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Boundary Value Problems for the Perturbed Dirac Equation

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Abstract: The perturbed Dirac operators yield a factorization for the well-known Helmholtz equation. In this paper, using the fundamental solution for the perturbed Dirac operator, we define Cauchy-type integral operators (singular integral operators with a Cauchy kernel). With the help of these operators, we investigate generalized Riemann and Dirichlet problems for the perturbed Dirac equation which is a higher-dimensional generalization of a Vekua-type equation. Furthermore, applying the generalized Cauchy-type integral operator $\tilde{F}_\lambda$, we construct the Mann iterative sequence and prove that the iterative sequence strongly converges to the fixed point of operator $\tilde{F}_\lambda$.

Keywords: perturbed Dirac equation; Cauchy-type integral operator; Clifford analysis

MSC: 30G35; 45J05

1. Introduction

A large number of interesting physical applications, for instance, problems in elasticity theory of shells and in gas dynamics, lead to so-called Vekua-type problems [1,2]. The Vekua-type problem is natural to generalize the function theory to higher dimensions. In Clifford algebra framework, the Dirac equation perturbed by a constant, i.e., $(\partial_x - \lambda)u = f$, where λ is a constant and u and f are Clifford algebra-valued functions [3]. In this paper, we study boundary value problems for Clifford algebra-valued partial differential equation.

Real Clifford analysis is a hyper-complex function theory defined in space $\mathbf{R}^m$ and taking values in Clifford algebra (named after William Kingdon Clifford [4]), referred to as the study of this extension of complex analysis [5–8]. Developing the corresponding theories in the Clifford analysis framework comparing with the theories in classical complex analysis is natural and interesting. In [9,10], some boundary value problems in Clifford analysis are studied, such as boundary value problems for plurigeneralized regular and pluri-Beltrami equations are considered, where pluriregular equations are related to the generalized Laplace equations, etc. In [11], Riemann-type boundary value problems are studied for the Helmholtz equations in Clifford analysis. Here, a Helmholtz equation is an elliptic partial differential equation which is regarded as the perturbed Laplace equation. It is a remarkable fact that the perturbed Dirac operators yield a factorization for the well-known Helmholtz equation. As far as we know, the related Riemann and Dirichlet problems for the perturbed Dirac equation have little studies in Clifford analysis. In [12], boundary value problems for the perturbed Dirac operators are studied based on Maxwell's system. In [13], applying Almansi-type expansions, the first author considered Riemann-type problems for the modified Dirac equation perturbed by a constant. In this paper, applying the theory of singular integral operators, we investigate generalized Riemann and Dirichlet problems for the perturbed Dirac equation.

The paper is organized as follows. In Section 2, we review some results on the theory of the perturbed Dirac operator in Clifford analysis, necessary for later. In Section 3, we introduce Cauchy-type integral operators (i.e., the integral operators with a Cauchy kernel) and the Plemelj formula for the perturbed operator. In Section 4, we introduce a integral

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operator to transform boundary value problems into integral equation problems. Then, we use the theory of integral equation and fixed point theory to prove the existence of solutions to boundary value problems and provide an integral expression for the solution. In Section 5, we investigate a kind of generalized Dirichlet problem for the perturbed Dirac equation by Cauchy-type integral operators. In Section 6, we define a Cauchy-type integral operator $\tilde{F}_\lambda$. Furthermore, we construct the Mann iterative sequence of operator $\tilde{F}_\lambda$ and prove that the iterative sequence strongly converges to the fixed point of operator $\tilde{F}_\lambda$.

2. Preliminaries

In this section, we briefly review some notions and basic facts about the perturbed Dirac operator in Clifford analysis which will be useful in the future. For more details, we refer the reader to the literature, e.g., [3–8].

2.1. Perturbed Dirac Operator

Clifford analysis is the study of the null solutions of Dirac operators. The Dirac operator is the generalization of the Cauchy Riemann operator in higher-dimensional spaces. When working over $\mathbf{R}^m$, this Dirac operator is defined by

$$\partial_x = \sum_{i=1}^{m} e_i \partial_{x_i},$$

where the e_i values form an orthonormal basis of $\mathbf{R}^m$ and satisfy the defining relations of the real 2^m-dimension Clifford algebra Cl_m, i.e.,

$$e_i e_j + e_j e_i = -2\delta_{ij}, \quad i, j = 1, \cdots, m.$$

One of the important elements is the vector variable $x = \sum_{i=1}^{m} x_i e_i$.

We let $\Omega \in \mathbf{R}^m$. Function f can be written as $f = \sum_A f_A(x) e_A$ with $f_A(x) \in C^k(\Omega)$. We denote $|f| = \left(\sum_A [f_A(x)]^2 \right)^{\frac{1}{2}}$ to be the norm of f.

The perturbed Dirac operator is defined by

$$\partial_x - \lambda = \sum_{i=1}^{m} e_i \partial_{x_i} - \lambda.$$

We denote $\ker(\partial_x - \lambda) = \{ f | (\partial_x - \lambda) f = 0, \quad f \in C^1(\Omega; Cl_m) \}$.

2.2. Fundamental Solutions for the Perturbed Dirac Operator

The fundamental solution for the Dirac operator is defined by

$$E(x) = -\frac{1}{\omega_m} \frac{x}{|x|^m},$$

where $\omega_m = \frac{(2\pi)^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}$ is the area of the unit sphere in $\mathbf{R}^m$.

Definition 1 ([3])**.** *The fundamental solution for operator $\partial_x - \lambda$, with $\lambda \in \mathbf{R}$, is defined by*

$$E_\lambda(x) = -\frac{\pi}{\omega_m \Gamma(\frac{m}{2})} \left(\frac{\lambda}{2} \right) |x|^{1-\frac{m}{2}} \left[Y_{(\frac{m}{2})-1}(\lambda |x|) - \frac{x}{|x|} Y_{\frac{m}{2}}(\lambda |x|) \right],$$

where $Y_\nu(x)$ and $J_\nu(x)$ are the Bessel functions of, respectively, the first and second kind ([14]), namely

$$Y_\nu(x) = \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi},$$

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{l=0}^{\infty} \frac{(\frac{ix}{2})^{2l}}{l!\Gamma(\nu+l+1)}.$$

Lemma 1 ([3])**.** *$E_\lambda(x)$ has the following properties:*

(i)

$$(\partial_x - \lambda)E_\lambda(x) = E_\lambda(x)(\partial_x - \lambda) = \delta(x),$$

where $\delta(x)$ is the Dirac distributional in $\mathbf{R}^m$.

