axioms

MDPI

Article

Binomial Series Involving Harmonic-like Numbers

Chunli Li [1] **and Wenchang Chu** [2,*]

[1] School of Mathematics and Statistics, Zhoukou Normal University, Zhoukou 466001, China; lichunlizk@outlook.com
[2] Department of Mathematics and Physics, University of Salento, 73100 Lecce, Italy
* Correspondence: hypergeometricx@outlook.com or chu.wenchang@unisalento.it

Abstract: By computing definite integrals, we shall examine binomial series of convergence rate $\pm 1/2$ and weighted by harmonic-like numbers. Several closed formulae in terms of the Riemann and Hurwitz zeta functions as well as logarithm and polylogarithm functions will be established, including a conjectured one made recently by Z.-W. Sun.

Keywords: binomial series; harmonic number; Riemann zeta function; dilogarithm; polylogarithm

MSC: Primary 11B65, 11G55; Secondary 11M06, 65B10

check for updates

Citation: Li, C.; Chu, W. Binomial Series Involving Harmonic-like Numbers. *Axioms* **2024**, *13*, 162. https://doi.org/10.3390/axioms13030162

Academic Editor: Mircea Merca

Received: 12 February 2024
Revised: 24 February 2024
Accepted: 27 February 2024
Published: 29 February 2024

1. Introduction and Outline

For $m, n \in \mathbb{N}_0$, define harmonic-like numbers by partial sums

$$\mathbf{H}_n^{\langle m\rangle} = \sum_{k=0}^{n-1} \frac{1}{(k+1)^m}, \qquad \mathbf{O}_n^{\langle m\rangle} = \sum_{k=0}^{n-1} \frac{1}{(2k+1)^m};$$
$$\bar{\mathbf{H}}_n^{\langle m\rangle} = \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)^m}, \qquad \bar{\mathbf{O}}_n^{\langle m\rangle} = \sum_{k=0}^{n-1} \frac{(-1)^k}{(2k+1)^m}.$$

When $m = 1$, it will be suppressed from these notations. By introducing a variable x, we can extend harmonic-like numbers to harmonic polynomials

$$\mathcal{H}_n(x) = \sum_{k=1}^{n} \frac{x^{k-1}}{k} \quad \text{and} \quad \mathcal{O}_n(x) = \sum_{k=1}^{n} \frac{x^{k-1}}{2k-1}.$$

There exist numerous infinite series identities involving harmonic numbers in the mathematical literature (see [1–13]). In combinatorial analysis and number theory (see [14–16]), the following binomial series is fundamental:

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{y^{2n}}{4^n} = \frac{1}{\sqrt{1-y^2}}.$$

The aim of this paper is to evaluate the weighted sums (of convergence rate $\pm 1/2$) obtained from this series by inserting harmonic numbers into the summands. This will be realized exclusively by transforming the series into definite integrals and then computing them in closed form. The results will be expressed in terms of the Riemann zeta and Hurwitz zeta functions for $m, z \in \mathbb{C}$ with $\Re(m) > 1$ and $z \neq 0$ by

$$\zeta(m) = \sum_{n=1}^{\infty} \frac{1}{n^m} \quad \text{and} \quad \zeta_m(z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^m},$$

as well as the polylogarithm function, which is defined (see Lewin [17]) for $m \in \mathbb{N}$ and $z \in \mathbb{C}$ by

$$\mathrm{Li}_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m}: \quad \mathrm{Li}_1(z) = -\ln(1-z).$$

The rest of this study will be organized as follows: In the next section, two general theorems will be shown for generating functions of harmonic polynomials that will be utilized, in turn, to review several binomial series involving harmonic-like numbers. Sections 3 and 4 will be devoted to evaluate binomial series weighted by quadratic harmonic numbers. Four further series containing cubic harmonic numbers will be examined in Section 5. Finally, this study will end with a brief concluding comment.

2. Generating Functions of Harmonic Polynomials

For harmonic polynomials $\mathcal{H}_n(x)$ and $\mathcal{O}_n(x)$, it is not hard to verify that they can be expressed by

$$\mathcal{H}_n(x) = \int_0^1 \frac{1-T^n x^n}{1-Tx} dT \quad \text{and} \quad \mathcal{O}_n(x) = \int_0^1 \frac{1-T^{2n} x^n}{1-T^2 x} dT.$$

We are going to derive their generating functions weighted by the central binomial coefficient. They will be utilized to review several previously known infinite series identities involving four harmonic-like numbers $\{\mathbf{H}_n, \bar{\mathbf{H}}_n, \mathbf{O}_n, \bar{\mathbf{O}}_n\}$.

2.1. Harmonic Polynomials $\mathcal{H}_n(x)$

For $|x| < 1$ and $|y| < \frac{1}{4}$, consider the generating function

$$\begin{aligned}\Phi(x,y) = \sum_{n=0}^{\infty} y^n \binom{2n}{n} \mathcal{H}_n(x) &= \sum_{n=0}^{\infty} y^n \binom{2n}{n} \int_0^1 \frac{1-T^n x^n}{1-Tx} dT \\ &= \int_0^1 \frac{dT}{1-Tx} \sum_{n=0}^{\infty} (-4y)^n \binom{-\frac{1}{2}}{n} \{1-(Tx)^n\} \\ &= \int_0^1 \frac{dT}{1-Tx} \left\{ \frac{1}{\sqrt{1-4y}} - \frac{1}{\sqrt{1-4Txy}} \right\}.\end{aligned}$$

Evaluating the two integrals in closed form

$$\int_0^1 \frac{dT}{(1-Tx)\sqrt{1-4y}} = \frac{-\ln(1-x)}{x\sqrt{1-4y}},$$
$$\int_0^1 \frac{dT}{(1-Tx)\sqrt{1-4Txy}} = \frac{-2}{x\sqrt{1-4y}} \ln\left(\frac{\sqrt{1-x}(1+\sqrt{1-4y})}{\sqrt{1-4y}+\sqrt{1-4xy}} \right);$$

then simplifying the resulting expression, we establish the following general theorem:

Theorem 1 ($|x| \le 1$ and $|y| < \frac{1}{4}$)**.**

$$\Phi(x,y) = \sum_{n=0}^{\infty} y^n \binom{2n}{n} \mathcal{H}_n(x) = \frac{2}{x\sqrt{1-4y}} \ln\left(\frac{1+\sqrt{1-4y}}{\sqrt{1-4y}+\sqrt{1-4xy}} \right).$$

We remark that the above theorem holds also when $|x| = 1$, since $x = \pm 1$ are not the singular points of the integrand

$$\frac{1}{1-Tx}\left\{ \frac{1}{\sqrt{1-4y}} - \frac{1}{\sqrt{1-4Txy}} \right\} = \frac{4y}{\sqrt{1-4y}\sqrt{1-4Txy}(\sqrt{1-4y}+\sqrt{1-4Txy})}.$$

This particular case was derived from the work by Boyadzhiev [2], where the reader can also find a few variants of binomial series and some examples displayed in the following corollaries.

Corollary 2 (Series of convergence rate $\pm 1/2$)**.**

$$\Phi(1,\frac{1}{8}) = \sum_{n=0}^{\infty}\binom{2n}{n}\frac{\mathbf{H}_n}{8^n} = 2\sqrt{2}\ln\Big(\frac{1+\sqrt{2}}{2}\Big),$$
$$\Phi(1,-\frac{1}{8}) = \sum_{n=0}^{\infty}\binom{2n}{n}\frac{\mathbf{H}_n}{(-8)^n} = \sqrt{\frac{2}{3}}\ln\Big(\frac{5+2\sqrt{6}}{12}\Big),$$
$$\Phi(-1,\frac{1}{8}) = \sum_{n=0}^{\infty}\binom{2n}{n}\frac{\bar{\mathbf{H}}_n}{8^n} = 2\sqrt{2}\ln\Big(\frac{1+\sqrt{3}}{1+\sqrt{2}}\Big),$$
$$\Phi(-1,-\frac{1}{8}) = \sum_{n=0}^{\infty}\binom{2n}{n}\frac{\bar{\mathbf{H}}_n}{(-8)^n} = 2\sqrt{\frac{2}{3}}\ln\Big(\frac{1+\sqrt{3}}{\sqrt{2}+\sqrt{3}}\Big).$$

Corollary 3 (Series of convergence rate $\pm 1/4$)**.**

$$\Phi(1,\frac{1}{16}) = \sum_{n=0}^{\infty}\binom{2n}{n}\frac{\mathbf{H}_n}{16^n} = \frac{4}{\sqrt{3}}\ln\Big(\frac{1}{2}+\frac{1}{\sqrt{3}}\Big),$$
$$\Phi(1,-\frac{1}{16}) = \sum_{n=0}^{\infty}\binom{2n}{n}\frac{\mathbf{H}_n}{(-16)^n} = \frac{4}{\sqrt{5}}\ln\Big(\frac{1}{2}+\frac{1}{\sqrt{5}}\Big),$$
$$\Phi(-1,\frac{1}{16}) = \sum_{n=0}^{\infty}\binom{2n}{n}\frac{\bar{\mathbf{H}}_n}{16^n} = \frac{4}{\sqrt{3}}\ln\Big(\frac{\sqrt{5}+\sqrt{3}}{2+\sqrt{3}}\Big),$$
$$\Phi(-1,-\frac{1}{16}) = \sum_{n=0}^{\infty}\binom{2n}{n}\frac{\bar{\mathbf{H}}_n}{(-16)^n} = \frac{4}{\sqrt{5}}\ln\Big(\frac{\sqrt{3}+\sqrt{5}}{2+\sqrt{5}}\Big).$$

