axioms

Article

Subclasses of Analytic Functions Subordinated to the Four-Leaf Function †

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† Dedicated to the memory of Professor Om Prakash Ahuja (1942–2024).
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Abstract: The purpose of this research is to unify and extend the study of the well-known concept of coefficient estimates for some subclasses of analytic functions. We define the new subclass $\mathcal{A}_4^{r,s}$ of analytic functions related to the four-leaf domain, to increase the adaptability of our investigation. The initial findings are the bound estimates for the coefficients $|a_n|$, $n = 2, 3, 4, 5$, among which the bound of $|a_2|$ is sharp. Also, we include the sharp-function illustration. Additionally, we obtain the upper-bound estimate for the second Hankel determinant for this subclass as well as those for the Fekete–Szegő functional. Finally, for these subclasses, we provide an estimation of the Krushkal inequality for the function class $\mathcal{A}_4^{r,s}$.

Keywords: analytic functions; subordination; four-leaf function; coefficient inequalities; Hankel determinant; Fekete–Szegő functional; Krushkal inequality

MSC: 30C45; 30C80

Citation: Gunasekar, S.; Sudharsanan, B.; Ibrahim, M.; Bulboacă, T. Subclasses of Analytic Functions Subordinated to the Four-Leaf Function. *Axioms* **2024**, *13*, 155. https://doi.org/10.3390/axioms13030155

Academic Editor: Behzad Djafari-Rouhani

Received: 13 February 2024
Revised: 20 February 2024
Accepted: 25 February 2024
Published: 27 February 2024

1. Introduction and Preliminaries

We let $\mathcal{A}$ denote the class of analytic functions defined in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, having the power-series expansion of the type

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in \mathbb{D}. \tag{1}$$

Also, we let $\mathcal{S}$ denote the class of all functions of $\mathcal{A}$ that are univalent in $\mathbb{D}$.

If F and G are analytic functions in $\mathbb{D}$, and if there exists a function w analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{D}$, such that $F = G \circ w$, then we say that F is *subordinated* to G, written $F(z) \prec G(z)$ (see, for example, [1] p. 368). Using the *Schwarz lemma*, it is easy to show that $F(z) \prec G(z)$ implies $F(0) = G(0)$ and $F(\mathbb{D}) \subset G(\mathbb{D})$, and assuming that G is univalent in $\mathbb{D}$ then the next equivalence holds:

$$F(z) \prec G(z) \Leftrightarrow F(0) = G(0) \text{ and } F(\mathbb{D}) \subset G(\mathbb{D}). \tag{2}$$

The classic Fekete–Szegő problem [2] involves finding the exact limits of the functional $\left|a_3 - \mu a_2^2\right|$ for a compact-function family or $f \in \mathcal{A}$ with any $\mu \in \mathbb{C}$; for further details, one may refer to [3].

Pommerenke provided the following *Hankel determinant* in [4,5], denoted by $\mathcal{D}_{q,n}(f)$, which contains the coefficients of a function $f \in \mathcal{S}$:

$$\mathcal{D}_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix},$$

with $q, n \in \mathbb{N} := \{1, 2, \dots\}$. Therefore, by altering the parameters q and n we obtain the following Hankel determinants:

$$\mathcal{D}_{2,1}(f) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2, \quad \mathcal{D}_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2,$$

$$\mathcal{D}_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2), \tag{3}$$

that denote the first, the second, and the third-order Hankel determinants. There are a few references in the literature to the Hankel determinant for functions in the general family $\mathcal{S}$. The best-known sharp inequality for the function $f \in \mathcal{S}$ is $\mathcal{D}_{2,n}(f) \leq \kappa\sqrt{n}$, where κ is a constant, and it is due to Hayman ([6] Theorem 1). Additionally, for the class $\mathcal{S}$, it was found in [7] that

$$|\mathcal{D}_{2,2}(f)| \leq \kappa, \quad \text{where} \quad 1 \leq \kappa \leq \frac{11}{3} \simeq 3.66\dots,$$

$$|\mathcal{D}_{3,1}(f)| \leq \nu, \quad \text{where} \quad \frac{4}{9} \leq \nu \leq \frac{32 + \sqrt{285}}{15} \simeq 3.258796\dots.$$

The precise bounds of Hankel determinants for a given family of functions have piqued the interest of several mathematicians. For the three well-known subfamilies of the set $\mathcal{S}$ that are $\mathcal{K}$, $\mathcal{S}^*$, and $\mathcal{R}$ (convex, starlike, and functions of a bounded turning, respectively), Janteng et al. [8,9] computed the sharp bounds of $|\mathcal{D}_{2,2}(f)|$. These bounds are provided by

$$|\mathcal{D}_{2,2}(f)| \leq \begin{cases} \frac{1}{8}, & \text{for} \quad f \in \mathcal{K}, \\ 1, & \text{for} \quad f \in \mathcal{S}^*, \\ \frac{4}{9}, & \text{for} \quad f \in \mathcal{R}. \end{cases}$$

Moreover, the sharp bounds of this determinant for a few subclasses $\mathcal{S}^*$ and $\mathcal{K}$ were found in [10] and subsequently studied in [11]. This problem was solved for various families of bi-univalent functions in [12–14].

Finding the bound of $|\mathcal{D}_{2,2}(f)|$ is significantly easier than calculating $|\mathcal{D}_{3,1}(f)|$, as is shown by Formula (3). In 2010, Babalola [15] was the first to study the third-order Hankel determinant for the classes $\mathcal{K}$, $\mathcal{S}^*$, and $\mathcal{R}$. The same approach was then used by several authors [16–20] to the values of $|\mathcal{D}_{3,1}(f)|$ for certain subclasses of univalent functions. The researchers became interested in Zaprawa's study [21] because he enhanced Babalola's findings by utilizing a novel technique to show that

$$|\mathcal{D}_{3,1}(f)| \leq \begin{cases} \frac{49}{540}, & \text{for} \quad f \in \mathcal{K}, \\ 1, & \text{for} \quad f \in \mathcal{S}^*, \\ \frac{41}{60}, & \text{for} \quad f \in \mathcal{R}, \end{cases}$$

and he also noted that the bounds are not sharp. For $f \in \mathcal{S}^*$, Kwon et al. [22] made a more agreeable finding in 2019 and proved that $|\mathcal{D}_{3,1}(f)| \leq \frac{8}{9}$. Zaprawa et al. [23] improved this limit even more, since they proved that for $f \in \mathcal{S}^*$ the inequality $|\mathcal{D}_{3,1}(f)| \leq \frac{5}{9}$ holds. In recent years, a sharp bound was obtained by Kowalczyk et al. [24] and Lecko et al. [25] for the third Hankel determinant, as below:

$$|\mathcal{D}_{3,1}(f)| \leq \begin{cases} \frac{4}{135}, & \text{for} \quad f \in \mathcal{K}, \\ \frac{1}{9}, & \text{for} \quad f \in \mathcal{S}^*\left(\frac{1}{2}\right), \end{cases}$$

where $S^*\left(\frac{1}{2}\right)$ is the family of starlike functions with order $\frac{1}{2}$.

Gandhi in [26] introduced a set of bounded turning functions connected to a *three-leaf function*. In 2022, in the articles [27,28] the authors introduced and studied different subclasses of analytic functions defined by subordination to the *four-leaf function* (see Figure 1, made with MAPLE™ 2023 computer software) that is given by

$$\mathcal{Q}_4(z) := 1 + \frac{5}{6}z + \frac{1}{6}z^5,\ z \in \mathbb{D}.$$

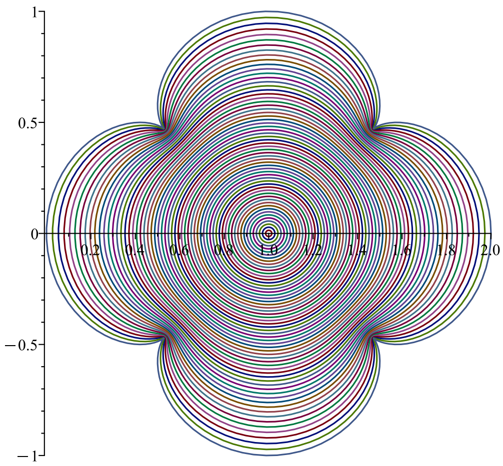


Figure 1. The image of $\mathcal{Q}_4(\mathbb{D})$.

With the aid of a four-leaf function, we define the following subclass of $\mathcal{A}$, using the notion of subordination, as follows:

Definition 1. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{A}_4^{r,s}$ if*

$$\Psi_{r,s}f(z) := (1-r)(1-s)\frac{f(z)}{z} + \big(s + r(1+s)\big)f'(z) + rs\big(zf''(z) - 2\big) \prec \mathcal{Q}_4(z), \qquad (4)$$

where $r \geq 0$ and $s \in [0,1]$.