(ii) $E_\lambda(x)$ behaves like $E(x)$ near $x = 0$, i.e.,

$$E(x) \sim -\frac{1}{\omega_m} \frac{x}{|x|^m}$$

for $|x| \to 0$.

Lemma 2 ([3])**.** *Assume that $\Omega \in \mathbf{R}^m$, where Ω is an $m-$dimensional compact differentiable and oriented manifold-with-boundary. If $f(x) \in C(\Omega; Cl_m)$, then*

$$\int_{\partial\Omega} E_\lambda(x-y)d\sigma_x f(x) - \int_\Omega E_\lambda(x-y)[(\partial_x - \lambda)f(x)]dx = \begin{cases} f(y), & y \in \Omega, \\ 0, & y \in \mathbf{R}^m \setminus \overline{\Omega}. \end{cases}$$

3. Cauchy-Type Integral Operator

Definition 2 ([15])**.** *A compact surface Γ is called a Liapunov surface with Hölder exponent α if the following conditions are satisfied:*

(i) At each point $x \in \Gamma$, there is a tangential plane.

(ii) There exists a number r, that for any point $x \in \Gamma$, the set $\Gamma \cap B_r(x)$ (Liapunov ball) is connected and parallel lines to the outer normal $n(x)$ intersect at not more than one point.

(iii) The normal $n(x)$ is Hölder continuous on Γ, i.e., there are constants $M > 0$ and $0 < \alpha < 1$, such that for $x, y \in \Gamma$,

$$|n(x) - n(y)| \leq M|x-y|^\alpha.$$

Definition 3 ([3])**.** *Assume that $\Omega \subset R^m$ with Liapunov boundary $\partial\Omega$. And let $0 < \alpha < 1$. If $f(x)$ satisfies*

$$|f(x_1) - f(x_2)| \leq M|x_1 - x_2|^\alpha, \quad x_1, x_2 \in \partial\Omega,$$

then $f(x) : \partial\Omega \to Cl_m$ is said to be Hölder continuous on $\partial\Omega$ with index α. We denote by $C_\alpha(\partial\Omega; Cl_m)$ the set of all Hölder-continuous function on $\partial\Omega$ with index α.

Remark 1. *The $(m-1)$ differential form $d\sigma = \sum_{i=1}^{m} e_i d\widehat{x}_i$, where $d\widehat{x}_i = dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots dx_m$. If dS stands for the classical surface element and $\overrightarrow{n} = \sum_{i=1}^{m} e_i n_i$, where n_i is the ith component of the unit outward normal vector, then the surface element $d\sigma = \overrightarrow{n} dS$.*

Definition 4. *The integral operator is defined by*

$$(F_\lambda f)(y) = \int_{\partial\Omega} E_\lambda(x-y)d\sigma_x f(x). \tag{1}$$

For $y \in \mathbf{R}^m \setminus \partial\Omega$, $f(y) \in C_\alpha(\partial\Omega; Cl_m)$, it is clear that the integral on the right side of Formula (1) is well defined. For $y \in \partial\Omega$, it is a singular integral. Thus, we offer the definition as follows.

Definition 5. *For $y_0 \in \partial\Omega$, the Cauchy principal value of a singular integral for a perturbed Dirac is defined by*

$$P.V.(F_\lambda f)(y) = \lim_{r\to 0} \int\limits_{\partial\Omega \setminus B_{\partial\Omega}(y_0,r)} E_\lambda(x-y_0)d\sigma_x f(x), \tag{2}$$

where $B_{\partial\Omega}(y_0, r)$ is denoted as the part of $\partial\Omega$ lying in the interior of sphere B with the center at y_0 and radius $r > 0$.

In the future, we will need the following lemma.

Lemma 3 ([4])**.** *Assume that $x, y \in R^m$, with $m \geq 2, l \geq 0$, are integers. Then,*

$$\left|\frac{x}{|x|^{l+2}} - \frac{y}{|y|^{l+2}}\right| \leq \begin{cases} |x-y| \sum\limits_{k=0}^{l} \frac{1}{|x|^{k+1}|y|^{l+1-k}}, & m \neq 0, \\ \frac{|x-y|}{|x|^{l+1}|y|^{l+1}}, & m = 0. \end{cases}$$

Now, we prove the following important lemma. The lemma offers an estimate of the fundamental solution for the perturbed Dirac operator.

Lemma 4. *If $y_1, y_2 \neq x$, then*

$$\begin{aligned} & |E_\lambda(x-y_1) - E_\lambda(x-y_2)| \\ \leq\ & C_1|y_2-y_1| \sum_{i=0}^{\frac{m}{2}-1} \frac{1}{|x-y_1|^{m-2-i}|x-y_2|^{1+i}} \\ & +C_2|y_2-y_1| \sum_{i=0}^{\frac{m}{2}-1} \frac{1}{|x-y_1|^{\frac{m}{2}-1-i}|x-y_2|^{\frac{m}{2}+i}} \\ & +C_3|y_1-y_2| \sum_{i=0}^{\frac{m}{2}} \frac{1}{|x-y_1|^{\frac{m}{2}-2-i}|x-y_2|^{\frac{m}{2}+1+i}} \\ & +C_4|y_1-y_2| \sum_{i=0}^{\frac{m}{2}} \frac{1}{|x-y_1|^{m-2-i}|x-y_2|^{1+i}}. \end{aligned}$$

Proof of Lemma 4. By Definition 1, we have

$$\begin{aligned} & E_\lambda(x-y_1) - E_\lambda(x-y_2) \\ = \ & -\frac{\pi\left(\frac{\lambda}{2}\right)^{\frac{m}{2}}}{\omega_m \Gamma\left(\frac{m}{2}\right)} |x-y_1|^{1-\frac{m}{2}} \left[Y_{\left(\frac{m}{2}\right)-1}(\lambda|x-y_1|) - \frac{x-y_1}{|x-y_1|} Y_{\frac{m}{2}}(\lambda|x-y_1|) \right] \\ & + \frac{\pi\left(\frac{\lambda}{2}\right)^{\frac{m}{2}}}{\omega_m \Gamma\left(\frac{m}{2}\right)} |x-y_2|^{1-\frac{m}{2}} \left[Y_{\left(\frac{m}{2}\right)-1}(\lambda|x-y_2|) - \frac{x-y_2}{|x-y_2|} Y_{\frac{m}{2}}(\lambda|x-y_2|) \right]. \end{aligned}$$