Corollary 4 (Series of convergence rate $\pm 1/8$)**.**

$$\Phi(1,\frac{1}{32}) = \sum_{n=0}^{\infty}\binom{2n}{n}\frac{\mathbf{H}_n}{32^n} = \frac{8}{\sqrt{14}}\ln\Big(\frac{1}{2}+\sqrt{\frac{2}{7}}\Big),$$
$$\Phi(1,-\frac{1}{32}) = \sum_{n=0}^{\infty}\binom{2n}{n}\frac{\mathbf{H}_n}{(-32)^n} = \frac{8}{3\sqrt{2}}\ln\Big(\frac{1}{2}+\frac{\sqrt{2}}{3}\Big),$$
$$\Phi(-1,\frac{1}{32}) = \sum_{n=0}^{\infty}\binom{2n}{n}\frac{\bar{\mathbf{H}}_n}{32^n} = \frac{8}{\sqrt{14}}\ln\Big(\frac{3+\sqrt{7}}{2\sqrt{2}+\sqrt{7}}\Big),$$
$$\Phi(-1,-\frac{1}{32}) = \sum_{n=0}^{\infty}\binom{2n}{n}\frac{\bar{\mathbf{H}}_n}{(-32)^n} = \frac{8}{3\sqrt{2}}\ln\Big(\frac{3+\sqrt{7}}{3+2\sqrt{2}}\Big).$$

2.2. Harmonic Polynomials $\mathcal{O}_n(x)$

For $|x| < 1$ and $|y| < \frac{1}{4}$, consider another generating function

$$\begin{aligned}\Psi(x,y) = \sum_{n=0}^{\infty} y^n\binom{2n}{n}\mathcal{O}_n(x) &= \sum_{n=0}^{\infty} y^n\binom{2n}{n}\int_0^1 \frac{1-T^{2n}x^n}{1-T^2x}dT\\ &= \int_0^1 \frac{dT}{1-T^2x}\sum_{n=0}^{\infty}(-4y)^n\binom{-\frac{1}{2}}{n}\{1-T^{2n}x^n\}\\ &= \int_0^1 \frac{dT}{1-T^2x}\Big\{\frac{1}{\sqrt{1-4y}}-\frac{1}{\sqrt{1-4T^2xy}}\Big\}.\end{aligned}$$

Evaluating the two integrals in closed form

$$\int_0^1 \frac{dT}{(1-T^2x)\sqrt{1-4y}} = \frac{\ln(1+\sqrt{x})}{2\sqrt{x-4xy}} - \frac{\ln(1-\sqrt{x})}{2\sqrt{x-4xy}},$$
$$\int_0^1 \frac{dT}{(1-T^2x)\sqrt{1-4T^2xy}} = \frac{-1}{2\sqrt{x-4xy}} \ln\left(\frac{\sqrt{1-4xy}-\sqrt{x-4xy}}{\sqrt{1-4xy}+\sqrt{x-4xy}}\right);$$

then simplifying the resulting expression, we establish another general theorem.

Theorem 5 ($|x| \leq 1$ and $|y| < \frac{1}{4}$)**.**

$$\Psi(x,y) = \sum_{n=0}^{\infty} y^n \binom{2n}{n} \mathcal{O}_n(x) = \frac{1}{\sqrt{x-4xy}} \ln\left(\frac{1+\sqrt{x}}{\sqrt{1-4xy}+\sqrt{x-4xy}}\right).$$

For the same reason as remarked after Theorem 1, the above infinite series identity is also valid when $|x| = 1$. The reader can refer to the work by Chen [3] for this reduced generating function and its consequences.

Corollary 6 (Series of convergence rate $\pm 1/2$)**.**

$$\Psi(1,\frac{1}{8}) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{\mathbf{O}_n}{8^n} = \frac{\ln 2}{\sqrt{2}},$$
$$\Psi(1,-\frac{1}{8}) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{\mathbf{O}_n}{(-8)^n} = \frac{\ln \frac{2}{3}}{\sqrt{6}},$$
$$\Psi(-1,\frac{1}{8}) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{\bar{\mathbf{O}}_n}{8^n} = \frac{\pi}{6\sqrt{2}},$$
$$\Psi(-1,-\frac{1}{8}) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{\bar{\mathbf{O}}_n}{(-8)^n} = \frac{-\pi}{6\sqrt{6}}.$$

Corollary 7 (Series of convergence rate $\pm 1/4$)**.**

$$\Psi(1,\frac{1}{16}) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{\mathbf{O}_n}{16^n} = \frac{\ln \frac{4}{3}}{\sqrt{3}},$$
$$\Psi(1,-\frac{1}{16}) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{\mathbf{O}_n}{(-16)^n} = \frac{\ln \frac{4}{5}}{\sqrt{5}},$$
$$\Psi(-1,\frac{1}{16}) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{\bar{\mathbf{O}}_n}{16^n} = \frac{\arcsin \frac{1}{4}}{\sqrt{3}},$$
$$\Psi(-1,-\frac{1}{16}) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{\bar{\mathbf{O}}_n}{(-16)^n} = \frac{\arcsin \frac{1}{4}}{-\sqrt{5}}.$$

Corollary 8 (Series of convergence rate $\pm 1/8$)**.**

$$\Psi(1,\frac{1}{32}) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{\mathbf{O}_n}{32^n} = \frac{2\ln \frac{8}{7}}{\sqrt{14}},$$
$$\Psi(1,-\frac{1}{32}) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{\mathbf{O}_n}{(-32)^n} = \frac{2\ln \frac{8}{9}}{3\sqrt{2}},$$

$$\Psi(-1, \frac{1}{32}) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{\bar{\mathbf{O}}_n}{32^n} = \frac{2\arcsin\frac{1}{8}}{\sqrt{14}},$$
$$\Psi(-1, -\frac{1}{32}) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{\bar{\mathbf{O}}_n}{(-32)^n} = \frac{2\arcsin\frac{1}{8}}{-3\sqrt{2}}.$$

3. Quadratic Harmonic Numbers

In this section, we shall evaluate, in closed form, four binomial series involving quadratic harmonic numbers $\mathbf{H}_n^{\langle 2\rangle}$ and $\bar{\mathbf{H}}_n^{\langle 2\rangle}$, that admit the following integral representations:

$$\mathbf{H}_n^{\langle 2\rangle} = \frac{\pi^2}{6} + \int_0^1 \frac{x^n \ln x}{1-x} dx \quad \text{and} \quad \bar{\mathbf{H}}_n^{\langle 2\rangle} = \frac{\pi^2}{12} + \int_0^1 \frac{(-x)^n \ln x}{1+x} dx.$$

3.1. Positive Series about $\mathbf{H}_n^{\langle 2\rangle}$

By expressing the series in terms of a definite integral

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{\mathbf{H}_n^{\langle 2\rangle}}{8^n} = \sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n \binom{-\frac{1}{2}}{n} \left\{ \frac{\pi^2}{6} + \int_0^1 \frac{x^n \ln x}{1-x} dx \right\}$$
$$= \frac{\pi^2}{3\sqrt{2}} + \int_0^1 \frac{\sqrt{2}\, \ln x}{(1-x)\sqrt{2-x}} dx,$$

we can show the closed formula as in the theorem below.

Theorem 9.
$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{\mathbf{H}_n^{\langle 2\rangle}}{8^n} = -2\sqrt{2}\mathrm{Li}_2(2\sqrt{2}-3).$$

Proof. Under the change of variable $\boxed{x \to \frac{1+6y+y^2}{(1+y)^2}}$, we can reformulate and then evaluate the integral below as

$$\int_0^1 \frac{\sqrt{2}\, \ln x}{(1-x)\sqrt{2-x}} dx = \sqrt{2} \int_0^{2\sqrt{2}-3} \frac{dy}{y} \ln\left(\frac{1+6y+y^2}{(1+y)^2}\right)$$
$$= \sqrt{2}\left\{ 2\mathrm{Li}_2(3-2\sqrt{2}) - \mathrm{Li}_2(17-12\sqrt{2}) - \frac{\pi^2}{6} \right\},$$

where we have appealed to the following three integral values:

$$\int_0^{2\sqrt{2}-3} \frac{\ln(1+y)}{y} dy = -\mathrm{Li}_2(3-2\sqrt{2}),$$
$$\int_0^{2\sqrt{2}-3} \frac{\ln(1+y(3+2\sqrt{2}))}{y} dy = -\mathrm{Li}_2(1) = -\frac{\pi^2}{6},$$
$$\int_0^{2\sqrt{2}-3} \frac{\ln(1+y(3-2\sqrt{2}))}{y} dy = -\mathrm{Li}_2(17-12\sqrt{2}).$$

Performing substitutions and then applying the equality

$$\mathrm{Li}_2(z^2) = 2\mathrm{Li}_2(z) + 2\mathrm{Li}_2(-z), \tag{1}$$

we confirm the formula in Theorem 9. □

3.2. Alternating Series about $\mathbf{H}_n^{\langle 2\rangle}$

By expressing the series in terms of a definite integral

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\mathbf{H}_n^{\langle 2\rangle}}{(-8)^n}=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\binom{-\frac{1}{2}}{n}\left\{\frac{\pi^2}{6}+\int_0^1\frac{x^n\ln x}{1-x}dx\right\}$$
$$=\frac{\pi^2}{3\sqrt{6}}+\int_0^1\frac{\sqrt{2}\,\ln x}{(1-x)\sqrt{2+x}}dx,$$

we can show the closed formula as in the theorem below.

