axioms

Article

Second-Order Damped Differential Equations with Superlinear Neutral Term: New Criteria for Oscillation

Asma Al-Jaser [1], Clemente Cesarano [2,*], Belgees Qaraad [3,*] and Loredana Florentina Iambor [4]

1 Department of Mathematical Science, College of Science, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia; ajaljaser@pnu.edu.sa
2 Section of Mathematics, International Telematic University Uninettuno, CorsoVittorio Emanuele II, 39, 00186 Roma, Italy
3 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
4 Department of Mathematics and Computer Science, University of Oradea, 1 University Street, 410087 Oradea, Romania; iambor.loredana@uoradea.ro
* Correspondence: clemente.cesarano@uninettunouniversity.net (C.C.); belgeesmath2016@gmail.com (B.Q.)

Abstract: This paper focuses on establishing new criteria to guarantee the oscillation of solutions for second-order differential equations with a superlinear and a damping term. New sufficient conditions are presented, aimed at analysing the oscillatory properties of the solutions to the equation under study. To prove these results, we employed various analysis methods, establishing new relationships to address certain problems that have hindered previous research. Consequently, by applying the principles of comparison and the Riccati transformation, we obtained findings that develop and complement those reported in earlier literature. The significance of our results is illustrated with several examples.

Keywords: second-order; oscillation; superlinear; damping; neutral

MSC: 34C10; 34K1

check for updates

Citation: Al-Jaser, A.; Cesarano, C.; Qaraad, B.; Iambor, L.F. Second-Order Damped Differential Equations with Superlinear Neutral Term: New Criteria for Oscillation. *Axioms* **2024**, *13*, 234. https://doi.org/10.3390/axioms13040234

Academic Editor: Behzad Djafari-Rouhani

Received: 18 February 2024
Revised: 26 March 2024
Accepted: 26 March 2024
Published: 1 April 2024

1. Introduction

In this work, we establish the new results of the asymptotic and oscillatory behavior of the solutions of the following differential equations with distributed deviating arguments:

$$\left(rg'\right)'(\top) + \eta(\top)g'(\top) + \int_a^b q(\top,s)\varkappa^{\lambda}(\sigma(\top,s))\mathrm{d}s = 0, \quad \top \geq \top_0 > 0, \tag{1}$$

where λ is quotient of odd positive integers,

$$g(\top) = \varkappa(\top) + \nu(\top)\varkappa^{\gamma}(\zeta(\top)),$$

$\gamma \in Q^+_{odd} := \{a_1/a_2 : a_1, a_2 \in \mathbb{Z}^+ \text{ are odd}\}$, $0 \leq a < b$ and

$$\gamma \geq 1. \tag{2}$$

Throughout this paper, we assume the following:

(C_1) $r \in C\left(I_0^+, (0,\infty)\right)$, $\eta \in C\left(I_0^+, (0,\infty)\right)$, where $I_i^+ = [\top_i, \infty)$, and

$$A(\top) := \int_{\top_0}^{\top} r^{-1}(u)\left(e^{-\int_{\top_0}^{u} \frac{\eta(s)}{r(s)}\mathrm{d}s}\right)\mathrm{d}u < \infty; \tag{3}$$

(C_2) $\zeta \in C\left(I_0^+, \mathbb{R}\right)$, $\sigma \in C\left(I_0^+ \times (a,b), \mathbb{R}\right)$, $q(\top, s) \in C\left(I_0^+ \times (a,b), \mathbb{R}\right)$, $\nu \in C\left(I_0^+, (0,\infty)\right)$, $\zeta(\top) \geq \top, \sigma(\top, s) \leq \top, q(\top, s) \geq 0, \lim_{\top\to\infty} \sigma(\top, s) = \infty$ and

$$\nu(\top) < 1. \tag{4}$$

Definition 1 ([1])**.** *By a solution of (1), we mean a function $\varkappa \in C(I_\varkappa^+, \mathbb{R})$ for some $\top_\varkappa \geq \top_0$, which has the property $rg' \in C^1(I_\varkappa^+, \mathbb{R})$. We consider only those solutions of (1) that will exist on some half-line $I_\varkappa^+$ and satisfy*

$$\sup\left\{|\varkappa(\top)| : \overline{T} \leq \top < \infty\right\} > 0$$

for any $\overline{T} \geq \top_\varkappa$.

Definition 2 ([2])**.** *If the set of zeros of solution $\varkappa$ of (1) is unbounded above, then we call the solution $\varkappa$ an oscillatory solution. Otherwise, we call it a nonoscillatory solution.*

Since Sturm [3] began studying the term oscillation, the researchers have shown remarkable interest in searching for different ways and methods to develop the qualitative theory of functional differential equations as an intermediate tool for transforming real-world phenomena into purely mathematical models that can be dealt with, solved, and studied easily. It is known that the qualitative approach does not seek explicit solutions but is concerned with the behaviour of solutions to differential equations. Since then, the asymptotic and oscillatory properties have attracted the attention of many researchers; see [4–13].

Delayed differential equations are among the most important equations closely related to modern sciences such as biology and mathematical biology. They are used in studying models of population growth and disease spread. These equations describe the time delay between contracting a disease and the appearance of symptoms, or between the birth of an individual and their ability to reproduce. In engineering and control systems, where there might be a time delay between issuing a command and achieving the desired result, their significance is evident. An example of this is the vehicle motion control system, which takes into account the time delay required for the driver's reaction. Their importance also extends to computer science, where delayed differential equations are used to analyse the performance of networks [14–16].

The study of the oscillations of solutions for second-order delay equations is fundamental across various disciplines to predict system behaviors, improve stability, enhance performance, and develop more accurate models reflecting real-world scenarios. Second-order delay equations can describe systems where the current state depends not only on current and past states but also on their rates of change. These include control systems, signal processing, and mechanical systems where inertia and damping are considered. By studying oscillations, engineers can design more stable and efficient systems, predict system responses under different conditions, and improve the performance of control algorithms. Notably, delay terms can represent incubation times, reaction delays, or other biological processes that are not instantaneous. Analysing the solutions and their oscillations can lead to better models for biological systems and improve our understanding of disease spread, drug kinetics, and physiological processes. On the other hand, understanding the behavior of these equations' solutions is crucial for developing stable digital methods and algorithms. This is important for accurately simulating real-world phenomena and solving practical problems where analytical solutions are not possible.