We let m be even. By Lemma 3, we have

$$\begin{aligned}
|I_1| &= \left||x-y_1|^{1-\frac{m}{2}}Y_{(\frac{m}{2})-1}(\lambda|x-y_1|) - |x-y_2|^{1-\frac{m}{2}}Y_{(\frac{m}{2})-1}(\lambda|x-y_2|)\right| \\
&\leq \left||x-y_1|^{1-\frac{m}{2}}Y_{(\frac{m}{2})-1}(\lambda|x-y_1|) - |x-y_2|^{1-\frac{m}{2}}Y_{(\frac{m}{2})-1}(\lambda|x-y_1|)\right| \\
&\quad + \left||x-y_2|^{1-\frac{m}{2}}Y_{(\frac{m}{2})-1}(\lambda|x-y_1|) - |x-y_2|^{1-\frac{m}{2}}Y_{(\frac{m}{2})-1}(\lambda|x-y_2|)\right| \\
&\leq \left||x-y_1|^{1-\frac{m}{2}} - |x-y_2|^{1-\frac{m}{2}}\right|\left|Y_{(\frac{m}{2})-1}(\lambda|x-y_1|)\right| \\
&\quad + |x-y_2|^{1-\frac{m}{2}}\left|Y_{(\frac{m}{2})-1}(\lambda|x-y_1|) - Y_{(\frac{m}{2})-1}(\lambda|x-y_2|)\right| \\
&\leq C_1|x-y_1|^{1-\frac{m}{2}}||y_2-y_1|\sum_{i=0}^{\frac{m}{2}-1}\frac{1}{|x-y_1|^{\frac{m}{2}-1-i}|x-y_2|^{1+i}} \\
&\quad + C_2|x-y_2|^{1-\frac{m}{2}}|y_2-y_1|\sum_{i=0}^{\frac{m}{2}-1}\frac{1}{|x-y_1|^{\frac{m}{2}-1-i}|x-y_2|^{1+i}} \\
&\leq C_1||y_2-y_1|\sum_{i=0}^{\frac{m}{2}-1}\frac{1}{|x-y_1|^{m-2-i}|x-y_2|^{1+i}} \\
&\quad + C_2|y_2-y_1|\sum_{i=0}^{\frac{m}{2}-1}\frac{1}{|x-y_1|^{\frac{m}{2}-1-i}|x-y_2|^{\frac{m}{2}+i}}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
|I_2| &= \left|\frac{x-y_2}{|x-y_2|^{\frac{m}{2}}}Y_{\frac{m}{2}}(\lambda|x-y_2|) - \frac{x-y_1}{|x-y_1|^{\frac{m}{2}}}Y_{\frac{m}{2}}(\lambda|x-y_1|)\right| \\
&\leq \left|\frac{x-y_2}{|x-y_2|^{\frac{m}{2}}}Y_{\frac{m}{2}}(\lambda|x-y_2|) - \frac{x-y_1}{|x-y_1|^{\frac{m}{2}}}Y_{\frac{m}{2}}(\lambda|x-y_2|)\right| \\
&\quad + \left|\frac{x-y_1}{|x-y_1|^{\frac{m}{2}}}Y_{\frac{m}{2}}(\lambda|x-y_2|) - \frac{x-y_1}{|x-y_1|^{\frac{m}{2}}}Y_{\frac{m}{2}}(\lambda|x-y_1|)\right| \\
&\leq \left|\frac{x-y_2}{|x-y_2|^{\frac{m}{2}}} - \frac{x-y_1}{|x-y_1|^{\frac{m}{2}}}\right|\left|Y_{\frac{m}{2}}(\lambda|x-y_2|)\right| \\
&\quad + \left|\frac{x-y_1}{|x-y_1|^{\frac{m}{2}}}\right|\left|Y_{\frac{m}{2}}(\lambda|x-y_2|) - Y_{\frac{m}{2}}(\lambda|x-y_1|)\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq C_3|x-y_2|^{-\frac{m}{2}}|y_1-y_2|\sum_{i=0}^{\frac{m}{2}}\frac{1}{|x-y_1|^{\frac{m}{2}-2-i}|x-y_2|^{1+i}}\\
&\quad+C_4|x-y_1|^{-\frac{m}{2}}|y_1-y_2|\sum_{i=0}^{\frac{m}{2}}\frac{1}{|x-y_1|^{\frac{m}{2}-2-i}|x-y_2|^{1+i}}\\
&\leq C_3|y_1-y_2|\sum_{i=0}^{\frac{m}{2}}\frac{1}{|x-y_1|^{\frac{m}{2}-2-i}|x-y_2|^{\frac{m}{2}+1+i}}\\
&\quad+C_4|y_1-y_2|\sum_{i=0}^{\frac{m}{2}}\frac{1}{|x-y_1|^{m-2-i}|x-y_2|^{1+i}}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&|E_\lambda(x-y_1)-E_\lambda(x-y_2)|\\
&\leq C_1|y_2-y_1|\sum_{i=0}^{\frac{m}{2}-1}\frac{1}{|x-y_1|^{m-2-i}|x-y_2|^{1+i}}\\
&\quad+C_2|y_2-y_1|\sum_{i=0}^{\frac{m}{2}-1}\frac{1}{|x-y_1|^{\frac{m}{2}-1-i}|x-y_2|^{\frac{m}{2}+i}}\\
&\quad+C_3|y_1-y_2|\sum_{i=0}^{\frac{m}{2}}\frac{1}{|x-y_1|^{\frac{m}{2}-2-i}|x-y_2|^{\frac{m}{2}+1+i}}\\
&\quad+C_4|y_1-y_2|\sum_{i=0}^{\frac{m}{2}}\frac{1}{|x-y_1|^{m-2-i}|x-y_2|^{1+i}}.
\end{aligned}
$$

By reduction of fractions to a common denominator and numerator rationalization, we have the conclusion that case m is odd. □

If $\Omega = \Omega^+$ and $\Omega^- = \mathbf{R}^m \setminus \overline{\Omega}$, then we consider the limit of the Cauchy principal value of a singular integral,

$$
(F_\lambda^+ f)(y) = \begin{cases} (F_\lambda f)(y), & y \in \Omega^+, \\ P.V.(F_\lambda f)(y) + \frac{1}{2}f(y), & y \in \partial\Omega. \end{cases}
$$

$$
(F_\lambda^- f)(y) = \begin{cases} (F_\lambda f)(y), & y \in \Omega^-, \\ P.V.(F_\lambda f)(y) - \frac{1}{2}f(y), & y \in \partial\Omega. \end{cases}
$$

where $P.V.(F_\lambda f)(y)$ is defined in Formula (2).