The reason for taking the above left-hand-side expression consisted in the fact that we could obtain a subordination condition where appeared the usual expressions $f(z)/z$, $f'(z)$, and $zf''(z)$. For special values of the parameters r and s, some of these functions vanished or the formula became more simple and, as we can see in the further Remark 2, we could simply obtain expressions subordinated to the four-leaf function.

Many results regarding some subclasses defined by subordinations with different functions with significant geometrical properties (e.g., the limaçon function, convex functions in one direction, the cosine function, the nephroid function, etc.) were studied by the fourth author in many papers (see, for example, [29–32]). The novelty of these subclasses and of this paper consists in the fact that such subordinations with similar expressions to the left-hand side of the subordination (4) were not studied in some other previous articles.

Throughout this paper, unless otherwise stated, we assume that

$$\tau_n := 1 + (n-1)(r+s) + \left(n^2+1\right)rs,\ n \in \mathbb{N} \setminus \{1\}, \tag{5}$$

where $r \geq 0$ and $s \in [0,1]$. Evidently, $\tau_n \geq 1$ and

$$\tau_{n+1} - \tau_n = \big(1 + (2n+1)s\big)r + s \geq 0.$$

Remark 1. *(i) If φ is an analytic function in $\mathbb{D}$ then φ is said to be a* starlike function *with respect to $w_0 = \varphi(0)$ if φ is univalent in $\mathbb{D}$ and $\varphi(\mathbb{D})$ is a starlike domain with respect to w_0—that is, the segment $[w_0, \varphi(z)]$ lies in $\varphi(\mathbb{D})$ for all $z \in \mathbb{D}$. It is well known that the function φ is starlike with respect to $w_0 = \varphi(0)$ if and only if $\varphi'(0) \neq 0$ and*

$$\operatorname{Re} \frac{z\varphi'(z)}{\varphi(z) - w_0} > 0,\ z \in \mathbb{D}.$$

Since $\mathcal{Q}_4(0) = 1$, $\mathcal{Q}_4'(0) = 5/6 \neq 0$ and

$$\operatorname{Re} \frac{z\mathcal{Q}_4'(z)}{\mathcal{Q}_4(z) - \mathcal{Q}_4(0)} = 5 \operatorname{Re} \frac{1+z^4}{5+z^4} > 0,\ z \in \mathbb{D},$$

it follows that the four-leaf function $\mathcal{Q}_4$ is starlike (univalent) in $\mathbb{D}$ with respect to $w_0 = \mathcal{Q}_4(0) = 1$. Moreover, from the fact that $\big(\mathcal{Q}_4(1) + \mathcal{Q}_4(-1)\big)/2 = 1$ it follows that the domain $\mathcal{Q}_4(\mathbb{D})$ is symmetric with respect to the point $w_0 = 1$, and because $\overline{\mathcal{Q}_4(z)} = \mathcal{Q}_4(\overline{z})$, $z \in \mathbb{D}$ the domain $\mathcal{Q}_4(\mathbb{D})$ is symmetric with respect to the real axis.

We have $\operatorname{Re} \mathcal{Q}_4(z) > 0$, $z \in \mathbb{D}$ because

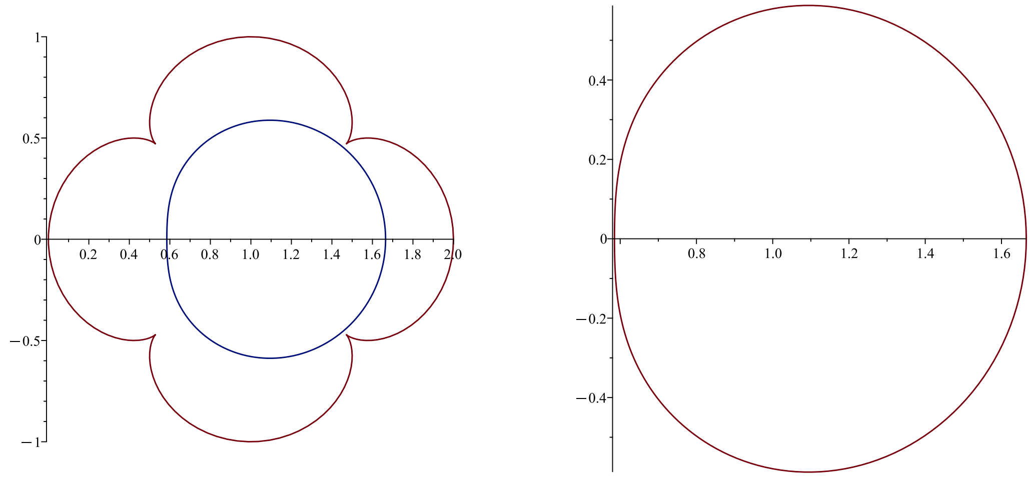
$$\begin{aligned} \operatorname{Re} \mathcal{Q}_4(z) &= \operatorname{Re}\left(1 + \frac{5}{6}z + \frac{1}{6}z^5\right) = 1 + \operatorname{Re}\left(\frac{5}{6}z + \frac{1}{6}z^5\right) \geq 1 - \left|\frac{5}{6}z + \frac{1}{6}z^5\right| \\ &\geq 1 - \frac{5}{6}|z| - \frac{1}{6}|z^5| > 1 - \frac{5}{6} - \frac{1}{6} = 0,\ z \in \mathbb{D}, \end{aligned}$$

hence, $\operatorname{Re} \mathcal{Q}_4(z) > 0$, $z \in \mathbb{D}$.

(ii) We will emphasize that the class $\mathcal{A}_4^{r,s}$ is not empty. Considering $\widetilde{f}(z) = z + az^2 + bz^3$, for the particular case $a = 0.08$, $b = 0.01$, $r = 1.5$, and $s = 0.5$, using the 2D plot *of the MAPLE™ computer software we obtain the images of the boundary $\partial\mathbb{D}$ by the functions $\Psi_{r,s}\widetilde{f}$ and $\mathcal{Q}_4$, shown in Figure 2a. Since $\mathcal{Q}_4$, as we showed above, is univalent in $\mathbb{D}$, the equivalence (2) yields that the subordination $\Psi_{r,s}\widetilde{f}(z) \prec \mathcal{Q}_4(z)$ holds whenever $\Psi_{r,s}\widetilde{f}(0) = \mathcal{Q}_4(0) = 1$ and $\Psi_{r,s}\widetilde{f}(\mathbb{D}) \subset \mathcal{Q}_4(\mathbb{D})$ (see Figure 2b). In conclusion, $\widetilde{f} \in \mathcal{A}_4^{r,s}$ for the above values of the parameters; hence, the class $\mathcal{A}_4^{r,s}$ is not empty for non-trivial values of the parameters.*

The following univalence theorem on the boundary is well known *(see, for example, [33] Lemma 1.1, p. 13): Let f be analytic in $\overline{\mathbb{D}}$ and injective on the boundary $\partial\mathbb{D}$. Then, f is univalent in $\mathbb{D}$ and maps $\mathbb{D}$ onto the inner domain of the (closed) Jordan curve $J = f(\partial\mathbb{D})$.*

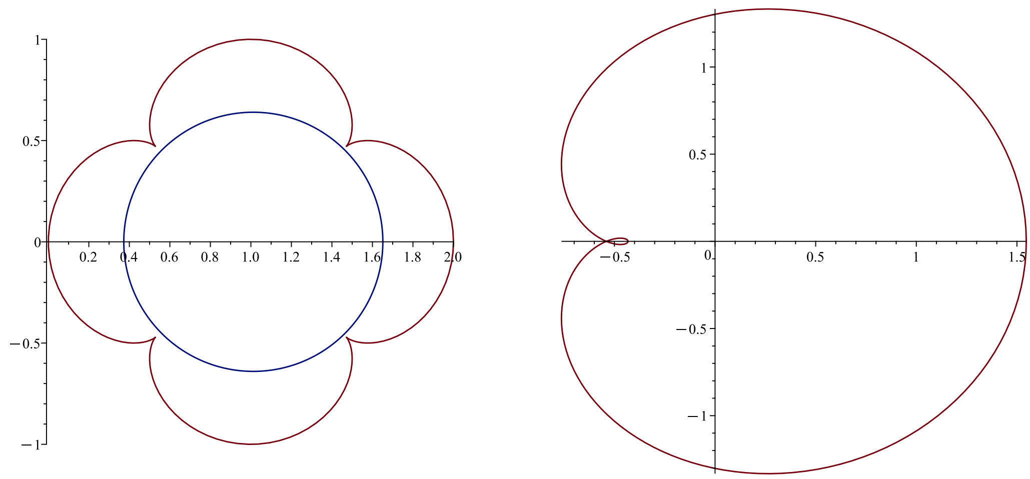
For the function $\widetilde{f}$ defined by the above item (ii), we have $\widetilde{f} \in \mathcal{A}_4^{r,s}$. Using the 2D plot *of the MAPLE™ computer software, the image of the boundary $\partial\mathbb{D}$ by the functions $\widetilde{f}$ (see Figure 2b), we see that $\widetilde{f}(\partial\mathbb{D})$ is a simple curve; hence, $\widetilde{f}$ is univalent on $\partial\mathbb{D}$. Therefore, according to the above result, we conclude that $\widetilde{f} \in \mathcal{S}$; hence, $\mathcal{A}_4^{r,s} \cap \mathcal{S} \neq \emptyset$ for some values of the parameters $r \geq 0$ and $s \in [0,1]$.*



