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A Class of Fractional Viscoelastic Kirchhoff Equations Involving Two Nonlinear Source Terms of Different Signs

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Abstract: A class of fractional viscoelastic Kirchhoff equations involving two nonlinear source terms of different signs are studied. Under suitable assumptions on the exponents of nonlinear source terms and the memory kernel, the existence of global solutions in an appropriate functional space is established by a combination of the theory of potential wells and the Galerkin approximations. Furthermore, the asymptotic behavior of global solutions is obtained by a combination of the theory of potential wells and the perturbed energy method.

Keywords: fractional viscoelastic Kirchhoff equations; existence; asymptotic behavior; potential wells; perturbed energy method

MSC: 35R11; 35A01; 35B40

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1. Introduction

In this paper, we deal with the initial boundary value problem for a class of fractional viscoelastic Kirchhoff equations involving two nonlinear source terms of different signs.

$$\begin{aligned} &u_{tt} + (a + b[u]_m^{2p-2})(-\Delta)^m u - \int_0^t g(t-\tau)(-\Delta)^m u(\tau)\,\mathrm{d}\tau \\ &+ u_t = |u|^{q-2}u - |u|^{r-2}u, \quad x \in \Omega,\ t > 0, \end{aligned} \tag{1}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{2}$$

$$u(x,t) = 0, \quad x \in \mathbb{R}^N \backslash \Omega,\ t > 0, \tag{3}$$

where $[u]_m$ is the Gagliardo seminorm defined by

$$[u]_m := \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x,t) - u(y,t)|^2}{|x-y|^{N+2m}}\,\mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}},$$

$[u]_m^{2p-2}$ means that $2p-2$ is a power of $[u]_m$, and $(-\Delta)^m$ is the fractional Laplacian with $0 < m < 1$, which (up to normalization factors) could be given by

$$(-\Delta)^m \varphi(x) := \int_{\mathbb{R}^N} \frac{2\varphi(x) - \varphi(x+y) - \varphi(x-y)}{|y|^{N+2m}}\,\mathrm{d}y, \quad x \in \mathbb{R}^N$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. In addition, $a > 0$, $b \ge 0$, $p > 1$, $\Omega \subset \mathbb{R}^N$ $(N \ge 1)$ is a bounded domain with a Lipschitz boundary, and the functions $u_0(x)$ and $u_1(x)$ are specified later.

The problem (1)–(3) can govern the motion of the viscoelastic string with a fractional length (see Ref. [1]). The unknown function $u = u(x,t)$ represents the vertical displacement, $-\int_0^t g(t-\tau)(-\Delta)^m u(\tau)\,\mathrm{d}\tau$ is the viscoelastic term, u_t is a weakly damped term, $|u|^{q-2}u$

and $-|u|^{r-2}u$ are two nonlinear source terms of different signs, and q, r, and the memory kernel g are introduced later.

In the case where $m = 1$, Equation (1) becomes a classical viscoelastic Kirchhoff wave equation. Concerning this class of equations, Torrejón and Yong [2] investigated

$$u_{tt} - h(\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,\mathrm{d}\tau = f,$$

where h is the Kirchhoff function. They obtained existence, uniqueness, and asymptotic behavior of global solutions. Wu and Tsai [3] treated a viscoelastic Kirchhoff wave equation with nonlinear weak damping and source terms:

$$u_{tt} - h(\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,\mathrm{d}\tau + f_2(u_t) = f_1(u).$$

They derived existence and blow-up of local solutions, and also established the estimates on the blow-up time. Liu et al. [4] studied a more general viscoelastic Kirchhoff wave equation:

$$u_{tt} - h(\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,\mathrm{d}\tau + f_2(x,t,u,u_t) = f_1(x,t,u).$$

They proved the nonexistence of global solutions.

Fractional partial differential equations arise in continuum mechanics [5], quantum mechanics [6,7], population dynamics [5], anomalous diffusion [8,9], fluid mechanics [10,11], and so on, and they have received considerable attention. Ambrosio and Isernia [12] investigated a fractional stationary Kirchhoff equation:

$$(a + b(1-m)[u]_m^2)(-\Delta)^m u = f(u),$$

where a and b are two parameters. By the minimax arguments, they established the multiplicity of solutions, provided b is sufficiently small. Nyamoradi and Ambrosio [13] dealt with a fractional stationary Kirchhoff equation involving two nonlinear source terms:

$$(a + b[u]_m^{2\theta-2})(-\Delta)^m u = \lambda f(x)|u|^{q-2}u + |u|^{2_m^*-2}u,$$

where θ and λ are also parameters, and 2_m^* represents the critical exponent of the fractional Sobolev space. Under suitable values of the parameters, they derived the existence and nonexistence of multiple solutions. Fiscella and Mishra [14] studied the following fractional stationary Kirchhoff equation involving two nonlinear source terms:

$$(a + b[u]_m^{2\theta-2})(-\Delta)^m u - \mu\frac{u}{|x|^{2m}} = \lambda f_1(x)u^{-p} + f_2(x)u^{2_m^*-1}.$$

They addressed the existence of at least two positive solutions depending on the parameters by exploiting the Nehari manifold. do Ó et al. [15] studied a fractional stationary Kirchhoff equation involving two nonlinear source terms of different signs:

$$(a + b[u]_m^{2\theta-2})(-\Delta)^m u = \lambda f(x)|u|^{q-2}u - |u|^{2_m^*-2}u.$$

By using a variational approach based on the Nehari manifold, they obtained the existence of two positive solutions for suitable values of the parameters. Zhang et al. [16] investigated a fractional stationary Kirchhoff equation involving the nonlocal integro-differential operator:

$$-h\left(\iint_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 K(x-y)\,\mathrm{d}x\mathrm{d}y\right)\mathcal{L}_K u = f(x,u), \tag{4}$$

where $K : \mathbb{R}^N \setminus \{0\} \to (0, \infty)$ is a function satisfying certain assumptions, and $\mathcal{L}_K$ denotes the nonlocal integral-differential operator. Equation (4) includes the following fractional Kirchhoff equation as a particular case.