Theorem 1. *If $f \in C_\alpha(\Omega; Cl_m)$, then $(F_\lambda^+ f)(y) \in C_\alpha(\overline{\Omega^+}; Cl_m)$ and $(F_\lambda^- f)(y) \in C_\alpha(\overline{\Omega^-}; Cl_m)$.*

Proof of Theorem 1. We only prove $(F_\lambda^+ f)(y) \in C_\alpha(\overline{\Omega^+}; Cl_m)$. $(F_\lambda^- f)(y) \in C_\alpha(\overline{\Omega^-}; Cl_m)$ may be proved analogously.

We first prove the fact that

$$
\int_{\partial\Omega} E_\lambda(x-y)d\sigma_x = \frac{1}{2}, \quad y \in \partial\Omega.
$$

For any $y \in \partial\Omega$, $B_{\partial\Omega}(y,\delta)$ is the restriction to $\partial\Omega$ of the ball centered at y and of radius δ. Then,

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{B_{\partial\Omega}(y,\delta)} E_\lambda(x-y)d\sigma_x \\
=\;& \lim_{\delta\to 0}\int_{B_{\partial\Omega}(y,\delta)} \left(E_\lambda(x-y)+\frac{1}{\omega_m}\frac{x-y}{|x-y|^m}\right)d\sigma_x - \lim_{\delta\to 0}\int_{B_{\partial\Omega}(y,\delta)} \frac{1}{\omega_m}\frac{x-y}{|x-y|^m}d\sigma_x \\
=\;& -\lim_{\delta\to 0}\int_{B_{\partial\Omega}(y,\delta)} \frac{1}{\omega_m}\frac{x-y}{|x-y|^m}d\sigma_x \\
=\;& -\lim_{\delta\to 0}\int_{B_{\partial\Omega}(y,\delta)} \frac{1}{\omega_m}\frac{x-y}{|x-y|^m}\frac{x-y}{|x-y|}dS = \frac{1}{2}.
\end{aligned}$$

For the different values of y, we divide three cases to prove the result.

Case 1. For $y_1, y_2 \in \partial\Omega$, it is enough to consider the case of $|y_1-y_2|$ being sufficiently small. We denote $|y_1-y_2| = \delta$.

$$\begin{aligned}
&|P.V.(F_\lambda f)(y_1) - P.V.(F_\lambda f)(y_2)| \\
=\;& \left|\int_{\partial\Omega} E_\lambda(x-y_1)d\sigma_x f(x) - \int_{\partial\Omega} E_\lambda(x-y_2)d\sigma_x f(x)\right| \\
\le\;& \left|\int_{\partial\Omega} E_\lambda(x-y_1)d\sigma_x[f(x)-f(y_1)] - \int_{\partial\Omega} E_\lambda(x-y_2)d\sigma_x[f(x)-f(y_2)]\right| \\
&+\left|\int_{\partial\Omega} E_\lambda(x-y_1)d\sigma_x[f(y_1)-f(y_2)]\right| + \left|\int_{\partial\Omega} E_\lambda(x-y_2)d\sigma_x[f(y_1)-f(y_2)]\right| \\
&+\left|\int_{\partial\Omega} E_\lambda(x-y_1)d\sigma_x f(y_2)\right| + \left|\int_{\partial\Omega} E_\lambda(x-y_2)d\sigma_x f(y_1)\right| \\
\le\;& \left|\int_{\partial\Omega} E_\lambda(x-y_1)d\sigma_x[f(x)-f(y_1)] - \int_{\partial\Omega} E_\lambda(x-y_2)d\sigma_x[f(x)-f(y_2)]\right| \\
&+\frac{1}{2}|y_1-y_2|^\alpha + \frac{1}{2}|y_1-y_2|^\alpha + \frac{1}{2}|f(y_1)| + \frac{1}{2}|f(y_2)|.
\end{aligned}$$

We calculate the two integrals on the above inequality:

$$
\begin{aligned}
I &= \left| \int_{\partial\Omega} E_\lambda(x-y_1)d\sigma_x[f(x)-f(y_1)] - \int_{\partial\Omega} E_\lambda(x-y_2)d\sigma_x[f(x)-f(y_2)] \right| \\
&\leq \left| \int_{\partial\Omega\setminus B_{\partial\Omega}(y_1,2\delta)} E_\lambda(x-y_1)d\sigma_x[f(x)-f(y_1)] - \int_{\partial\Omega\setminus B_{\partial\Omega}(y_2,2\delta)} E_\lambda(x-y_2)d\sigma_x[f(x)-f(y_2)] \right| \\
&\quad + \left| \int_{B_{\partial\Omega}(y_1,4\delta)} E_\lambda(x-y_1)d\sigma_x[f(x)-f(y_1)] \right| \\
&\quad + \left| \int_{B_{\partial\Omega}(y_2,4\delta)} E_\lambda(x-y_2)d\sigma_x[f(x)-f(y_2)] \right| \\
&\leq | \int_{\partial\Omega\setminus B_{\partial\Omega}(y_1,2\delta)} E_\lambda(x-y_1)d\sigma_x[f(x)-f(y_1)] \\
&\quad - \int_{\partial\Omega\setminus B_{\partial\Omega}(y_2,2\delta)} E_\lambda(x-y_2)d\sigma_x[f(x)-f(y_2)]| \\
&\quad + \left| \int_{B_{\partial\Omega}(y_1,4\delta)} E(x-y_1)d\sigma_x[f(x)-f(y_1)] \right| \\
&\quad + \left| \int_{B_{\partial\Omega}(y_2,4\delta)} E(x-y_2)d\sigma_x[f(x)-f(y_2)] \right|.
\end{aligned}
$$