(**a**) (**b**)

Figure 2. Figures for Remark 1 (ii): (**a**) The images of $\Psi_{r,s}\widetilde{f}\left(e^{i\theta}\right)$ (blue color) and $\mathcal{Q}_4\left(e^{i\theta}\right)$ (red color), $\theta \in [0, 2\pi)$. (**b**) The image of $\widetilde{f}(\partial\mathbb{D})$.

(iii) Let us consider the function $\widehat{f}(z) = z + az^2 + bz^3$ *for* $a = 0.58$, $b = 0.01$, *and let us take* $r = 0.05$ *and* $s = 0.06$. *From the* 2D plot *of the MAPLE™ computer software we represent the images of the boundary* $\partial\mathbb{D}$ *by the functions* $\Psi_{r,s}\widehat{f}$ *and* $\mathcal{Q}_4$ *in Figure 3a. For similar reasons, like item (ii) we have* $\Psi_{r,s}\widehat{f}(z) \prec \mathcal{Q}_4(z)$. *In conclusion,* $\widehat{f} \in \mathcal{A}_4^{r,s}$ *for the above given values of the parameters. But, representing with a* 2D plot *of the MAPLE™ computer software the image of the circle* $|z| = 0.98$ *by the functions* $\widehat{f}$ *(see Figure 3b), we see that* $\widehat{f}\left(0.98\, e^{i\theta}\right)$, $\theta \in [0, 2\pi)$ *is not a simple curve; hence,* $\widehat{f}$ *is not univalent in* $\mathbb{D}$. *Consequently, we have* $\mathcal{A}_4^{r,s} \not\subset \mathcal{S}$ *for the general choices of the parameters* $r \geq 0$ *and* $s \in [0, 1]$.



(**a**) (**b**)

Figure 3. Figures for Remark 1 (iii): (**a**) The images of $\Psi_{r,s}\widehat{f}\left(e^{i\theta}\right)$ (blue color) and $\mathcal{Q}_4\left(e^{i\theta}\right)$ (red color), $\theta \in [0, 2\pi)$. (**b**) The image of $\widehat{f}\left(0.98\, e^{i\theta}\right)$, $\theta \in [0, 2\pi)$.

(iv) Not only polynomial functions belong to these classes $\mathcal{A}_4^{r,s}$, *as can we see in the next examples. Taking* $f_c(z) = z \cdot \dfrac{1 + az}{1 + bz}$ *for the particular case* $a = 0.58$, $b = 0.001$, $r = 0.05$, *and* $s = 0.06$, *we similarly obtain the images of the boundary* $\partial\mathbb{D}$ *by the functions* $\Psi_{r,s}f_c$ *and* $\mathcal{Q}_4$, *shown in Figure 4a, and, for the same reasons as in the above item, we conclude that* $f_c \in \mathcal{A}_4^{r,s}$ *for these values of the parameters. We could mention the same property for the transcendental function* $f_e(z) = z\, e^{az}$ *with* $a = 0.38$, *where for* $r = 0.05$ *and* $s = 0.06$, *using a proof similar to those of item (ii) (see Figure 4b), we obtain* $f_e \in \mathcal{A}_4^{r,s}$.

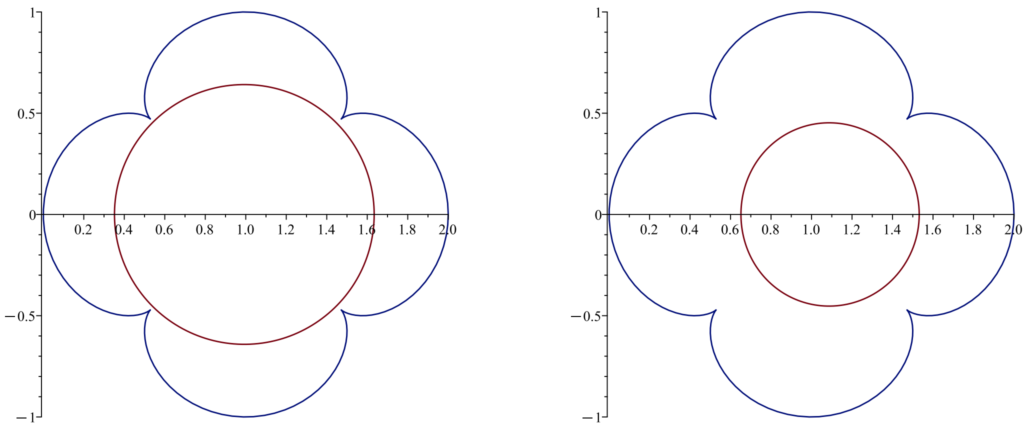
(v) For all $n \in \mathbb{N} \setminus \{1\}$ *and* $\gamma \in \mathbb{C}$ *with* $|\gamma| = 1$*, if we define the functions*

$$f_{n-2}(z) := z + \frac{5}{6\tau_n}\gamma^{n-1}z^n + \frac{1}{6\tau_{5n-4}}\gamma^{5n-5}z^{5n-4}, \ z \in \mathbb{D}, \tag{6}$$

using the fact that

$$\Psi_{r,s}f_{n-2}(z) = 1 + \frac{5}{6}(\gamma z)^{n-1} + \frac{1}{6}(\gamma z)^{5n-5} = \mathcal{Q}_4\Big((\gamma z)^{n-1}\Big) \prec \mathcal{Q}_4(z)$$

it follows that $f_{n-2} \in \mathcal{A}_4^{r,s}$ *for all* $r \geq 0$*,* $s \in [0,1]$ *and* $n \in \mathbb{N} \setminus \{1\}$*.*



(**a**) (**b**)

Figure 4. Figures for Remark 1 (iv): (**a**) The images of $\Psi_{r,s}f_c\big(e^{i\theta}\big)$ (red color) and $\mathcal{Q}_4\big(e^{i\theta}\big)$ (blue color), $\theta \in [0, 2\pi)$. (**b**) The image of $\Psi_{r,s}f_e\big(e^{i\theta}\big)$ (red color) and $\mathcal{Q}_4\big(e^{i\theta}\big)$ (blue color), $\theta \in [0, 2\pi)$.

(vi) Definition 1 of the class $\mathcal{A}_4^{r,s}$ *generates the next natural question: whether for every function* $f \in \mathcal{A}$ *there exists* $r \geq 0$ *and* $s \in [0,1]$*, such that the function* f *belongs to the class* $\mathcal{A}_4^{r,s}$*.*

We will provide below a negative answer to this question, i.e., there exists a function $g \in \mathcal{A}$*, such that for any* $r \geq 0$ *and* $s \in [0,1]$ *we have* $\Psi_{r,s}g(z) \not\prec \mathcal{Q}_4(z)$*. The proof of this fact will be presented below, where we provide an example of such a function.*