$$h([u]_m^2)(-\Delta)^m u = f(x, u).$$

By computing the critical groups at zero and at infinity, they derived the existence of at least one nontrivial solution via Morse theory. Molica Bisci and Vilasi [17] considered the following fractional stationary Kirchhoff equation involving two nonlinear source terms:

$$-h([u]_m^2)\mathcal{L}_K u = \lambda f_1(x, u) + \mu f_2(x, u).$$

By exploiting an abstract critical point theorem for smooth functionals, they obtained the existence of at least three solutions for suitable values of the parameters. Additionally, in the autonomous case, they gave a precise estimate for the range of these parameters by using some properties of the fractional calculus on a specific family of test functions. Concerning the fractional evolution Kirchhoff equations, Xiang et al. [18] studied

$$u_t + h([u]_m^2)(-\Delta)^m u = |u|^{p-2}u.$$

They obtained the existence of non-negative local solutions by using the Galerkin approximations and proved the blow-up of non-negative local solutions with suitable initial data by virtue of a differential inequality technique. Lin et al. [19] studied a fractional evolution Kirchhoff equation of the form

$$u_{tt} + [u]_m^{2\theta-2}(-\Delta)^m u = f(u).$$

They utilized the concavity arguments to obtain the blow-up of solutions. Pan et al. [20] considered the following fractional evolution Kirchhoff equation with nonlinear weak damping:

$$u_{tt} + [u]_m^{2\theta-2}(-\Delta)^m u + |u_t|^{r-2}u_t + u = |u|^{q-2}u.$$

They obtained the global existence, vacuum isolation, asymptotic behavior, and blow-up of solutions by using the theory of potential wells. Recently, Xiang and Hu [21] conducted an investigation on the following fractional viscoelastic Kirchhoff equation:

$$u_{tt} + h([u]_m^2)(-\Delta)^m u - \int_0^t g(t-\tau)(-\Delta)^m u(\tau)\,d\tau + (-\Delta)^s u_t = \lambda |u|^{q-2}u.$$

They utilized the Galerkin approximations to establish the existence of local and global solutions, and they employed the concavity arguments to derive the blow-up of solutions.

Inspired by the above works, we deal with the problem (1)–(3). There are two features about our results. On the one hand, our results are independent of any parameters. On the other hand, we focus on the effects of the nonlinear source terms of different signs on solutions. From a physical perspective, the two nonlinear source terms of different signs actually represent the two opposing external forces acting on the viscoelastic string with fractional length. In the absence of the external force $|u|^{q-2}u$, it is trivial that the problem (1)–(3) has a global solution with the asymptotic behavior by the arguments similar to Ref. [22]. Our results show that even though the external force $|u|^{q-2}u$ appears, the problem (1)–(3) can still have a global solution with the asymptotic behavior. In addition, our main technical tool is the theory of potential wells that has been widely employed to analyze the qualitative properties of solutions of evolution equations. In this regard, in addition to the work of Ref. [20] mentioned above, we refer to the following work. Xu et al. [23] used the theory of potential wells to investigate the global existence and blow-up of solutions of a sixth-order nonlinear hyperbolic equation. Cavalcanti and Domingos Cavalcanti [24] modified the theory of potential wells to study the global existence and asymptotic behaviour of solutions of a nonlinear evolution equation. Gazzola and Squassina [25] improved the theory of potential wells to obtain the global existence, asymptotic behaviour, and blow-up of solutions of a damped semilinear wave equation. Liu et al. [26] applied the idea of Ref. [25] to deal with a fourth-order damped nonlinear hyperbolic equation. Xu and Su [27], Luo et al. [28], and Liu and Li [29] introduced a family of potential wells to study the equation under consideration, respectively. Liu et al. [30] introduced a family of potential wells that were different from those in Refs. [27–29]. In contrast, the definition of the potential well in this paper differs from those in the above studies.

This paper is organized as follows. Section 1 serves as an introduction. Section 2 provides a comprehensive overview of the relevant functional spaces. Additionally, we define a potential well and give its properties. Section 3 is devoted to the proof of the existence of global solutions by using the theory of potential wells and the Galerkin approximations. In Section 4, we prove the asymptotic behavior of global solutions by using the theory of potential wells and the perturbed energy method [31–33]. In Section 5, we summarize our main results.

2. Preliminaries

As in Refs. [34,35], we denote by X the linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function u in X belongs to $L^2(\Omega)$ and

$$\iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2m}}\,\mathrm{d}x\mathrm{d}y < \infty,$$

where $Q := \mathbb{R}^{2N}\backslash(\mathcal{C}\Omega \times \mathcal{C}\Omega)$ and $\mathcal{C}\Omega := \mathbb{R}^N\backslash\Omega$. The space X is endowed with

$$\|u\|_X := \|u\|_{L^2(\Omega)} + \left(\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2m}}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}.$$

It is straightforward to verify that $\|\cdot\|_X$ is a norm on X. We introduce the following closed linear subspace of X,

$$X_0 := \{u \in X | u = 0 \text{ a.e. in } \mathcal{C}\Omega\}.$$

This is a Hilbert space equipped with the inner product

$$(u,v)_{X_0} := \iint_Q \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2m}}\,\mathrm{d}x\mathrm{d}y$$

and the norm

$$\|u\|_{X_0} := \left(\iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2m}}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}$$

which is equivalent to $\|u\|_X$. Moreover, the embedding $X_0 \hookrightarrow L^q(\Omega)$ is continuous for any $1 \le q \le 2_m^*$ and compact for any $1 \le q < 2_m^*$, where

$$2_m^* = \begin{cases} \dfrac{2N}{N-2m} & \text{if } 2m < N, \\ \infty & \text{if } 2m \ge N. \end{cases}$$

In this paper, the exponents q and r satisfy the following assumption:

$$(\mathrm{A}_1)\ \ 2 < r < q < 2_m^*.$$

In addition, as in Ref. [33], the memory kernel g satisfies

$$(\mathrm{A}_2)\ \ g \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+),\ g(t) \geq 0,\ g'(t) \leq 0 \text{ for all } t \in [0,\infty),\ \text{and}$$
$$\kappa := a - \int_0^\infty g(t)\,\mathrm{d}t > 0.$$

For simplicity, we denote

$$\|\cdot\|_p := \|\cdot\|_{L^p(\Omega)},\ (u,v) := \int_\Omega uv\,\mathrm{d}x$$

and

$$(g \circ u)(t) := \int_0^t g(t-\tau)\|u(t)-u(\tau)\|_{X_0}^2\,\mathrm{d}\tau.$$

Moreover, $C > 0$ denotes a generic constant.

Definition 1. *A function $u \in L^\infty(0,T;X_0)$ with $u_t \in L^\infty(0,T;L^2(\Omega))$ is called the weak solution of the problem (1)–(3), if $u(0) = u_0$ in X_0, $u_t(0) = u_1$ in $L^2(\Omega)$, and*

$$\begin{aligned}(u_t(t),w) &+ \int_0^t (a + b\|u(\tau)\|_{X_0}^{2p-2})(u(\tau),w)_{X_0}\,\mathrm{d}\tau - \int_0^t\int_0^s g(s-\tau)(u(\tau),w)_{X_0}\,\mathrm{d}\tau\mathrm{d}s \\ &+ (u(t),w) = (u_1,w) + (u_0,w) + \int_0^t (|u|^{q-2}u,w)\,\mathrm{d}\tau - \int_0^t (|u|^{r-2}u,w)\,\mathrm{d}\tau\end{aligned}$$

for any $w \in X_0$ and $t \in (0,T]$.