We calculate the first absolute value inequality on the above inequality:

$$
\begin{aligned}
I_1 &= \left| \int_{\partial\Omega\setminus B_{\partial\Omega}(y_1,2\delta)} E_\lambda(x-y_1)d\sigma_x[f(x)-f(y_1)] - \int_{\partial\Omega\setminus B_{\partial\Omega}(y_2,2\delta)} E_\lambda(x-y_2)d\sigma_x[f(x)-f(y_2)] \right| \\
&\leq \left| \int_{\partial\Omega\setminus B_{\partial\Omega}(y_1,2\delta)} [E_\lambda(x-y_1)-E_\lambda(x-y_2)]d\sigma_x[f(x)-f(y_1)] \right| \\
&\quad + \left| \int_{\partial\Omega\setminus B_{\partial\Omega}(y_2,2\delta)} E_\lambda(x-y_2)d\sigma_x[f(y_2)-f(y_1)] \right| \\
&= I_{11} + I_{12}.
\end{aligned}
$$

Thus, by Lemma 4 and in view of $\partial\Omega$ being compact and $f \in C_\alpha(\partial\Omega; Cl_m)$, there exists a positive constant L_0 independent of y_1, y_2 such that

$$\begin{aligned}
I_{11} &= \left| \int\limits_{\partial\Omega \setminus B_{\partial\Omega}(y_1, 2\delta)} [E_\lambda(x-y_1) - E_\lambda(x-y_2)] d\sigma_x [f(x) - f(y_1)] \right| \\
&\leq C_1(m,\lambda) 2\delta \int\limits_{2\delta}^{L_0} \frac{1}{\rho^{m-1}} \rho^{m-2} \rho^\alpha d\rho, \\
&\leq C_1(m,\lambda) 2\delta \int\limits_{2\delta}^{L_0} \frac{1}{\rho^{2-\alpha}} d\rho, \\
I_{12} &= \left| \int\limits_{\partial\Omega \setminus B_{\partial\Omega}(y_2, 2\delta)} E_\lambda(x-y_2) d\sigma_x [f(y_2) - f(y_1)] \right| \\
&\leq C_2(m,\lambda) \delta^\alpha \int\limits_{2\delta}^{L_0} \frac{1}{\rho^{m-1}} \rho^{m-2} d\rho \\
&= C_2(m,\lambda) \delta^\alpha \int\limits_{2\delta}^{L_0} \frac{1}{\rho} d\rho.
\end{aligned}$$

So we have

$$I_1 \leq C_3(m,\lambda) \delta^\alpha \ln \delta.$$

Now we calculate the second absolute value inequality on the above inequality:

$$\begin{aligned}
I_2 &= \left| \int\limits_{B_{\partial\Omega}(y_1, 4\delta)} E_\lambda(x-y_1) d\sigma_x [f(x) - f(y_1)] \right| \\
&\leq C_4(m,\lambda) \int\limits_0^{4\delta} \frac{1}{\rho^{m-1}} \rho^{m-2} \rho^\alpha d\rho \\
&\leq C_4(m,\lambda) \delta^\alpha.
\end{aligned}$$

Similarly, we have

$$I_3 = \left| \int\limits_{B_{\partial\Omega}(y_2, 4\delta)} E_\lambda(x-y_2) d\sigma_x [f(x) - f(y_2)] \right| \leq C_5(m,\lambda) \delta^\alpha.$$

Thus, $P.V.(F_\lambda f)(y) \in C_\alpha(\partial\Omega; Cl_m)$. It is easy to see that $(F_\lambda^+ f)(y) \in C_\alpha(\overline{\Omega^+}; Cl_m)$ for $y_1, y_2 \in \partial\Omega$.

Case 2. For $y_1 \in \Omega, y_2 \in \partial\Omega$, we prove

$$\mid (F_\lambda^+ f)(y_1) - (F_\lambda^+ f)(y_2) \mid \leq |y_1 - y_2|^\alpha.$$

Since $\partial\Omega$ is compact, for all $y_1 \in \Omega$, there exists point $y_{\tilde{1}}$ such that

$$|y_1 - y_{\tilde{1}}| = \inf_{y\in\partial\Omega} |y_1 - y|.$$

We also call $y_{\tilde{1}}$ the nearest distance point between y_1 and $\partial\Omega$. Obviously, $y_{\tilde{1}}$ exists, but it is not unique. It satisfies

$$|y_2 - y_{\tilde{1}}| \leq 2|y_2 - y_1| = 2\delta.$$

Thus, we have

$$\begin{aligned} & | \ (F_\lambda^+ f)(y_1) - (F_\lambda^+ f)(y_2) \ | \\ \leq & \ | \ (F_\lambda^+ f)(y_1) - (F_\lambda^+ f)(y_{\tilde{1}}) \ | + | \ (F_\lambda^+ f)(y_{\tilde{1}}) - (F_\lambda^+ f)(y_2)|. \end{aligned}$$

We calculate the first part in the above inequality:

$$\begin{aligned} & | \ (F_\lambda^+ f)(y_1) - (F_\lambda^+ f)(y_{\tilde{1}}) \ | \\ \leq & \left| \int\limits_{\partial\Omega\setminus B_{\partial\Omega}(y_1,2\delta)} [E_\lambda(x-y_1) - E_\lambda(x-y_{\tilde{1}})]d\sigma_x[f(x) - f(y_{\tilde{1}})] \right| \\ & + \left| \int\limits_{B_{\partial\Omega}(y_1,2\delta)} E(x-y_1)d\sigma_x[f(x) - f(y_{\tilde{1}})] \right| \\ & + \left| \int\limits_{B_{\partial\Omega}(y_1,2\delta)} E(x-y_{\tilde{1}})d\sigma_x[f(x) - f(y_{\tilde{1}})] \right| = J_1 + J_2 + J_3. \end{aligned}$$

In view of $|y_1 - y_{\tilde{1}}| \leq \delta$, and for $x \in \partial\Omega \setminus B_{\partial\Omega}(y_{\tilde{1}})$, it holds

$$|x - y_1| \leq 2|x - y_{\tilde{1}}|, \quad |x - y_{\tilde{1}}| \leq 2|x - y_1|.$$

Thus, by Lemma 4 and in view of $\partial\Omega$ being compact and $f \in C_\alpha(\partial\Omega; Cl_m)$, there exists a constant δ independent of y_1, y_2 such that

$$J_1 \leq C_6(m,\lambda)\delta^\alpha \ln\delta.$$

In view of $y_{\tilde{1}}$ being a nearest distance point between y_1 and $\partial\Omega$, $|x - y_{\tilde{1}}| \leq 2|x - y_1|$ holds for $x \in \partial\Omega$. Thus, we have

$$J_2 \leq C_7(m,\lambda)\delta^\alpha.$$

Similarly, it is easy to see that

$$J_3 \leq C_8(m,\lambda)\delta^\alpha.$$

From Case 1, we have

$$| \ (F_\lambda^+ f)(y_{\tilde{1}}) - (F_\lambda^+ f)(y_2)| \leq C_9(m,\lambda)\delta^\beta.$$

Thus, we have

$$| \ (F_\lambda^+ f)(y_1) - (F_\lambda^+ f)(y_2) \ | \leq |y_1 - y_2|^\alpha,$$

where $y_1 \in \Omega, y_2 \in \partial\Omega$.