Neutral delay differential equations are defined as the equations in which the highest-order derivative of the unknown function appears with and without a delay. The study of the properties of the oscillation of solutions to this type of equation is currently receiving great attention. These equations are used in many fields, such as problems related to dealing with masses attached to a Shaky supple rod. They also occur in some electrical network applications seen in high-speed computers, where they interconnect switching circuits using lossless transmission lines, and in solving problems with time delays. The references [17–19] can be referred to for more applications in science and technology.

The study of the oscillatory behavior of solutions of various types of second-order differential equations that do not contain the term with the first derivative is considered one of the studies in which the most publications have appeared. We note that the fact that the derivative of the coefficients is non-negative (due to the coefficients being positive and the solution being on the semi-axis) is used in most studies of second-order delay differential equations. Adding the first derivative in the equation explicitly—in other words, including the damping term in the equation—makes the study of the oscillation properties of its solutions more complicated as it is difficult for the solution to specify a derivative sign, so this type of equation is much less studied compared to the equations without the damping term.

Now, we briefly discuss some relevant findings that motivated our study. Bohner and Saker in [8] derived some oscillatory results related to the equation

$$(r\varkappa')'(\top) + \eta(\top)\varkappa'(\top) + q(\top)\varkappa^{\lambda}(\sigma(\top)) = 0.$$

Also, the authors studied similar equations in the references [7,20] and obtained results close to those found in [8].

Tunc and Kaymaz, in [21,22], presented some criteria to guarantee the oscillation of solutions to the two differential equations

$$g''(\top) + \eta(\top)g'(\top) + q(\top)f(\top, \varkappa(\sigma(\top))) = 0,$$

and

$$\left((rg')^{\lambda}\right)'(\top) + \eta(\top)(g'(\top))^{\lambda} + q(\top)f(\top, \varkappa(\sigma(\top))) = 0, \tag{5}$$

respectively, in the presence of the conditions

$$\nu(\top) \geq 1, \tag{6}$$

and

$$\gamma = 1. \tag{7}$$

Said et al. [23] provided some results about the oscillation criteria for solutions of the most general equation

$$\left(r(g')^{\lambda}\right)'(\top) + \eta(\top)(g'(\top))^{\lambda} + \int_a^b f\left(\top, \varkappa^{\lambda}(\sigma(\top, s))\right)q(\top, s)\mathrm{d}s = 0$$

where $A(\top) < \infty$, (4) and (7) hold, and

$$\sigma' < 0.$$

Tunç and Ozdemin [24] used integral criterion and Riccati transformation and revealed some important results regarding the oscillation criteria of the equation

$$g''(\top) + \eta(\top)g'(\top) + q(t)\varkappa^{\lambda}(\sigma(\top)) = 0,$$

also, it is assumed that (2) and (6) hold, and

$$\sigma(\top) \leq \zeta(\top) \leq \top. \tag{8}$$

Motivation

Most of the previous studies that highlighted the study of second-order differential equations with the damping term are without the neutral term (i.e., $\nu(\top) = 0$); see [7,8,20,25]. Some results are also available with both the damping term and the neutral term together, for instance, see [26–33], but these results are subject to the conditions (7) and (6), or at least one of them. Therefore, all of these results cannot be applied when (2) and (4) hold. On the other hand, we do not need more additional conditions, such as (8). By utilising the results provided in the reference [34], we establish new oscillation criteria for Equation (1). Based on the above, we aim in this paper to complete, simplify, and develop previous results. Therefore, we believe that this paper will be a good contribution to the study of the oscillatory behavior of the Equation (1) and its special cases.

We organize this paper as follows: In the first section (Introduction), we offer the studied equation and the general conditions necessary to reach the main results of the paper. We also provide an overview of related topics and the motivation behind this study. In Section 2, we offer some relationships and results that will be used to reach the oscillation results discussed in the subsection titled "Oscillation Results". In Section 3, we provide some examples to illustrate the significance of the obtained results. Finally, in Section 4, we summarise the main results of the paper and highlight an open question that may be of interest to researchers in this field.

2. Main Results

2.1. Auxiliary Lemmas

The study of the oscillation of first-order equations went through stages of development over the years until it became clearer and more understandable from both a theoretical and scientific perspective. The behavior of the solutions to delay first-order differential equations differs entirely from that of the homogeneous ordinary differential equations of the first-order, where we notice that the presence of deviating arguments can cause oscillations in the solutions, whereas ordinary equations do not possess oscillatory solutions. The study of the oscillation of this type of equation has been utilized to arrive at oscillation criteria for equations of higher order. Consider the first order differential equation

$$g'(\top) + F(\top)g(\sigma(\top)) = 0, \tag{9}$$

where

$$F \in C[\top_0, \infty), F(\top) > 0, \sigma \in C^1[\top_0, \infty), \sigma(\top) < \top, \sigma'(\top) \geq 0, \lim_{t\to\infty} h(\top) = \infty. \tag{10}$$

Now, we present some important relationships and conditions that we will employ to obtain the main results.

Lemma 1 ([35])**.** *Assume that (10) holds. If the first-order delay differential inequality*

$$g'(\top) + F(\top)g(\sigma(\top)) \leq 0$$

has a positive solution, then the delay differential Equation (9) also has a positive solution.

Lemma 2 ([36])**.** *Suppose that*

$$\int_{\top}^{\top+\sigma} F(s)ds > 0 \ \textit{for}\ \top \geq \top_0$$

for some $\top_0 > 0$ *and*

$$\int_{\top_0}^{\infty} F(t) \ln\left(e \int_{\sigma}^{\top+\sigma} F(s)ds\right) d\top = \infty. \tag{11}$$

Then every solution of (9) oscillates.

Lemma 3. *Suppose that* $\varkappa > 0$ *is a solution of (1). Then, one of the following cases is valid:*

$$(\Upsilon_1)\ g(\top) > 0 \ \textit{and}\ \ g'(\top) > 0$$

or

$$(Y_2)\ \ g(\top) > 0\ \ \textit{and}\ g'(\top) < 0$$

for $\top \geq \top_1 \geq \top_0$ *with sufficiently large* $\top_1$.