Letting $g(z) := ze^z \in \mathcal{A}$*, from Formula (4) we easily obtain*

$$H(z) := \Psi_{r,s}g(z) = \Big\{\big[\big(z^2 + 3z + 2\big)s + z\big]r + zs + 1\Big\}e^z - 2rs,$$

that is, an entire function (analytic in the whole complex plane $\mathbb{C}$*), and from the* theorem of the maximum of the module *it follows that*

$$\sup_{z \in \mathbb{D}}|H(z)| = \max_{z \in \overline{\mathbb{D}}}|H(z)| = \max_{|z|=1}|H(z)| \\ \geq |H(1)| = \big[(6e - 2)s + e\big]r + (1+s)e =: L(r,s), \tag{7}$$

with $L : [0, +\infty) \times [0,1] \to \mathbb{R}$*. Since*

$$\frac{\partial L(r,s)}{\partial r} = (6e - 2)s + e \geq e > 0, \ (r,s) \in [0, +\infty) \times [0,1],$$

it follows that $L(\cdot, s)$ *is a strictly increasing function on* $[0, +\infty)$ *for all* $s \in [0,1]$*, therefore,*

$$L(r,s) \geq L(0,s) = (1+s)e \geq e, \ (r,s) \in [0, +\infty) \times [0,1], \tag{8}$$

and combining the inequalities (7) and (8) we deduce that

$$\sup_{z\in\mathbb{D}}|H(z)| \geq \mathrm{e} \simeq 2.7182\ldots \tag{9}$$

On the other hand, the function $\mathcal{Q}_4$ is also an entire function and it is easy to check that

$$\sup_{z\in\mathbb{D}}|\mathcal{Q}_4(z)| = \mathcal{Q}_4(1) = 2, \tag{10}$$

consequently, from (9) and (10) we obtain

$$\sup_{z\in\mathbb{D}}|\Psi_{r,s}g(z)| \geq \mathrm{e} > 2 = \sup_{z\in\mathbb{D}}|\mathcal{Q}_4(z)|,$$

which implies $\Psi_{r,s}g(z) \not\prec \mathcal{Q}_4(z)$. Thus, for the function $g(z) = z\mathrm{e}^z \in \mathcal{A}$, there does not exist $(r,s) \in [0,+\infty) \times [0,1]$, such that $g \in \mathcal{A}_4^{r,s}$; hence,

$$\mathcal{A} \not\subset \left\{\mathcal{A}_4^{r,s} : (r,s) \in [0,+\infty) \times [0,1]\right\}.$$

Remark 2. *Some relevant special cases of the class $\mathcal{A}_4^{r,s}$ could be obtained as follows:*

(i) For $s = 0$ and $r \geq 0$, the class $\mathcal{A}_4^{r,0}$ will be

$$\mathcal{A}_4^{r,0} = \left\{f \in \mathcal{A} : (1-r)\frac{f(z)}{z} + rf'(z) \prec \mathcal{Q}_4(z)\right\}.$$

(ii) Putting $s = 0$ and $r = 1$ in (4), we obtain the class $\mathcal{A}_4^{1,0}$, which was introduced and studied by Sunthrayuth et al. [27], which is

$$\mathcal{A}_4^{1,0} = \{f \in \mathcal{A} : f'(z) \prec \mathcal{Q}_4(z)\}.$$

To prove our main results, we will use the next preliminary results.

We say a function p belongs to the class $\mathcal{P}$ of *Carathéodory functions* (see [34,35]) if and only if it has the series expansion

$$p(z) = 1 + \sum_{k=1}^{\infty} c_n z^k,\ z \in \mathbb{D}, \tag{11}$$

and $\operatorname{Re} p(z) > 0$ for all $z \in \mathbb{D}$.

Lemma 1. *Let $p \in \mathcal{P}$ be of the form (11). Then:*

(i) For $n \geq 1$

$$|c_n| \leq 2. \tag{12}$$

The inequality holds for all $n \geq 1$ if and only if $p(z) = (1+\lambda z)/(1-\lambda z)$, $|\lambda| = 1$.

(ii) Also, if $\mu \geq 0$ then

$$|c_{n+k} - \mu c_n c_k| \leq 2\max\{1; |2\mu - 1|\} = \begin{cases} 2, & \text{if } \ 0 \leq \mu \leq 1, \\ 2|2\mu-1|, & \text{otherwise.} \end{cases} \tag{13}$$

If $0 < \mu < 1$ the inequality is sharp for the function $p(z) = \left(1+z^{n+k}\right)/\left(1-z^{n+k}\right)$. In the other cases, the inequality is sharp for the function $p(z) = (1+z)/(1-z)$.

(iii) Moreover, if $\mathsf{B} \in [0,1]$ with $\mathsf{B}(2\mathsf{B}-1) \leq \mathsf{D} \leq \mathsf{B}$, we have

$$\left|c_3 - 2\mathsf{B}c_1c_2 + \mathsf{D}c_1^3\right| \leq 2. \tag{14}$$

We note that inequality (12) is the well-known result of the *Carathéodory lemma* [34] (see also ([33] Corollary 2.3, p. 41), ([36] Carathéodory's Lemma, p. 41)). Inequality (13) represents Lemma 2.3 of [37], that for $\mu = 1$ was proved in a more general form for $p(0) = c_0$ in Lemma 1 of ([38] p. 546). Inequality (14) refers to Lemma 3 of ([39] p. 66).

Lemma 2. *If $p \in \mathcal{P}$ is given by (11) then*

$$2c_2 = c_1^2 + x\left(4 - |c_1|^2\right), \tag{15}$$

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - \left(4 - c_1^2\right)c_1 x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)\eta, \quad \text{if} \quad c_1 \geq 0, \tag{16}$$

for some x, η with $|x| \leq 1$, $|\eta| \leq 1$.

Formula (15) for c_2 can be found in relation (10) of ([33] p. 166), while (16) for c_3 was originally derived by Libera and Złotkiewicz, as referenced in equalities (3.9) and (3.10) of ([40] p. 229) and ([41] p. 254), respectively.

Lemma 3 ([37] Lemma 2.1)**.** *Let ϑ, ε, ς, and a satisfy that a, $\vartheta \in (0,1)$ and*

$$8a(1-a)\left((\vartheta\varepsilon - 2\varsigma)^2 + (\vartheta(a+\vartheta) - \varepsilon)^2\right) + \vartheta(1-\vartheta)(\varepsilon - 2a\vartheta)^2 \leq 4a\vartheta^2(1-\vartheta)^2(1-a). \tag{17}$$

If $p \in \mathcal{P}$ and is given by (11) then

$$\left|\varsigma c_1^4 + a c_2^2 + 2\vartheta c_1 c_3 - \frac{3}{2}\varepsilon c_1^2 c_2 - c_4\right| \leq 2. \tag{18}$$

2. Initial Coefficient Estimates for Class $\mathcal{A}_4^{r,s}$

The first theorem gives us the upper bounds for the first five coefficients $|a_n|$ for the functions belonging to $\mathcal{A}_4^{r,s}$ as follows:

Theorem 1. *If the function $f \in \mathcal{A}_4^{r,s}$ is given by (1) then*

$$|a_n| \leq \frac{5}{6\tau_n}, \; n = 2,3,4,5, \tag{19}$$

where τ_n is given by (5).

For $n = 2$ the bound is the best possible, and the inequality $|a_2| \leq \dfrac{5}{6\tau_2}$ is sharp for the function

$$f_*(z) := z + \frac{5}{6\tau_2}\gamma z^2 + \frac{1}{6\tau_6}\gamma^5 z^6,$$

with $\gamma \in \mathbb{C}$, $|\gamma| = 1$.

Proof. Supposing that $f \in \mathcal{A}_4^{r,s}$ has the form (1), then there exists a function w analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{D}$ satisfying

$$(1-r)(1-s)\frac{f(z)}{z} + \big(s + r(1+s)\big)f'(z) + rs\big(zf''(z) - 2\big) = \mathcal{Q}_4(w(z)), \; z \in \mathbb{D}. \tag{20}$$

It is easy to check that

$$(1-r)(1-s)\frac{f(z)}{z} + \big(s + r(1+s)\big)f'(z) + rs\big(zf''(z) - 2\big) = 1 + \sum_{n=2}^{\infty} \tau_n a_n z^{n-1}, \; z \in \mathbb{D}, \tag{21}$$

where $\tau_n a_n = \dfrac{\left(\Psi_{r,s} f\right)^{(n-1)}(0)}{(n-1)!}$, with $\Psi_{r,s} f$ and τ_n given by (4) and (5), respectively, for $n \in \{2,3,4,5\}$.

Letting the function l defined by

$$l(z) := \frac{1+w(z)}{1-w(z)} = 1 + \sum_{n=1}^{\infty} l_n z^n, \; z \in \mathbb{D},$$

since $|w(z)| < 1$ in $\mathbb{D}$, it follows that $l \in \mathcal{P}$.

