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Solvability Criterion for a System Arising from Monge–Ampère Equations with Two Parameters

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Abstract: Monge–Ampère equations have important research significance in many fields such as geometry, convex geometry and mathematical physics. In this paper, under some superlinear and sublinear conditions, the existence of nontrivial solutions for a system arising from Monge–Ampère equations with two parameters is investigated based on the Guo–Krasnosel'skii fixed point theorem. In the end, two examples are given to illustrate our theoretical results.

Keywords: fixed point theorem; Monge–Ampère equations; boundary value problem

MSC: 35J60; 34B15; 47H10

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1. Introduction

In this paper, we concentrate on the existence of nontrivial solutions for the boundary value problem:

$$\begin{cases} ((u'(s))^N)' = \lambda N r^{N-1} f(-u(s), -v(s)), \ \ 0 < s < 1, \\ ((v'(s))^N)' = \mu N r^{N-1} g(-u(s), -v(s)), \ \ 0 < s < 1, \\ u'(0) = u(1) = 0, \ v'(0) = v(1) = 0, \end{cases} \tag{1}$$

where $N \geq 1$, $f, g : [0,1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ are continuous, λ and μ are two positive parameters. Problem (1) emerges when considering the existence of nontrivial solutions for the following Dirichlet problem related to Monge–Ampère equations:

$$\begin{cases} \det(D^2 u) = \lambda f(-u, -v) \text{ in } B, \\ \det(D^2 v) = \mu g(-u, -v) \text{ in } B, \\ u = v = 0 \ \text{ on } \ \partial B, \end{cases}$$

where $D^2 u = (\frac{\partial^2 u}{\partial x_i \partial x_j})$ is the Hessian matrix of u, $D^2 v = (\frac{\partial^2 v}{\partial x_i \partial x_j})$ is the Hessian matrix of v, $B = \{x \in R^N : |x| < 1\}$.

Monge–Ampère equations play a crucial role in the exploration of mathematical physics, engineering, biological sciences and other hot application disciplines (see [1]). As is known, Figalli was awarded the Fields Medal in 2018 for his contribution to the Monge–Ampère equation, e.g., see [2]. Caffarelli received the Abel Prize in 2023 for his pioneering contributions to the understanding of the regularity theory of nonlinear partial differential equations, including the Monge–Ampère equation, e.g., see [3]. On the basis of their research, an increasing number of researchers have conducted some investigations associated with Monge–Ampère equations. For example, Mohammed and Mooney studied the singular problems of the Monge–Ampère equation, see [4,5]; Son, Wang, Aranda and Godoy substituted the p-Laplacian operator for the Monge–Ampère operator, thus offering a new conclusion to the corresponding singular problem, which can be found in [6,7]. Recently, Feng [8] considered the singular problems of p-Monge–Ampère equations. In addition, some scholars have studied the existence of nontrivial radial convex solutions for a single Monge–Ampère equation or systems of such equations, utilizing the theory of topological degree, bifurcation techniques, the upper and lower solutions method, and so on. For further details, see [2–5,8–25] and the references therein.

For example, in [18], Ma and Gao investigated the following boundary value problem:

$$\begin{cases} ((u_1'(t))^n)' = \lambda n t^{n-1} f(-u(t)),\ 0 < t < 1, \\ u'(0) = u(1) = 0. \end{cases} \tag{2}$$

Boundary value Problem (2) arose from the following Monge–Ampère equation:

$$\begin{cases} \det(D^2 u) = \lambda f(-u) \text{ in } B, \\ u = 0 \ \text{ on } \ \partial B, \end{cases} \tag{3}$$

where $D^2 u = (\frac{\partial^2 u}{\partial x_i \partial x_j})$ is the Hessian matrix of u, $B = \{x \in R^n : |x| < 1\}$. The global bifurcation technique was applied to ascertain the optimal intervals of parameter λ, thereby further guaranteeing the existence of single or multiple solutions to Problem (2).

In [21], Wang established two solvability criteria for a weakly coupled system:

$$\begin{cases} ((u_1'(t))^N)' = N t^{N-1} f(-u_2(t)),\ 0 < t < 1, \\ ((u_2'(t))^N)' = N t^{N-1} g(-u_1(t)),\ 0 < t < 1, \\ u_1'(0) = u_2'(0) = 0,\ \ u_1(1) = u_2(1) = 0, \end{cases} \tag{4}$$

where $N \geq 1$. System (4) arose from the following Monge–Ampère equations:

$$\begin{cases} \det(D^2 u_1) = f(-u_2) \text{ in } B, \\ \det(D^2 u_2) = g(-u_1) \text{ in } B, \\ u_1 = u_2 = 0 \ \text{ on } \ \partial B, \end{cases}$$

where $B = \{x \in R^N : |x| < 1\}$, and $D^2 u_i$ is the determinant of the Hessian matrix $(\frac{\partial^2 u_i}{\partial x_m \partial x_n})$ of u_i. The existence of convex radial solutions for Problem (4) was established in both the superlinear and sublinear instances, utilizing fixed point theorems within a cone.

In [20], Wang and An discussed the following system of Monge–Ampère equations:

$$\begin{cases} \det(D^2 u_1) = f_1(-u_1, \cdots, -u_n) \text{ in } B, \\ \cdots \\ \det(D^2 u_n) = f_n(-u_1, \cdots, -u_n) \text{ in } B, \\ u(x) = 0 \ \text{ on } \ \partial B, \end{cases} \tag{5}$$

where $D^2 u_i = (\frac{\partial^2 u_i}{\partial x_i \partial x_j})$ is the Hessian matrix of u_i, $B = \{x \in R^N : |x| < 1\}$. Obviously, System (5) can readily be changed into the subsequent boundary value problem:

$$\begin{cases} ((u_1'(r))^N)' = N r^{N-1} f_1(-u_1, \cdots, -u_n),\ 0 < r < 1, \\ \cdots \\ ((u_n'(r))^N)' = N r^{N-1} f_n(-u_1, \cdots, -u_n),\ 0 < r < 1, \\ u_i'(0) = u_i(1) = 0,\ \ i = 1, \cdots, n, \end{cases}$$

where $N \geq 1$. The existence of triple nontrivial radial convex solutions was obtained through the application of the Leggett–Williams fixed point theorem.