The energy function for the problem (1)–(3) is defined as follows:

$$\begin{aligned}E(t) :=& \frac{1}{2}\|u_t(t)\|_2^2 + \frac{b}{2p}\|u(t)\|_{X_0}^{2p} + \frac{1}{2}\left(a - \int_0^t g(\tau)\,\mathrm{d}\tau\right)\|u(t)\|_{X_0}^2 \\ &+ \frac{1}{2}(g \circ u)(t) - \frac{1}{q}\|u(t)\|_q^q + \frac{1}{r}\|u(t)\|_r^r.\end{aligned}$$

Following the idea from Ref. [36], the potential well and its boundary are defined by

$$\mathcal{W} := \left\{ u \in X_0 \,\middle|\, \|u\|_{X_0} < \left(\frac{2q}{(q-2)\kappa}d\right)^{\frac{1}{2}} \right\} \tag{5}$$

and

$$\partial\mathcal{W} := \left\{ u \in X_0 \,\middle|\, \|u\|_{X_0} = \left(\frac{2q}{(q-2)\kappa}d\right)^{\frac{1}{2}} \right\}, \tag{6}$$

where the depth of the potential well is

$$d := \frac{q-2}{2q}\kappa^{\frac{q}{q-2}}\mathfrak{C}_1^{-\frac{2q}{q-2}}, \tag{7}$$

and $\mathfrak{C}_1$ is the best Sobolev constant for the embedding $X_0 \hookrightarrow L^q(\Omega)$, i.e.,

$$\mathfrak{C}_1 := \sup_{u \in X_0 \setminus \{0\}} \frac{\|u\|_q}{\|u\|_{X_0}}.$$

Lemma 1. *Let (A_1) and (A_2) be satisfied. Then,*

(i) if $u \in \mathcal{W}$ and $\|u\|_{X_0} \neq 0$, then $\kappa\|u\|_{X_0}^2 > \|u\|_q^q$;

(ii) if $u \in \partial\mathcal{W}$, then $\kappa\|u\|_{X_0}^2 \geq \|u\|_q^q$.

Proof. (i) By $u \in \mathcal{W}$ and (5), we obtain

$$\|u\|_{X_0} < \left(\frac{2q}{(q-2)\kappa}d\right)^{\frac{1}{2}}.$$

From (7), we obtain

$$\|u\|_{X_0} < \kappa^{\frac{1}{q-2}}\mathfrak{C}_1^{-\frac{q}{q-2}}.$$

Noting that $\|u\|_{X_0} \neq 0$, we have

$$\kappa\|u\|_{X_0}^2 > \mathfrak{C}_1^q\|u\|_{X_0}^q.$$

Therefore,

$$\kappa\|u\|_{X_0}^2 > \|u\|_q^q.$$

(ii) By $u \in \partial\mathcal{W}$ and (6), we obtain

$$\|u\|_{X_0} = \left(\frac{2q}{(q-2)\kappa}d\right)^{\frac{1}{2}}.$$

By adopting similar arguments in the proof of (i), it is evident that

$$\kappa\|u\|_{X_0}^2 \geq \|u\|_q^q.$$

□

3. Existence of Global Solutions

Theorem 1. *Let* (A_1) *and* (A_2) *be satisfied. If* $u_0 \in \mathcal{W}$, $u_1 \in L^2(\Omega)$, *and* $E(0) < d$, *then the problem* (1)–(3) *has a global solution* $u(t) \in \overline{\mathcal{W}} := \mathcal{W} \cup \partial\mathcal{W}$ *for all* $t \in (0, \infty)$.

Proof. We divide the proof of this theorem into three steps.

Step I. Galerkin approximations. Let $\{\omega_j\}_{j=1}^{\infty}$ be the orthogonal basis of X_0 and an orthonormal basis of $L^2(\Omega)$ given by eigenfunctions of $(-\Delta)^m$ with boundary condition (3) (see Ref. [34], Proposition 9, for details). Denote $W_n = \mathrm{Span}\{\omega_1, \omega_2, \cdots, \omega_n\}$, $n = 1, 2, \cdots$. We seek the approximate solutions of the problem (1)–(3)

$$u_n(t) = \sum_{j=1}^{n} \xi_{jn}(t)\omega_j, \ \ n = 1, 2, \cdots, \tag{8}$$

which satisfy

$$\begin{aligned}(u_{ntt}(t), w) + (a + b\|u_n(t)\|_{X_0}^{2p-2})(u_n(t), w)_{X_0} - \int_0^t g(t-\tau)(u_n(\tau), w)_{X_0}\,\mathrm{d}\tau \\ + (u_{nt}(t), w) = (|u_n(t)|^{q-2}u_n(t), w) - (|u_n(t)|^{r-2}u_n(t), w), \ \ t > 0,\end{aligned} \tag{9}$$

$$u_n(0) = \sum_{j=1}^{n} \xi_{jn}(0)\omega_j \to u_0 \ \text{ in } X_0, \tag{10}$$

$$u_{nt}(0) = \sum_{j=1}^{n} \xi'_{jn}(0)\omega_j \to u_1 \ \text{ in } L^2(\Omega), \tag{11}$$

for any $w \in W_n$. Let $\xi_n(t) = (\xi_{1n}(t), \xi_{2n}(t), \cdots, \xi_{nn}(t))^T$. Then, the vector function ξ_n solves

$$\xi''_n(t) + \xi'_n(t) + \mathcal{L}_n(t, \xi_n(t)) = 0, \ \ t > 0, \tag{12}$$

$$\xi_n(0) = ((u_0, \omega_1), (u_0, \omega_2), \cdots, (u_0, \omega_n))^T, \tag{13}$$

$$\xi_n'(0) = ((u_1, \omega_1), (u_1, \omega_2), \cdots, (u_1, \omega_n))^T, \tag{14}$$

where

$$\begin{aligned} \mathcal{L}_n(t, \xi_n(t)) &= (\mathcal{L}_{1n}(t, \xi_n(t)), \mathcal{L}_{2n}(t, \xi_n(t)), \cdots, \mathcal{L}_{nn}(t, \xi_n(t)))^T, \\ \mathcal{L}_{in}(t, \xi_n(t)) &= \left(a + b \left\| \sum_{j=1}^{n} \xi_{jn}(t)\omega_j \right\|_{X_0}^{2p-2} \right) \left(\sum_{j=1}^{n} \xi_{jn}(t)\omega_j, \omega_i \right)_{X_0} \\ &\quad - \int_0^t g(t-\tau) \left(\sum_{j=1}^{n} \xi_{jn}(\tau)\omega_j, \omega_i \right)_{X_0} \mathrm{d}\tau \\ &\quad + \left(\left| \sum_{j=1}^{n} \xi_{jn}(t)\omega_j \right|^{q-2} \sum_{j=1}^{n} \xi_{jn}(t)\omega_j, \omega_i \right) \\ &\quad - \left(\left| \sum_{j=1}^{n} \xi_{jn}(t)\omega_j \right|^{r-2} \sum_{j=1}^{n} \xi_{jn}(t)\omega_j, \omega_i \right). \end{aligned}$$

In light of the standard theory of ordinary differential equations, the problem (12)–(14) has a solution $\xi_n \in C^2[0, T_n)$, with $T_n \le T$. Consequently, $u_n(t)$ defined by (8) satisfies the problem (9)–(11).

