4x_2x_3^2 - 96x_1x_2 + x_3^2 - 48x_1\right).$$

By the power transformation

$$x_1 = y_1y_3^2, x_2 = y_2, x_3 = y_3. \tag{121}$$

We have

$$ftr1923pow = 254803968y_3^3\left(2y_2^2 - 2y_2 - 1\right)^3\left(96y_1y_2^2 - 96y_1y_2 + 4y_2^2 - 48y_1 - 4y_2 + 1\right) =$$
$$= 254803968y_3^3\left(2y_2^2 - 2y_2 - 1\right)^3 \cdot F_{60}(y_1, y_2),$$

where $F_{60}(y_1, y_2) = 96y_1y_2^2 - 96y_1y_2 + 4y_2^2 - 48y_1 - 4y_2 + 1$. The curve $F_{60} = 0$ has genus 0, and parameterization

$$\{y_1, y_2\} = \{b_1(t), b_2(t)\} = \{-\frac{(2t-1)^2}{48(2t^2 - 2t - 1)}, t\}. \tag{122}$$

It is shown in Figure 50.

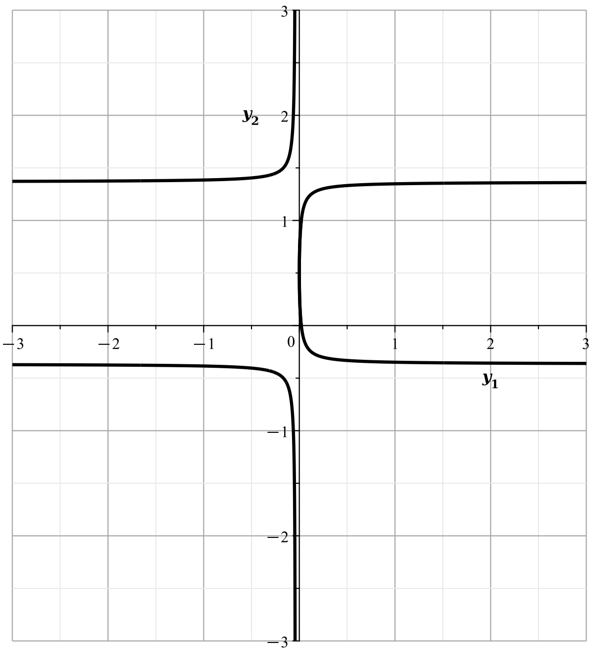

Figure 50. Curve $F_{60}(y_1, y_2) = 0$.

In (122), the denominator in $b_1(t)$ has 2 real roots t_1, t_2 given by (113). In fact, the parametric expansion of the manifold Ω can also be calculated here. To do this, we do a power transformation (121) in the polynomial $S_9(\mathbf{x})$ and get the polynomial

$$-y_3^3P(\mathbf{y}) = S_9(\mathbf{x}) = -y_3^3\sum_{k=0} P_k(y_1, y_2)y_3^k.$$

Into the polynomials $P_k(y_1, y_2)$ we substitute

$$y_1 = b_1(t) + \varepsilon,\ y_2 = b_2(t) + \varepsilon = t + \varepsilon$$

according to (122).

We obtain the polynomial $u(\varepsilon,\ y_3) = P(y_1, y_2, y_3)$ with coefficients depending on t through $b_1(t)$ and $b_2(t)$. In this polynomial

$$u(\varepsilon, y_3) = \sum_{k=0}^{m} P_k(b_1 + \varepsilon, b_2 + \varepsilon) y_3^k = \sum_{p,q \geq 0} u_{pq} \varepsilon^p y_3^q,$$

where $u_{pq} = \sum_{p_1+p_2=p\geq 1} \frac{1}{p_1! p_2!} \cdot \frac{\partial^p P_q}{\partial y_1^{p_1} \partial y_2^{p_2}}$ where $y_i = b_i = b_i(t)$, $i = 1, 2$, $p_1, p_2 \geq 0$, $p \geq 1$. Specifically

$$\begin{aligned} u_{00} \equiv& 0, \\ u_{10} =& \frac{\partial P_0(y_1, y_2)}{\partial y_1} + \frac{\partial P_0(y_1, y_2)}{\partial y_2} = -509607936\left(2y_2^2 - 2y_2 - 1\right)^2 \times \\ &\times \left(768y_1y_2^3 + 96y_2^4 - 1152y_1y_2^2 - 160y_2^3 - 48y_2^2 + 192y_1 + 114y_2 + 23\right) = \\ =& -1528823808(2t-3)\left(16t^3 - 8t^2 - 12t - 3\right)\left(2t^2 - 2t - 1\right)^2 \stackrel{\text{def}}{=} H(b_1(t), b_2(t)), \\ u_{01} =& P_1(b_1, b_2) = -31850496(2t+5)(2t-1)\left(2t^2 - 2t - 1\right)^2 \stackrel{\text{def}}{=} G(b_1(t), b_2(t)). \end{aligned} \tag{123}$$