In [22], the author studied the following system:

$$\begin{cases} ((u_1'(r))^N)' = \lambda N r^{N-1} f_1(-u_1, \cdots, -u_n),\ 0 < r < 1, \\ \cdots \\ ((u_n'(r))^N)' = \lambda N r^{N-1} f_n(-u_1, \cdots, -u_n),\ 0 < r < 1, \\ u_i'(0) = u_i(1) = 0,\ \ i = 1, \cdots, n, \end{cases} \tag{6}$$

where $N \geq 1$. System (6) arose from the following system:

$$\begin{cases} \det(D^2u_1) = \lambda f_1(-u_1, \cdots, -u_n) \text{ in } B, \\ \cdots \\ \det(D^2u_n) = \lambda f_n(-u_1, \cdots, -u_n) \text{ in } B, \\ u_i = 0 \text{ on } \partial B, i = 1, \cdots, n, \end{cases}$$

where $D^2u_i = (\frac{\partial^2 u_i}{\partial x_i \partial x_j})$ is the Hessian matrix of u_i, $B = \{x \in R^N : |x| < 1\}$.

Using fixed point theorems and considering sublinear and superlinear conditions, Wang explored the existence of two nontrivial radial solutions for System (6) with a carefully selected parameter.

In [14], Gao and Wang considered the following boundary value problem:

$$\begin{cases} ((u_1'(r))^N)' = \lambda_1 N r^{N-1} f_1(-u_1, -u_2, \cdots, -u_n), \\ ((u_2'(r))^N)' = \lambda_2 N r^{N-1} f_2(-u_1, -u_2, \cdots, -u_n), \\ \cdots \\ ((u_n'(r))^N)' = \lambda_n N r^{N-1} f_n(-u_1, -u_2, \cdots, -u_n), \\ u_i'(0) = u_i(1) = 0, \; i = 1, 2, \cdots, n, \; 0 < r < 1, \end{cases} \tag{7}$$

where $N \geq 1$. System (7) arose from the following system:

$$\begin{cases} \det(D^2u_1) = \lambda_1 f_1(-u_1, \cdots, -u_n) \text{ in } B, \\ \det(D^2u_2) = \lambda_2 f_2(-u_1, \cdots, -u_n) \text{ in } B, \\ \cdots \\ \det(D^2u_n) = \lambda_n f_n(-u_1, \cdots, -u_n) \text{ in } B, \\ u_i = 0 \text{ on } \partial B, i = 1, \cdots, n, \end{cases}$$

where $D^2u_i = (\frac{\partial^2 u_i}{\partial x_i \partial x_j})$ is the Hessian matrix of u_i, and $B = \{x \in R^N : |x| < 1\}$. By using the method of upper and lower solutions and the fixed point index theory, they established the existence, nonexistence, and multiplicity of convex solutions for Problem (7).

In [12], Feng continued to consider the uniqueness and existence of nontrivial radial convex solutions of System (3). And the author also studied the following system:

$$\begin{cases} \det(D^2u_1) = \lambda_1 f_1(-u_2) \text{ in } B, \\ \det(D^2u_2) = \lambda_2 f_2(-u_3) \text{ in } B, \\ \cdots \\ \det(D^2u_n) = \lambda_n f_n(-u_1) \text{ in } B, \\ u_1 = u_2 \cdots = u_n = 0 \text{ on } \partial B, \end{cases} \tag{8}$$

where $\lambda_i (i = 1, 2, \cdots, n)$ are positive parameters. The author derived novel existence results for nontrivial radial convex solutions of System (8) via employing the eigenvalue theory in a cone and defining composite operators.

In addition, in recent decades, some authors have investigated the existence of nontrivial solutions to other differential equations with parameters. For example, in [26], by employing the Guo–Krasnosel'skii fixed point theorem, Hao et al. considered the existence of positive solutions for a class of nonlinear fractional differential systems, specifically nonlocal boundary value problems with parameters and a p-Laplacian operator. In [27], Yang studied the existence of positive solutions for the Dirichlet boundary value problem of certain nonlinear differential systems using the upper and lower solution method and the fixed point index theory. In [28], Jiang and Zhai investigated a coupled system of nonlinear fourth-order equations based on the Guo–Krasnosel'skii fixed point theorem and Green's functions.

Inspired by literatures [12,14,20–22,26–28], we consider Problem (1). In this paper, under some different combinations of superlinearity and sublinearity of the nonlinear terms, we use the Guo–Krasnosel'skii fixed point theorem to investigate the existence results of System (1) and establish some existence results of nontrivial solutions based on various different values values of λ and μ. Here, we extend the study in literature [21], and the main results differ from those in literatures [12,14,21].

2. Preliminaries

In this section, we list some basic preliminaries to be used in Section 3. For further background knowledge of cone, we refer readers to papers [21,29] for more details.

Lemma 1 (see [29])**.** *Let E be a Banach space, and $P \subset E$ be a cone. Assume that Ω_1 and Ω_2 are bounded open sets in E, $\theta \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$; operator $A : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is completely continuous. If the following conditions are satisfied,*

$$\text{(i)} \|Ax\| \leq \|x\|, \forall x \in P \cap \partial\Omega_1, \|Ax\| \geq \|x\|, \forall x \in P \cap \partial\Omega_2, or$$
$$\text{(ii)} \|Ax\| \geq \|x\|, \forall x \in P \cap \partial\Omega_1, \|Ax\| \leq \|x\|, \forall x \in P \cap \partial\Omega_2,$$

then operator A has at least one fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

In order to solve System (1), we offer a simple transformation, $x(s) = -u(s), y(s) = -v(s)$, in System (1); then, System (1) can be changed to the following system:

$$\begin{cases} ((-x'(s))^N)' = \lambda N s^{N-1} f(x(s), y(s)), & 0 < s < 1, \\ ((-y'(s))^N)' = \mu N s^{N-1} g(x(s), y(s)), & 0 < s < 1, \\ x'(0) = x(1) = 0, \ y'(0) = y(1) = 0. \end{cases} \tag{9}$$

In the following, we treat the existence of positive solutions of System (9).

We let $E = C[0,1] \times C[0,1]$ with norm $\|(x,y)\|_E = \|x\| + \|y\|$, where $\|x\| = \max\limits_{s\in[0,1]} |x(s)|$ and $\|y\| = \max\limits_{s\in[0,1]} |y(s)|$.

We define

$$P = \{(x,y) \in E : x(s) \geq 0, y(s) \geq 0, \forall s \in [0,1], \min_{s\in[\frac{1}{4},\frac{3}{4}]} (x(s) + y(s)) \geq \frac{1}{4}\|(x,y)\|_E\}.$$

Then, P is a cone of E.

According to literature [21], now, we denote operators A_1, A_2 and A by

$$A_1(x,y)(s) = \int_s^1 (\int_0^u \lambda N \tau^{N-1} f(x(\tau), y(\tau)) d\tau)^{\frac{1}{N}} du, \ \ s \in [0,1],$$

$$A_2(x,y)(s) = \int_s^1 (\int_0^u \mu N \tau^{N-1} g(x(\tau), y(\tau)) d\tau)^{\frac{1}{N}} du, \ \ s \in [0,1].$$

and $A(x,y) = (A_1(x,y), A_2(x,y)), (x,y) \in E$. Thus, it is easy to see that the fixed points of operator A correspond to solutions of System (9).