Proof. Assume that $\varkappa(\top) > 0$, that is, $\varkappa(\zeta(\top)) > 0$ and $\varkappa(\sigma(\top, s)) > 0$ for all $\top \geq \top_1 \geq \top_0$. Then, $g(\top)$ is positive and either $g'(\top)$ is nonoscillatory or $g'(\top)$ is oscillatory. Assume that $g'(\top)$ is oscillatory; then, from (1) we find that

$$(rg')'(\top) + \eta(\top)g'(\top) = -\int_a^b q(\top, s)\varkappa^\lambda(\sigma(\top, s))\mathrm{d}s.$$

Set

$$U(\top) = \exp\left(\int_{\top_0}^{\top} \frac{\eta(\top)}{r(\top)}\mathrm{d}s\right).$$

This yields

$$\big(U(\top)r(\top)g'(\top)\big)' = -U(\top)\int_a^b q(\top, s)\varkappa^\lambda(\sigma(\top, s))\mathrm{d}s. \tag{12}$$

Thus, $U(\top)r(\top)g'(\top)$ has one sign eventually, that is, $g'(\top)$ has a fixed sign. □

Lemma 4. *Suppose that* $\varkappa > 0$ *is a solution of (1) and* (Y_1) *satisfies. Then,*

$$\frac{g(\top)}{g'(\top)} > A(\top)U(\top)r(\top),\ \ \top \geq \top_1 \tag{13}$$

and

$$\left(\frac{g}{A}\right)'(\top) < 0.$$

Proof. Assume that (Y_1) holds. That is, $g(\top) > 0$, $g'(\top) > 0$ and $(rg')'(\top) \leq 0$ for $\top \geq \top_1$. From (1), we see that

$$(rg')'(\top) + \eta(\top)g'(\top) = -\int_a^b q(\top, s)\varkappa^\lambda(\sigma(\top, s))\mathrm{d}s.$$

which leads to

$$\big(U(\top)r(\top)g'(\top)\big)' = -U(\top)\int_a^b q(\top, s)\varkappa^\lambda(\sigma(\top, s))\mathrm{d}s \leq 0.$$

That is,

$$\big(U(\top)r(\top)g'(\top)\big)' \leq 0.$$

Since $U(\top)r(\top)g'(\top)$ is decreasing, we have

$$\begin{aligned} g(\top) &= g(\top_1) + \int_{\top_1}^{\top} \frac{U(s)r(s)g'(s)}{U(s)r(s)}\mathrm{d}s \\ &> A(\top)U(\top)r(\top)g'(\top), \end{aligned}$$

thus,

$$g(\top) > A(\top)U(\top)r(\top)g'(\top),$$

which yields

$$A(\top)U(\top)r(\top)g'(\top) - g(\top) < 0. \tag{14}$$

Since

$$\begin{aligned}\frac{g(\top)A'(\top)}{A^2(\top)} &= \frac{r(\top)U(\top)g(\top)A'(\top)}{r(\top)U(\top)A^2(\top)} \\ &> \frac{g(\top)}{r(\top)U(\top)A^2(\top)}.\end{aligned}$$

It followes that ,

$$\frac{g(\top)A'(\top)}{A^2(\top)} > \frac{g(\top)}{r(\top)U(\top)A^2(\top)}. \tag{15}$$

From (14) and (15), we obtain

$$\begin{aligned}\left(\frac{g}{A}\right)'(\top) &= (g'(\top)A(\top) - g(\top)A'(\top))\frac{1}{A^2(\top)} \\ &= (g'(\top)A(\top)r(\top)U(\top) - g(\top))\frac{1}{r(\top)U(\top)A^2(\top)} \\ &< 0.\end{aligned}$$

Thus, it proves that $(g/A)' < 0$ is decreasing on $[\top_1, \infty)$. The proof is complete. □

Lemma 5. *Suppose that* $\varkappa > 0$ *is a solution of (1), and*

$$\lim_{\top\to\infty} \nu(\top)\widetilde{A}(\top) = 0, \text{ where } \widetilde{A}(\top) = \frac{A^\gamma(\zeta(\top))}{A(\top)}. \tag{16}$$

If

$$\int_{\top_0}^{\infty} U(s)\left(\int_a^b q(\top,u)du\right)\mathrm{ds} = \infty, \tag{17}$$

then (Y_2) *holds.*

Proof. Assume that $\varkappa(\top) > 0$,, that is, $\varkappa(\zeta(\top)) > 0$ and $\varkappa(\sigma(\top,s)) > 0$ for all $\top \geq \top_1 \geq \top_0$. From Lemma 3, we note that g satisfies one of the two cases (Y_1) or (Y_2). Let us assume that (Y_1) holds. Then, from Lemma 4, we find $(g(\top)/A(\top))' < 0$ for $\top \geq \top_2 \geq \top_1$. Since $(g/A)' < 0$ and $g' > 0$, we obtain

$$g(\top) \geq M_1 \text{ and } g(\top) \leq M_2 A(\top)\text{ , for } \top > \top_3,$$

where M_1, $M_2 > 0$, and $\top_3 > \top_2$. Let $\varepsilon \in (0,1)$. By (16), we note that $\top_4 \geq \top_3$ exists such that

$$\nu(\top)A^\gamma(\zeta(\top)) \leq A(\top)M_2^{1-\gamma}(1-\varepsilon) \text{ for } \top \geq \top_4.$$