Theorem 10.
$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\mathbf{H}_n^{\langle 2\rangle}}{(-8)^n}=-\sqrt{\frac{8}{3}}\mathrm{Li}_2(5-2\sqrt{6}).$$

Proof. Under the change of variable $\boxed{x\to\frac{1-10y+y^2}{(1+y)^2}}$, we can evaluate the integral as

$$\int_0^1\frac{\sqrt{2}\ln x}{(1-x)\sqrt{2+x}}dx=\sqrt{\frac{2}{3}}\int_0^{5-2\sqrt{6}}\frac{dy}{y}\ln\left(\frac{1-10y+y^2}{(1+y)^2}\right)$$
$$=\sqrt{\frac{2}{3}}\left\{2\mathrm{Li}_2(2\sqrt{6}-5)-\mathrm{Li}_2(49-20\sqrt{6})-\frac{\pi^2}{6}\right\},$$

where we have employed the following three integral values:

$$\int_0^{5-2\sqrt{6}}\frac{\ln(1+y)}{y}dy=-\mathrm{Li}_2(2\sqrt{6}-5),$$
$$\int_0^{5-2\sqrt{6}}\frac{\ln(1-y(5+2\sqrt{6}))}{y}dy=-\mathrm{Li}_2(1)=-\frac{\pi^2}{6},$$
$$\int_0^{5-2\sqrt{6}}\frac{\ln(1-y(5-2\sqrt{6}))}{y}dy=-\mathrm{Li}_2(49-20\sqrt{6}).$$

Then, the formula in Theorem 10 follows by substitution and simplification via equality (1). □

3.3. Positive Series about $\bar{\mathbf{H}}_n^{\langle 2\rangle}$

By expressing the series in terms of a definite integral

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\bar{\mathbf{H}}_n^{\langle 2\rangle}}{8^n}=\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n\binom{-\frac{1}{2}}{n}\left\{\frac{\pi^2}{12}+\int_0^1\frac{(-x)^n\ln x}{1+x}dx\right\}$$
$$=\frac{\pi^2}{6\sqrt{2}}+\int_0^1\frac{\sqrt{2}\,\ln x}{(1+x)\sqrt{2+x}}dx,$$

we can show the closed formula as in the theorem below.

Theorem 11.
$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\bar{\mathbf{H}}_n^{\langle 2\rangle}}{8^n}=\sqrt{2}\left\{2\mathrm{Li}_2(2\sqrt{2}-3)+2\mathrm{Li}_2(2-\sqrt{3})-\frac{\pi^2}{12}\right.$$
$$\left.+\mathrm{Li}_2\left(\frac{3-2\sqrt{2}}{2-\sqrt{3}}\right)-\mathrm{Li}_2\left(\frac{3-2\sqrt{2}}{2+\sqrt{3}}\right)+\frac{1}{2}\ln^2\left(\frac{3+2\sqrt{2}}{2+\sqrt{3}}\right)\right\}.$$

Proof. Under the change of variable $\boxed{x \to -\frac{1+6y+y^2}{(1+y)^2}}$, we can evaluate the integral as

$$\begin{aligned}
&\int_0^1 \frac{\sqrt{2}\,\ln x}{(1+x)\sqrt{2+x}}dx = \sqrt{2}\int_{2\sqrt{2}-3}^{\sqrt{3}-2} \frac{dy}{y}\ln\Big(-\frac{1+6y+y^2}{(1+y)^2}\Big)\\
&= \sqrt{2}\Bigg\{2\mathrm{Li}_2\Big(2-\sqrt{3}\Big) - 2\mathrm{Li}_2\Big(3-2\sqrt{2}\Big) + \mathrm{Li}_2\Big(17-12\sqrt{2}\Big)\\
&- \mathrm{Li}_2\left(\frac{3-2\sqrt{2}}{2+\sqrt{3}}\right) + \mathrm{Li}_2\left(1-\frac{3+2\sqrt{2}}{2+\sqrt{3}}\right) + \ln\left(\frac{3+2\sqrt{2}}{2+\sqrt{3}}\right)\ln\left(\frac{3+2\sqrt{2}}{2+\sqrt{3}}-1\right)\Bigg\},
\end{aligned}$$

where we have made use of the following three integral values:

$$\begin{aligned}
&\int_{2\sqrt{2}-3}^{\sqrt{3}-2}\frac{\ln(1+y)}{y}dy = \mathrm{Li}_2\Big(3-2\sqrt{2}\Big) - \mathrm{Li}_2\Big(2-\sqrt{3}\Big),\\
&\int_{2\sqrt{2}-3}^{\sqrt{3}-2}\frac{\ln(1+y(3-2\sqrt{2}))}{y}dy = \mathrm{Li}_2\Big(17-12\sqrt{2}\Big) - \mathrm{Li}_2\left(\frac{3-2\sqrt{2}}{2+\sqrt{3}}\right),\\
&\int_{2\sqrt{2}-3}^{\sqrt{3}-2}\frac{\ln(-1-y(3+2\sqrt{2}))}{y}dy = \int_0^{\frac{3+2\sqrt{2}}{2+\sqrt{3}}-1}\frac{\ln T}{1+T}dT\\
&\qquad = \mathrm{Li}_2\left(1-\frac{3+2\sqrt{2}}{2+\sqrt{3}}\right) + \ln\left(\frac{3+2\sqrt{2}}{2+\sqrt{3}}\right)\ln\left(\frac{3+2\sqrt{2}}{2+\sqrt{3}}-1\right).
\end{aligned}$$

Recalling dilogarithm Equation (1) and

$$\mathrm{Li}_2(1-x) - \mathrm{Li}_2\Big(\frac{1}{x}\Big) = \frac{\ln x}{2}\ln\frac{x}{(x-1)^2} - \frac{\pi^2}{6}, \quad \text{where} \quad x > 1, \tag{2}$$

we derive the following expression after substitution and simplification:

$$\begin{aligned}
\int_0^1 \frac{\sqrt{2}\,\ln x}{(1+x)\sqrt{2+x}}dx = \sqrt{2}\Bigg\{&2\mathrm{Li}_2(2\sqrt{2}-3) + 2\mathrm{Li}_2(2-\sqrt{3}) - \frac{\pi^2}{6}\\
&+ \mathrm{Li}_2\left(\frac{3-2\sqrt{2}}{2-\sqrt{3}}\right) - \mathrm{Li}_2\left(\frac{3-2\sqrt{2}}{2+\sqrt{3}}\right) + \frac{1}{2}\ln^2\left(\frac{3+2\sqrt{2}}{2+\sqrt{3}}\right)\Bigg\},
\end{aligned}$$

which leads us to the value stated in Theorem 11 after substitution. □

3.4. Alternating Series about $\bar{\mathbf{H}}_n^{\langle 2\rangle}$

By expressing the series in terms of a definite integral

$$\begin{aligned}
\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\bar{\mathbf{H}}_n^{\langle 2\rangle}}{(-8)^n} &= \sum_{n=0}^{\infty}\Big(\frac{1}{2}\Big)^n\binom{-\frac{1}{2}}{n}\left\{\frac{\pi^2}{12} + \int_0^1\frac{(-x)^n\ln x}{1+x}dx\right\}\\
&= \frac{\pi^2}{6\sqrt{6}} + \int_0^1\frac{\sqrt{2}\,\ln x}{(1+x)\sqrt{2-x}}dx,
\end{aligned}$$

we can show the closed formula as in the theorem below.

Theorem 12.