Step II. A priori estimates. Taking $w = u_{nt}(t)$ in (9), we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|u_{nt}(t)\|_2^2 + \frac{a}{2}\|u_n(t)\|_{X_0}^2 + \frac{b}{2p}\|u_n(t)\|_{X_0}^{2p} \right) \\ &- \int_0^t g(t-\tau)(u_n(\tau), u_{nt}(t))_{X_0}\,\mathrm{d}\tau + \|u_{nt}(t)\|_2^2 \\ &= \frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|u_n(t)\|_q^q - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\|u_n(t)\|_r^r. \end{aligned} \tag{15}$$

Concerning the fourth term on the left hand side of (15), we have

$$\begin{aligned} &\int_0^t g(t-\tau)(u_n(\tau), u_{nt}(t))_{X_0}\,\mathrm{d}\tau \\ &= \int_0^t g(t-\tau)(u_n(\tau) - u_n(t), u_{nt}(t))_{X_0}\,\mathrm{d}\tau + \int_0^t g(t-\tau)(u_n(t), u_{nt}(t))_{X_0}\,\mathrm{d}\tau \\ &= -\frac{1}{2}\int_0^t g(t-\tau)\frac{\mathrm{d}}{\mathrm{d}t}\|u_n(\tau) - u_n(t)\|_{X_0}^2\,\mathrm{d}\tau + \frac{1}{2}\int_0^t g(t-\tau)\frac{\mathrm{d}}{\mathrm{d}t}\|u_n(t)\|_{X_0}^2\,\mathrm{d}\tau \\ &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left((g \circ u_n)(t) - \int_0^t g(\tau)\,\mathrm{d}\tau \|u_n(t)\|_{X_0}^2 \right) + \frac{1}{2}(g' \circ u_n)(t) - \frac{1}{2}g(t)\|u_n(t)\|_{X_0}^2. \end{aligned}$$

Substituting this equality into (15) and performing integration with respect to t, we obtain

$$E_n(t) + \int_0^t \left(\|u_{n\tau}(\tau)\|_2^2 - \frac{1}{2}(g' \circ u_n)(\tau) + \frac{1}{2}g(\tau)\|u_n(\tau)\|_{X_0}^2 \right) \mathrm{d}\tau = E_n(0) \tag{16}$$

for all $t \in [0, T]$, where

$$\begin{aligned} E_n(t) =& \frac{1}{2}\|u_{nt}(t)\|_2^2 + \frac{b}{2p}\|u_n(t)\|_{X_0}^{2p} + \frac{1}{2}\left(a - \int_0^t g(\tau)\,\mathrm{d}\tau \right)\|u_n(t)\|_{X_0}^2 \\ &+ \frac{1}{2}(g \circ u_n)(t) - \frac{1}{q}\|u_n(t)\|_q^q + \frac{1}{r}\|u_n(t)\|_r^r. \end{aligned} \tag{17}$$

Based on the observations from (10) and (11), we conclude that $E_n(0) < d$ and $u_n(0) \in \mathcal{W}$ for a sufficiently large n. We claim that

$$u_n(t) \in \mathcal{W} \tag{18}$$

for all $t \in [0, T]$ and a sufficiently large n. Suppose that $u_n(t) \notin \mathcal{W}$ for some $0 < t < T$. Then, there exists a time $0 < t_0 < T$ such that $u_n(t_0) \in \partial\mathcal{W}$ and $u_n(t) \in \mathcal{W}$ for all $t \in [0, t_0)$. Therefore,

$$\|u_n(t_0)\|_{X_0} = \left(\frac{2q}{(q-2)\kappa}d\right)^{\frac{1}{2}}.$$

Utilizing (17) in conjunction with part (ii) in Lemma 1, we are able to derive

$$\begin{aligned}
E_n(t_0) \geq & \frac{1}{2}\kappa\|u_n(t_0)\|_{X_0}^2 - \frac{1}{q}\|u_n(t_0)\|_q^q \\
= & \frac{q-2}{2q}\kappa\|u_n(t_0)\|_{X_0}^2 + \frac{1}{q}\left(\kappa\|u_n(t_0)\|_{X_0}^2 - \|u_n(t_0)\|_q^q\right) \\
\geq & \frac{q-2}{2q}\kappa\|u_n(t_0)\|_{X_0}^2 \\
= & d.
\end{aligned}$$

In view of (16), we obtain a contradiction with $E_n(0) < d$.

By (17), assertion (18) and (i) in Lemma 1, we can conclude that

$$\begin{aligned}
E_n(t) \geq & \frac{1}{2}\|u_{nt}(t)\|_2^2 + \frac{1}{2}\kappa\|u_n(t)\|_{X_0}^2 - \frac{1}{q}\|u_n(t)\|_q^q \\
= & \frac{1}{2}\|u_{nt}(t)\|_2^2 + \frac{q-2}{2q}\kappa\|u_n(t)\|_{X_0}^2 + \frac{1}{q}\left(\kappa\|u_n(t)\|_{X_0}^2 - \|u_n(t)\|_q^q\right) \\
\geq & \frac{1}{2}\|u_{nt}(t)\|_2^2 + \frac{q-2}{2q}\kappa\|u_n(t)\|_{X_0}^2,
\end{aligned} \tag{19}$$

which, together with (16), gives

$$\frac{1}{2}\|u_{nt}(t)\|_2^2 + \frac{q-2}{2q}\kappa\|u_n(t)\|_{X_0}^2 < d$$

for all $t \in [0, T]$. Thus,

$$\|u_{nt}(t)\|_2^2 < 2d \tag{20}$$

and

$$\|u_n(t)\|_{X_0}^2 < \frac{2q}{(q-2)\kappa}d. \tag{21}$$

Further, it can be derived from (21) that

$$\||u_n(t)|^{q-2}u_n(t)\|_{\frac{q}{q-1}} = \|u_n(t)\|_q^{q-1} \leq C\|u_n(t)\|_{X_0}^{q-1} < C\left(\frac{2q}{(q-2)\kappa}d\right)^{\frac{q-1}{2}} \tag{22}$$