Case 3. For $y_1, y_2 \in \Omega$, we denote $|y_1 - y_2| = \delta$. Since segment $\overline{y_1 y_2}$ and $\partial\Omega$ are compact, there exist $\widetilde{y} \in \overline{y_1 y_2}$, $\widetilde{y}_{\partial\Omega} \in \partial\Omega$ such that

$$|y_1 - y_2| = \inf_{y' \in \overline{y_1 y_2}, y'' \in \partial\Omega} |y' - y''|.$$

We denote $| \widetilde{y} - \widetilde{y}_{\partial\Omega} |= \delta_0$.

(i) If $\delta_0 = 0$, then $\widetilde{y} \in \partial\Omega$. By Case 2, we can see that

$$\begin{aligned} & | (F_\lambda^+ f)(y_1) - (F_\lambda^+ f)(y_2) | \\ \leq \;& | (F_\lambda^+ f)(y_1) - (F_\lambda^+ f)(\widetilde{y}) | + | (F_\lambda^+ f)(\widetilde{y}) - (F_\lambda^+ f)(y_2) | \\ \leq \;& C_{10}(m, \lambda)|y_1 - y_2|^\alpha. \end{aligned}$$

(ii) If $\delta_0 > 0, \delta \geq \frac{1}{2}\delta_0$, then $| \widetilde{y_1} - \widetilde{y}_{\partial\Omega} |\leq 3\delta, | \widetilde{y_2} - \widetilde{y}_{\partial\Omega} |\leq 3\delta, \widetilde{y} \in \partial\Omega$. By Case 2, we have

$$\begin{aligned} & | (F_\lambda^+ f)(y_1) - (F_\lambda^+ f)(y_2) | \\ \leq \;& | (F_\lambda^+ f)(y_1) - (F_\lambda^+ f)(\widetilde{y}) | + | (F_\lambda^+ f)(\widetilde{y}) - (F_\lambda^+ f)(y_2) | \\ \leq \;& C_{11}(m, \lambda)|y_1 - y_2|^\alpha. \end{aligned}$$

(iii) If $\delta_0 > 0, \delta < \frac{1}{2}\delta_0$, then for any $x \in \partial\Omega$,

$$| x - y_1 |\leq 2 | x - y_2 |, | x - y_2 |\leq 2 | x - y_1 |,$$

and

$$| x - \widetilde{y}_{\partial\Omega} |\leq 3 | x - y_1 |, | x - \widetilde{y}_{\partial\Omega} |\leq 2 | x - y_2 | .$$

By Case 2, we obtain

$$\begin{aligned} & | (F_\lambda^+ f)(y_1) - (F_\lambda^+ f)(y_2) | \\ \leq \;& C_{12}(m, \lambda)|y_1 - y_2|^\alpha \ln |y_1 - y_2|. \end{aligned}$$

Thus, we find that

$$| (F_\lambda^+ f)(y_1) - (F_\lambda^+ f)(y_2) |\leq |y_1 - y_2|^\alpha,$$

where $y_1, y_2 \in \Omega$.

To sum up, we have the result. □

Applying Theorem 1, we have the following theorem (i.e., Plemelj formula). For more details on Plemelj formula, you can refer to [7].

Theorem 2. *For $f(x) \in C_\alpha(\partial\Omega; Cl_m)$ and $y_0 \in \partial\Omega$,*

$$\begin{cases} (F_\lambda^+ f)(y_0) = P.V.(F_\lambda f)(y_0) + \frac{1}{2} f(y_0), \\ (F_\lambda^- f)(y_0) = P.V.(F_\lambda f)(y_0) - \frac{1}{2} f(y_0), \end{cases}$$

or

$$\begin{cases} (F_\lambda^+ f)(y_0) + (F_\lambda^- f)(y_0) = 2P.V.(F_\lambda f)(y_0), \\ (F_\lambda^+ f)(y_0) - (F_\lambda^- f)(y_0) = f(y_0), \end{cases}$$

where $(F_\lambda^+ f)(y_0)((F_\lambda^- f)(y_0))$ is the limit of $(F_\lambda f)(y)$ as $y \to y_0$ in $\Omega^+(\Omega^-)$.

4. A Generalized Riemann Problem for the Perturbed Dirac Equation

First, we offer the following definitions.

For $f \in C_\alpha(\partial\Omega; Cl_m)$,

$$\|f\|_\alpha = M(f, \partial\Omega) + H(f, \partial\Omega, \alpha),$$

where

$$M(f, \partial\Omega) = \sup_{t \in \partial\Omega} |f(t)|,$$

and

$$H(f, \partial\Omega, \alpha) = \sup_{t_1 \neq t_2, t_1, t_2 \in \partial\Omega} \frac{|\, f(t_1) - f(t_2)\,|}{|t_1 - t_2|^\alpha}.$$

Thus, we consider the following boundary value problem: find a function $\Phi(x) \in C^1(\Omega; Cl_m)$, such that

$$\begin{cases} (\partial_x - \lambda)\Phi(x) = 0, \quad x \in \mathbf{R}^{\mathbf{m}} \backslash \Omega, \\ a(t)\Phi^+(t) + b(t)\Phi^-(t) = g(t), \quad t \in \partial\Omega, \\ \Phi^-(\infty) = 0, \end{cases} \tag{3}$$

where $a(t), b(t), g(t) \in C_\alpha(\partial\Omega; Cl_m)$ are given functions.

Theorem 3. *Suppose that* $\left|2^m J \|a(t) + b(t)\|_\alpha + 2^m \|1 - \frac{a(t)+b(t)}{2}\|_\alpha\right| < 1$, *where J is a constant and $a(t), b(t), g(t) \in C_\alpha(\partial\Omega; Cl_m)$. Then Problem (11) has a unique solution*

$$\varphi(y) = \int_{\partial\Omega} E_\lambda(x - y) d\sigma_x \varphi(x),$$

for $\varphi(x) \in C_\alpha(\partial\Omega; Cl_m)$.