Similar to the proof of Lemma 2.3 in literature [21], we can easily obtain the lemma as follows.

Lemma 2. *$A : P \to P$ is completely continuous.*

3. Main Results

We denote

$$f_0 = \limsup_{x+y\to 0^+} \frac{f(x,y)}{(x+y)^N}, \quad g_0 = \limsup_{x+y\to 0^+} \frac{g(x,y)}{(x+y)^N},$$

$$f_\infty = \liminf_{x+y\to\infty} \frac{f(x,y)}{(x+y)^N}, \quad g_\infty = \liminf_{x+y\to\infty} \frac{g(x,y)}{(x+y)^N},$$

$$\widehat{f}_0 = \liminf_{x+y\to 0^+} \frac{f(x,y)}{(x+y)^N}, \quad \widehat{g}_0 = \liminf_{x+y\to 0^+} \frac{g(x,y)}{(x+y)^N},$$

$$\widehat{f}_\infty = \limsup_{x+y\to\infty} \frac{f(x,y)}{(x+y)^N}, \quad \widehat{g}_\infty = \limsup_{x+y\to\infty} \frac{g(x,y)}{(x+y)^N}.$$

$$F = \int_0^1 (\int_0^u N\tau^{N-1}d\tau)^{\frac{1}{N}} du, \quad G = \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^u N\tau^{N-1}d\tau)^{\frac{1}{N}} du.$$

For $f_0, g_0, f_\infty, g_\infty \in (0,\infty)$, we define the symbols below:

$$M_1 = \frac{2^N}{G^N f_\infty}, \quad M_2 = \frac{1}{2^N F^N f_0},$$

$$M_3 = \frac{2^N}{G^N g_\infty}, \quad M_4 = \frac{1}{2^N F^N g_0}.$$

Theorem 1. *(1) Assume that $f_0, g_0, f_\infty, g_\infty \in (0,\infty), M_1 < M_2, M_3 < M_4$; then, for $\lambda \in (M_1, M_2)$ and $\mu \in (M_3, M_4)$, System (9) has at least one positive solution.*

(2) Assume that $f_0 = 0, g_0, f_\infty, g_\infty \in (0,\infty), M_3 < M_4$; then, for $\lambda \in (M_1,\infty)$ and $\mu \in (M_3, M_4)$, System (9) has at least one positive solution.

(3) Assume that $f_0, f_\infty, g_\infty \in (0,\infty), g_0 = 0, M_1 < M_2$; then, for $\lambda \in (M_1, M_2)$ and $\mu \in (M_3,\infty)$, System (9) has at least one positive solution.

(4) Assume that $f_0 = g_0 = 0, f_\infty, g_\infty \in (0,\infty)$; then, for $\lambda \in (M_1,\infty)$ and $\mu \in (M_3,\infty)$, System (9) has at least one positive solution.

(5) Assume that $f_0, g_0 \in (0,\infty), f_\infty = \infty$ or $f_0, g_0 \in (0,\infty), g_\infty = \infty$; then, for $\lambda \in (0, M_2)$ and $\mu \in (0, M_4)$, System (9) has at least one positive solution.

(6) Assume that $f_0 = 0, g_0 \in (0,\infty), g_\infty = \infty$ or $f_0 = 0, g_0 \in (0,\infty), f_\infty = \infty$; then, for $\lambda \in (0,\infty)$ and $\mu \in (0, M_4)$, System (9) has at least one positive solution.

(7) Assume that $f_0 \in (0,\infty), g_0 = 0, g_\infty = \infty$ or $f_0 \in (0,\infty), g_0 = 0, f_\infty = \infty$; then, for $\lambda \in (0, M_2)$ and $\mu \in (0,\infty)$, System (9) has at least one positive solution.

(8) Assume that $f_0 = g_0 = 0, g_\infty = \infty$ or $f_0 = g_0 = 0, f_\infty = \infty$; then, for $\lambda \in (0,\infty)$ and $\mu \in (0,\infty)$, System (9) has at least one positive solution.

Proof. Due to the similarity in the proofs of the above cases, we demonstrate Case (1) and Case (6).

(1) For each $\lambda \in (M_1, M_2)$ and $\mu \in (M_3, M_4)$, there exists $\varepsilon > 0$ such that

$$\frac{2^N}{G^N(f_\infty - \varepsilon)} \le \lambda \le \frac{1}{2^N F^N (f_0 + \varepsilon)},$$

$$\frac{2^N}{G^N(g_\infty - \varepsilon)} \le \mu \le \frac{1}{2^N F^N (g_0 + \varepsilon)}.$$

It follows from the definitions of f_0 and g_0 that there exists $r_1 > 0$ such that

$$f(x,y) < (f_0 + \varepsilon)(x+y)^N, \ \ 0 \le x+y \le r_1,$$

$$g(x,y) < (g_0 + \varepsilon)(x+y)^N, \ \ 0 \le x+y \le r_1.$$

Further, we choose the set $\Omega_1 = \{(x,y) \in E : \|(x,y)\|_E < r_1\}$; then, for any $(x,y) \in P \cap \partial\Omega_1$, we obtain

$$0 \le x(s) + y(s) \le \|x\| + \|y\| = \|(x,y)\|_E = r_1, \ \forall s \in [0,1],$$

by simple calculation, we have

$$\begin{aligned} A_1(x,y)(s) &= \int_s^1 (\int_0^u \lambda N\tau^{N-1} f(x(\tau),y(\tau))d\tau)^{\frac{1}{N}} du \\ &\le \int_0^1 (\int_0^u \lambda N\tau^{N-1} f(x(\tau),y(\tau))d\tau)^{\frac{1}{N}} du \\ &\le \int_0^1 (\int_0^u \lambda N\tau^{N-1} (f_0+\varepsilon)(x(\tau)+y(\tau))^N d\tau)^{\frac{1}{N}} du \\ &\le (f_0+\varepsilon)^{\frac{1}{N}} \int_0^1 (\int_0^u \lambda N\tau^{N-1} (\|x\|+\|y\|)^N d\tau)^{\frac{1}{N}} du \\ &= (f_0+\varepsilon)^{\frac{1}{N}} \lambda^{\frac{1}{N}} \int_0^1 (\int_0^u N\tau^{N-1} d\tau)^{\frac{1}{N}} du \cdot \|(x,y)\|_E \\ &\le \frac{\|(x,y)\|_E}{2}. \end{aligned}$$

Next, we show that

$$\|A_1(x,y)\| \le \frac{1}{2}\|(x,y)\|_E, \ \forall\ (x,y) \in P \cap \partial\Omega_1. \tag{10}$$