and

$$\||u_n(t)|^{r-2}u_n(t)\|_{\frac{r}{r-1}} = \|u_n(t)\|_r^{r-1} \leq C\|u_n(t)\|_{X_0}^{r-1} < C\left(\frac{2q}{(q-2)\kappa}d\right)^{\frac{r-1}{2}} \tag{23}$$

for all $t \in [0, T]$. Estimates (20)–(23) imply that

$$\{u_n\} \text{ is bounded in } L^\infty(0, T; X_0), \tag{24}$$

$$\{u_{nt}\} \text{ is bounded in } L^\infty(0, T; L^2(\Omega)), \tag{25}$$

$$\{|u_n|^{q-2}u_n\} \text{ is bounded in } L^\infty(0, T; L^{\frac{q}{q-1}}(\Omega)), \tag{26}$$

$$\{|u_n|^{r-2}u_n\} \text{ is bounded in } L^\infty(0, T; L^{\frac{r}{r-1}}(\Omega)). \tag{27}$$

Step III. Passage to the limit. By virtue of (24)–(27), there is a subsequence of $\{u_n\}$ (still represented by $\{u_n\}$) and a function u such that as $n \to \infty$,

$$u_n \rightharpoonup u \text{ weakly star in } L^\infty(0, T; X_0), \tag{28}$$

$$u_{nt} \rightharpoonup u_t \text{ weakly star in } L^\infty(0, T; L^2(\Omega)), \tag{29}$$

$$|u_n|^{q-2}u_n \rightharpoonup |u|^{q-2}u \text{ weakly star in } L^\infty(0, T; L^{\frac{q}{q-1}}(\Omega)),$$

$$|u_n|^{r-2}u_n \rightharpoonup |u|^{r-2}u \text{ weakly star in } L^\infty(0, T; L^{\frac{r}{r-1}}(\Omega)).$$

Integrating (9) with respect to t, we obtain

$$\begin{aligned}
&(u_{nt}(t), w) + \int_0^t (a + b\|u_n(\tau)\|_{X_0}^{2p-2})(u_n(\tau), w)_{X_0}\,\mathrm{d}\tau \\
&- \int_0^t \int_0^s g(s-\tau)(u_n(\tau), w)_{X_0}\,\mathrm{d}\tau\mathrm{d}s + (u_n(t), w) \\
&= (u_{nt}(0), w) + (u_n(0), w) + \int_0^t (|u_n(\tau)|^{q-2}u_n(\tau), w)\,\mathrm{d}\tau \\
&- \int_0^t (|u_n(\tau)|^{r-2}u_n(\tau), w)\,\mathrm{d}\tau.
\end{aligned}$$

Taking $n \to \infty$, we obtain

$$\begin{aligned}
&(u_t(t), w) + \int_0^t (a + b\|u(\tau)\|_{X_0}^{2p-2})(u(\tau), w)_{X_0}\,\mathrm{d}\tau - \int_0^t \int_0^s g(s-\tau)(u(\tau), w)_{X_0}\,\mathrm{d}\tau\mathrm{d}s \\
&+ (u(t), w) = (u_1, w) + (u_0, w) + \int_0^t (|u(\tau)|^{q-2}u(\tau), w)\,\mathrm{d}\tau - \int_0^t (|u(\tau)|^{r-2}u(\tau), w)\,\mathrm{d}\tau.
\end{aligned}$$

From (10) and (11), we obtain $u(0) = u_0$ in X_0 and $u_t(0) = u_1$ in $L^2(\Omega)$. Hence, u is a global solution of the problem (1)–(3). Furthermore, from (28), we obtain

$$\|u(t)\|_{X_0} \le \liminf_{n\to\infty} \|u_n(t)\|_{X_0},$$

which, together with (21), gives

$$\|u(t)\|_{X_0} \le \left(\frac{2q}{(q-2)\kappa}d\right)^{\frac{1}{2}},$$

that is, $u(t) \in \overline{W}$ for all $t \in (0, \infty)$. □

4. Asymptotic Behavior of Global Solutions

Theorem 2. *In addition to all the assumptions of Theorem 1, assume that there exists a constant $\rho > 0$ such that $g'(t) \le -\rho g(t)$ for all $t \in [0, \infty)$. Then,*

$$\|u(t)\|_{X_0}^2 + \|u_t(t)\|_2^2 \le \alpha e^{-\beta t}, \quad \forall t \in [0, \infty), \tag{30}$$

for some constants $\alpha, \beta > 0$.

Proof. For the approximate solutions given in the proof of Theorem 1, we construct

$$L(t) = E_n(t) + \varepsilon\Psi(t), \quad \forall t \in [0, \infty), \tag{31}$$

where $\Psi(t) = (u_n(t), u_{nt}(t))$, and $\varepsilon > 0$ is a constant to be determined later.

Step I. We now claim that there exist two constants $\gamma_1, \gamma_2 > 0$, depending on ε, such that

$$\gamma_1 E_n(t) \le L(t) \le \gamma_2 E_n(t), \quad \forall t \in [0, \infty). \tag{32}$$

In fact, according to Cauchy's inequality, we obtain

$$|\Psi(t)| \leq \frac{1}{2}\|u_n(t)\|_2^2 + \frac{1}{2}\|u_{nt}(t)\|_2^2,$$

and so,

$$|\Psi(t)| \leq \frac{\mathfrak{C}_2^2}{2}\|u_n(t)\|_{X_0}^2 + \frac{1}{2}\|u_{nt}(t)\|_2^2, \tag{33}$$

where $\mathfrak{C}_2$ is the best Sobolev constant for the embedding $X_0 \hookrightarrow L^2(\Omega)$. Combining (33) and (19), we obtain $|\Psi(t)| \leq ME_n(t)$ for some constant $M > 0$ independent of n, which, together with (31), yields that assertion (32) holds.