With $\varkappa < g$, it follows that

$$\begin{aligned}\varkappa(\top) &= g(\top) - \nu(\top)\varkappa^\gamma(\zeta(\top)) \\ &\geq g(\top) - \nu(\top)g^\gamma(\zeta(\top)) \\ &= g(\top) - \nu(\top)\left(\frac{g(\zeta(\top))}{A(\zeta(\top))}\right)^\gamma A^\gamma(\zeta(\top)) \\ &\geq g(\top) - \nu(\top)\left(\frac{g(\top)}{A(\top)}\right)^\gamma A^\gamma(\zeta(\top)) \\ &= g(\top)\left(1 - \left(\nu(\top)\frac{A^\gamma(\zeta(\top))}{A(\top)}\right)\left(\frac{g(\top)}{A(\top)}\right)^{\gamma-1}\right),\end{aligned}$$

which implies

$$\begin{aligned} \varkappa(\top) &\geq M_1\left(1 - M_2^{1-\gamma}(1-\varepsilon)M_2^{\gamma-1}\right) = M_1\varepsilon \\ &= :M, \ \text{ for } \top \geq \top_4 \end{aligned}$$

or

$$\varkappa(\top, s) \geq M.$$

That is, M is positive. Using this in (12), we obtain

$$\left(U(\top)r(\top)g'(\top)\right)' + U(\top)M^{\lambda}\int_a^b q(\top, s)\mathrm{d}s \leq 0.$$

Integrating the above inequality, (17) leads to

$$\begin{aligned} 0 &< U(\top)r(\top)g'(\top) \\ &\leq U(\top_4)r(\top_4)g'(\top_4) - M^{\lambda}\int_{\top_4}^{\top} U(s)\left(\int_a^b q(\top, u)\mathrm{d}u\right)\mathrm{d}s \to -\infty. \end{aligned}$$

The proof is complete. □

Lemma 6. *Suppose that* $\varkappa > 0$ *is a solution of (1) such that* (Υ_2) *holds and (16) satisfies. If an increasing function* $\rho \in C^1\left(I_0^+, \mathbb{R}^+\right)$ *exists such that*

$$\int_{\top_0}^{\infty} \frac{1}{r(\top)U(\top)\rho(\top)}\left(\int_{\top_0}^{\top} \rho(s)U(s)\int_a^b q(\top, u)\mathrm{d}u\mathrm{d}s\right)\mathrm{d}\top = \infty, \tag{18}$$

then

$$\lim_{\top\to\infty} \varkappa(\top) = \lim_{\top\to\infty} g(\top) = 0. \tag{19}$$

Proof. Let $\varkappa(\top) > 0$, that is, $\varkappa(\zeta(\top)) > 0$ and $\varkappa(\sigma(\top, s)) > 0$ for all $\top \geq \top_1 \geq \top_0$. Since $g'(\top) < 0$, then

$$g(\top) \leq M_3,$$

where $M_3 > 0$ is constant and $\top_2 \geq \top_1$. In view of (16) and by increasing and bounded property of the function $A(\top)$, we see that

$$\lim_{\top\to\infty} \nu(\top) = 0,$$

$\top_3 \geq \top_2$ exists such that

$$\nu(\top)\frac{1}{(1-\varepsilon_2)} \leq M_3^{1-\gamma}, \ \varepsilon_2 \in (0,1) \text{ for } \top \geq \top_3.$$

Since $\varkappa < g$, we obtain

$$\begin{aligned} \varkappa(\top) &= g(\top) - \nu(\top)\varkappa^{\gamma}(\zeta(\top)) \\ &\geq g(\top) - \nu(\top)g^{\gamma}(\zeta(\top)) \\ &\geq g(\top) - \nu(\top)g^{\gamma}(\top) \\ &= g(\top)\left(1 - \nu(\top)g^{\gamma-1}(\top)\right) \\ &\geq g(\top)\left(1 - M_3^{1-\gamma}(1-\varepsilon_2)M_3^{\gamma-1}(\top)\right) \\ &= \varepsilon_2 g(\top). \end{aligned}$$

This implies that

$$\varkappa(\top) \geq \varepsilon_2 g(\top).$$

In (12), we have

$$\left(U(\top)r(\top)g'(\top)\right)' + \varepsilon U(\top)g^{\lambda}(\sigma(\top,s))\int_a^b q(\top,u)\mathrm{d}u \leq 0, \tag{20}$$

for $\top \geq \top_3$ and $\varepsilon := \varepsilon_2^{\lambda}$. Since $g' < 0$, we find that

$$\lim_{\top\to\infty} g(\top) =: \varrho,\ \varrho \text{ is nonnegative.}$$

Let $\varrho > 0$. Then, there is $\top_4 \geq \top_3$ such that $g(\sigma(\top,s)) \geq \varrho$ for $\top \geq \top_4$, and

$$\left(U(\top)r(\top)g'(\top)\right)' + \varrho_1 U(\top)\int_a^b q(\top,s)\mathrm{d}s \leq 0,\ \varrho_1 := \varepsilon\varrho^{\lambda} > 0 \tag{21}$$

for $\top \geq \top_4$. Set

$$w(\top) := \rho(\top)U(\top)r(\top)g'(\top).$$

By using (21), we obtain

$$\begin{aligned} w'(\top) &= \rho(\top)\left(Urg'\right)'(\top) + \rho'(\top)U(\top)r(\top)g'(\top) \\ &\leq -\varrho_1\rho(\top)U(\top)\int_a^b q(\top,s)\mathrm{d}s + \rho'(\top)U(\top)r(\top)g'(\top) \\ &\leq -\varrho_1\rho(\top)U(\top)\int_a^b q(\top,s)\mathrm{d}s,\ \top \geq \top_4. \end{aligned} \tag{22}$$

Integrating (22), we see that

$$\begin{aligned} w(\top) &\leq w(\top_4) - \varrho_1\int_{\top_4}^{\top} U(s)\rho(s)\int_a^b q(\top,u)\mathrm{d}u\mathrm{d}s \\ &\leq -\varrho_1\int_{\top_4}^{\top} U(s)\rho(s)\int_a^b q(\top,u)\mathrm{d}u\mathrm{d}s. \end{aligned}$$

That is,

$$g'(\top) \leq -\varrho_1\frac{1}{\rho(\top)U(\top)r(\top)}\int_{\top_4}^{\top} U(s)\rho(s)\int_a^b q(\top,u)\mathrm{d}u\mathrm{d}s.$$

Now, integrating from $\top_4$ to $\top$, we obtain

$$g(\top) \leq g(\top_4) - \varrho_1\int_{\top_4}^{\top}\frac{1}{r(s)U(s)\rho(s)}\left(\int_{\top_4}^{s}\rho(u)U(u)\int_a^b q(\top,u)\mathrm{d}u\mathrm{d}s\right)\mathrm{d}s \to -\infty$$

as $t \to \infty$, we find

$$g(\top) \leq g(\top_4) - \varrho_1\int_{\top_0}^{\infty}\frac{1}{r(s)U(s)\rho(s)}\left(\int_{\top_0}^{s}\rho(u)U(u)\int_a^b q(\top,u)\mathrm{d}u\mathrm{d}s\right)\mathrm{d}s \to -\infty.$$