By applying the same method, we deduce

$$\begin{aligned} A_2(x,y)(s) &= \int_s^1 (\int_0^u \mu N\tau^{N-1} g(x(\tau),y(\tau))d\tau)^{\frac{1}{N}} du \\ &\le \int_0^1 (\int_0^u \mu N\tau^{N-1} g(x(\tau),y(\tau))d\tau)^{\frac{1}{N}} du \\ &\le \int_0^1 (\int_0^u \mu N\tau^{N-1} (g_0+\varepsilon)(x(\tau)+y(\tau))^N d\tau)^{\frac{1}{N}} du \\ &\le (g_0+\varepsilon)^{\frac{1}{N}} \int_0^1 (\int_0^u \mu N\tau^{N-1} (\|x\|+\|y\|)^N d\tau)^{\frac{1}{N}} du \\ &= (g_0+\varepsilon)^{\frac{1}{N}} \mu^{\frac{1}{N}} \int_0^1 (\int_0^u N\tau^{N-1} d\tau)^{\frac{1}{N}} du \cdot \|(x,y)\|_E \\ &\le \frac{\|(x,y)\|_E}{2}. \end{aligned}$$

Next, we show that

$$\|A_2(x,y)\| \le \frac{1}{2}\|(x,y)\|_E, \ \forall (x,y) \in P \cap \partial\Omega_1. \tag{11}$$

Thus, by (10) and (11), we have

$$\|A(x,y)\|_E = \|A_1(x,y)\| + \|A_2(x,y)\| \le \|(x,y)\|_E, \ \forall (x,y) \in P \cap \partial\Omega_1. \tag{12}$$

On the other hand, considering the definitions of f_∞ and g_∞, it is easy to see that there exists $\bar{r}_2 > 0$ such that

$$f(x,y) \ge (f_\infty - \varepsilon)(x+y)^N, \ \ x+y \ge \bar{r}_2,$$

$$g(x,y) \ge (g_\infty - \varepsilon)(x+y)^N, \ \ x+y \ge \bar{r}_2.$$

Further, we choose $r_2 = \max\{2r_1, 4\bar{r}_2\}$ and denote $\Omega_2 = \{(x,y) \in E : \|(x,y)\|_E < r_2\}$; then, for any $(x,y) \in P \cap \partial\Omega_2$, we obtain

$$\min_{s\in[\frac{1}{4},\frac{3}{4}]} (x(s)+y(s)) \ge \frac{1}{4}\|(x,y)\|_E = \frac{1}{4}r_2 \ge \bar{r}_2,$$

in the following, we deduce

$$\begin{aligned}
A_1(x,y)(\frac{1}{4}) &= \int_{\frac{1}{4}}^{1}(\int_0^u \lambda N\tau^{N-1}f(x(\tau),y(\tau))d\tau)^{\frac{1}{N}}du \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u \lambda N\tau^{N-1}f(x(\tau),y(\tau))d\tau)^{\frac{1}{N}}du \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u \lambda N\tau^{N-1}(f_\infty-\varepsilon)(x(\tau)+y(\tau))^N d\tau)^{\frac{1}{N}}du \\
&\geq (f_\infty-\varepsilon)^{\frac{1}{N}}\int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u \lambda N\tau^{N-1}(\frac{1}{4}\|(x,y)\|_E)^N d\tau)^{\frac{1}{N}}du \\
&= \frac{1}{4}(f_\infty-\varepsilon)^{\frac{1}{N}}\lambda^{\frac{1}{N}}\int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u N\tau^{N-1}d\tau)^{\frac{1}{N}}du\cdot\|(x,y)\|_E \\
&\geq \frac{\|(x,y)\|_E}{2}.
\end{aligned}$$

Now, we know that

$$\|A_1(x,y)\| \geq \frac{1}{2}\|(x,y)\|_E, \ \ \forall (x,y)\in P\cap\partial\Omega_2. \tag{13}$$

In a similar manner, for any $(x,y)\in P\cap\partial\Omega_2$, we obtain

$$\begin{aligned}
A_2(x,y)(\frac{1}{4}) &= \int_{\frac{1}{4}}^{1}(\int_0^u \mu N\tau^{N-1}g(x(\tau),y(\tau))d\tau)^{\frac{1}{N}}du \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u \mu N\tau^{N-1}g(x(\tau),y(\tau))d\tau)^{\frac{1}{N}}du \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u \mu N\tau^{N-1}(g_\infty-\varepsilon)(x(\tau)+y(\tau))^N d\tau)^{\frac{1}{N}}du \\
&\geq (g_\infty-\varepsilon)^{\frac{1}{N}}\int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u \mu N\tau^{N-1}(\frac{1}{4}\|(x,y)\|_E)^N d\tau)^{\frac{1}{N}}du \\
&= \frac{1}{4}(g_\infty-\varepsilon)^{\frac{1}{N}}\mu^{\frac{1}{N}}\int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u N\tau^{N-1}d\tau)^{\frac{1}{N}}du\cdot\|(x,y)\|_E \\
&\geq \frac{\|(x,y)\|_E}{2}.
\end{aligned}$$

Now, we know that

$$\|A_2(x,y)\| \geq \frac{1}{2}\|(x,y)\|_E, \ \ \forall (x,y)\in P\cap\partial\Omega_2. \tag{14}$$

Consequently, by means of (13) and (14), we show that

$$\|A(x,y)\|_E = \|A_1(x,y)\| + \|A_2(x,y)\| \geq \|(x,y)\|_E, \ \ \forall (x,y)\in P\cap\partial\Omega_2. \tag{15}$$

Obviously, it follows from (12), (15) and Lemma 1 that A has at least one fixed point $(x,y)\in P\cap(\overline{\Omega}_2\backslash\Omega_1)$ such that $r_1\leq\|(x,y)\|_E\leq r_2$. Thus, System (9) has at least one positive solution. The proof of Case (1) is completed.