Step II. We next claim that

$$L'(t) \leq -\varepsilon\eta E_n(t) \tag{34}$$

for sufficiently small η and ε. Indeed, since

$$E_n'(t) = \frac{1}{2}(g' \circ u_n)(t) - \frac{1}{2}g(t)\|u_n(t)\|_{X_0}^2 - \|u_{nt}(t)\|_2^2$$

and

$$\begin{aligned}\Psi'(t) =& \|u_{nt}(t)\|_2^2 - b\|u_n(t)\|_{X_0}^{2p} - a\|u_n(t)\|_{X_0}^2 + \int_0^t g(t-\tau)(u_n(\tau), u_n(t))_{X_0}\,\mathrm{d}\tau \\ &- (u_n(t), u_{nt}(t)) + \|u_n(t)\|_q^q - \|u_n(t)\|_r^r,\end{aligned}$$

it follows from (31) that

$$\begin{aligned}L'(t) =& \frac{1}{2}(g' \circ u_n)(t) - \frac{1}{2}g(t)\|u_n(t)\|_{X_0}^2 - \|u_{nt}(t)\|_2^2 + \varepsilon\|u_{nt}(t)\|_2^2 \\ &- \varepsilon b\|u_n(t)\|_{X_0}^{2p} - \varepsilon a\|u_n(t)\|_{X_0}^2 + \varepsilon\int_0^t g(t-\tau)(u_n(\tau), u_n(t))_{X_0}\,\mathrm{d}\tau \\ &- \varepsilon(u_n(t), u_{nt}(t)) + \varepsilon\|u_n(t)\|_q^q - \varepsilon\|u_n(t)\|_r^r.\end{aligned} \tag{35}$$

For the seventh term on the right hand side of (35), we can derive from Schwarz's inequality and Cauchy's inequality with $\epsilon_1 > 0$ that

$$\begin{aligned}&\int_0^t g(t-\tau)(u_n(\tau), u_n(t))_{X_0}\,\mathrm{d}\tau \\ =& \int_0^t g(t-\tau)\|u_n(t)\|_{X_0}^2\,\mathrm{d}\tau + \int_0^t g(t-\tau)(u_n(\tau) - u_n(t), u_n(t))_{X_0}\,\mathrm{d}\tau \\ \leq& \int_0^t g(\tau)\,\mathrm{d}\tau\|u_n(t)\|_{X_0}^2 + \epsilon_1\int_0^t g(\tau)\,\mathrm{d}\tau\|u_n(t)\|_{X_0}^2 + \frac{1}{4\epsilon_1}(g \circ u_n)(t) \\ \leq& (a-\kappa)\|u_n(t)\|_{X_0}^2 + \epsilon_1(a-\kappa)\|u_n(t)\|_{X_0}^2 + \frac{1}{4\epsilon_1}(g \circ u_n)(t).\end{aligned} \tag{36}$$

For the eighth term on the right-hand side of (35), we deduce from Cauchy's inequality with $\epsilon_2 > 0$ that

$$\begin{aligned}-(u_n(t), u_{nt}(t)) \leq& \epsilon_2\|u_n(t)\|_2^2 + \frac{1}{4\epsilon_2}\|u_{nt}(t)\|_2^2 \\ \leq& \epsilon_2\mathfrak{C}_2^2\|u_n(t)\|_{X_0}^2 + \frac{1}{4\epsilon_2}\|u_{nt}(t)\|_2^2.\end{aligned} \tag{37}$$

Plugging (36) and (37) into (35), we obtain

$$\begin{aligned} L'(t) \leq & \left(\varepsilon + \frac{\varepsilon}{4\epsilon_2} - 1\right)\|u_{nt}(t)\|_2^2 - \varepsilon b\|u_n(t)\|_{X_0}^{2p} \\ & + \varepsilon(\epsilon_1(a-\kappa) + \epsilon_2\mathfrak{C}_2^2 - \kappa)\|u_n(t)\|_{X_0}^2 + \left(\frac{\varepsilon}{4\epsilon_1} - \frac{\rho}{2}\right)(g \circ u_n)(t) \\ & + \varepsilon\|u_n(t)\|_q^q - \varepsilon\|u_n(t)\|_r^r, \end{aligned}$$

and so,

$$\begin{aligned} L'(t) \leq & -\varepsilon\eta E_n(t) + \left(\varepsilon + \frac{\varepsilon}{4\epsilon_2} + \frac{\varepsilon\eta}{2} - 1\right)\|u_{nt}(t)\|_2^2 + \varepsilon b\left(\frac{\eta}{2p} - 1\right)\|u_n(t)\|_{X_0}^{2p} \\ & + \varepsilon\left(\epsilon_1(a-\kappa) + \epsilon_2\mathfrak{C}_2^2 + \frac{\eta a}{2} - \kappa\right)\|u_n(t)\|_{X_0}^2 \\ & + \left(\frac{\varepsilon}{4\epsilon_1} + \frac{\varepsilon\eta}{2} - \frac{\rho}{2}\right)(g \circ u_n)(t) + \varepsilon\|u_n(t)\|_q^q - \varepsilon\|u_n(t)\|_r^r \\ & - \frac{\varepsilon\eta}{q}\|u_n(t)\|_q^q + \frac{\varepsilon\eta}{r}\|u_n(t)\|_r^r, \end{aligned} \tag{38}$$

where $\eta > 0$ is a constant to be determined later. We conclude from (16) and (19) that

$$E_n(0) \geq \frac{q-2}{2q}\kappa\|u_n(t)\|_{X_0}^2.$$

This leads to

$$\|u_n(t)\|_{X_0} \leq \left(\frac{2q}{(q-2)\kappa}E_n(0)\right)^{\frac{1}{2}}.$$

Hence,

$$\|u_n(t)\|_q^q \leq \mathfrak{C}_1^q\|u_n(t)\|_{X_0}^{q-2}\|u_n(t)\|_{X_0}^2 \leq \mathfrak{C}_1^q\left(\frac{2q}{(q-2)\kappa}E_n(0)\right)^{\frac{q-2}{2}}\|u_n(t)\|_{X_0}^2.$$

The substitution of this inequality into (38) yields

$$\begin{aligned} L'(t) \leq & -\varepsilon\eta E_n(t) + \left(\varepsilon + \frac{\varepsilon}{4\epsilon_2} + \frac{\varepsilon\eta}{2} - 1\right)\|u_{nt}(t)\|_2^2 + \varepsilon b\left(\frac{\eta}{2p} - 1\right)\|u_n(t)\|_{X_0}^{2p} \\ & + \varepsilon\left(\epsilon_1(a-\kappa) + \epsilon_2\mathfrak{C}_2^2 + \frac{\eta a}{2} + \mathfrak{C}_1^q\left(\frac{2q}{(q-2)\kappa}E_n(0)\right)^{\frac{q-2}{2}} - \kappa\right)\|u_n(t)\|_{X_0}^2 \\ & + \left(\frac{\varepsilon}{4\epsilon_1} + \frac{\varepsilon\eta}{2} - \frac{\rho}{2}\right)(g \circ u_n)(t) - \varepsilon\|u_n(t)\|_r^r + \frac{\varepsilon\eta}{r}\|u_n(t)\|_r^r. \end{aligned} \tag{39}$$