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\bar{\mathbf{H}}_n^{\langle 2\rangle}}{(-8)^n}=\sqrt{\frac{2}{3}}\bigg\{2\mathrm{Li}_2\big(-5-2\sqrt{6}\big)-2\mathrm{Li}_2\big(-2-\sqrt{3}\big)+\mathrm{Li}_2\big(49-20\sqrt{6}\big)-\frac{\pi^2}{12}$$
$$+2\ln^2\big(5+2\sqrt{6}\big)+\mathrm{Li}_2\left(\frac{5-2\sqrt{6}}{2-\sqrt{3}}\right)-\mathrm{Li}_2\left(\frac{5-2\sqrt{6}}{2+\sqrt{3}}\right)-\frac{1}{2}\ln^2\left(\frac{5+2\sqrt{6}}{2-\sqrt{3}}\right)\bigg\}.$$

Proof. Under the change of variable $\boxed{x\to-\frac{1-10y+y^2}{(1+y)^2}}$, we can evaluate the integral as

$$\int_0^1\frac{\sqrt{2}\,\ln x\,dx}{(1+x)\sqrt{2-x}}=\sqrt{\frac{2}{3}}\int_{2+\sqrt{3}}^{5+2\sqrt{6}}\frac{dy}{y}\ln\left(-\frac{1-10y+y^2}{(1+y)^2}\right)$$
$$=\sqrt{\frac{2}{3}}\bigg\{\ln\big(49+20\sqrt{6}\big)\ln\big(48+20\sqrt{6}\big)-\frac{\pi^2}{6}$$
$$+2\mathrm{Li}_2\big(-5-2\sqrt{6}\big)-2\mathrm{Li}_2\big(-2-\sqrt{3}\big)+\mathrm{Li}_2\Big(-4\big(12+5\sqrt{6}\big)\Big)$$
$$+\mathrm{Li}_2\left(\frac{5-2\sqrt{6}}{2-\sqrt{3}}\right)-\mathrm{Li}_2\left(1-\frac{5+2\sqrt{6}}{2-\sqrt{3}}\right)-\ln\left(\frac{5+2\sqrt{6}}{2-\sqrt{3}}\right)\ln\left(\frac{5+2\sqrt{6}}{2-\sqrt{3}}-1\right)\bigg\},$$

where we have employed the following three integral values:

$$\int_{2+\sqrt{3}}^{5+2\sqrt{6}}\frac{\ln(1+y)}{y}dy=\mathrm{Li}_2\big(-2-\sqrt{3}\big)-\mathrm{Li}_2\big(-5-2\sqrt{6}\big),$$
$$\int_{2+\sqrt{3}}^{5+2\sqrt{6}}\frac{\ln(1+y(2\sqrt{6}-5))}{y}dy=\mathrm{Li}_2\left(\frac{5-2\sqrt{6}}{2-\sqrt{3}}\right)-\frac{\pi^2}{6},$$
$$\int_{2+\sqrt{3}}^{5+2\sqrt{6}}\frac{\ln(-1+y(5+2\sqrt{6}))}{y}dy=\int_{\frac{5+2\sqrt{6}}{2-\sqrt{3}}-1}^{(5+2\sqrt{6})^2-1}\frac{\ln T}{1+T}dT$$
$$=\ln\big(49+20\sqrt{6}\big)\ln\big(48+20\sqrt{6}\big)+\mathrm{Li}_2\Big(-4\big(12+5\sqrt{6}\big)\Big)$$
$$-\ln\left(\frac{5+2\sqrt{6}}{2-\sqrt{3}}\right)\ln\left(\frac{5+2\sqrt{6}}{2-\sqrt{3}}-1\right)-\mathrm{Li}_2\left(1-\frac{5+2\sqrt{6}}{2-\sqrt{3}}\right).$$

Performing substitutions and applying (2), we find the integral value below

$$\int_0^1\frac{\sqrt{2}\,\ln x}{(1+x)\sqrt{2-x}}dx=\sqrt{\frac{2}{3}}\bigg\{2\mathrm{Li}_2\big(-5-2\sqrt{6}\big)-2\mathrm{Li}_2\big(-2-\sqrt{3}\big)+\mathrm{Li}_2\Big(49-20\sqrt{6}\Big)$$
$$+2\ln^2\big(5+2\sqrt{6}\big)-\frac{\pi^2}{6}-\frac{1}{2}\ln^2\left(\frac{5+2\sqrt{6}}{2-\sqrt{3}}\right)+\mathrm{Li}_2\left(\frac{5-2\sqrt{6}}{2-\sqrt{3}}\right)-\mathrm{Li}_2\left(\frac{5-2\sqrt{6}}{2+\sqrt{3}}\right)\bigg\}.$$

This leads us to the value stated in Theorem 12 after substitution. □

4. Quadratic Skew Harmonic Numbers

In this section, we shall examine four binomial series containing quadratic skew harmonic numbers $\mathbf{O}_n^{\langle 2\rangle}$ and $\bar{\mathbf{O}}_n^{\langle 2\rangle}$, that admit the following integral expressions:

$$\mathbf{O}_n^{\langle 2\rangle}=\frac{\pi^2}{8}+\int_0^1\frac{x^{2n}\ln x}{1-x^2}dx\quad\text{and}\quad\bar{\mathbf{O}}_n^{\langle 2\rangle}=G+\int_0^1\frac{(-x^2)^n\ln x}{1+x^2}dx;$$

where G is the Catalan constant, defined by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.915965594.$$

4.1. Positive Series about $\mathbf{O}_n^{\langle 2\rangle}$

By writing the series in terms of the integral below

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{2n}{n} \frac{\mathbf{O}_n^{\langle 2\rangle}}{8^n} &= \sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n \binom{-\frac{1}{2}}{n} \left\{ \frac{\pi^2}{8} + \int_0^1 \frac{x^{2n} \ln x}{1-x^2} dx \right\} \\ &= \frac{\pi^2}{4\sqrt{2}} + \int_0^1 \frac{\sqrt{2}\, \ln x}{(1-x^2)\sqrt{2-x^2}} dx,\end{aligned}$$

we can show the following theorem:

Theorem 13 (cf. [18] (Remark 2.2))**.**

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{\mathbf{O}_n^{\langle 2\rangle}}{8^n} = \frac{\pi^2}{16\sqrt{2}}.$$

Proof. This identity can be validated by making the change of variable $\boxed{x \to \frac{1-y}{\sqrt{1+y^2}}}$ in the integral below, and then evaluating it explicitly

$$\int_0^1 \frac{\sqrt{2}\, \ln x}{(1-x^2)\sqrt{2-x^2}} dx = \int_0^1 \ln\left(\frac{1-y}{\sqrt{1+y^2}}\right) \frac{dy}{y\sqrt{2}} = \frac{-3\pi^2}{16\sqrt{2}},$$

where we have invoked the following two integral values:

$$\begin{aligned}\int_0^1 \frac{\ln(1-y)}{y} dy &= -\mathrm{Li}_2(1) = -\frac{\pi^2}{6}, \\ \int_0^1 \frac{\ln(1+y^2)}{y} dy &= -\frac{1}{2}\mathrm{Li}_2(-1) = \frac{\pi^2}{24}.\end{aligned}$$

This completes the proof of Theorem 13. □

4.2. Alternating Series about $\mathbf{O}_n^{\langle 2\rangle}$

By writing the series in terms of the integral below

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{2n}{n} \frac{\mathbf{O}_n^{\langle 2\rangle}}{(-8)^n} &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \binom{-\frac{1}{2}}{n} \left\{ \frac{\pi^2}{8} + \int_0^1 \frac{x^{2n} \ln x}{1-x^2} dx \right\} \\ &= \frac{\pi^2}{4\sqrt{6}} + \int_0^1 \frac{\sqrt{2}\, \ln x}{(1-x^2)\sqrt{2+x^2}} dx,\end{aligned}$$

we can show the following theorem.

Theorem 14.

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{\mathbf{O}_n^{\langle 2\rangle}}{(-8)^n} = -\frac{\ln^2(2+\sqrt{3})}{4\sqrt{6}}.$$

Proof. Under the change of variable $\boxed{x \to \frac{1-y}{\sqrt{1+4y+y^2}}}$, we can reformulate and then evaluate the following integral:

$$\int_0^1 \frac{\sqrt{2}\,\ln x}{(1-x^2)\sqrt{2+x^2}}dx = \int_0^1 \frac{dy}{y\sqrt{6}} \ln\left(\frac{1-y}{\sqrt{1+4y+y^2}}\right)$$
$$= \frac{1}{2\sqrt{6}}\left\{\mathrm{Li}_2(\sqrt{3}-2) + \mathrm{Li}_2(-2-\sqrt{3}) - \frac{\pi^2}{3}\right\},$$

where we have made use of the following three integral values:

$$\int_0^1 \frac{\ln(1-y)}{y}dy = -\mathrm{Li}_2(1) = -\frac{\pi^2}{6},$$
$$\int_0^1 \frac{\ln(1+y(2+\sqrt{3}))}{y}dy = -\mathrm{Li}_2(-2-\sqrt{3}),$$
$$\int_0^1 \frac{\ln(1+y(2-\sqrt{3}))}{y}dy = -\mathrm{Li}_2(\sqrt{3}-2).$$

Taking into account the equation

$$\mathrm{Li}_2(-x) + \mathrm{Li}_2(-x^{-1}) = -\frac{\pi^2}{6} - \frac{\ln^2 x}{2}, \quad \text{where} \quad x > 0, \tag{3}$$

we can further simplify

$$\int_0^1 \frac{\sqrt{2}\,\ln x}{(1-x^2)\sqrt{2+x^2}}dx = -\frac{\pi^2 + \ln^2(2+\sqrt{3})}{4\sqrt{6}},$$

which leads us to the formula in Theorem 14. □

4.3. Positive Series about $\bar{\mathbf{O}}_n^{\langle 2\rangle}$

By writing the series in terms of the integral below

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\bar{\mathbf{O}}_n^{\langle 2\rangle}}{8^n} = \sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n\binom{-\frac{1}{2}}{n}\left\{G + \int_0^1 \frac{(-x^2)^n \ln x}{1+x^2}dx\right\}$$
$$= \sqrt{2}G + \int_0^1 \frac{\sqrt{2}\,\ln x}{(1+x^2)\sqrt{2+x^2}}dx,$$

we can show the following theorem.