A simple computation gives

$$w(z) = \frac{l(z)-1}{l(z)+1} = \frac{1}{2} l_1 z + \frac{1}{2}\left(l_2 - \frac{1}{2} l_1^2\right) z^2 + \frac{1}{2}\left(l_3 - l_1 l_2 + \frac{1}{4} l_1^3\right) z^3 + \ldots, \; z \in \mathbb{D}, \tag{22}$$

and by replacing the power series expansion of (22) in relation (20) we obtain

$$\begin{aligned}(1-r)(1-s)\frac{f(z)}{z} + \big(s + r(1+s)\big) f'(z) + rs\big(z f''(z) - 2\big) = 1 + \frac{5}{12} l_1 z \\ + \left(\frac{5 l_2}{12} - \frac{5 l_1^2}{24}\right) z^2 + \left(\frac{5}{12} l_3 - \frac{5}{12} l_1 l_2 + \frac{5}{48} l_1^3\right) z^3 \\ + \left(\frac{5}{12} l_4 - \frac{5}{12} l_1 l_3 - \frac{5}{24} l_2^2 + \frac{5}{16} l_2 l_1^2 - \frac{5}{96} l_1^4\right) z^4 + \ldots, \; z \in \mathbb{D}.\end{aligned} \tag{23}$$

Equating the first five coefficients of (21) and (23) we obtain

$$a_2 = \frac{5}{12\tau_2} l_1, \tag{24}$$

$$a_3 = \frac{1}{\tau_3}\left(-\frac{5}{24} l_1^2 + \frac{5}{12} l_2\right), \tag{25}$$

$$a_4 = \frac{1}{\tau_4}\left(-\frac{5}{12} l_1 l_2 + \frac{5}{12} l_3 + \frac{5}{48} l_1^3\right), \tag{26}$$

and

$$a_5 = \frac{1}{\tau_5}\left(\frac{5}{12} l_4 + \frac{5}{16} l_1^2 l_2 - \frac{5}{96} l_1^4 - \frac{5}{24} l_2^2 - \frac{5}{12} l_1 l_3\right). \tag{27}$$

Using the inequality (12) for $n = 2$ in (24) we obtain

$$|a_2| \le \frac{5}{6\tau_2}. \tag{28}$$

Since (25) can be written as

$$a_3 = \frac{5}{12\tau_3}\left(l_2 - \frac{1}{2} l_1^2\right),$$

using inequality (13) for $n = k = 1$ and $\mu = \dfrac{1}{2}$ we obtain

$$|a_3| \le \frac{5}{6\tau_3}.$$

From (26), it follows that

$$|a_4| = \frac{5}{12\tau_4}\left| l_3 - 2 \cdot \frac{1}{2} \cdot l_1 l_2 + \frac{1}{4} l_1^3 \right|$$

and we will compare the right-hand side of the above relation to (14). Therefore, since

$$0 \le \mathrm{B} = \frac{1}{2} \le 1, \quad \mathrm{B} = \frac{1}{2} \ge \mathrm{D} = \frac{1}{4}, \quad \mathrm{B}(2\mathrm{B}-1) = 0 \le \mathrm{D} = \frac{1}{4},$$

all the requirements of Lemma 1 (iii) are satisfied; hence, (14) leads us to

$$|a_4| \le \frac{5}{6\tau_4}.$$

Equality (27) implies that

$$|a_5| = \frac{5}{12\tau_5}\left|\frac{1}{8}l_1^4 + \frac{1}{2}l_2^2 + 2\cdot\frac{1}{2}\cdot l_1 l_3 - \frac{3}{2}\cdot\frac{1}{2}\cdot l_1^2 l_2 - l_4\right|, \tag{29}$$

and by comparing the right-hand side of (29) with the left-hand side of (18) we obtain

$$\varsigma = \frac{1}{8},\ a = \frac{1}{2},\ \vartheta = \frac{1}{2},\ \varepsilon = \frac{1}{2}$$

Since

$$8a(1-a)\Big((\vartheta\varepsilon - 2\varsigma)^2 + (\vartheta(a+\vartheta) - \varepsilon)^2\Big) + \vartheta(1-\vartheta)(\varepsilon - 2a\vartheta)^2 = 0 \le \frac{1}{16} = 4a\vartheta_2(1-\vartheta)^2(1-a),$$

the assumption inequality (17) holds; consequently, (18) combined with (29) implies

$$|a_5| \le \frac{5}{6\tau_5},$$

and the proof of the theorem is complete.

To prove the sharpness for $n = 2$, we will use the fact $f_* \equiv f_0$, given by (6). From Remark 1 (iv) we obtain $f_* \in \mathcal{A}_4^{r,s}$ and for $n = 2$ the equality holds in (19). □

Fekete and Szegő [2] proved the well-known result,

$$\max\left\{\left|a_3 - \mu a_2^2\right| : f \in \mathcal{S}\right\} = 1 + 2e^{-\frac{2\mu}{1-\mu}},\ \mu \in [0,1],$$

and in the next result we consider the corresponding problem for the family $\mathcal{A}_4^{r,s}$:

Theorem 2. *If the function $f \in \mathcal{A}_4^{r,s}$ has the form* (1) *and $\mu \in \mathbb{R}$ then*

$$\left|a_3 - \mu a_2^2\right| \le \max\left\{\frac{5}{6\tau_3}; \frac{25|\mu|}{36\tau_2^2}\right\}.$$

Proof. If $f \in \mathcal{A}_4^{r,s}$ has the form (1), as in the proof of the previous theorem, using (24) and (25), we obtain

$$\left|a_3 - \mu a_2^2\right| = \left|\frac{1}{\tau_3}\left(\frac{5}{12}l_2 - \frac{5}{24}l_1^2\right) - \mu\,\frac{25}{144\tau_2^2}\,l_1^2\right| = \frac{5}{12\tau_3}\left|l_2 - \frac{6\tau_2^2 + 5\mu\,\tau_3}{12\tau_2^2}\,l_1^2\right|.$$

Using inequality (13) for the right-hand side of the above equality, if $n = k = 1$ we obtain

$$\left|a_3 - \mu a_2^2\right| \le \frac{5}{6\tau_3}\max\left\{1; \left|\frac{6\tau_2^2 + 5\mu\tau_3}{6\tau_2^2} - 1\right|\right\} = \max\left\{\frac{5}{6\tau_3}; \frac{25|\mu|}{36\tau_2^2}\right\}.$$

□

Another three estimations of the differences of the coefficients modules for the functions of the class $\mathcal{A}_4^{r,s}$ will be presented as follows.

Theorem 3. *If the function $f \in \mathcal{A}_4^{r,s}$ has the form* (1) *then*

$$|a_2a_3 - a_4| \leq \frac{5}{6\tau_4}.$$

Proof. Since $f \in \mathcal{A}_4^{r,s}$ is of the form (1), as in the proof of Theorem 1 according to (24)–(26) we obtain

$$|a_2a_3 - a_4| = \frac{5}{12\tau_4}\left|l_3 - 2\left(\frac{5\tau_4}{24\tau_2\tau_3} + \frac{1}{2}\right)l_1l_2 + \left(\frac{5\tau_4}{24\tau_2\tau_3} + \frac{1}{4}\right)l_1^3\right|. \tag{30}$$

If we compare the right-hand side of the above equality with the left-hand side of (14) we obtain

$$\mathsf{B} = \frac{5\tau_4}{24\tau_2\tau_3} + \frac{1}{2}, \quad \mathsf{D} = \frac{5\tau_4}{24\tau_2\tau_3} + \frac{1}{4}.$$

Since

$$\mathsf{B} - 1 = -\frac{600r^2s^2 + 240r^2s + 240r\,s^2 + 24r^2 + 143rs + 24s^2 + 21r + 21s + 7}{24(5rs + r + s + 1)(10rs + 2r + 2s + 1)} < 0$$

and

$$\begin{aligned}\mathsf{B}(2\mathsf{B}-1) - \mathsf{D} = -&\frac{1}{288(5rs + r + s + 1)^2(10rs + 2r + 2s + 1)^2}\Big(180000r^4s^4\\ &+ 144000r^4s^3 + 144000r^3s^4 + 43200r^4s^2 + 194400r^3s^3 + 43200r^2s^4\\ &+ 5760r^4s + 82080r^3s^2 + 82080r^2s^3 + 5760r\,s^4 + 288r^4 + 14112r^3s\\ &+ 43823r^2s^2 + 14112r\,s^3 + 288s^4 + 864r^3 + 9402r^2s + 9402r\,s^2 + 864s^3\\ &+ 711r^2 + 2732rs + 711s^2 + 282r + 282s + 47\Big) < 0\end{aligned}$$

for all $r \geq 0$ and $s \in [0,1]$, using also (5), it follows that

$$0 \leq \mathsf{B} = \frac{5\tau_4}{24\tau_2\tau_3} + \frac{1}{2} \leq 1, \quad \mathsf{B} = \frac{5\tau_4}{24\tau_2\tau_3} + \frac{1}{2} \geq \mathsf{D} = \frac{5\tau_4}{24\tau_2\tau_3} + \frac{1}{4},$$
$$\mathsf{B}(2\mathsf{B}-1) = \frac{5\tau_4}{12\tau_2\tau_3}\left(\frac{5\tau_4}{24\tau_2\tau_3} + \frac{1}{2}\right) \leq \mathsf{D} = \frac{5\tau_4}{24\tau_2\tau_3} + \frac{1}{4}.$$