Since

$$\mathfrak{C}_1^q\left(\frac{2q}{(q-2)\kappa}E_n(0)\right)^{\frac{q-2}{2}} < \mathfrak{C}_1^q\left(\frac{2q}{(q-2)\kappa}d\right)^{\frac{q-2}{2}} = \kappa,$$

the values of ϵ_1, ϵ_2, and η can be selected to be sufficiently small, such that $\eta < \min\{2p, r\}$ and

$$\epsilon_1(a-\kappa) + \epsilon_2\mathfrak{C}_2^2 + \frac{\eta a}{2} + \mathfrak{C}_1^q\left(\frac{2q}{(q-2)\kappa}E_n(0)\right)^{\frac{q-2}{2}} - \kappa < 0.$$

Consequently, it follows from (39) that

$$L'(t) \le -\varepsilon\eta E_n(t) + \left(\varepsilon + \frac{\varepsilon}{4\epsilon_2} + \frac{\varepsilon\eta}{2} - 1\right)\|u_{nt}(t)\|_2^2 + \left(\frac{\varepsilon}{4\epsilon_1} + \frac{\varepsilon\eta}{2} - \frac{\rho}{2}\right)(g \circ u_n)(t).$$

Thus, for fixed ϵ_1, ϵ_2, and η, we can choose

$$\varepsilon < \min\left\{\frac{1}{M}, \frac{4\epsilon_2}{4\epsilon_2 + 1 + 2\eta\epsilon_2}, \frac{2\rho\epsilon_1}{1 + 2\eta\epsilon_1}\right\}$$

such that assertion (34) holds.

Step III. We prove (30). To accomplish this, by combining (34) with the second inequality stated in assertion (32), we obtain

$$L'(t) \le -\frac{\varepsilon\eta}{\gamma_2} L(t).$$

Hence,

$$L(t) \le C e^{-\frac{\varepsilon\eta}{\gamma_2}t}, \quad \forall t \in [0, \infty).$$

The first inequality in assertion (32) allows us to further deduce that

$$E_n(t) \le \frac{C}{\gamma_1} e^{-\frac{\varepsilon\eta}{\gamma_2}t}, \quad \forall t \in [0, \infty). \tag{40}$$

It can be inferred from (28) and (29) that

$$\|u(t)\|_{X_0}^2 + \|u_t(t)\|_2^2 \le \liminf_{n\to\infty}\left(\|u_n(t)\|_{X_0}^2 + \|u_{nt}(t)\|_2^2\right),$$

which, together with (19) and (40), gives

$$\liminf_{n\to\infty}\left(\|u_n(t)\|_{X_0}^2 + \|u_{nt}(t)\|_2^2\right) \le \liminf_{n\to\infty} C E_n(t) \le \frac{C}{\gamma_1} e^{-\frac{\varepsilon\eta}{\gamma_2}t},$$

and so, (30) is derived, where $\alpha = C/\gamma_1$ and $\beta = \varepsilon\eta/\gamma_2$. Theorem 2 is proved. □

Theorem 2 provides a sufficient condition for the asymptotic behavior of solutions of the problem (1)–(3). For example, we take $N = 1$, $m = 1/2$, $a = 1$, $b = 0$, $q = 4$, $r = 3$, and $g(t) = g(0)e^{-t}$ with $g(0) < 1$. Then, it is easy to verify that (A_1), (A_2), and $g'(t) \le -\rho g(t)$ hold, where $\kappa = 1 - g(0)$ and $\rho = 1$. By virtue of (7) and (5), we obtain

$$d = \frac{1}{4}(1 - g(0))^2 \mathfrak{C}_1^{-4}$$

and

$$\mathcal{W} = \left\{u \in X_0 \,\middle|\, \|u\|_{X_0} < (1 - g(0))^{\frac{1}{2}} \mathfrak{C}_1^{-2}\right\}.$$

Therefore, if u_0 and u_1 satisfy

$$\|u_0\|_{X_0} < (1 - g(0))^{\frac{1}{2}} \mathfrak{C}_1^{-2}$$

and

$$\frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|u_0\|_{X_0}^2 - \frac{1}{4}\|u_0\|_4^4 + \frac{1}{3}\|u_0\|_3^3 < \frac{1}{4}(1 - g(0))^2 \mathfrak{C}_1^{-4},$$

then, we conclude from Theorem 2 that the norm of global solutions of the problem (1)–(3) in the phase space $X_0 \times L^2(\Omega)$ decays exponentially to zero when the time tends to infinity.

5. Conclusions

The initial boundary value problem for a class of fractional viscoelastic Kirchhoff equations involving two nonlinear source terms of different signs is studied by this paper. In the case where the two opposing external forces $|u|^{q-2}u$ and $-|u|^{r-2}u$ appear simultaneously, the existence and asymptotic behavior of global solutions are derived, namely, Theorems 1 and 2. More specifically, Theorem 1 shows that if the initial data u_0 lie in the potential well, and the initial energy is less than the depth of the potential well, then the problem (1)–(3) has a global solution. Theorem 2 shows us that if the memory kernel decays exponentially, then the global solutions of the problem (1)–(3) also do this.

Our main technical tool is the theory of potential wells, which is different from the classical. We describe the potential well by a sphere, whose radius is expressed by d. Although the depth of the potential well d is smaller than the classical, the spatial structure of the potential well is clearer so that it is not necessary to introduce the Nehari functional and the Nehari manifold.

In the future, we will focus on the study of the qualitative properties of solutions of fractional viscoelastic Kirchhoff equations involving more general Kirchhoff functions and nonlinear source terms.

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References

1. Fiscella, A.; Valdinoci, E. A critical Kirchhoff type problem involving a nonlocal operator. *Nonlinear Anal.* **2014**, *94*, 156–170. [CrossRef]
2. Torrejón, R.; Yong, J.M. On a quasilinear wave equation with memory. *Nonlinear Anal.* **1991**, *16*, 61–78. [CrossRef]
3. Wu, S.T.; Tsai, L.Y. Blow-up of positive-initial-energy solutions for an integro-differential equation with nonlinear damping. *Taiwan. J. Math.* **2010**, *14*, 2043–2058. [CrossRef]
4. Liu, G.W.; Hou, C.S.; Guo, X.L. Global nonexistence for nonlinear Kirchhoff systems with memory term. *Z. Angew. Math. Phys.* **2014**, *65*, 925–940. [CrossRef]
5. Caffarelli, L. Non-local diffusions, drifts and games. In Proceedings of the Nonlinear Partial Differential Equations The Abel Symposium 2010, Oslo, Norway, 28 September–1 October 2010; Springer: Berlin/Heidelberg, Germany, 2012; pp. 37–52. [CrossRef]
6. Applebaum, D. Lévy processes-from probability to finance and quantum groups. *Not. Amer. Math. Soc.* **2004**, *51*, 1336–1347.
7. Laskin, N. Fractional quantum mechanics and Lévy path integrals. *Phys. Lett. A* **2000**, *268*, 298–305. [CrossRef]
8. Valdinoci, E. From the long jump random walk to the fractional Laplacian. *arXiv* **2009**, arXiv:0901.3261.
9. Vázquez, J.L. Recent progress in the theory of nonlinear diffusion with fractional laplacian operators. *Discret. Contin. Dyn. Syst.* **2014**, *7*, 857–885. [CrossRef]
10. Nguyen, H.T.; Tuan, N.A.; Yang, C. Global well-posedness for fractional Sobolev-Galpern type equations. *Discret. Contin. Dyn. Syst.* **2022**, *42*, 2637–2665. [CrossRef]
11. Nguyen, A.T.; Tuan, N.H.; Yang, C. On Cauchy problem for fractional parabolic-elliptic Keller-Segel model. *Adv. Nonlinear Anal.* **2023**, *12*, 97–116. [CrossRef]
12. Ambrosio, V.; Isernia, T. A multiplicity result for a fractional Kirchhoff equation in $\mathbb{R}^N$ with a general nonlinearity. *Commun. Contemp. Math.* **2018**, *20*, 1750054. [CrossRef]
13. Nyamoradi, N.; Ambrosio, V. Existence and non-existence results for fractional Kirchhoff Laplacian problems. *Anal. Math. Phys.* **2021**, *11*, 125. [CrossRef]
14. Fiscella, A.; Mishra, P.K. Fractional Kirchhoff Hardy problems with singular and critical Sobolev nonlinearities. *Manuscr. Math.* **2022**, *168*, 257–301. [CrossRef]