Proof of Theorem 3. We let $\varphi(x) \in C_\alpha(\partial\Omega; Cl_m)$. Applying Theorem 2, Problem (3) is equivalent to solving the following singular integral equation in φ:

$$-[a(t) + b(t)]P.V.\varphi(t) + \left(1 - \frac{a(t) + b(t)}{2}\right)\varphi(t) + g(t) = \varphi(t).$$

We assume that the integral operator G is defined by

$$G\varphi(t) = -[a(t) + b(t)]P.V.\varphi(t) + \left(1 - \frac{a(t) + b(t)}{2}\right)\varphi(t) + g(t).$$

From Theorem 1, we have $P.V.\varphi(t) \in C_\alpha(\partial\Omega; Cl_m)$. It is easy to prove that $C_\alpha(\partial\Omega; Cl_m)$ is a real Banach space. Note that

$$\|f + g\|_\alpha \le \|f\|_\alpha + \|g\|_\alpha, \quad \|fg\|_\alpha \le 2^m \|f\|_\alpha \|g\|_\alpha, \quad f, g \in C_\alpha(\partial\Omega; Cl_m).$$

By the fixed point theorem, we determine that G is a contraction operator on $C_\alpha(\partial\Omega; Cl_m)$. Furthermore, we have the result. □

5. A Generalized Dirichlet Problem for the Perturbed Dirac Equation

Applying Lemma 2, we define operator T_λ as follows:

$$(T_\lambda f)(y) = \int_\Omega E_\lambda(x - y) f(x) dx,$$

where $E_\lambda(x)$ is defined as Definition 1.

Applying Lemma 2, we can define orthoprojections P and Q as follows:

$$(Pf)(y) = \int_{\partial\Omega} E_\lambda(x-y)d\sigma_x f(x), \tag{4}$$

$$(Qf)(y) = \int_{\Omega} E_\lambda(x-y)[(\partial_x - \lambda)f(x)]dx. \tag{5}$$

Theorem 4. *If $f \in C^2(\Omega; Cl_m)$ and $g \in C^2(\partial\Omega; Cl_m)$, then the generalized Dirichlet problem*

$$\begin{cases} (\partial_x - \lambda)^2\Phi(x) = f(x), & x \in \Omega, \\ \Phi(x) = g(x), & x \in \partial\Omega, \end{cases} \tag{6}$$

has solution $\Phi \in C^2(\Omega; Cl_m)$ of the form

$$\Phi = F_\lambda g + T_\lambda P(\partial_x - \lambda)h + T_\lambda Q T_\lambda f,$$

where $h \in C^2(\Omega; Cl_m)$ is a extension of g.

Proof of Theorem 4. First, we consider the following boundary value problem:

$$\begin{cases} (\partial_x - \lambda)^2\Phi_1(x) = f(x), & x \in \Omega, \\ \Phi_1(x) = 0, & x \in \partial\Omega. \end{cases} \tag{7}$$

Using Formulas (4) and (5), we have $P + Q = I$. That is to say, $Qf = f - Pf$, where $Pf \in \ker[(\partial_x - \lambda)^2]$. Let $\Psi = T_\lambda Q T_\lambda f$, then $(\partial_x - \lambda)\Psi = QT_\lambda f$. Using $\partial_x - \lambda$ again, we determine that $(\partial_x - \lambda)^2\Psi = f(x)$. Because $T_\lambda Qf$ has vanishing boundary values, we determine that Ψ is a solution of (7). Moreover, we can use this result to solve boundary value problem

$$\begin{cases} (\partial_x - \lambda)^2\Phi_2(x) = 0, & x \in \Omega, \\ \Phi_2(x) = g(x), & x \in \partial\Omega. \end{cases} \tag{8}$$

We let $g(x) \in C^2(\partial\Omega; Cl_m)$. Thus, we can determine that $h(x) \in C^2(\Omega; Cl_m)$ is a extension of $g(x)$. We assume that $\Phi_2(x) = \Psi + h$, where Ψ is a solution of (7). Th, we transform the last boundary value problem into the following problem:

$$\begin{cases} (\partial_x - \lambda)^2\Psi(x) = (\partial_x - \lambda)^2 h(x), & x \in \Omega, \\ \Psi(x) = 0, & x \in \partial\Omega. \end{cases} \tag{9}$$

Applying our solution of boundary value Problem (9), we determine that $\Psi = T_\lambda Q T_\lambda (\partial_x - \lambda)^2 h$. Note that $P = I - Q$. We can see that

$$\Psi = -T_\lambda Q(\partial_x - \lambda)h + T_\lambda Q F_\lambda (\partial_x - \lambda)h = -h + T_\lambda P(\partial_x - \lambda)h + F_\lambda h.$$

Furthermore, we have $\Phi_2 = \Psi + h = F_\lambda g + T_\lambda P(\partial_x - \lambda)h$. On noting that if Φ_1, Φ_2 are solutions of boundary value Problems (7) and (8), $\Phi = \Phi_1 + \Phi_2$ is a solution of boundary value Problem (6). Thus, it can be seen that

$$\Phi = F_\lambda g + T_\lambda P(\partial_x - \lambda)h + T_\lambda Q T_\lambda f,$$

which completes the proof. □

6. The Mann Iterative Sequence of Operator $\tilde{F}_\lambda$

Definition 6. *The integral operator $\tilde{F}$ is defined by*

$$(\tilde{F}_\lambda f)(y) = \mu(F_\lambda f)(y) = \mu \int_{\partial\Omega} E_\lambda(x-y)d\sigma_x f(x),$$

where $\Omega \in \mathbf{R}^m$ is an m-dimensional compact differentiable and oriented manifold with boundary.