Theorem 15.

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\bar{\mathbf{O}}_n^{\langle 2\rangle}}{8^n} = \sqrt{2}G + \frac{5\sqrt{2}\mathbf{i}}{16}\left\{\mathrm{Li}_2\left(\frac{1+\sqrt{3}\mathbf{i}}{2}\right) - \mathrm{Li}_2\left(\frac{1-\sqrt{3}\mathbf{i}}{2}\right)\right\}.$$

Proof. Under the change of variable $\boxed{x \to \sin y\sqrt{\frac{2}{\cos(2y)}}}$, we can reformulate the integral

$$\int_0^1 \frac{\ln x}{(1+x^2)\sqrt{2+x^2}}dx = \int_0^{\frac{\pi}{6}} \ln\left(\sin y\sqrt{\frac{2}{\cos(2y)}}\right)dy$$
$$= \frac{\pi\ln 2}{12} + \int_0^{\frac{\pi}{6}} \ln\left(\sin y\right)dy - \frac{1}{2}\int_0^{\frac{\pi}{6}} \ln\left(\cos(2y)\right)dy.$$

In view of two Fourier series expansions

$$\ln(\sin y) = -\ln 2 - \sum_{n=1}^{\infty} \frac{\cos(2ny)}{n}, \qquad 0 < y < \pi;$$
$$\ln(\cos y) = -\ln 2 - \sum_{n=1}^{\infty} (-1)^n \frac{\cos(2ny)}{n}, \quad -\frac{\pi}{2} < y < \frac{\pi}{2};$$

we can evaluate the two integrals

$$\begin{aligned}\int_0^{\frac{\pi}{6}} \ln\Big(\sin y\Big)dy &= -\frac{\pi}{6}\ln 2 - \sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{6}}\cos(2ny)dy \\ &= -\frac{\pi}{6}\ln 2 - \sum_{n=1}^{\infty}\frac{1}{2n^2}\sin\Big(\frac{n\pi}{3}\Big) \\ &= -\frac{\pi}{6}\ln 2 - \frac{1}{4\mathbf{i}}\sum_{n=1}^{\infty}\frac{1}{n^2}\Big\{e^{\frac{n\pi\mathbf{i}}{3}} - e^{\frac{-n\pi\mathbf{i}}{3}}\Big\} \\ &= -\frac{\pi}{6}\ln 2 - \frac{1}{4\mathbf{i}}\Big\{\mathrm{Li}_2\Big(\frac{1+\sqrt{3}\mathbf{i}}{2}\Big) - \mathrm{Li}_2\Big(\frac{1-\sqrt{3}\mathbf{i}}{2}\Big)\Big\},\end{aligned}$$

$$\begin{aligned}\int_0^{\frac{\pi}{6}} \ln\Big(\cos(2y)\Big)dy &= -\frac{\pi}{6}\ln 2 - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{6}}\cos(4ny)dy \\ &= -\frac{\pi}{6}\ln 2 - \sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2}\sin\Big(\frac{2n\pi}{3}\Big) \\ &= -\frac{\pi}{6}\ln 2 - \frac{1}{8\mathbf{i}}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\Big\{e^{\frac{2n\pi\mathbf{i}}{3}} - e^{\frac{-2n\pi\mathbf{i}}{3}}\Big\} \\ &= -\frac{\pi}{6}\ln 2 - \frac{1}{8\mathbf{i}}\Big\{\mathrm{Li}_2\Big(\frac{1-\sqrt{3}\mathbf{i}}{2}\Big) - \mathrm{Li}_2\Big(\frac{1+\sqrt{3}\mathbf{i}}{2}\Big)\Big\}.\end{aligned}$$

By substitution, we have

$$\int_0^1 \frac{\ln x}{(1+x^2)\sqrt{2+x^2}}dx = \frac{5\mathbf{i}}{16}\Big\{\mathrm{Li}_2\Big(\frac{1+\sqrt{3}\mathbf{i}}{2}\Big) - \mathrm{Li}_2\Big(\frac{1-\sqrt{3}\mathbf{i}}{2}\Big)\Big\},$$

which leads us to the formula in Theorem 15. □

4.4. Alternating Series about $\bar{\mathbf{O}}_n^{\langle 2\rangle}$

By writing the series in terms of the integral below

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\bar{\mathbf{O}}_n^{\langle 2\rangle}}{(-8)^n} &= \sum_{n=0}^{\infty}\Big(\frac{1}{2}\Big)^n\binom{-\frac{1}{2}}{n}\Big\{G + \int_0^1\frac{(-x^2)^n\ln x}{1+x^2}dx\Big\} \\ &= \sqrt{\frac{2}{3}}G + \int_0^1\frac{\sqrt{2}\,\ln x}{(1+x^2)\sqrt{2-x^2}}dx,\end{aligned}$$

we can show the following theorem.

Theorem 16.

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\bar{\mathbf{O}}_n^{\langle 2\rangle}}{(-8)^n} = \sqrt{\frac{2}{3}}G - \frac{\pi}{3\sqrt{6}}\ln(2+\sqrt{3}) + \frac{\sqrt{2}}{36}\Big\{\zeta_2\Big(\frac{2}{3}\Big) - \zeta_2\Big(\frac{1}{3}\Big)\Big\} \\ + \frac{\mathbf{i}}{2\sqrt{6}}\Big\{\mathrm{Li}_2\Big(\frac{1+\sqrt{3}\mathbf{i}}{4+2\sqrt{3}}\Big) - \mathrm{Li}_2\Big(\frac{1-\sqrt{3}\mathbf{i}}{4+2\sqrt{3}}\Big)\Big\}.\end{aligned}$$

Proof. Under the change of variable $\boxed{x \to \frac{\sqrt{2}\tan y}{\sqrt{3+\tan^2 y}}}$, we can reformulate the following integral:

$$\int_0^1 \frac{\sqrt{2}\,\ln x}{(1+x^2)\sqrt{2-x^2}}dx = \sqrt{\frac{2}{3}}\int_0^{\frac{\pi}{3}} \ln\left(\frac{\sqrt{2}\tan y}{\sqrt{3+\tan^2 y}}\right)dy = \sqrt{\frac{2}{3}}\int_0^{\frac{\pi}{3}} \ln\left(\frac{\sqrt{2}\sin y}{\sqrt{2+\cos(2y)}}\right)dy$$
$$= \frac{\pi}{3\sqrt{6}}\ln 2 + \sqrt{\frac{2}{3}}\int_0^{\frac{\pi}{3}} \ln\Big(\sin y\Big)dy - \frac{1}{\sqrt{6}}\int_0^{\frac{\pi}{3}} \ln\Big(2+\cos(2y)\Big)dy.$$

Taking into account the factorization

$$2+\cos(2y) = \frac{2+\sqrt{3}}{2}\Big\{1+(2-\sqrt{3})e^{2y\mathbf{i}}\Big\}\Big\{1+(2-\sqrt{3})e^{-2y\mathbf{i}}\Big\}$$

and the Fourier series expansions for $0<y<\pi$

$$\ln(\sin y) = -\ln 2 - \sum_{n=1}^{\infty}\frac{\cos(2ny)}{n},$$
$$\ln\big(1+(2-\sqrt{3})e^{2y\mathbf{i}}\big) + \ln\big(1+(2-\sqrt{3})e^{-2y\mathbf{i}}\big) = -2\sum_{n=1}^{\infty}\frac{(\sqrt{3}-2)^n}{n}\cos(2ny);$$

we can evaluate the two integrals

$$\int_0^{\frac{\pi}{3}} \ln\Big(\sin y\Big)dy = -\frac{\pi}{3}\ln 2 - \sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{3}}\cos(2ny)dy$$
$$= -\frac{\pi}{3}\ln 2 - \sum_{n=1}^{\infty}\frac{1}{2n^2}\sin\Big(\frac{2n\pi}{3}\Big)$$
$$= -\frac{\pi}{3}\ln 2 - \frac{\sqrt{3}}{4}\Big\{\sum_{n=1}^{\infty}\frac{1}{(3n-2)^2} - \sum_{n=1}^{\infty}\frac{1}{(3n-1)^2}\Big\}$$
$$= -\frac{\pi}{3}\ln 2 - \frac{\sqrt{3}}{36}\Big\{\zeta_2\Big(\frac{1}{3}\Big) - \zeta_2\Big(\frac{2}{3}\Big)\Big\},$$
$$\int_0^{\frac{\pi}{3}} \ln\Big(2+\cos(2y)\Big)dy = \frac{\pi}{3}\ln\frac{2+\sqrt{3}}{2} - 2\sum_{n=1}^{\infty}\frac{(\sqrt{3}-2)^n}{n}\int_0^{\frac{\pi}{3}}\cos(2ny)dy$$
$$= \frac{\pi}{3}\ln\frac{2+\sqrt{3}}{2} - \frac{1}{2\mathbf{i}}\sum_{n=1}^{\infty}\frac{(\sqrt{3}-2)^n}{n^2}\Big(e^{\frac{2n\pi\mathbf{i}}{3}} - e^{\frac{-2n\pi\mathbf{i}}{3}}\Big)$$
$$= \frac{\pi}{3}\ln\frac{2+\sqrt{3}}{2} + \frac{\mathbf{i}}{2}\Big\{\mathrm{Li}_2\Big(\frac{1-\sqrt{3}\mathbf{i}}{4+2\sqrt{3}}\Big) - \mathrm{Li}_2\Big(\frac{1+\sqrt{3}\mathbf{i}}{4+2\sqrt{3}}\Big)\Big\}.$$