The function $u_{10}(t)$ has real roots (114). By the Implicit Function Theorem [1] (Theorem 1), the equation $u(\varepsilon, y_3) = 0$ has a solution as a power series on y_3

$$\varepsilon = \sum_{k=1}^{\infty} c_k(t) y_3^k,$$

where $c_k(t)$ are rational functions of t, which are expressed through the coefficients $u_{pq}(t)$, which in turn are expressed through $b_1(t)$ and $b_2(t)$ according to (122). This expansion is valid for all values of t, except maybe the neighborhood of the roots of the polynomial (123). In particular,

$$c_1(t) = -\left(\frac{u_{01}}{u_{10}}\right) = -\frac{G}{H} = -\frac{(2t+5)(2t-1)}{48(2t-3)(16t^3 - 8t^2 - 12t - 3)},$$

where the denominator has 2 real roots t_1 and t_4 of (114). We get an approximation

$$\varepsilon \approx c_1(t) y_3.$$

Let's return to the previous coordinates, which, for small $|y_3|$ on Ω, are approximately equal to

$$y_1 = b_1(t) + c_1(t) y_3, \quad y_2 = b_2(t) + c_1(t) y_3. \tag{124}$$

We substitute the expressions (124) into the transformation (121) and get

$$x_1 = y_1 y_3^2 = b_1(t) y_3^2 + c_1(t) y_3^3, x_2 = y_2 = b_2(t) + c_1(t) y_3, x_3 = y_3. \tag{125}$$

We substitute the expressions (125) into the transformation (109) and obtain

$$\begin{aligned} D_1 =& \frac{1}{4}x_1 + \frac{1}{4}x_3 = \frac{1}{4}b_1(t) y_3^2 + \frac{1}{4}c_1(t) y_3^3 + \frac{1}{4}y_3, \\ D_2 =& x_2 = b_2(t) + c_1(t) y_3, \\ D_3 =& -\frac{1}{6}x_1 + \frac{1}{6}x_3 = -\frac{1}{6}b_1(t) y_3^2 - \frac{1}{6}c_1(t) y_3^3 + \frac{1}{6}y_3. \end{aligned} \tag{126}$$

We substitute the expressions (126) into the transformation (108) and get the following results

$$
\begin{aligned}
C_1 =& D_2 = b_2(t) + c_1(t)y_3, \\
C_2 =& D_1 - D_2 + 1 = \frac{1}{4}b_1(t)y_3^2 + \frac{1}{4}c_1(t)y_3^3 + \frac{1}{4}y_3 - b_2(t) - c_1(t)y_3 + 1, \\
C_3 =& D_3 = -\frac{1}{6}b_1(t)y_3^2 - \frac{1}{6}c_1(t)y_3^3 + \frac{1}{6}y_3.
\end{aligned} \tag{127}
$$

We substitute the expressions (127) into the transformation (65) and obtain

$$
\begin{aligned}
B_1 =& C_1 = b_2(t) + c_1(t)y_3, \\
B_2 =& C_2 = \frac{1}{4}c_1(t)y_3^3 + \frac{1}{4}b_1(t)y_3^2 + \frac{1}{4}y_3 - c_1(t)y_3 - b_2(t) + 1, \\
B_3 =& C_3^{-1} = \frac{1}{C_3} = \frac{1}{-\frac{1}{6}b_1(t)y_3^2 - \frac{1}{6}c_1(t)y_3^3 + \frac{1}{6}y_3}.
\end{aligned} \tag{128}
$$