Thus, $\varrho = 0$. Therefore, (19) holds. The proof is complete. □

2.2. Oscillation Results

Theorem 1. *Let (17) hold. If there exists increasing function $\rho \in C^1\left(I_0^+, \mathbb{R}^+\right)$ and (18) holds, then any solution of (1) is either oscillatory or $\lim_{\top\to\infty} \varkappa(\top) = 0$.*

Proof. Let $\varkappa(\top) > 0$, that is, $\varkappa(\zeta(\top))$ and $\varkappa(\sigma(\top,s))$ are positive on $[\top_1,\infty)$ for $\top_1 \geq \top_0$. From Lemma 3, *either Case* (Y_1) or *Case* (Y_2) holds. By Lemma 5, it is easy to see that (Y_2) holds. From Lemma 6, it follows that any solution of (1) is either oscillatory or satisfies (19). This completes the proof. □

Theorem 2. *Assume that (16) and (17) are satisfied. If*

$$\int_{\top_0}^{\infty} \frac{\left(\int_{\top_0}^{\top}\left(\int_a^b q(\top,u)\mathrm{d}u\right)U(v)A^{\lambda}(\sigma(v,s))\mathrm{d}v\right)}{r(\top)U(\top)}\mathrm{d}\top = \infty, \tag{23}$$

then all solutions of (1) are oscillatory.

Proof. Let $\varkappa(\top) > 0$, that is, $\varkappa(\zeta(\top))$ and $\varkappa(\sigma(\top,s))$ are positive on $[\top_1,\infty)$ for $\top_1 \geq \top_0$. By (17), from Lemma 5, we see that g satisfies (Y_2) for all $\top \geq \top_2 \geq \top_1$. Similarly to the proof of the Lemma 6, we obtain (20) and since $(Urg')' \leq 0$, we have

$$\begin{aligned} g(\top) &\geq -\int_{\top}^{\infty} \frac{1}{r(s)U(s)}U(s)r(s)g'(s)\mathrm{d}s \\ &\geq -A(\top)U(\top)r(\top)g'(\top), \end{aligned}$$

hence

$$\left(\frac{g}{A}\right) \text{ is nondecreasing}$$

and

$$\frac{g(\top)}{A(\top)} \geq \overline{M} \text{ for } \top \geq T_x,\ \overline{M} > 0.$$

(20) yields

$$(Urg')'(\top) + M_4 U(\top)\left(\int_a^b q(\top,u)\mathrm{d}u\right)A^{\lambda}(\sigma(\top,s)) \leq 0,\ \ M_4 := \varepsilon\overline{M}^{\lambda} \ \text{ for } \top \geq T_x. \tag{24}$$

From (24), we obtain

$$\begin{aligned} U(\top)r(\top)g'(\top) &\leq U(T_x)r(T_x)g'(T_x) - M_4\int_{T_x}^{\top} U(s)\left(\int_a^b q(\top,u)\mathrm{d}u\right)A^{\lambda}(\sigma(\top,s))\mathrm{d}s \\ &\leq -M_4\int_{T_x}^{\top} U(s)q(\top,s)A^{\lambda}(\sigma(\top,s))\mathrm{d}s. \end{aligned}$$

Integrating from T_x to $\top$, and according to (1), we find

$$g(\top) \leq g(T_x) - M_4\int_{T_x}^{\top} \frac{\left(\int_{T_x}^{s} U(u)\left(\int_a^b q(\top,v)\mathrm{d}v\right)A^{\lambda}(\sigma(\top,u))\mathrm{d}u\right)}{U(s)r(s)}\mathrm{d}s \to -\infty$$

as $\top \to \infty$, we see that

$$g(\top) \leq g(T_x) - M_4\int_{\top_0}^{\infty} \frac{\left(\int_{\top_0}^{\top} U(u)\left(\int_a^b q(\top,v)\mathrm{d}v\right)A^{\lambda}(\sigma(\top,u))\mathrm{d}u\right)}{U(\top)r(\top)}\mathrm{d}\top \to -\infty,$$

this contradicts (23). The proof is complete. □

Theorem 3. *Let (16) and (17) be held. If* $\lambda = 1$ *and*

$$\limsup_{\top\to\infty}\left(A(\top) \times \int_{\top_0}^{\top} U(s)\int_a^b q(\top,u)\mathrm{d}u\mathrm{d}s\right) > 1, \tag{25}$$

then all solutions of (1) are oscillatory.

Proof. Let $\varkappa(\top) > 0$, that is, $\varkappa(\zeta(\top)) > 0$ and $\varkappa(\sigma(\top,s)) > 0$ on $[\top_1,\infty)$ for some $\top_1 \geq \top_0$. From (4) and Lemma 5, it is easy to see that g satisfies (Y_2) for all $\top \geq \top_2 \geq \top_1$. Also, by Lemma 6, g satisfies (20). Integrating (20) from $\top$ to $\top_3$, we have

$$\begin{aligned} -U(\top)r(\top)g'(\top) &\geq \varepsilon \int_{\top_3}^{\top} U(s)\left(\int_a^b q(\top,u)\mathrm{d}u\right) g^{\lambda}(\sigma(\top,s))\mathrm{d}s \\ &\geq \varepsilon g^{\lambda}(\sigma(\top,s)) \int_{\top_3}^{\top} U(s) \int_a^b q(\top,u)\mathrm{d}u\mathrm{d}s, \end{aligned} \tag{26}$$

where $\varepsilon \in (0,1)$ and $\top \geq \top_3$ for some $\top_3 \in [\top_2,\infty)$. Using (20) in the latter inequality, we have

$$\begin{aligned} -U(\top)r(\top)g'(\top) &\geq \varepsilon g^{\lambda}(\top) \int_{\top_3}^{\top} U(s) \int_a^b q(\top,u)\mathrm{d}u\mathrm{d}s \\ &\geq \varepsilon A^{\lambda}(\top)\left(-U(\top)r(\top)g'(\top)\right)^{\lambda} \int_{\top_3}^{\top} U(s) \int_a^b q(\top,u)\mathrm{d}u\mathrm{d}s. \end{aligned}$$