(6) We assume $f_0=0, g_0\in(0,\infty), g_\infty=\infty$; then, for each $\lambda\in(0,\infty)$ and $\mu\in(0,M_4)$, there exists $\varepsilon>0$ such that

$$0<\lambda<\frac{1}{2^N F^N\varepsilon}, \ \ \frac{4^N\varepsilon}{G^N}<\mu<\frac{1}{2^N F^N(g_0+\varepsilon)}.$$

Notice that the definitions of f_0 and g_0, and there exists $r_3 > 0$ such that

$$f(x,y) < \varepsilon(x+y)^N,\ \ 0 \le x+y \le r_3,$$

$$g(x,y) < (g_0+\varepsilon)(x+y)^N,\ \ 0 \le x+y \le r_3.$$

Further, we choose the set $\Omega_3 = \{(x,y) \in E : \|(x,y)\|_E < r_3\}$; then, for any $(x,y) \in P \cap \partial\Omega_3$, we have

$$\begin{aligned} A_1(x,y)(s) &= \int_s^1 (\int_0^u \lambda N\tau^{N-1} f(x(\tau),y(\tau))d\tau)^{\frac{1}{N}} du \\ &\le \int_0^1 (\int_0^u \lambda N\tau^{N-1} f(x(\tau),y(\tau))d\tau)^{\frac{1}{N}} du \\ &\le \int_0^1 (\int_0^u \lambda N\tau^{N-1} \varepsilon(x(\tau)+y(\tau))^N d\tau)^{\frac{1}{N}} du \\ &\le \varepsilon^{\frac{1}{N}} \int_0^1 (\int_0^u \lambda N\tau^{N-1} (\|x\|+\|y\|)^N d\tau)^{\frac{1}{N}} du \\ &= \varepsilon^{\frac{1}{N}} \lambda^{\frac{1}{N}} \int_0^1 (\int_0^u N\tau^{N-1} d\tau)^{\frac{1}{N}} du \cdot \|(x,y)\|_E \\ &< \frac{\|(x,y)\|_E}{2}. \end{aligned} \tag{16}$$

Therefore,

$$\|A_1(x,y)\| \le \frac{1}{2}\|(x,y)\|_E,\ \ \forall (x,y) \in P \cap \partial\Omega_3.$$

Similarly, we have

$$\|A_2(x,y)\| \le \frac{1}{2}\|(x,y)\|_E,\ \ \forall (x,y) \in P \cap \partial\Omega_3;$$

clearly,

$$\|A(x,y)\|_E \le \|(x,y)\|_E,\ \ \forall (x,y) \in P \cap \partial\Omega_3. \tag{17}$$

On the other hand, since $g_\infty = \infty$, we know that there exists $\bar{r}_4 > 0$ such that

$$g(x,y) \ge \frac{1}{\varepsilon}(x+y)^N,\ \ x,y \ge 0,\ \ x+y \ge \bar{r}_4.$$

Further, we choose $r_4 = \max\{2r_3, 4\bar{r}_4\}$ and denote $\Omega_4 = \{(x,y) \in E : \|(x,y)\|_E < r_4\}$; then, for any $(x,y) \in P \cap \partial\Omega_4$, we have $\min_{s\in[\frac{1}{4},\frac{3}{4}]}(x(s)+y(s)) \ge \frac{1}{4}\|(x,y)\|_E = \frac{1}{4}r_4 \ge \bar{r}_4$, Now, we deduce that

$$\begin{aligned} A_2(x,y)(\frac{1}{4}) &= \int_{\frac{1}{4}}^1 (\int_0^u \mu N\tau^{N-1} g(x(\tau),y(\tau))d\tau)^{\frac{1}{N}} du \\ &\ge \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^u \mu N\tau^{N-1} g(x(\tau),y(\tau))d\tau)^{\frac{1}{N}} du \\ &\ge \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^u \mu N\tau^{N-1} \frac{1}{\varepsilon}(x(\tau)+y(\tau))^N d\tau)^{\frac{1}{N}} du \\ &\ge (\frac{1}{\varepsilon})^{\frac{1}{N}} \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^u \mu N\tau^{N-1} (\frac{1}{4}\|(x,y)\|_E)^N d\tau)^{\frac{1}{N}} du \\ &= \frac{1}{4}(\frac{1}{\varepsilon})^{\frac{1}{N}} \mu^{\frac{1}{N}} \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^u N\tau^{N-1} d\tau)^{\frac{1}{N}} du \cdot \|(x,y)\|_E \\ &> \|(x,y)\|_E. \end{aligned}$$

Then, it is easy to see that

$$\|A(x,y)\|_E \geq \|A_2(x,y)\| \geq \|(x,y)\|_E, \quad (x,y) \in P \cap \partial\Omega_4. \tag{18}$$

Hence, it follows from (17), (18) and Lemma 1 that A has at least one fixed point $(x,y) \in P \cap (\overline{\Omega}_4 \backslash \Omega_3)$ such that $r_3 \leq \|(x,y)\|_E \leq r_4$, namely (x,y) is a positive solution for System (9), so the proof is completed. □

For $\widehat{f}_0, \widehat{g}_0, \widehat{f}_\infty, \widehat{g}_\infty \in (0,\infty)$, we define the symbols below:

$$Q_1 = \frac{2^N}{G^N \widehat{f}_0}, \quad Q_2 = \frac{1}{2^N F^N \widehat{f}_\infty},$$

$$Q_3 = \frac{2^N}{G^N \widehat{g}_0}, \quad Q_4 = \frac{1}{2^N F^N \widehat{g}_\infty}.$$

Theorem 2. *(1) Assume that $\widehat{f}_0, \widehat{g}_0, \widehat{f}_\infty, \widehat{g}_\infty \in (0,\infty)$, $Q_1 < Q_2$, $Q_3 < Q_4$; then, for $\lambda \in (Q_1, Q_2)$ and $\mu \in (Q_3, Q_4)$, System (9) has at least one positive solution.*

(2) Assume that $\widehat{f}_0, \widehat{g}_0, \widehat{f}_\infty \in (0,\infty)$, $\widehat{g}_\infty = 0$, and $Q_1 < Q_2$; then, for each $\lambda \in (Q_1, Q_2)$ and $\mu \in (Q_3, \infty)$, System (9) has at least one positive solution.

(3) Assume that $\widehat{f}_0, \widehat{g}_0, \widehat{g}_\infty \in (0,\infty)$, $\widehat{f}_\infty = 0$, and $Q_3 < Q_4$; then, for each $\lambda \in (Q_1, \infty)$ and $\mu \in (Q_3, Q_4)$, System (9) has at least one positive solution.

(4) Assume that $\widehat{f}_0, \widehat{g}_0 \in (0,\infty)$, $\widehat{f}_\infty = \widehat{g}_\infty = 0$; then, for each $\lambda \in (Q_1, \infty)$ and $\mu \in (Q_3, \infty)$, System (9) has at least one positive solution.

(5) Assume that $\widehat{f}_\infty, \widehat{g}_\infty \in (0,\infty)$, $\widehat{f}_0 = \infty$ or $\widehat{f}_\infty, \widehat{g}_\infty \in (0,\infty)$, $\widehat{g}_0 = \infty$; then, for each $\lambda \in (0, Q_2)$ and $\mu \in (0, Q_4)$, System (9) has at least one positive solution.