Finally, we substitute the expressions (128) into the transformation of (64) and obtain

$$
A_1 = B_1B_3 = \frac{b_2(t) + c_1(t)y_3}{-\frac{1}{6}b_1(t)y_3^2 - \frac{1}{6}c_1(t)y_3^3 + \frac{1}{6}y_3} = -\frac{6(b_2(t) + c_1(t)y_3)}{c_1(t)y_3^3 + b_1(t)y_3^2 - y_3}, \tag{129}
$$

$$
A_2 = B_2B_3 = \frac{\frac{1}{4}c_1(t)y_3^3 + \frac{1}{4}b_1(t)y_3^2 + \frac{1}{4}y_3 - c_1(t)y_3 - b_2(t) + 1}{-\frac{1}{6}b_1(t)y_3^2 - \frac{1}{6}c_1(t)y_3^3 + \frac{1}{6}y_3} \tag{130}
$$

$$
= -\frac{\frac{3}{2}c_1(t)y_3^3 + \frac{3}{2}b_1(t)y_3^2 + \frac{3}{2}y_3 - 6c_1(t)y_3 - 6b_2(t) + 6}{c_1(t)y_3^3 + b_1(t)y_3^2 - y_3}, \tag{131}
$$

$$
A_3 = B_3 = -\frac{6}{c_1(t)y_3^3 + b_1(t)y_3^2 - y_3}.
$$

In Figures 51 and 52, show the curves (129) and (131) for values $y_3 = 1$ and $y_3 = -1$, respectively.

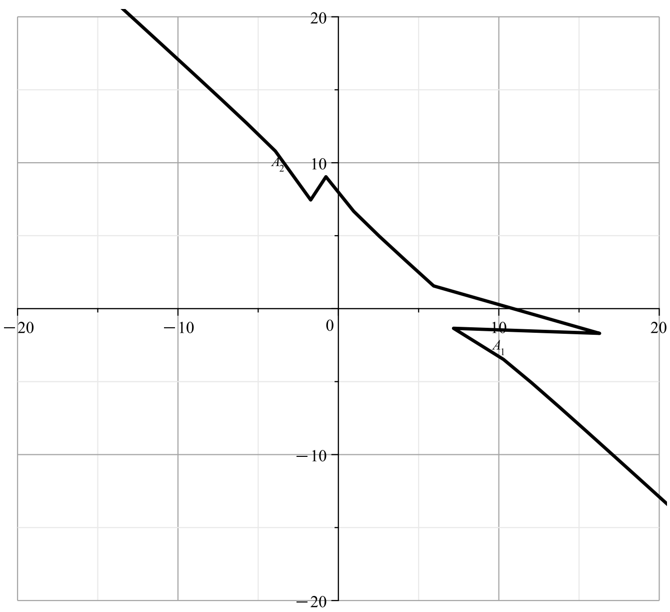

Figure 51. Curve (129) and (131) at $y_1 = 1$.

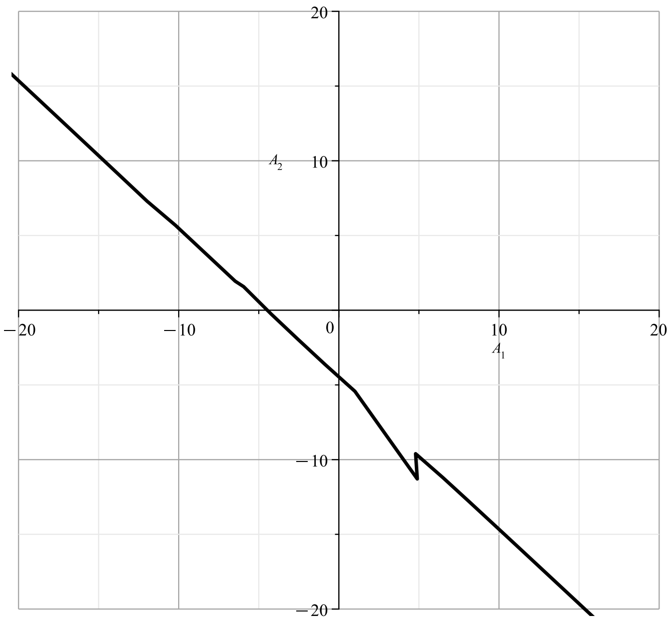

Figure 52. Curve of (129) and (131) at $y_1 = -1$.

6. Conclusions

In the paper we show that all parametric expansions of variety Ω near its singularities and infinity can be computed with any accuracy and compute their first terms. We consider a very rich set of cases and find the ways to finish computations in all of them.

We do not intend to explain our results for original problems of Ricci flows and Einstein's metrics. Let it will be done by authors of [14–22], who are specialists in the problem.

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