By substitution, we have

$$\int_0^1 \frac{\sqrt{2}\,\ln x}{(1+x^2)\sqrt{2-x^2}}dx = -\frac{\pi}{3\sqrt{6}}\ln(2+\sqrt{3}) + \frac{\sqrt{2}}{36}\Big\{\zeta_2\Big(\frac{2}{3}\Big) - \zeta_2\Big(\frac{1}{3}\Big)\Big\}$$
$$+ \frac{\mathbf{i}}{2\sqrt{6}}\Big\{\mathrm{Li}_2\Big(\frac{1+\sqrt{3}\mathbf{i}}{4+2\sqrt{3}}\Big) - \mathrm{Li}_2\Big(\frac{1-\sqrt{3}\mathbf{i}}{4+2\sqrt{3}}\Big)\Big\},$$

which leads us to the formula in Theorem 16. □

5. Further Summation Formulae

Finally, we are going to investigate four binomial series weighted by harmonic numbers of the third order $\{\mathbf{H}_n^{\langle 3\rangle}, \mathbf{O}_n^{\langle 3\rangle}\}$, which can be expressed, respectively, by the following integrals:

$$\mathbf{H}_n^{\langle 3\rangle} = \zeta(3) - \frac{1}{2}\int_0^1 \frac{x^n \ln^2 x}{1-x}dx \quad \text{and} \quad \mathbf{O}_n^{\langle 3\rangle} = \frac{7}{8}\zeta(3) - \frac{1}{2}\int_0^1 \frac{x^{2n} \ln^2 x}{1-x^2}dx.$$

Since most definite integrals appearing in this section are rather involved, we have to compute them simply by invoking *Mathematica* commands. Otherwise, it would be a challenging task to handle them manually. The reader can consult Vălean's monograph [19] for similar integrals containing both logarithm and polylogarithm functions.

5.1. Positive Series about $\mathbf{H}_n^{\langle 3\rangle}$

By expressing the series in terms of the definite integral

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\mathbf{H}_n^{\langle 3\rangle}}{8^n} &= \sum_{n=0}^{\infty}\Big(\frac{-1}{2}\Big)^n\binom{-\frac{1}{2}}{n}\Big\{\zeta(3) - \frac{1}{2}\int_0^1 \frac{x^n \ln^2 x}{1-x}dx\Big\}\\ &= \sqrt{2}\zeta(3) - \frac{1}{2}\int_0^1 \frac{\sqrt{2}\,\ln^2 x}{(1-x)\sqrt{2-x}}dx,\end{aligned}$$

we can show the infinite series identity as in the theorem below.

Theorem 17.

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\mathbf{H}_n^{\langle 3\rangle}}{8^n} &= \frac{1}{\sqrt{2}}\ln(1+\sqrt{2})\ln^2(24-16\sqrt{2}) - 2\sqrt{2}\zeta(3)\\ &\quad - \sqrt{2}\ln(24-16\sqrt{2})\Big\{\mathrm{Li}_2\Big(\frac{\sqrt{2}-1}{2\sqrt{2}}\Big) - \mathrm{Li}_2\Big(\frac{1+\sqrt{2}}{2\sqrt{2}}\Big)\Big\}\\ &\quad - 2\sqrt{2}\Big\{2\mathrm{Li}_3\big(-(\sqrt{2}-1)^2\big) + \mathrm{Li}_3\Big(\frac{\sqrt{2}-1}{2\sqrt{2}}\Big) - \mathrm{Li}_3\Big(\frac{1+\sqrt{2}}{2\sqrt{2}}\Big)\Big\}.\end{aligned}$$

Proof. It suffices to evaluate the integral below and simplify the output by *Mathematica*

$$\begin{aligned}\int_0^1 \frac{\sqrt{2}\,\ln^2 x}{(1-x)\sqrt{2-x}}dx &= \int_0^{2\sqrt{2}-3}\frac{\sqrt{2}\,dy}{y}\ln^2\Big(\frac{1+6y+y^2}{(1+y)^2}\Big) \quad \boxed{x \to \frac{1+6y+y^2}{(1+y)^2}}\\ &= 6\sqrt{2}\zeta(3) - \sqrt{2}\ln(1+\sqrt{2})\ln^2(24-16\sqrt{2})\\ &\quad + 2\sqrt{2}\ln(24-16\sqrt{2})\Big\{\mathrm{Li}_2\Big(\frac{\sqrt{2}-1}{2\sqrt{2}}\Big) - \mathrm{Li}_2\Big(\frac{1+\sqrt{2}}{2\sqrt{2}}\Big)\Big\}\\ &\quad + 4\sqrt{2}\Big\{2\mathrm{Li}_3\big(-(\sqrt{2}-1)^2\big) + \mathrm{Li}_3\Big(\frac{\sqrt{2}-1}{2\sqrt{2}}\Big) - \mathrm{Li}_3\Big(\frac{1+\sqrt{2}}{2\sqrt{2}}\Big)\Big\}.\end{aligned}$$

Then, the formula in Theorem 17 follows after substitution. □

5.2. Alternating Series about $\mathbf{H}_n^{\langle 3\rangle}$

By expressing the series in terms of the definite integral

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\mathbf{H}_n^{\langle 3\rangle}}{(-8)^n}=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\binom{-\frac{1}{2}}{n}\left\{\zeta(3)-\frac{1}{2}\int_0^1\frac{x^n\ln^2 x}{1-x}dx\right\}$$
$$=\sqrt{\frac{2}{3}}\zeta(3)-\frac{1}{2}\int_0^1\frac{\sqrt{2}\,\ln^2 x}{(1-x)\sqrt{2+x}}dx,$$

we can show the infinite series identity.

Theorem 18.

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\mathbf{H}_n^{\langle 3\rangle}}{(-8)^n}=-2\sqrt{\frac{2}{3}}\,\zeta(3)-\sqrt{\frac{2}{27}}\ln^2\left(\frac{4}{2+\sqrt{6}}\right)\ln\left(\frac{8}{4801+1960\sqrt{6}}\right)$$
$$-2\sqrt{\frac{2}{3}}\ln\left(2\sqrt{6}-4\right)\left\{\mathrm{Li}_2\left(\frac{2-\sqrt{6}}{4}\right)+\mathrm{Li}_2\left(2\sqrt{6}-4\right)\right\}$$
$$-2\sqrt{\frac{2}{3}}\left\{2\mathrm{Li}_3\left(5-2\sqrt{6}\right)-\mathrm{Li}_3\left(2\sqrt{6}-4\right)+\mathrm{Li}_3\left(\frac{2-\sqrt{6}}{4}\right)\right\}.$$

Proof. It is enough to evaluate the integral below and simplify the output by *Mathematica*

$$\int_0^1\frac{\sqrt{2}\,\ln^2 x}{(1-x)\sqrt{2+x}}dx=\sqrt{\frac{2}{3}}\int_0^{5-2\sqrt{6}}\frac{dy}{y}\ln^2\left(\frac{1-10y+y^2}{(1+y)^2}\right)\quad\boxed{x\to\frac{1-10y+y^2}{(1+y)^2}}$$
$$=2\sqrt{6}\,\zeta(3)+\sqrt{\frac{8}{27}}\ln^2\left(\frac{4}{2+\sqrt{6}}\right)\ln\left(\frac{8}{4801+1960\sqrt{6}}\right)$$
$$+4\sqrt{\frac{2}{3}}\ln\left(2\sqrt{6}-4\right)\left\{\mathrm{Li}_2\left(\frac{2-\sqrt{6}}{4}\right)+\mathrm{Li}_2\left(2\sqrt{6}-4\right)\right\}$$
$$+4\sqrt{\frac{2}{3}}\left\{2\mathrm{Li}_3\left(5-2\sqrt{6}\right)-\mathrm{Li}_3\left(2\sqrt{6}-4\right)+\mathrm{Li}_3\left(\frac{2-\sqrt{6}}{4}\right)\right\}.$$

This leads us to the identity in Theorem 18. □

5.3. Positive Series about $\mathbf{O}_n^{\langle 3\rangle}$

By expressing the series in terms of a definite integral

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\mathbf{O}_n^{\langle 3\rangle}}{8^n}=\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n\binom{-\frac{1}{2}}{n}\left\{\frac{7}{8}\zeta(3)-\frac{1}{2}\int_0^1 x^{2n}\frac{\ln^2 x}{1-x^2}dx\right\}$$
$$=\frac{7\sqrt{2}}{8}\zeta(3)-\frac{1}{2}\int_0^1\frac{\ln^2 x}{1-x^2}dx\sum_{n=0}^{\infty}\left(\frac{-x^2}{2}\right)^n\binom{-\frac{1}{2}}{n}$$
$$=\frac{7\sqrt{2}}{8}\zeta(3)-\frac{1}{\sqrt{2}}\int_0^1\frac{\ln^2 x}{(1-x^2)\sqrt{2-x^2}}dx,$$

we can confirm the formula stated in the theorem below.