(6) Assume that $\widehat{f}_0 = \infty$, $\widehat{f}_\infty \in (0,\infty)$, $\widehat{g}_\infty = 0$ or $\widehat{f}_\infty \in (0,\infty)$, $\widehat{g}_\infty = 0$, $\widehat{g}_0 = \infty$; then, for each $\lambda \in (0, Q_2)$ and $\mu \in (0,\infty)$, System (9) has at least one positive solution.

(7) Assume that $\widehat{f}_0 = \infty$, $\widehat{g}_\infty \in (0,\infty)$, $\widehat{f}_\infty = 0$ or $\widehat{g}_\infty \in (0,\infty)$, $\widehat{g}_0 = \infty$, $\widehat{f}_\infty = 0$; then, for each $\lambda \in (0,\infty)$ and $\mu \in (0, Q_4)$, System (9) has at least one positive solution.

(8) Assume that $\widehat{f}_\infty = \widehat{g}_\infty = 0$, $\widehat{f}_0 = \infty$ or $\widehat{f}_\infty = \widehat{g}_\infty = 0$, $\widehat{g}_0 = \infty$; then, for each $\lambda \in (0,\infty)$ and $\mu \in (0,\infty)$, System (9) has at least one positive solution.

Proof. Due to the similarity in the proofs of the above cases, we demonstrate Case (1) and Case (6).

(1) For each $\lambda \in (Q_1, Q_2)$ and $\mu \in (Q_3, Q_4)$, there exists $\varepsilon > 0$ such that

$$\frac{2^N}{G^N(\widehat{f}_0 - \varepsilon)} \leq \lambda \leq \frac{1}{2^N F^N(\widehat{f}_\infty + \varepsilon)},$$

$$\frac{2^N}{G^N(\widehat{g}_0 - \varepsilon)} \leq \mu \leq \frac{1}{2^N F^N(\widehat{g}_\infty + \varepsilon)}.$$

It follows from the definitions of $\widehat{f}_0$ and $\widehat{g}_0$ that there exists $r_1 > 0$ such that

$$f(x,y) \geq (\widehat{f}_0 - \varepsilon)(x+y)^N, \quad x, y \geq 0, x+y \leq r_1,$$

$$g(x,y) \geq (\widehat{g}_0 - \varepsilon)(x+y)^N, \quad x, y \geq 0, x+y \leq r_1.$$

Further, we define the set $\Omega_1 = \{(x,y) \in E : \|(x,y)\|_E < r_1\}$; then, for any $(x,y) \in P \cap \partial\Omega_1$, we obtain

$$\begin{aligned}
A_1(x,y)(\frac{1}{4}) &= \int_{\frac{1}{4}}^{1}(\int_0^u \lambda N\tau^{N-1} f(x(\tau),y(\tau))d\tau)^{\frac{1}{N}}du \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u \lambda N\tau^{N-1} f(x(\tau),y(\tau))d\tau)^{\frac{1}{N}}du \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u \lambda N\tau^{N-1} (\widehat{f}_0-\varepsilon)(x(\tau)+y(\tau))^N d\tau)^{\frac{1}{N}}du \\
&\geq (\widehat{f}_0-\varepsilon)^{\frac{1}{N}} \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u \lambda N\tau^{N-1} (\frac{1}{4}\|(x,y)\|_E)^N d\tau)^{\frac{1}{N}}du \\
&= \frac{1}{4}(\widehat{f}_0-\varepsilon)^{\frac{1}{N}}\lambda^{\frac{1}{N}} \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u N\tau^{N-1} d\tau)^{\frac{1}{N}}du \cdot \|(x,y)\|_E \\
&\geq \frac{\|(x,y)\|_E}{2}.
\end{aligned}$$

Next, we show that

$$\|A_1(x,y)\| \geq \frac{1}{2}\|(x,y)\|_E, \ \ \forall (x,y) \in P \cap \partial\Omega_1. \tag{19}$$

In a similar manner, for any $(x,y) \in P \cap \partial\Omega_1$, we deduce

$$\begin{aligned}
A_2(x,y)(\frac{1}{4}) &= \int_{\frac{1}{4}}^{1}(\int_0^u \mu N\tau^{N-1} g(x(\tau),y(\tau))d\tau)^{\frac{1}{N}}du \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u \mu N\tau^{N-1} g(x(\tau),y(\tau))d\tau)^{\frac{1}{N}}du \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u \mu N\tau^{N-1} (\widehat{g}_0-\varepsilon)(x(\tau)+y(\tau))^N d\tau)^{\frac{1}{N}}du \\
&\geq (\widehat{g}_0-\varepsilon)^{\frac{1}{N}} \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u \mu N\tau^{N-1} (\frac{1}{4}\|(x,y)\|_E)^N d\tau)^{\frac{1}{N}}du \\
&= \frac{1}{4}(\widehat{g}_0-\varepsilon)^{\frac{1}{N}}\mu^{\frac{1}{N}} \int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^u N\tau^{N-1} d\tau)^{\frac{1}{N}}du \cdot \|(x,y)\|_E \\
&\geq \frac{\|(x,y)\|_E}{2}.
\end{aligned}$$

Next, we show that

$$\|A_2(x,y)\| \geq \frac{1}{2}\|(x,y)\|_E, \ \ \forall (x,y) \in P \cap \partial\Omega_1. \tag{20}$$

Thus, from (19) and (20) we deduce

$$\|A(x,y)\|_E = \|A_1(x,y)\| + \|A_2(x,y)\| \geq \|(x,y)\|_E, \ \ \forall (x,y) \in P \cap \partial\Omega_1. \tag{21}$$

We let $f^*(u) = \max_{0\leq x+y\leq u} f(x,y)$, $g^*(u) = \max_{0\leq x+y\leq u} g(x,y)$; then, we have

$$f(x,y) \leq f^*(u), \ \ x,y \geq 0, \ \ x+y \leq u,$$

$$g(x,y) \leq g^*(u), \ \ x,y \geq 0, \ \ x+y \leq u.$$

Similar to the proof of [26], we have

$$\limsup_{u\to+\infty} \frac{f^*(u)}{u^N} \leq \widehat{f}_\infty, \ \ \limsup_{u\to+\infty} \frac{g^*(u)}{u^N} \leq \widehat{g}_\infty.$$

According to the above inequality, there exists $\bar{r}_2 > 0$ such that

$$\frac{f^*(u)}{u^N} \leq \limsup_{u \to +\infty} \frac{f^*(u)}{u^N} + \varepsilon \leq \widehat{f}_\infty + \varepsilon, \ \ u \geq \bar{r}_2,$$

$$\frac{g^*(u)}{u^N} \leq \limsup_{u \to +\infty} \frac{g^*(u)}{u^N} + \varepsilon \leq \widehat{g}_\infty + \varepsilon, \ \ u \geq \bar{r}_2;$$

consequently, we have

$$f^*(u) \leq (\widehat{f}_\infty + \varepsilon)u^N, \ g^*(u) \leq (\widehat{g}_\infty + \varepsilon)u^N, \ u \geq \bar{r}_2.$$