Since all the conditions of Lemma 1 (iii) are satisfied, using (14) we obtain from (30) the required conclusion. □

Theorem 4. *If the function $f \in \mathcal{A}_4^{r,s}$ is given by* (1) *then*

$$|a_5 - a_2a_4| \leq \frac{5}{6\tau_5}.$$

Proof. Similarly, as in the proof of the previous theorems, since $f \in \mathcal{A}_4^{r,s}$ has the form (1) from (24), (26), and (27), we obtain

$$a_5 - a_2a_4 = -\frac{5}{12\tau_5}\left[\frac{l_2^2}{2} - l_4 + \left(\frac{5\tau_5}{12\tau_2\tau_4} + 1\right)l_1l_3 - \left(\frac{5\tau_5}{12\tau_2\tau_4} + \frac{3}{4}\right)^2_1 l_2 + \left(\frac{5\tau_5}{48\tau_2\tau_4} + \frac{1}{8}\right)l_1^4\right],$$

hence,

$$|a_5 - a_2 a_4| = \frac{5}{12\tau_4}\left|\left(\frac{1}{8} + \frac{5\tau_5}{48\tau_2\tau_4}\right) l_1^4 + \frac{1}{2} l_2^2 + 2\left(\frac{1}{2} + \frac{5\tau_5}{24\tau_2\tau_4}\right) l_1 l_3 - \frac{3}{2}\left(\frac{1}{2} + \frac{5\tau_5}{18\tau_2\tau_4}\right) l_1^2 l_2 - l_4\right|. \tag{31}$$

Comparing the right side of (31) with the left-hand side of (18) we obtain

$$\varsigma = \frac{1}{8} + \frac{5\tau_5}{48\tau_2\tau_4}, \quad a = \frac{1}{2}, \quad \vartheta = \frac{1}{2} + \frac{5\tau_5}{24\tau_2\tau_4}, \quad \varepsilon = \frac{1}{2} + \frac{5\tau_5}{18\tau_2\tau_4},$$

and denoting

$$U := 8a(1-a)\Big((\vartheta\varepsilon - 2\varsigma)^2 + (\vartheta(a+\vartheta) - \varepsilon)^2\Big) + \vartheta(1-\vartheta)(\varepsilon - 2a\vartheta)^2, \tag{32}$$

$$V := 4a\vartheta_2(1-\vartheta)^2(1-a), \tag{33}$$

it follows that

$$\begin{aligned}
V - U = & \frac{1}{373248(5rs + r + s + 1)^4(17rs + 3r + 3s + 1)^4}\Big(1217736180000r^8s^8 \\
& + 1833767424000r^8s^7 + 1833767424000r^7s^8 + 1207454947200r^8s^6 \\
& + 3675624998400r^7s^7 + 1207454947200r^6s^8 + 454063656960r^8s^5 \\
& + 3015278369280r^7s^6 + 3015278369280r^6s^7 + 454063656960r^5s^8 \\
& + 106659488448r^8s^4 + 1355054095872r^7s^5 + 2992998111048r^6s^6 \\
& + 1355054095872r^5s^7 + 106659488448r^4s^8 + 16025776128r^8s^3 \\
& + 369638318592r^7s^4 + 1586405248416r^6s^5 + 1586405248416r^5s^6 \\
& + 369638318592r^4s^7 + 16025776128r^3s^8 + 1504096128r^8s^2 \\
& + 63159906816r^7s^3 + 500006872080r^6s^4 + 967753705944r^5s^5 \\
& + 500006872080r^4s^6 + 63159906816r^3s^7 + 1504096128r^2s^8 \\
& + 80621568r^8s + 6634483200r^7s^2 + 97102192320r^6s^3 + 345393269808r^5s^4 \\
& + 345393269808r^4s^5 + 97102192320r^3s^6 + 6634483200r^2s^7 + 80621568r\,s^8 \\
& + 1889568r^8 + 393030144r^7s + 11439040392r^6s^2 + 75185189568r^5s^3 \\
& + 135240643162r^4s^4 + 75185189568r^3s^5 + 11439040392r^2s^6 + 393030144r\,s^7 \\
& + 1889568s^8 + 10077696r^7 + 751686048r^6s + 9858848928r^5s^2 \\
& + 31767073568r^4s^3 + 31767073568r^3s^4 + 9858848928r^2s^5 + 751686048r\,s^6 \\
& + 10077696s^7 + 21184416r^6 + 717170664r^5s + 4488452100r^4s^2 \\
& + 7822440500r^3s^3 + 4488452100r^2s^4 + 717170664r\,s^5 + 21184416s^6 \\
& + 22266000r^5 + 354292592r^4s + 1153024056r^3s^2 + 1153024056r^2s^3 \\
& + 354292592r\,s^4 + 22266000s^5 + 12045058r^4 + 96743788r^3s + 174849942r^2s^2 \\
& + 96743788r\,s^3 + 12045058s^4 + 3603400r^3 + 15476952r^2s + 15476952r\,s^2 \\
& + 3603400s^3 + 643974r^2 + 1497212rs + 643974s^2 + 73648r + 73648s + 4603\Big).
\end{aligned}$$

Therefore, $V > U$ for all $r \geq 0$ and $s \in [0,1]$; hence, assumption (17) of Lemma 3 is satisfied, and by combining (31) with (18) we obtain our result. □

Theorem 5. *If the function $f \in \mathcal{A}_4^{r,s}$ is given by* (1) *then*

$$\left|a_5 - a_3^2\right| \leq \frac{5}{6\tau_5}.$$

Proof. If $f \in \mathcal{A}_4^{r,s}$ has the form (1), from (25) and (27) we obtain

$$a_5 - a_3^2 = -\frac{5}{12\tau_5}\left[\left(\frac{1}{8}+\frac{5\tau_5}{48\tau_3^2}\right)l_1^4 + \left(\frac{1}{2}+\frac{5\tau_5}{12\tau_3^2}\right)l_2^2 + l_1l_3 - \frac{3}{2}\left(\frac{1}{2}+\frac{5\tau_5}{18\tau_3^2}\right)l_1^2l_2 - l_4\right],$$

hence, it follows that

$$\left|a_5 - a_3^2\right| = \frac{5}{12\tau_5}\left|\left(\frac{1}{8}+\frac{5\tau_5}{48\tau_3^2}\right)l_1^4 + \left(\frac{1}{2}+\frac{5\tau_5}{12\tau_3^2}\right)l_2^2 + 2\cdot\frac{1}{2}\cdot l_1l_3 - \frac{3}{2}\left(\frac{1}{2}+\frac{5\tau_5}{18\tau_3^2}\right)l_1^2l_2 - l_4\right|. \tag{34}$$

Comparing the right-hand side of (34) with the left-hand side of inequality (18), that is,

$$\left|\varsigma c_1^4 + ac_2^2 + 2\vartheta c_1c_3 - \frac{3}{2}\varepsilon c_1^2c_2 - c_4\right|,$$

we obtain

$$\varsigma = \frac{1}{8}+\frac{5\tau_5}{48\tau_3^2},\quad a = \frac{1}{2}+\frac{5\tau_5}{12\tau_3^2},\quad \vartheta = \frac{1}{2},\quad \varepsilon = \frac{1}{2}+\frac{5\tau_5}{18\tau_3^2}.$$