15. do Ó, J.M.; He, X.M.; Mishra, P.K. Fractional Kirchhoff problem with critical indefinite nonlinearity. *Math. Nachr*. **2019**, *292*, 615–632. [CrossRef]
16. Zhang, B.L.; Rădulescu, V.D.; Wang, L. Existence results for Kirchhoff-type superlinear problems involving the fractional Laplacian. *Proc. R. Soc. Edinb. Sect. A* **2019**, *149*, 1061–1081. [CrossRef]
17. Bisci, G.M.; Vilasi, L. On a fractional degenerate Kirchhoff-type problem. *Commun. Contemp. Math.* **2017**, *19*, 1550088. [CrossRef]
18. Xiang, M.Q.; Rădulescu, V.D.; Zhang, B.L. Nonlocal Kirchhoff diffusion problems: Local existence and blow-up of solutions. *Nonlinearity* **2018**, *31*, 3228–3250. [CrossRef]
19. Lin, Q.; Tian, X.T.; Xu, R.Z.; Zhang, M.N. Blow up and blow up time for degenerate Kirchhoff- type wave problems involving the fractional Laplacian with arbitrary positive initial energy. *Discret. Contin. Dyn. Syst. Ser. S* **2020**, *13*, 2095–2107. [CrossRef]
20. Pan, N.; Pucci, P.; Xu, R.Z.; Zhang, B.L. Degenerate Kirchhoff-type wave problems involving the fractional Laplacian with nonlinear damping and source terms. *J. Evol. Equ.* **2019**, *19*, 615–643. [CrossRef]
21. Xiang, M.Q.; Hu, D. Existence and blow-up of solutions for fractional wave equations of Kirchhoff type with viscoelasticity. *Discret. Contin. Dyn. Syst. Ser.* **2021**, *14*, 4609–4629. [CrossRef]
22. Cavalcanti, M.M.; Cavalcanti, V.N.D.; Ma, T.F. Exponential decay of the viscoelastic Euler–Bernoulli equation with a nonlocal dissipation in general domains. *Differ. Integral Equ.* **2004**, *17*, 495–510. [CrossRef]
23. Xu, R.Z.; Zhang, M.Y.; Chen, S.H.; Yang, Y.B.; Shen, J.H. The initial-boundary value problems for a class of six order nonlinear wave equation. *Discret. Contin. Dyn. Syst.* **2017**, *37*, 5631–5649. [CrossRef]
24. Cavalcanti, M.M.; Cavalcanti, V.N.D. Existence and asymptotic stability for evolution problems on manifolds with damping and source terms. *J. Math. Anal. Appl* **2004**, *291*, 109–127. [CrossRef]
25. Gazzola, F.; Squassina, M. Global solutions and finite time blow up for damped semilinear wave equations. *Ann. Inst. Henri Poincare Anal. Non Lineaire* **2006**, *3*, 185–207. [CrossRef]
26. Liu, Y.; Mu, J.; Jiao, Y.J. A class of fourth order damped wave equations with arbitrary positive initial energy. *Proc. Edinb. Math. Soc.* **2019**, *62*, 165–178. [CrossRef]
27. Xu, R.Z.; Su, J. Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations. *J. Funct. Anal.* **2013**, *264*, 2732–2763. [CrossRef]
28. Luo, Y.B.; Xu, R.Z.; Yang, C. Global well-posedness for a class of semilinear hyperbolic equations with singular potentials on manifolds with conical singularities. *Calc. Var. Partial. Differ. Equ.* **2022**, *61*, 210. [CrossRef]
29. Liu, Y.; Li, W.K. A class of fourth-order nonlinear parabolic equations modeling the epitaxial growth of thin films. *Discret. Contin. Dyn. Syst. Ser. S* **2021**, *14*, 4367–4381. [CrossRef]
30. Liu, Y.; Yu, T.; Li, W.K. Global well-posedness, asymptotic behavior and blow-up of solutions for a class of degenerate parabolic equations. *Nonlinear Anal.* **2020**, *196*, 111759. [CrossRef]
31. Haraux, A.; Zuazua, E. Decay estimates for some semilinear damped hyperbolic problems. *Arch. Ration. Mech. Anal.* **1988**, *100*, 191–206. [CrossRef]
32. Liu, Y. Global attractors for a nonlinear plate equation modeling the oscillations of suspension bridges. *Commun. Anal. Mech.* **2023**, *15*, 436–456. [CrossRef]
33. Liu, Y.; Moon, B.; Rădulescu, V.D.; Xu, R.Z.; Yang, C. Qualitative properties of solution to a viscoelastic Kirchhoff-like plate equation. *J. Math. Phys.* **2023**, *64*, 051511. [CrossRef]
34. Servadei, R.; Valdinoci, E. Variational methods for non-local operators of elliptic type. *Discret. Contin. Dyn. Syst.* **2013**, *33*, 2105–2137. [CrossRef]
35. Servadei, R.; Valdinoci, E. The Brezis-Nirenberg result for the fractional Laplacian. *Trans. Amer. Math. Soc.* **2015**, *367*, 67–102. [CrossRef]
36. Liu, Y.; Lv, P.J.; Da, C.J. Blow-up of a nonlocal p-Laplacian evolution equation with critical initial energy. *Ann. Polon. Math.* **2016**, *117*, 89–99. [CrossRef]

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