This implies that

$$\left(-U(\top)r(\top)g'(\top)\right)^{1-\lambda} \geq \varepsilon A^{\lambda}(\top) \int_{\top_3}^{\top} U(s) \int_a^b q(\top,u)\mathrm{d}u\mathrm{d}s, \tag{27}$$

for any $0 < \varepsilon < 1$ and $\top \geq \top_3$. If $\lambda = 1$, then (27) implies

$$\varepsilon A(\top) \int_{\top_3}^{\top} U(s) \int_a^b q(\top,u)\mathrm{d}u\mathrm{d}s \leq 1.$$

The proof is complete. □

Theorem 4. *Let (16) be satisfied and (17) hold. Assume that $\sigma(\top,s)$ has nonnegative partial derivatives and $\sigma(\top,s) < \top$. Then, (1) exhibits oscillatory behavior if any one of the following conditions is true:*

$$\liminf_{\top\to\infty} \int_{\sigma(\top,s)}^{\top} \frac{1}{r(s)U(s)}\left(\int_{\top_0}^{s} U(u)\left(\int_a^b q(\top,v)\mathrm{d}v\right)\mathrm{d}u\right)\mathrm{d}s > \frac{1}{e} \ \textit{when } \lambda = 1 \tag{28}$$

or

$$\int_{\top_0}^{\infty} \frac{1}{r(\top)U(\top)}\left(\int_{\top_0}^{\top} U(s) \int_a^b q(\top,u)\mathrm{d}u\mathrm{d}s\right)\mathrm{d}\top = \infty \ \textit{when } \lambda < 1. \tag{29}$$

Proof. Let $\varkappa(\top) > 0$, that is, $\varkappa(\zeta(\top)) > 0$ and $\varkappa(\sigma(\top,s)) > 0$ on $[\top_1,\infty)$ for some $\top_1 \geq \top_0$. By (17) and Lemma 5, g satisfies (Y_2) for all $\top \geq \top_2 \geq \top_1$. By virtue of Theorem 3, we see that (26) holds; thus, $g > 0$ is considered a solution to inequality

$$g'(\top) + \left(\varepsilon \frac{\int_{\top_3}^{\top} U(s) \int_a^b q(\top,u)\mathrm{d}u\mathrm{d}s}{r(\top)U(\top)}\right) g^{\lambda}(\sigma(\top,s)) \leq 0, \text{ for all } \varepsilon \in (0,1),$$

for $\top \geq \top_3$. By ([6] Theorem 5.1.1), we note that the associated delay differential equation

$$g'(\top) + \left(\varepsilon \frac{\int_{\top_3}^{\top} U(s) \int_a^b q(\top,u)\mathrm{d}u\mathrm{d}s}{r(\top)U(\top)}\right) g^{\lambda}(\sigma(\top,s)) = 0 \tag{30}$$

also possesses a positive solution. Consequently, by Lemma 2, the conditions (28) or (29) confirm the oscillation of (30) if $\lambda = 1$ or $\lambda < 1$, respectively. Therefore, (1) cannot have an eventually positive solution; this inconsistency concludes the proof. □

Corollary 1. *Let (16) and (4) be satisfied. Assume that* $\lambda = 1$, $\sigma'(\top, s) \geq 0$ *and* $\sigma(\top, s) < \top$. *If*

$$\int_{\top}^{\top+\zeta} \left(\frac{\varepsilon}{r(\top)U(\top)} \int_{\top_3}^{\top} U(s) \int_a^b q(\top, u) \mathrm{d}u \mathrm{d}s \right) \mathrm{d}\top > 0 \tag{31}$$

and

$$\int_{\top_0}^{\infty} \left(\frac{\varepsilon}{r(\top)U(\top)} \int_{\top_3}^{\top} U(s) \int_a^b q(\top, u) \mathrm{d}u \mathrm{d}s \right) \ln\left(e \int_{\top}^{\top+\zeta} p(s) \mathrm{d}s \right) \mathrm{d}\top = \infty, \tag{32}$$

then all of the solutions of (1) are oscillatory.

Proof. In view of Lemma (2), we find that conditions (31) and (32) imply the oscillation of equation

$$g'(\top) + F(\top) \left(\varepsilon \frac{\int_{\top_3}^{\top} U(s) \int_a^b q(\top, u) \mathrm{d}u \mathrm{d}s}{r(\top)U(\top)} \right) g(\sigma(\top, s)) = 0,$$

where

$$F(\top) = \left(\varepsilon \frac{\int_{\top_3}^{\top} U(s) \int_a^b q(\top, u) \mathrm{d}u \mathrm{d}s}{r(\top)U(\top)} \right).$$

The proof is complete. □

3. Applications

Example 1. *Consider the following equation*

$$\left(\top^2 \left(\varkappa(\top) + \frac{1}{\top^2} \varkappa^3(2\top) \right)' \right)' + \top g'(\top) + \int_0^1 s^6 \varkappa^3 \left(\frac{s}{3} \right) \mathrm{d}s = 0, \quad \top \in [1, \infty). \tag{33}$$

From (33), we note that $\gamma = \lambda = 3, r(\top) = \top^2$, $\nu(\top) = \frac{1}{\top^2}$, $q(\top, s) = \top^6$, $\zeta(\top) = 2\top$, $\sigma(\top, s) = \frac{\top}{3}$, *which satisfy conditions* (C_1) *and* (C_2). *Furthermore, (17) holds and*

$$\lim_{\top \to \infty} \frac{A^{\gamma}(\zeta(\top))}{A(\top)} \nu(\top) = 0.$$

Hence, (16) satisfies. Also,

$$\begin{aligned} &\int_{\top_0}^{\infty} \frac{\left(\int_{\top_0}^{\top} q(\top, s) U(s) A^{\lambda}(\sigma(\top, s)) \mathrm{d}s \right)}{r(\top)U(\top)} \mathrm{d}\top \\ = \; &\int_1^{\infty} \top^{-3} \int_1^{\top} \frac{729}{8} s \mathrm{d}s \\ = \; &729(16)^{-1} \int_1^{\infty} \left(\top^{-1} - \top^{-3} \right) \mathrm{d}\top = \infty. \end{aligned}$$

Note that (23) holds. By Theorem 2, Equation (33) is oscillatory.