If we use notations (32) and (33), for the above values of the parameters we obtain

$$\begin{aligned}
V - U = &\frac{1}{46656(10rs+2r+2s+1)^8}\Big(291600000000r^8s^8 + 466560000000r^8s^7 \\
&+ 466560000000r^7s^8 + 326592000000r^8s^6 + 886464000000r^7s^7 \\
&+ 326592000000r^6s^8 + 130636800000r^8s^5 + 718502400000r^7s^6 + 718502400000r^6s^7 \\
&+ 130636800000r^5s^8 + 32659200000r^8s^4 + 326592000000r^7s^5 + 648219600000r^6s^6 \\
&+ 326592000000r^5s^7 + 32659200000r^4s^8 + 5225472000r^8s^3 + 91445760000r^7s^4 \\
&+ 322600320000r^6s^5 + 322600320000r^5s^6 + 91445760000r^4s^7 + 5225472000r^3s^8 \\
&+ 522547200r^8s^2 + 16198963200r^7s^3 + 98225568000r^6s^4 + 171272304000r^5s^5 \\
&+ 98225568000r^4s^6 + 16198963200r^3s^7 + 522547200r^2s^8 + 29859840r^8s \\
&+ 1776660480r^7s^2 + 18871856640r^6s^3 + 54977126400r^5s^4 + 54977126400r^4s^5 \\
&+ 18871856640r^3s^6 + 1776660480r^2s^7 + 29859840r\,s^8 + 746496r^8 + 110481408r^7 \\
&+ 2242816128r^6s^2 + 11129803776r^5s^3 + 18355915120r^4s^4 + 11129803776r^3s^5 \\
&+ 2242816128r^2s^6 + 110481408r\,s^7 + 746496s^8 + 2985984r^7 + 151054848r^6s \\
&+ 1403345664r^5s^2 + 3861088640r^4s^3 + 3861088640r^3s^4 + 1403345664r^2s^5 \\
&+ 151054848r\,s^6 + 2985984s^7 + 4419072r^6 + 101308032r^5s + 510650880r^4s^2 \\
&+ 838703840r^3s^3 + 115053120r^3s^2 + 510650880r^2s^4 + 101308032r\,s^5 + 4419072s^6 \\
&+ 3209472r^5 + 39359360r^4s + 115053120r^2s^3 + 39359360r\,s^4 + 3209472s^5 \\
&+ 1359520r^4 + 9404800r^3s + 16218960r^2s^2 + 9404800r\,s^3 + 1359520s^4 \\
&+ 357568r^3 + 1389024r^2s + 1389024r\,s^2 + 357568s^3 + 58992r^2 + 126464rs \\
&+ 58992s^2 + 6256r + 6256s + 391\Big) > 0,\ r \geq 0,\ s \in [0,1],
\end{aligned}$$

because all the terms of the sum are positive and, thus, assumption (17) of Lemma 3 is satisfied. Therefore, the required inequality follows from (34) and (18). □

3. Krushkal Inequalities for the Class $\mathcal{A}_4^{r,s}$

In this section, we will show that for the well-known inequality

$$\left|a_n^p - a_2^{p(n-1)}\right| \le 2^{p(n-1)-n^p} \tag{35}$$

we can find smaller upper bounds for the subclass $\mathcal{A}_4^{r,s}$ and for the specific couples of values $n = 4$, $p = 1$ and $n = 5$, $p = 1$. This inequality was originally introduced and proved by Krushkal for the entire class of normalized univalent functions $\mathcal{S}$ and integers $n > 3$, $p \ge 1$, while it is sharp and the equality occurs for the *Koebe function* (as cited in [42] Theorem 6.1, p. 17).

First, for $n = 4$ and $p = 1$ we obtain the following first upper bound for the left-hand side of (35), while the second result deals with the same problem for $n = 5$, $p = 1$. According to the fact that $\tau_5 \ge \tau_4 > 1$, it is obvious that these bounds are smaller than the right-hand side of (35) for these values of n and p.

Theorem 6. *If the function $f \in \mathcal{A}_4^{r,s}$ has the form (1), then*

$$\left|a_4 - a_2^3\right| \le \frac{5}{6\tau_4}.$$

Proof. If $f \in \mathcal{A}_4^{r,s}$, from (24) and (26) we obtain

$$a_4 - a_2^3 = \frac{5l_3}{12\tau_4} - \frac{5}{12}\frac{l_2}{\tau_4}l_1 + \left(\frac{5}{48\tau_4} - \frac{125}{1728\tau_2^3}\right)l_1^3,$$

hence,

$$\left|a_4 - a_2^3\right| = \frac{5}{12\tau_4}\left|l_3 - 2\cdot\frac{1}{2}\cdot l_2 l_3 + \left(\frac{1}{4} - \frac{25\tau_4}{144\tau_2^3}\right)l_1^3\right|. \tag{36}$$

Comparing the right-hand side of the above relation (14), since $r \ge 0$ and $s \in [0,1]$ and according to (5), we obtain

$$0 \le \mathsf{B} = \frac{1}{2} \le 1, \quad \mathsf{B} = \frac{1}{2} \ge \mathsf{D} = \frac{1}{4} - \frac{25\tau_4}{144\tau_2^3}.$$

Moreover,

$$\mathsf{B}(2\mathsf{B}-1) = 0 \le \mathsf{D} = \frac{1}{4} - \frac{25\tau_4}{144\tau_2^3},$$

because

$$\begin{aligned}\mathsf{D} = \frac{1}{144(5rs+r+s+1)^3}\Big(&4500r^3s^3 + 2700r^3s^2 + 2700r^2s^3 + 540r^3s + 3780r^2s^2 \\ &+540rs^3 + 36r^3 + 1188r^2s + 1188rs^2 + 36s^3 + 108r^2 + 331rs + 108s^2 + 33r \\ &+33s + 11\Big) > 0,\ r \ge 0,\ s \in [0,1],\end{aligned}$$

and using (14) together with (36) we obtain the desired result. □

Theorem 7. *If the function $f \in \mathcal{A}_4^{r,s}$ is given by (1) then*

$$\left|a_5 - a_2^4\right| \le \frac{5}{6\tau_5}.$$

Proof. If $f \in \mathcal{A}_4^{r,s}$ has the power expansion series (1), from (24) and (27) we obtain

$$a_5 - a_2^4 = -\frac{5\left(4l_2^2 - 8l_4\right)}{96\tau_5} - \frac{5}{12}\frac{l_3}{\tau_5}l_1 + \frac{5}{16}\frac{l_2}{\tau_5}l_1^2 - \left(\frac{625}{20736\tau_2^4} + \frac{5}{96\tau_5}\right)l_1^4,$$

hence,

$$\left|a_5 - a_2^4\right| = \frac{5}{12\tau_5}\left|\left(\frac{1}{8} + \frac{125\tau_5}{1728\tau_2^4}\right)l_1^4 + \frac{1}{2}\cdot l_2^2 + 2\cdot\frac{1}{2}\cdot l_1 l_3 - \frac{3}{2}\cdot\frac{1}{2}\cdot l_1^2 l_2 - l_4\right|. \qquad (37)$$

After we compare the right-hand side of (37) to

$$\left|\varsigma c_1^4 + a c_2^2 + 2\vartheta c_1 c_3 - \frac{3}{2}\varepsilon c_1^2 c_2 - c_4\right|$$

we obtain

$$\varsigma = \frac{1}{8} + \frac{125\tau_5}{1728\tau_2^4}, \quad a = \frac{1}{2}, \quad \vartheta = \frac{1}{2}, \quad \varepsilon = \frac{1}{2}.$$

Letting U and V be defined by (32) and (33), it follows that

$$\begin{aligned} V - U = &\frac{1}{373248(5rs + r + s + 1)^8}\Big(9112500000r^8s^8 + 14580000000r^8s^7 \\ &+ 14580000000r^7s^8 + 10206000000r^8s^6 + 34992000000r^7s^7 + 10206000000r^6s^8 \\ &+ 4082400000r^8s^5 + 32659200000r^7s^6 + 32659200000r^6s^7 + 4082400000r^5s^8 \\ &+ 1020600000r^8s^4 + 16329600000r^7s^5 + 40824000000r^6s^6 + 16329600000r^5s^7 \\ &+ 1020600000r^4s^8 + 163296000r^8s^3 + 4898880000r^7s^4 + 26127360000r^6s^5 \\ &+ 26127360000r^5s^6 + 4898880000r^4s^7 + 163296000r^3s^8 + 16329600r^8s^2 \\ &+ 914457600r^7s^3 + 9634464000r^6s^4 + 21555072000r^5s^5 + 9634464000r^4s^6 \\ &+ 914457600r^3s^7 + 16329600r^2s^8 + 933120r^8s + 104509440r^7s^2 \\ &+ 2142443520r^6s^3 + 9993715200r^5s^4 + 9993715200r^4s^5 + 2142443520r^3s^6 \\ &+ 104509440r^2s^7 + 933120r\,s^8 + 23328r^8 + 6718464r^7s + 284788224r^6s^2 \\ &+ 2712019968r^5s^3 + 5888453760r^4s^4 + 2712019968r^3s^5 + 284788224r^2s^6 \\ &+ 6718464r\,s^7 + 23328s^8 + 186624r^7 + 20901888r^6s + 428488704r^5s^2 \\ &+ 1998743040r^4s^3 + 1998743040r^3s^4 + 428488704r^2s^5 + 20901888r\,s^6 \\ &+ 186624s^7 + 653184r^6 + 36578304r^5s + 385378560r^4s^2 + 862202880r^3s^3 \\ &+ 385378560r^2s^4 + 36578304r\,s^5 + 653184s^6 + 1306368r^5 + 39191040r^4s \\ &+ 209018880r^3s^2 + 209018880r^2s^3 + 39191040r\,s^4 + 1306368s^5 + 1632960r^4 \\ &+ 26127360r^3s + 54755900r^2s^2 + 26127360r\,s^3 + 1632960s^4 + 1306368r^3 \\ &+ 7200944r^2s + 7200944r\,s^2 + 1306368s^3 + 403184r^2 + 926988rs + 403184s^2 \\ &+ 61624r + 61624s + 7703\Big), \ r \geq 0, \ s \in [0,1], \end{aligned}$$

using the fact that all the terms are positive, and from (18) combined with (37) we obtain our result. □

Next, for the class $f \in \mathcal{A}_4^{r,s}$ we will determine an upper bound for the Hankel determinant of order two.