Further, we define $r_2 = \max\{2r_1, \bar{r}_2\}$ and denote $\Omega_2 = \{(x,y) \in E : \|(x,y)\|_E < r_2\}$; then, for any $(x,y) \in P \cap \partial\Omega_2$, we obtain

$$f(x(s) + y(s)) \leq f^*(\|(x,y)\|_E), \ \ g(x(s) + y(s)) \leq g^*(\|(x,y)\|_E),$$

by simple calculation, we have

$$\begin{aligned} A_1(x,y)(s) &\leq \int_0^1 (\int_0^u \lambda N \tau^{N-1} f^*(\|(x,y)\|_E) d\tau)^{\frac{1}{N}} du \\ &\leq \int_0^1 (\int_0^u \lambda N \tau^{N-1} (\widehat{f}_\infty + \varepsilon)(\|(x,y)\|_E)^N d\tau)^{\frac{1}{N}} du \\ &= (\widehat{f}_\infty + \varepsilon)^{\frac{1}{N}} \lambda^{\frac{1}{N}} \int_0^1 (\int_0^u N \tau^{N-1} d\tau)^{\frac{1}{N}} du \cdot \|(x,y)\|_E \\ &\leq \frac{\|(x,y)\|_E}{2}. \end{aligned}$$

Now, we know that

$$\|A_1(x,y)\| \leq \frac{1}{2}\|(x,y)\|_E, \ \ \forall (x,y) \in P \cap \partial\Omega_2. \tag{22}$$

In a similar manner, for any $(x,y) \in P \cap \partial\Omega_2$, we have

$$\begin{aligned} A_2(x,y)(s) &\leq \int_0^1 (\int_0^u \mu N \tau^{N-1} g^*(\|(x,y)\|_E) d\tau)^{\frac{1}{N}} du \\ &\leq \int_0^1 (\int_0^u \mu N \tau^{N-1} (\widehat{g}_\infty + \varepsilon)(\|(x,y)\|_E)^N d\tau)^{\frac{1}{N}} du \\ &= (\widehat{g}_\infty + \varepsilon)^{\frac{1}{N}} \mu^{\frac{1}{N}} \int_0^1 (\int_0^u N \tau^{N-1} d\tau)^{\frac{1}{N}} du \cdot \|(x,y)\|_E \\ &\leq \frac{\|(x,y)\|_E}{2}. \end{aligned}$$

Now, we know that

$$\|A_2(x,y)\| \leq \frac{1}{2}\|(x,y)\|_E, \ \ \forall (x,y) \in P \cap \partial\Omega_2. \tag{23}$$

Clearly, by means of (22) and (23), we deduce that

$$\|A(x,y)\|_E = \|A_1(x,y)\| + \|A_2(x,y)\| \leq \|(x,y)\|_E, \ \ \forall (x,y) \in P \cap \partial\Omega_2. \tag{24}$$

Consequently, by using (21), (24) and Lemma 1, we conclude that A has at least one fixed point $(x,y) \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$ such that $r_1 \leq \|(x,y)\| \leq r_2$.

(6) We assume $\widehat{f}_0 = \infty, \widehat{f}_\infty \in (0, \infty), \widehat{g}_\infty = 0$; then, for any $\lambda \in (0, Q_2)$ and $\mu \in (0, \infty)$, there exists $\varepsilon > 0$ such that

$$\frac{4^N \varepsilon}{G^N} < \lambda < \frac{1}{2^N F^N (\widehat{f}_\infty + \varepsilon)}, \ \ 0 < \mu < \frac{1}{2^N F^N \varepsilon}.$$

Since $\widehat{f}_0 = \infty$, there exists $r_3 > 0$ such that

$$f(x, y) \geq \frac{1}{\varepsilon}(x + y)^N, \ \ x, y \geq 0, \ \ 0 \leq x + y \leq r_3.$$

Further, we define the set $\Omega_3 = \{(x, y) \in E : \|(x, y)\|_E < r_3\}$; then, for any $(x, y) \in P \cap \partial\Omega_3$, we have

$$\begin{aligned} A_1(x, y)(\frac{1}{4}) &\geq \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^{u} \lambda N \tau^{N-1} f(x(\tau), y(\tau)) d\tau)^{\frac{1}{N}} du \\ &\geq \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^{u} \lambda N \tau^{N-1} \frac{1}{\varepsilon} (x(\tau) + y(\tau))^N d\tau)^{\frac{1}{N}} du \\ &\geq (\frac{1}{\varepsilon})^{\frac{1}{N}} \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^{u} \lambda N \tau^{N-1} (\frac{1}{4} \|(x, y)\|_E)^N d\tau)^{\frac{1}{N}} du \\ &= \frac{1}{4} (\frac{1}{\varepsilon})^{\frac{1}{N}} \lambda^{\frac{1}{N}} \int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^{u} N \tau^{N-1} d\tau)^{\frac{1}{N}} du \cdot \|(x, y)\|_E \\ &\geq \|(x, y)\|_E. \end{aligned}$$

Obviously,

$$\|A(x, y)\|_E \geq \|A_1(x, y)\| \geq \|(x, y)\|_E, \ \ \forall (x, y) \in P \cap \partial\Omega_3. \tag{25}$$

We let $f^*(u) = \max\limits_{0 \leq x+y \leq u} f(x, y)$, $g^*(u) = \max\limits_{0 \leq x+y \leq u} g(x, y)$. Similar to the proof of [26], we have

$$\limsup_{u \to +\infty} \frac{f^*(u)}{u^N} \leq \widehat{f}_\infty, \ \ \limsup_{u \to +\infty} \frac{g^*(u)}{u^N} = 0.$$

Moreover, for above $\varepsilon > 0$, it is easy to see that there exists $\bar{r}_4 > 0$ such that

$$\frac{f^*(u)}{u^N} \leq \limsup_{u \to +\infty} \frac{f^*(u)}{u^N} + \varepsilon \leq \widehat{f}_\infty + \varepsilon, \ \ u \geq \bar{r}_4,$$

$$\frac{g^*(u)}{u^N} \leq \limsup_{u \to +\infty} \frac{g^*(u)}{u^N} + \varepsilon = \varepsilon, \ \ u \geq \bar{r}_4;$$

consequently, we obtain

$$f^*(u) \leq (\widehat{f}_\infty + \varepsilon) u^N, \ g^*(u) \leq \varepsilon u^N, \ u \geq \bar{r}_4.$$