Example 2. *Consider the following equation:*

$$\left(\top^{3/2} \left(\varkappa(\top) + \frac{1}{\top} \varkappa^3(2\top) \right)' \right)' + \top^{1/2} g'(\top) + \int_0^1 b s^{-1/2} \varkappa(s/2) \mathrm{d}s = 0, \tag{34}$$

$\top \in [1,\infty)$, $\lambda = 1, \gamma = 3$, and $b > 0$. By (34), we note that $r(\top) = \top^{3/2}$, $\nu(\top) = \frac{1}{\top}$, $\eta(\top) = \top^{1/2}$, $q(\top,s) = b\top^{-1/2}$, $\zeta(\top) = 2\top$, $\sigma(\top,s) = \frac{\top}{2}$, which satisfy conditions (C_1) and (C_2). Furthermore, (17) holds,

$$A(\top) = 2\left(1 - \top^{-\frac{1}{2}}\right)$$

and

$$\lim_{\top\to\infty} \frac{1}{A(\top)}\nu(\top)A^{\gamma}(\zeta(\top)) = \lim_{\top\to\infty} \frac{4}{\left(\top - \top^{\frac{1}{2}}\right)}\left(1 - (2\top)^{-\frac{1}{2}}\right)^3 = 0$$

which yields (16) satisfies. Moreover,

$$\begin{aligned} \limsup_{\top\to\infty}\left\{\frac{2}{3\top^{3/2}}\int_1^{\top} U(s)q(\top,s)\mathrm{d}s\right\} &= \lim_{\top\to\infty}\frac{2}{3}\top^{-3/2}\int_1^{\top} bs^{1/2}\mathrm{d}s \\ &= \lim_{\top\to\infty}\frac{4}{9}\left(b - b\top^{-3/2}\right)^3 \\ &= \frac{4}{9}b. \end{aligned}$$

It's clear that (25) holds if $b > 9/4$. By Theorem 3, we see that Equation (34) is oscillatory if

$$b > \frac{9}{4}.$$

Example 3. *Consider Equation (34). As in Example 2, it satisfies conditions (C_1) and (C_2). Since (17) holds and*

$$\liminf_{\top\to\infty}\int_{\sigma(\top,s)}^{\top}\frac{1}{r(s)U(s)}\left(\int_{\top_0}^{s} U(u)q(\top,u)\mathrm{d}u\right)\mathrm{d}s > \frac{1}{e}$$

it follows that

$$\liminf_{\top\to\infty}\int_{\frac{\top}{2}}^{\top}\frac{1}{s^{5/2}}\left(b(s)^{3/2} - b\right)\mathrm{d}s > \frac{1}{e}$$

Thus, by Theorem 4, we see that all of the solutions of (34) are oscillatory if

$$b > \frac{1}{e\ln 2}.$$

4. Conclusions

Through this work, the oscillatory properties of a class of second-order differential equations with distributed deviating arguments were studied. We present some properties related to non-oscillatory solutions of the types (Y_1) and (Y_2), and then we employ these properties to reach the oscillation criteria of the Equation (1). The oscillation criteria mentioned in this paper do not require additional conditions. We find that most of the previous literature has addressed results that cannot be applied to our more general equation, which is in case of a U neutral term. Additionally, there is a scarcity of results that study second-order differential equations with the damping term and a U neutral term; see, for example, see [7,8,20,25–33]. Based on the above, the results of this paper are an improvement, extension, and completion of the previous results.
Studying the following equation:

$$\left(rg'(\top)\right)'(\top) + d(\top)g'(\top) + \sum_{i=1}^{m} q_i(\top,s)\varkappa^{\lambda}(\sigma_i(\top,s)) = 0$$

is expected to significantly contribute to the enhancement and development of oscillation theory in future scientific fields. In addition, discussing the results of Equation (1) when $\sigma(\top,s) > \top$ or if the damping function $d \in C(I_0^+,\mathbb{R})$ will be an interesting research point for researchers.

Author Contributions: Conceptualization, A.A.-J. and B.Q.; methodology, B.Q., A.A.-J. and C.C.; investigation, A.A.-J. and B.Q.; writing-original draft preparation, A.A.-J., C.C. and B.Q.; and writing—review and editing, A.A.-J., C.C., B.Q. and L.F.I. All authors have read and agreed to the published version of the manuscript.