Theorem 8. *If the function $f \in \mathcal{A}_4^{r,s}$ is given by (1) then*

$$|\mathcal{D}_{2,2}(f)| = \left|a_2a_4 - a_3^2\right| \leq \frac{25}{36\tau_3^2}.$$

Proof. If $f \in \mathcal{A}_4^{r,s}$, from (24), (25), and (26) we obtain

$$a_2a_4 - a_3^2 = \frac{25}{144\tau_2\tau_4}l_1l_3 - \left(\frac{25}{144\tau_2\tau_4} - \frac{25}{288\tau_3^2}\right)l_1^2l_2 + \left(\frac{25}{576\tau_2\tau_4} - \frac{25}{576\tau_3^2}\right)l_1^4 - \frac{25}{144\tau_3^2}l_2^2. \tag{38}$$

Using (15) and (16) to express l_2 and l_3 in terms of l_1, and noting that without loss in generality and using (12) we can write $l_1 := l \in [0,2]$, from (38) we obtain

$$\begin{aligned} a_2a_4 - a_3^2 &= \frac{25\left(4-l^2\right)\left[-2k^2\eta\, l\tau_3^2 - l\left(l\,x^2 - 2\eta\right)\tau_3^2 - \tau_2\tau_4x^2\left(4-l^2\right)\right]}{576\tau_2\tau_4\tau_3^2} \\ &= -\frac{25}{576\tau_2\tau_4}l^2\left(4-l^2\right)x^2 + \frac{25}{288\tau_2\tau_4}l\left(4-l^2\right)\left(1-k^2\right)\eta - \frac{25}{576\tau_3^2}x^2\left(4-l^2\right)^2, \end{aligned}$$

where $|x| = k \leq 1$ and $|\eta| \leq 1$. Using the triangle inequality in the above relation, since $l \in [0,2]$ and k, $|\eta| \in [0,1]$ we obtain

$$\left|a_2a_4 - a_3^2\right| \leq \frac{25}{576\tau_2\tau_4}l^2\left(4-l^2\right)k^2 + \frac{25}{288\tau_2\tau_4}l\left(4-l^2\right)\left(1-k^2\right) + \frac{25}{576\tau_3^2}k^2\left(4-l^2\right)^2 =: \phi(l,k), \tag{39}$$

and we need to determine

$$\max\left\{\phi(l,k) : (l,k) \in [0,2]\times[0,1]\right\}.$$

For this purpose, a simple computation shows that

$$\begin{aligned} \frac{\partial\phi(l,k)}{\partial k} &= \frac{25l^2\left(4-l^2\right)k}{288\tau_2\tau_4} - \frac{25l\left(4-l^2\right)k}{144\tau_2\tau_4} + \frac{25k\left(4-l^2\right)^2}{288\tau_3^2} \\ &= \frac{25(l-2)^2(l+2)\,k\left[l\left(\tau_2\tau_4 - \tau_3^2\right) + 2\tau_2\tau_4\right]}{288\tau_2\tau_4\tau_3^2}. \end{aligned} \tag{40}$$

Since

$$\tau_2\tau_4 - \tau_3^2 = -\left(15s^2 + 8s + 1\right)r^2 - 8r\,s^2 - s^2,$$

a simple computation leads to

$$\begin{aligned} l\left(\tau_2\tau_4 - \tau_3^2\right) + 2\tau_2\tau_4 = &\left[5(34-3l)s^2 + 8(8-l)s + 6 - l\right]r^2 \\ &+ \left[8 + 56s + 8(8-l)s^2\right]r + (6-l)s^2 + 2(1+4s) \geq 0 \end{aligned}$$

for all $r \geq 0$, because all the coefficients of r from the above equality are non-negative whenever $(l,k) \in [0,2]\times[0,1]$. That is,

$$l\left(\tau_2\tau_4 - \tau_3^2\right) + 2\tau_2\tau_4 \geq 0,$$

and from (40) we obtain

$$\frac{\partial\phi(l,k)}{\partial k}=\frac{25(l-2)^2(l+2)\,k\left[l\left(\tau_2\tau_4-\tau_3^2\right)+2\tau_2\tau_4\right]}{288\tau_2\tau_4\tau_3^2}\geq 0,\ (l,k)\in[0,2]\times[0,1],$$

therefore, the function $\phi(l,\cdot)$ is increasing on $[0,1]$; hence,

$$\phi(l,k)\leq\phi(l,1)=\frac{25}{576\tau_2\tau_4}l^2\left(4-l^2\right)+\frac{25}{576\tau_3^2}\left(4-l^2\right)^2=:\psi(l). \tag{41}$$

Using the fact that

$$\psi'(l)=-\frac{25l}{144(5rs+r+s+1)(17rs+3r+3s+1)(10rs+2r+2s+1)^2}\Big(15l^2r^2s^2 +8l^2r^2s+8l^2r\,s^2+l^2r^2+l^2s^2+140r^2s^2+48r^2s+48r\,s^2+4r^2+56rs +4s^2+8r+8s+2\Big)\leq 0,\ l\in[0,2],$$

for all $r\geq 0$ and $s\in[0,1]$, the function ψ will be decreasing on $[0,2]$, which implies that

$$\psi(l)\leq\psi(0)=\frac{25}{36\tau_3^2}. \tag{42}$$

According to inequalities (41) and (42) we deduce

$$\max\left\{\phi(l,k):(l,k)\in[0,2]\times[0,1]\right\}=\phi(1,0)=\frac{25}{36\tau_3^2},$$

and from (39) we obtain our final result. □

Remark 3. *The results presented in this paper specifically for the case when $r=1$ and $s=0$ were previously obtained by Sunthrayuth et al. [27].*

4. Conclusions

In this study, we focused on a subclass of bounded turning functions associated with a four-leaf-type domain. We made some useful findings for this class, including the bounds of the first four initial coefficients, the Fekete–Szegő-type inequality, the Zalcman inequality, the Krushkal inequality, and the estimation of the second-order Hankel determinant.

Related results for subclasses defined by subordinations with the limaçon function, convex functions in one direction, the cosine function, the nephroid function, etc., were studied in the last period by the fourth author. The actual results do not overlap any of these, nor the structure of the subclasses, because the subordinations by expressions as the left-hand side of subordination (4) had not already appeared.

All of the obtained results have been confirmed to be the best possible. This work has been applied to derive higher-order Hankel determinants, such as when investigating the boundaries of fourth- and fifth-order Hankel determinants. Furthermore, this novel methodology can be used to obtain precise bounds on the third-order Hankel determinant for various subclasses of univalent functions.

Taking into account the upper bounds given in Theorem 1, an interesting open problem that could start a real challenge is to prove that the inequality $|a_n|\leq\dfrac{5}{6\tau_n}$ holds for all $n\in\mathbb{N}\setminus\{1\}$ for the function class $\mathcal{A}_4^{r,s}$, where τ_n is given by (5).

Author Contributions: Conceptualization, S.G., B.S., M.I. and T.B.; methodology, S.G., B.S., M.I. and T.B.; software, S.G., B.S., M.I. and T.B.; validation, S.G., B.S., M.I. and T.B.; formal analysis, S.G., B.S., M.I. and T.B.; investigation, S.G., B.S., M.I. and T.B.; resources, S.G., B.S., M.I. and T.B.; data curation,

S.G., B.S., M.I. and T.B.; writing—original draft preparation, S.G., B.S., M.I. and T.B.; writing—review and editing, S.G., B.S., M.I. and T.B.; visualization, S.G., B.S., M.I. and T.B.; supervision, S.G., B.S., M.I. and T.B.; project administration, S.G., B.S., M.I. and T.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The authors are grateful to the reviewers of this article, who gave valuable remarks, comments, and advice for improving the quality of the paper.

Conflicts of Interest: The authors declare no conflicts of interest.

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