Further, we define $r_4 = \max\{2r_3, \bar{r}_4\}$ and denote $\Omega_4 = \{(x, y) \in E : \|(x, y)\|_E < r_4\}$; then, for any $(x, y) \in P \cap \partial\Omega_4$, we have

$$f(x(s) + y(s)) \leq f^*(\|(x, y)\|_E), \ \ g(x(s) + y(s)) \leq g^*(\|(x, y)\|_E),$$

Now, we deduce that

$$\begin{aligned}
A_1(x,y)(s) &\leq \int_0^1 (\int_0^u \lambda N\tau^{N-1} f^*(\|(x,y)\|_E) d\tau)^{\frac{1}{N}} du \\
&\leq \int_0^1 (\int_0^u \lambda N\tau^{N-1} (\widehat{f}_\infty + \varepsilon)(\|(x,y)\|_E)^N d\tau)^{\frac{1}{N}} du \\
&= (\widehat{f}_\infty + \varepsilon)^{\frac{1}{N}} \lambda^{\frac{1}{N}} \int_0^1 (\int_0^u N\tau^{N-1} d\tau)^{\frac{1}{N}} du \cdot \|(x,y)\|_E \\
&\leq \frac{\|(x,y)\|_E}{2}.
\end{aligned}$$

Therefore,

$$\|A_1(x,y)\| \leq \frac{1}{2}\|(x,y)\|_E, \ \ \forall (x,y) \in P \cap \partial\Omega_4. \tag{26}$$

Likewise, for any $(x,y) \in P \cap \partial\Omega_4$, we have

$$\begin{aligned}
A_2(x,y)(s) &\leq \int_0^1 (\int_0^u \mu N\tau^{N-1} g^*(\|(x,y)\|_E) d\tau)^{\frac{1}{N}} du \\
&\leq \int_0^1 (\int_0^u \mu N\tau^{N-1} \varepsilon(\|(x,y)\|_E)^N d\tau)^{\frac{1}{N}} du \\
&= \varepsilon^{\frac{1}{N}} \mu^{\frac{1}{N}} \int_0^1 (\int_0^u N\tau^{N-1} d\tau)^{\frac{1}{N}} du \cdot \|(x,y)\|_E \\
&\leq \frac{\|(x,y)\|_E}{2}.
\end{aligned}$$

That is,

$$\|A_2(x,y)\| \leq \frac{1}{2}\|(x,y)\|_E, \ \ \forall (x,y) \in P \cap \partial\Omega_4. \tag{27}$$

Obviously, from (26) and (27), we deduce

$$\|A(x,y)\|_E = \|A_1(x,y)\| + \|A_2(x,y)\| \leq \|(x,y)\|_E, \ \ \forall (x,y) \in P \cap \partial\Omega_4. \tag{28}$$

Hence, by using (25), (28) and Lemma 1, we conclude that A has at least one fixed point $(x,y) \in P \cap (\overline{\Omega}_4 \backslash \Omega_3)$ such that $r_3 \leq \|(x,y)\|_E \leq r_4$, namely (x,y) is a positive solution for System (9). □

4. Applications

Example 1. *We consider the following boundary value problem:*

$$\begin{cases}
((-x'(s))^3)' = 3\lambda s^2 f(x(s),y(s)), \ \ 0 < s < 1, \\
((-y'(s))^3)' = 3\mu s^2 g(x(s),y(s)), \ \ 0 < s < 1, \\
x'(0) = x(1) = 0, \ y'(0) = y(1) = 0,
\end{cases} \tag{29}$$

We take $f(x,y) = (x+y)^{N+2}$, $g(x,y) = (x+y)^N + (x+y)^N e^{x+y}$, where $N = 3$. By simple calculation, we obtain $M_4 \approx 0.0625$, and

$$f_0 = \limsup_{x+y \to 0^+} \frac{f(x,y)}{(x+y)^N} = \limsup_{x+y \to 0^+} (x+y)^2 = 0,$$

$$g_0 = \limsup_{x+y \to 0^+} \frac{g(x,y)}{(x+y)^N} = \limsup_{x+y \to 0^+} (1 + e^{x+y}) = 2,$$

$$f_\infty = \liminf_{x+y \to \infty} \frac{f(x,y)}{(x+y)^N} = \liminf_{x+y \to \infty} (x+y)^2 = \infty.$$

Then, for each $\lambda \in (0, \infty)$ and $\mu \in (0, 0.0625)$, by Theorem 1 (6), we determine that System (29) has at least one positive solution.

Example 2. *We consider the following boundary value problem:*

$$\begin{cases} ((-x'(s))^3)' = 3\lambda s^2 f(x(s), y(s)), \ \ 0 < s < 1, \\ ((-y'(s))^3)' = 3\mu s^2 g(x(s), y(s)), \ \ 0 < s < 1, \\ x'(0) = x(1) = 0, \ y'(0) = y(1) = 0, \end{cases} \tag{30}$$

We take $f(x, y) = \frac{(x+y)^N}{\tan(x+y)^N}$, $g(x, y) = \frac{1}{x+y}$, where $N = 3$. By simple calculation, we obtain $Q_2 \approx 0.1962$, and

$$\widehat{f}_0 = \liminf_{x+y \to 0^+} \frac{f(x, y)}{(x+y)^N} = \liminf_{x+y \to 0^+} \frac{1}{\arctan(x+y)^N} = \infty,$$

$$\widehat{g}_\infty = \limsup_{x+y \to \infty} \frac{g(x, y)}{(x+y)^N} = \limsup_{x+y \to \infty} \frac{1}{(x+y)^{N+1}} = 0,$$

$$\widehat{f}_\infty = \limsup_{x+y \to \infty} \frac{f(x, y)}{(x+y)^N} = \limsup_{x+y \to \infty} \frac{1}{\arctan(x+y)^N} = \frac{2}{\pi}.$$

Then, for each $\lambda \in (0, 0.1962)$ and $\mu \in (0, \infty)$, by Theorem 2 (6), we determine that System (30) has at least one positive solution.

5. Conclusions

The system of Monge–Ampère equations is significant in various fields of study, including geometry, mathematical physics, materials science, and others. In this paper, by considering some combinations of superlinearity and sublinearlity of functions f and g, we use the Guo–Krasnosel'skii fixed point theorem to study the existence of nontrivial solutions for a system of Monge–Ampère equations with two parameters and establish diverse existence outcomes for nontrivial solutions based on various values of λ and μ which enrich the theories for the system of Monge–Ampère equations. The research in this paper is different from reference [21]. When $\lambda = \mu = 1$ in System (1), System (1) can be reduced to System (4) of reference [21]; then, it can be simply seen that System (4) is a special case of this paper, so this paper can be said to be a generalization of reference [21].

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