Funding: Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2024R406), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The authors would like to thank Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2024R406), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Aldiaiji, M.; Qaraad, B.; Iambor, L.F.; Rabie, S.S.; Elabbasy, E.M. Oscillation of Third-Order Differential Equations with Advanced Arguments. *Mathematics* **2024**, *12*, 93. [CrossRef]
2. Qaraad, B.; AL Nuwairan, M. Asymptotic behavior of solutions of the third-order nonlinear advanced differential equations. *AIMS Math.* **2023**, *8*, 23800–23814. [CrossRef]
3. Al-Jaser, A.; Qaraad, B.; Ramos, H.; Serra-Capizzano, S. New Conditions for Testing the Oscillation of Solutions of Second-Order Nonlinear Differential Equations with Damped Term. *Axioms* **2024**, *13*, 105. [CrossRef]
4. Agarwal, R.P.; Grace, S.R.; O'Regan, D. *Oscillation Theory for Difference and Functional Differential Equations*; Springer Science & Business Media: New York, NY, USA, 2013.
5. Dosly, O.; Rehak, P. *Half-Linear Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2005; Volume 202.
6. Erbe, L.H.; Kong, Q.; Zhang, B. *Oscillation Theory for Functional Differential Equations*; Marcel Dekker, Inc.: New York, NY, USA, 1995; p. 190.
7. Bohner, M.; Li, T. Kamenev-type criteria for nonlinear damped dynamic equations. *Sci. China Math.* **2015**, *58*, 1445–1452. [CrossRef]
8. Bohner, M.; Saker, S.H. Oscillation of damped second order nonlinear delay differential equations of Emden-Fowler type. *Adv. Dyn. Syst. Appl.* **2006**, *1*, 163–182.
9. Al Themairi, A.; Qaraad, B.; Bazighifan, O.; Nonlaopon, K. Third-order neutral differential equations with damping and distributed delay: New asymptotic properties of solutions. *Symmetry* **2022** , 14, 2192. [CrossRef]
10. Liu, Q.; Bohner, M.; Grace, R.S.; Li, T. Asymptotic behavior of even-order damped differential equations with p-Laplacian like operators and deviating arguments. *J. Inequal. Appl.* **2016**, *2016*, 321. [CrossRef]
11. Iskandarov, S.; Komartsova, E. On the influence of integral perturbations on the boundedness of solutions of a fourth-order linear differential equation. *TWMS J. Pure Appl. Math.* **2022**, *13*, 3–9.
12. Pankov, P.S.; Zheentaeva, Z.K.; Shirinov, T. Asymptotic reduction of solution space dimension for dynamical systems. *TWMS J. Pure Appl. Math.* **2021**, *12*, 243–253.
13. Musaev, H.K. The Cauchy problem for degenerate parabolic convolution equation. *TWMS J. Pure Appl. Math.* **2021**, *12*, 278–288.
14. Li, T.; Baculíková, B.; Dzurina, J.; Zhang, C. Oscillation of fourth-order neutral differential equations with p-Laplacian like operators bound. *Chaos* **2014**, *2014*, 56.
15. Moaaz, O.; Fahd, M.; Clemente, C.; Shami, A.M.; Alsallami, E.M.K.; Mohamed, L.B. Neutral Differential Equations of Second-Order: Iterative Monotonic Properties. *Mathematics* **2022**, *10*, 1356. [CrossRef]
16. Zhang, C.; Agarwal, R.P.; Li, T. Oscillation and asymptotic behavior of higher-order delay differential equations with p-Laplacian like operators. *J. Math. Analy. Appl.* **2014**, *409*, 1093–1106. [CrossRef]
17. Hale, J.K. *Functional Differential Equations*; Springer: New York, NY, USA, 1971; p. 3. [CrossRef]
18. Jayakumar, C.; Santra, S.S.; Baleanu, D.; Edwan, R.; Govindan, V.; Murugesan, A.; Altanji, M. Oscillation result for half-linear delay difference equations of second-order. *Math. Biosci. Eng.* **2022**, *19*, 3879–3891. [CrossRef] [PubMed]
19. Shokri, A. The symmetric P-stable hybrid Obrenchkoff methods for the numerical solution of second order IVPs. *TWMS J. Pure Appl. Math.* **2012**, *5*, 28–35.
20. Agarwal, R.P.; Bohner, M.; Li, T. Oscillatory behavior of second-order half-linear damped dynamic equations. *Appl. Math. Comput.* **2015**, *254*, 408–418. [CrossRef]
21. Tunç, E.; Kaymaz, A. On oscillation of second-order linear neutral differential equations with damping term. *Dynam. Syst. Appl.* **2019**, *28*, 289–301. [CrossRef]
22. Tunç, E.; Kaymaz, A. Oscillatory behavior of second-order half-linear neutral differential equations with damping. *Adv. Dyn. Syst. Appl.* **2019**, *14*, 213–227. [CrossRef]
23. Grace, S.R.; Graef, J.R.; Tunç, E. Oscillatory behavior of second order damped neutral differential equations with distributed deviating arguments. *Miskolc Math. Notes.* **2017**, *18*, 759–769. [CrossRef]

24. Tunç, E.; Osman Özdemir, O. Oscillatory behavior of second-order damped differential equations with a *U* neutral term. *Opusc. Math.* **2020**, *40*, 629–639. [CrossRef]
25. Agarwal, R.P.; Bohner, M.; Li, W.-T. *Nonoscillation and Oscillation Theory for Functional Differential Equations*; CRC Press: Boca Raton, FL, USA, 2004. [CrossRef]
26. Baculíkova, B.; Dzurina, J. Oscillation theorems for second-order nonlinear neutral differential equations. *Comput. Math. Appl.* **2011**, 62, 4472–4478. [CrossRef]
27. Li, T.; Agarwal, R.P.; Bohner, M. Some oscillation results for second-order neutral dynamic equations. *Hacet. J. Math. Stat.* **2012**, *41*, 715–721.
28. Li, T.; Rogovchenko, Y.V. Oscillation theorems for second-order nonlinear neutral delay differential equations. *Abstr. Appl. Anal.* **2014**, 1–5. . [CrossRef]
29. Li, T.; Rogovchenko, Y.V. Oscillation of second-order neutral differential equations. *Math. Nachr.* **2015**, *288*, 1150–1162. [CrossRef]
30. Li, T.; Rogovchenko, Y.V. Oscillation criteria for second-order *U* Emden-Fowler neutral differential equations. *Monatsh. Math.* **2017**, *184*, 489–500. [CrossRef]
31. Graef, J. R.; Li, T.; Thandapani, E.; Tunc, E. Oscillation of secondorder Emden-Fowler neutral differential equations. *Nonlinear Stud.* **2013**, *20*, 1–8.
32. Thandapani, E.; Padmavathi, S.; Pinelas, S. Classifications of solutions of second-order nonlinear neutral differential equations of mixed type. *Adv. Differ. Equ.* **2012**, *2012*, 1–13. . [CrossRef]
33. Wong, S.W.J. Necessary and suffcient conditions for oscillation of second order neutral differential equations. *J. Math. Anal. Appl.* **2000**, *252*, 342–352. [CrossRef]
34. Aldiaiji, M.; Qaraad, B.; Lambor, L.F.; Elabbasy, E.M. New oscillation theorems for second-order superlinear neutral differential equations with variable damping terms. *Symmetry* **2023**, *15*, 1630. [CrossRef]
35. Philos, C.G. On the existence of nonoscillatory solutions tending to zero at â^ž for differential equations with positive delays. *Arch. Math.* **1981**, *36*, 168–178.
36. Li, B. Oscillation of first order delay differential equations. *Proc. Amer. Math. Soc.* **1996**, 124, 3729–3737. [CrossRef]