Theorem 19 (Conjectured by Sun [18] (Equation (2.4)))**.**

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\mathbf{O}_n^{\langle 3\rangle}}{8^n}=\frac{35\sqrt{2}}{64}\zeta(3)-\frac{\pi G}{4\sqrt{2}}.$$

As suggested by an anonymous reviewer, we made a numerical test for the positive series in Theorem 19. Let $\mathrm{S}(m)$ be the partial sum of the series

$$\mathrm{S}(m)=\sum_{n=0}^{m}\binom{2n}{n}\frac{\mathbf{O}_n^{\langle 3\rangle}}{8^n}:\quad \mathrm{S}(\infty)=\frac{35\sqrt{2}}{64}\zeta(3)-\frac{\pi G}{4\sqrt{2}}\approx 0.420977466.$$

Table 1 shows the discrepancy between the real value $\mathrm{S}(\infty)$ and the partials sums $\mathrm{S}(m)$. In order to compute the series precisely, the partial sum $\mathrm{S}(m)$ adds three valid decimal digits when the number of terms is augmented by ten.

Table 1. The discrepancy between the real value $\mathrm{S}(\infty)$ and the partials sums $\mathrm{S}(m)$.

m	$\mathbf{S}(\infty)-\mathbf{S}(m)$
0	0.420977466
10	0.000166194041
20	1.20122224×10^{-7}
30	$9.73764143\times 10^{-11}$
40	$8.30614158\times 10^{-14}$
50	$7.29318580\times 10^{-17}$
60	$6.52470966\times 10^{-20}$
70	$5.91416751\times 10^{-23}$
80	$5.41293066\times 10^{-26}$
90	$4.99124388\times 10^{-29}$
100	$4.62971458\times 10^{-32}$

Proof. Even though this series looks simple, its proof is not as easy as expected. It is obvious that Sun's conjectured formula is equivalent to the integral identity below

$$\int_0^1\frac{\ln^2 x}{(1-x^2)\sqrt{2-x^2}}dx=\frac{\pi G}{4}+\frac{21}{32}\zeta(3).$$

By making the change of variable $\boxed{x\to\frac{1-y}{\sqrt{1+y^2}}}$, we can reformulate the integral and then evaluate it in closed form as

$$\int_0^1\frac{\ln^2 x}{(1-x^2)\sqrt{2-x^2}}dx=\int_0^1\frac{dy}{2y}\ln^2\left(\frac{1-y}{\sqrt{1+y^2}}\right)=\frac{\pi G}{4}+\frac{21}{32}\zeta(3).$$

This is justified by expanding the logarithm function

$$\ln^2\left(\frac{1-y}{\sqrt{1+y^2}}\right)=\ln^2(1-y)-\ln(1-y)\ln(1+y^2)+\frac{1}{4}\ln^2(1+y^2)$$

and then evaluating (manually or by *Mathematica*) the three corresponding integrals

$$\int_0^1\frac{\ln^2(1+y^2)}{y}dy=\frac{\zeta(3)}{8},$$
$$\int_0^1\frac{\ln^2(1-y)}{y}dy=\int_0^1\frac{\ln^2 y}{1-y}dy=2\zeta(3),$$
$$\int_0^1\frac{\ln(1-y)\ln(1+y^2)}{y}dy=\frac{23}{32}\zeta(3)-\frac{\pi G}{2}.$$

Thus, we have completed the proof of Theorem 19. □

5.4. Alternating Series about $\mathbf{O}_n^{\langle 3\rangle}$

Analogously, we can rewrite the series by

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\mathbf{O}_n^{\langle 3\rangle}}{(-8)^n}=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\binom{-\frac{1}{2}}{n}\left\{\frac{7}{8}\zeta(3)-\frac{1}{2}\int_0^1\frac{x^{2n}\ln^2 x}{1-x^2}dx\right\}$$
$$=\frac{7}{4\sqrt{6}}\zeta(3)-\frac{1}{\sqrt{2}}\int_0^1\frac{\ln^2 x}{(1-x^2)\sqrt{2+x^2}}dx.$$

Then, we have the formula as in the following theorem:

Theorem 20.

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\mathbf{O}_n^{\langle 3\rangle}}{(-8)^n}=\frac{1}{24\sqrt{6}}\left\{\begin{array}{c}24\zeta(3)-2\pi^2\ln(2+\sqrt{3})-24\mathrm{Li}_3(2-\sqrt{3})\\ -12\mathrm{Li}_2(2-\sqrt{3})\ln\left(2+\sqrt{3}\right)-\ln^3\left(2+\sqrt{3}\right)\end{array}\right\}.$$

For Theorem 20, let $\mathcal{S}(\infty)$ stand for the expression on the right and $\mathcal{S}(m)$ for the partial sum

$$\mathcal{S}(m)=\sum_{n=0}^{m}\binom{2n}{n}\frac{\mathbf{O}_n^{\langle 3\rangle}}{(-8)^n}:\qquad \mathcal{S}(\infty)\approx -0.181230005.$$

Then, a numerical test for this alternating series is reported in Table 2.

Table 2. The discrepancy between the real value $\mathcal{S}(\infty)$ and the partials sums $\mathcal{S}(m)$.

m	$\mathcal{S}(\infty)-\mathcal{S}(m)$
1	0.0687699948
12	−0.0000134242504
23	4.82032558×10^{-9}
34	$-1.94824782\times 10^{-12}$
45	$8.29600998\times 10^{-16}$
56	$-3.63844789\times 10^{-19}$
67	$1.62638769\times 10^{-22}$
78	$-7.36718900\times 10^{-26}$
89	$3.37006801\times 10^{-29}$
100	$-1.55327774\times 10^{-32}$

Proof. Under the change of variable $\boxed{x\to\frac{1-y}{\sqrt{1+4y+y^2}}}$, we can reformulate and then evaluate the integral as

$$\int_0^1\frac{\sqrt{2}\ln^2 x}{(1-x^2)\sqrt{2+x^2}}dx=\int_0^1\frac{dy}{y\sqrt{6}}\ln^2\left(\frac{1-y}{\sqrt{1+4y+y^2}}\right)$$
$$=\frac{1}{12\sqrt{6}}\left\{\begin{array}{c}18\zeta(3)+2\pi^2\ln(2+\sqrt{3})+24\mathrm{Li}_3(2-\sqrt{3})\\ +12\mathrm{Li}_2(2-\sqrt{3})\ln\left(2+\sqrt{3}\right)+\ln^3\left(2+\sqrt{3}\right)\end{array}\right\}.$$

This result is deduced after a laboriously manual manipulations from a lengthy output expression produced by *Mathematica*. Then, the identity in Theorem 20 follows from substitution. □

6. Concluding Comments

By computing definite integrals, we have reviewed a few and derived more closed formulae for binomial series involving harmonic numbers. Most of the identities recorded as "Theorems" (except for Theorems 13 and 19) do not seem to have appeared previously. In view of Sun's numerous conjectures on harmonic sums [18], these identities would have further significant applications and implications to both infinite series evaluations and congruences of finite sums in combinatorial number theory.

However, there will be increasing difficulties in integral evaluations and polylogarithmic simplifications as the order of harmonic numbers becomes higher. In this case, it seems more plausible to deal with these series through the "coefficient extraction method" based on summation formulae of classical hypergeometric series (cf. [8,9,20–22]).

We are going to illustrate this approach to proving the following identity detected experimentally by Sun [23] (Conjecture 1.43):

$$\frac{5\pi}{18}\zeta(3)=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}\mathbf{O}_{n+1}^{\langle 3\rangle}}{16^n(2n+1)}. \tag{4}$$

Following the same procedure as in Section 5.3, it is not difficult to express the last series as

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}\mathbf{O}_{n+1}^{\langle 3\rangle}}{16^n(2n+1)}=\frac{7\pi}{24}\zeta(3)-\int_0^1\frac{x\ln^2 x}{1-x^2}\arcsin\Big(\frac{x}{2}\Big)dx,$$

which reduces proving (4) to confirming the integral identity

$$\frac{\pi}{72}\zeta(3)=\int_0^1\frac{x\ln^2 x}{1-x^2}\arcsin\Big(\frac{x}{2}\Big)dx.$$

Unfortunately, we did not succeed in evaluating this integral in closed form both manually, and by *Mathematica* (Version 11).

Recalling the hypergeometric series formula due to Gessel and Stanton [24]

$$\begin{aligned}{}_5F_4\left[\begin{matrix}2a,\ 1+\frac{2a}{3},\ 2b,\ 1-2b,\ a-d\\ \frac{2a}{3},1+a-b,\frac{1}{2}+a+b,1+2d\end{matrix}\,\middle|\,\frac{1}{4}\right]&=\sum_{n=0}^{\infty}\Big(\frac{1}{4}\Big)^n\frac{(2a)_n(1+\frac{2a}{3})_n(2b)_n(1-2b)_n(a-d)_n}{n!(\frac{2a}{3})_n(1+a-b)_n(\frac{1}{2}+a+b)_n(1+2d)_n}\\&=\frac{\Gamma(1+d)\Gamma(\frac{1}{2}+d)\Gamma(1+a-b)\Gamma(\frac{1}{2}+a+b)}{\Gamma(1+a)\Gamma(\frac{1}{2}+a)\Gamma(1-b+d)\Gamma(\frac{1}{2}+b+d)}.\end{aligned}$$

Under the parameter replacements

$$a\to\frac{3}{4}+3x,\ b\to\frac{1}{4}+2x,\ d\to\frac{1}{4}+x,$$

we can reformulate the above equality as below:

$$4^{1+2x}\Gamma\left[\begin{matrix}\frac{1}{2}+x,\frac{1}{2}+2x,\frac{1}{2}+5x\\ 1+3x,1-x,\frac{1}{2}+6x\end{matrix}\right]=\sum_{n=0}^{\infty}\frac{(\frac{1}{2}+2x)_n(\frac{1}{2}+4x)_n(\frac{1}{2}-4x)_n(\frac{1}{2}+6x)_{n+1}}{4^n n!(\frac{1}{2}+x)_{n+1}(\frac{1}{2}+2x)_{n+1}(\frac{1}{2}+5x)_{n+1}}\Big\{3+12x+6n\Big\}. \tag{5}$$

Both sides of the above equation are analytic in the neighborhood of $x=0$ and can be expanded into Maclaurin series in x. Considering the constant terms of (5), we recover the identity (see [15]) below

$$\frac{\pi}{3}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{16^n(2n+1)}.$$

The next two identities are obtained, respectively, from the coefficients of x and x^2:

$$\begin{aligned}\frac{\pi}{3}\ln 2&=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{16^n(2n+1)}\Big\{\mathbf{O}_n+\frac{2}{3(1+2n)}\Big\},\\ \frac{\pi^3}{108}&=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{16^n(2n+1)}\Big\{\mathbf{O}_n^{\langle 2\rangle}+\frac{5}{21(1+2n)^2}\Big\}.\end{aligned}$$

Finally, Sun's conjectured identity (4) follows by extracting coefficients of x^3 across (5) and then performing some simplifications. More formulae of similar nature can be found in [18,21,22].

In order to realize the preceding operations, it is necessary to know how to extract coefficients from factorial fractions and the Γ-function quotients, which are briefly recorded as follows. Denoting by $[x^m]\phi(x)$ the coefficient of x^m in the formal power series $\phi(x)$, we can express the skew-harmonic numbers in terms of the following coefficients:

$$[y]\frac{(\frac{1}{2}+y)_n}{(\frac{1}{2})_n} = 2\mathbf{O}_n, \qquad [y]\frac{(\frac{1}{2})_n}{(\frac{1}{2}-y)_n} = 2\mathbf{O}_n;$$
$$[y^2]\frac{(\frac{1}{2}+y)_n}{(\frac{1}{2})_n} = 2(\mathbf{O}_n^2 - \mathbf{O}_n^{\langle 2\rangle}), \qquad [y^2]\frac{(\frac{1}{2})_n}{(\frac{1}{2}-y)_n} = 2(\mathbf{O}_n^2 + \mathbf{O}_n^{\langle 2\rangle});$$
$$[y^3]\frac{(\frac{1}{2}+y)_n}{(\frac{1}{2})_n} = \frac{4}{3}(\mathbf{O}_n^3 - 3\mathbf{O}_n\mathbf{O}_n^{\langle 2\rangle} + 2\mathbf{O}_n^{\langle 3\rangle}), \quad [y^3]\frac{(\frac{1}{2})_n}{(\frac{1}{2}-y)_n} = \frac{4}{3}(\mathbf{O}_n^3 + 3\mathbf{O}_n\mathbf{O}_n^{\langle 2\rangle} + 2\mathbf{O}_n^{\langle 3\rangle}).$$

Alternatively, the power series expansions of the Γ-function (see [20] and [25] (Section 2.10)) are given by

$$\Gamma(1-y) = \exp\Big\{\sum_{k\geq 1}\frac{\sigma_k}{k}y^k\Big\} \quad \text{and} \quad \Gamma(\tfrac{1}{2}-y) = \sqrt{\pi}\exp\Big\{\sum_{k\geq 1}\frac{\tau_k}{k}y^k\Big\},$$

where σ_k and τ_k are defined, respectively, by

$$\begin{aligned} \sigma_1 &= \gamma && \text{and} \quad \sigma_m = \zeta(m) && \text{for} \quad m \geq 2; \\ \tau_1 &= \gamma + 2\ln 2 && \text{and} \quad \tau_m = (2^m-1)\zeta(m) && \text{for} \quad m \geq 2; \end{aligned}$$

with the usual symbol γ for the Euler–Mascheroni constant.

Author Contributions: Writing & editing, formal analysis, C.L.; original draft and review, W.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: The authors express their sincere gratitude to the two reviewers for the careful reading, critical comments, and valuable suggestions that contributed significantly to improving the manuscript during the revision.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Adegoke, K.; Frontczak, R.; Goy, T. Combinatorial sums, series and integrals involving odd harmonic numbers. *arXiv* **2024**, arXiv:2401.02470v1.
2. Boyadzhiev, K.N. Series with central binomial coefficients, Catalan numbers, and harmonic numbers. *J. Integer Seq.* **2012**, *15*, 12.1.7.
3. Chen, H. Interesting series associated with central binomial coefficients, Catalan numbers and harmonic numbers. *J. Integer Seq.* **2016**, *19*, 16.1.5.
4. Chen, K.W. Generalized harmonic numbers and Euler sums. *Int. J. Number Theory* **2017**, *13*, 513–528. [CrossRef]
5. Chen, K.W.; Chen, Y.H. Infinite series containing generalized harmonic functions. *Notes Number Theory Discret. Math.* **2020**, *26*, 85–104. [CrossRef]
6. Choi, J. Certain summation formulas involving harmonic numbers and generalized harmonic numbers. *Appl. Math. Comput.* **2011**, *218*, 734–740. [CrossRef]
7. Choi, J. Finite summation formulas involving binomial coefficients, harmonic numbers and generalized harmonic numbers. *J. Inequal. Appl.* **2013**, *2013*, 49. [CrossRef]
8. Choi, J. Summation formulas involving binomial coefficients, harmonic numbers, and generalized harmonic numbers. *Abstr. Appl. Anal.* **2014**, *2014*, 501906. [CrossRef]
9. Chu, W. Hypergeometric approach to Apéry–like series. *Integral Transform. Spec. Funct.* **2017**, *28*, 505–518. [CrossRef]

10. Chu, W. Three symmetric double series by telescoping. *Amer. Math. Monthly* **2023**, *130*, 468–477. [CrossRef]
11. Frontczak, R. Binomial sums with skew-harmonic numbers. *Palest. J. Math.* **2021**, *10*, 756–763.
12. Genčev, M. Binomial sums involving harmonic numbers. *Math. Slovaca* **2011**, *61*, 215–226. [CrossRef]
13. Nimbran, A.S.; Levrie, P.; Sofo, A. Harmonic-binomial Euler-like sums via expansions of $(\arcsin x)^p$. *RACSAM* **2022**, *116*, 23. [CrossRef]
14. Elsner, C. On sums with binomial coefficient. *Fibonacci Quart.* **2005**, *43*, 31–45.
15. Lehmer, D.H. Interesting series involving the central binomial coefficient. *Amer. Math. Mon.* **1985**, *92*, 449–457. [CrossRef]
16. Zucker, I.J. On the series $\sum_{k=1}^{\infty}\binom{2k}{k}^{-1}k^{-n}$. *J. Number Theory* **1985**, *20*, 92–102. [CrossRef]
17. Lewin, L. *Polylogarithms and Associated Functions*; North-Holland: New York, NY, USA, 1981.
18. Sun, Z.-W. Series with summands involving higher harmonic numbers. *arXiv* **2023**, arXiv:2210.07238v8.
19. Vălean, C.I. *(Almost) Impossible Integrals, Sums, and Series*; Springer Nature AG: Cham, Switzerland, 2019.
20. Chu, W. Hypergeometric series and the Riemann zeta function. *Acta Arith.* **1997**, *82*, 103–118. [CrossRef]
21. Li, C.L.; Chu, W. Infinite series about harmonic numbers Inspired by Ramanujan–like formulae. *Electron. Res. Arch.* **2023**, *31*, 4611–4636. [CrossRef]
22. Li, C.L.; Chu, W. Series of convergence rate $-1/4$ containing harmonic numbers. *Axioms* **2023**, *12*, 513. [CrossRef]
23. Sun, Z.-W. List of conjectural series for powers of π and other constants. *arXiv* **2014**, arXiv:1102.5649v47.
24. Gessel, I.; Stanton, D. Strange evaluations of hypergeometric series. *SIAM J. Math. Anal.* **1982**, *13*, 295–308. [CrossRef]
25. Luke, Y.L. *Special Functions and Their Approximations*; Academic Press: London, UK; New York, NY, USA, 1969; Volume I.