Definition 7 ([16]). *Suppose that $A : E \to E$ is a mapping, where E is a linear subspace. A is defined as the Mann iterative sequence, if for given $x_1 \in E$, A satisfies $x_{m+1} = (1-\beta_m)x_m + \beta_m A x_m$, $\{\beta_m\} \subset [0,1]$, $m \geq 1$.*

Lemma 5 ([17]). *Let $\{a_m\}, \{b_m\}, \{c_m\}$ be non-negative real sequences. Suppose that $\sum_{m=1}^{\infty} t_m = \infty$, where $t_m \in [0,1]$. If*

$$\begin{cases} a_{m+1} \leq (1-t_m)a_m + b_m + c_m, \\ 1 \leq (1-t_m)a_m + b_m + c_m, \\ b_m = o(t_m), \\ \sum_{m=1}^{\infty} c_m < \infty, \end{cases}$$

then

$$\lim_{m\to\infty} a_m = 0.$$

Lemma 6. *If $\|f\|_\alpha \leq l_1$, then $\|\tilde{F}_\lambda f\|_\alpha \leq l\|f\|_\alpha$, where $l_1, l > 0$.*

Proof of Lemma 6. Note that

$$\int_{\partial\Omega} E_\lambda(x-y)d\sigma_x = \frac{1}{2}, \quad y \in \partial\Omega.$$

Thus, we have

$$\begin{aligned} |(\tilde{F}_\lambda^- f)(y)| &= \left| \int_{\partial\Omega} E_\lambda(x-y)d\sigma_x[f(x)-f(y)] + \int_{\partial\Omega} E_\lambda(x-y)d\sigma_x f(y) \right| \\ &\leq \left| \int_{\partial\Omega} E_\lambda(x-y)d\sigma_x[f(x)-f(y)] \right| + \left| \int_{\partial\Omega} E_\lambda(x-y)d\sigma_x f(y) \right| \\ &\leq l_2[M(f,\partial\Omega) + H(f,\partial\Omega,\alpha)] \\ &\leq l_3\|f\|_\alpha, \end{aligned}$$

where $l_2, l_3 > 0$. From Theorem 2,

$$(\tilde{F}_\lambda^+ f)(y) = (\tilde{F}_\lambda^- f)(y) + f(y).$$

It leads us to have

$$|(\tilde{F}_\lambda^+ f)(y)| \leq |(\tilde{F}_\lambda^- f)(y)| + |f(y)|.$$

It is easy to obtain $|(\tilde{F}_\lambda^- f)(y)| \leq l_4\|f\|_\alpha$, where $l_4 > 0$. Note that

$$(\tilde{F}_\lambda f)(y) = \frac{(\tilde{F}_\lambda^+ f)(y) + (\tilde{F}_\lambda^- f)(y)}{2}.$$

Thus, we obtain

$$\sup_{y_1 \neq y_2, y_1, y_2 \in \partial\Omega} \frac{|\,(\tilde{F}_\lambda f)(y_1) - (\tilde{F}_\lambda f)(y_2)\,|}{\|y_1 - y_2\|^\alpha}$$

$$\leq \frac{1}{2}\left[\sup_{y_1 \neq y_2, y_1, y_2 \in \partial\Omega} \frac{|\,(\tilde{F}^+_\lambda f)(y_1) - (\tilde{F}^+_\lambda f)(y_2)\,|}{|y_1 - y_2|^\alpha} + \sup_{y_1 \neq y_2, y_1, y_2 \in \partial\Omega} \frac{|\,(\tilde{F}^-_\lambda f)(y_1) - (\tilde{F}^-_\lambda f)(y_2)\,|}{|y_1 - y_2|^\alpha}\right].$$

Moreover, we have

$$\sup_{y \in \partial\Omega} |(\tilde{F}_\lambda f)(y)| \leq \frac{1}{2}\left[\sup_{y \in \partial\Omega} |(\tilde{F}^+_\lambda f)(y)| + \sup_{y \in \partial\Omega} |(\tilde{F}^-_\lambda f)(y)|\right].$$

Thus, we have

$$\|\tilde{F}_\lambda f\|_\alpha \leq \frac{1}{2}[\|\tilde{F}^+_\lambda f\|_\alpha + \|\tilde{F}^-_\lambda f\|_\alpha] \leq l\|f\|.$$

Therefore, we finish the proof. □

Theorem 5. *Assume that* $E = \{f | f \in C_\beta(\partial\Omega; Cl_m), \|f\|_\beta \leq l_1, 0 < \beta < 1\}$. *Then* $\tilde{F}_\lambda$ *has a unique fixed point in* E.

Proof of Theorem 5. Note that $E \subset C_\alpha(\partial\Omega; Cl_m)$, and E is a closed subspace. Then we obtain E is a Banach space. It is obvious that

$$(\tilde{F}_\lambda f)(y) = \frac{(\tilde{F}^+_\lambda f)(y) + (\tilde{F}^-_\lambda f)(y)}{2}.$$

From Theorem 3, we obtain $F_\lambda f \in C_\alpha(\partial\Omega; Cl_m)$. By Definition 5, we have $F_\lambda f \in C_\alpha(\partial\Omega; Cl_m)$. By Lemma 6, we have

$$\|\tilde{F}_\lambda f\|_\alpha = \|\mu F_\lambda f\|_\alpha < \|f\|_\alpha \leq l_1.$$

So $\tilde{F}_\lambda : E \to E$. For any $f_1, f_2 \in E$,

$$\|\tilde{F}_\lambda f_1 - \tilde{F}_\lambda f_2\|_\alpha < l\|f_1 - f_2\|_\alpha,$$

which completes the proof. □

That is to say, $\tilde{F}_\lambda$ is a contraction mapping in E. From the contraction mapping principle in Banach space, we determine that operator $\tilde{F}_\lambda$ has a unique fixed point.

Theorem 6. *Assume that* $\{\beta_m\} \subset [0,1]$, $\sum_{m=1}^{\infty} \beta_m = \infty$, $\lim_{m\to\infty} \beta_m = 0$. *For a given* $f_1 \in E$,

$$f_{m+1} = (1 - \beta_m)f_m + \beta_m \tilde{F}_\lambda f_m, \quad m \geq 1.$$

Then the iterative sequence $\{f_m\}$ *strongly converges to the fixed point of operator* $\tilde{F}_\lambda$.

Proof of Theorem 6. From Theorem 5, we determine that $\tilde{F}_\lambda$ has a unique fixed point in E. We let $\tilde{F}_\lambda f = f$.

$$\begin{aligned} \|f_{m+1} - f\|_\alpha &= \|(1 - \beta_m)f_m + \beta_m \tilde{F}_\lambda[f_m]\|_\alpha] \\ &\leq (1 - \beta_m)\|f_m - f\|_\alpha + \beta_m l\|f_m - f\|_\alpha \\ &= [1 - \beta_m(1 - l)]\|f_m - f\|_\alpha. \end{aligned}$$

We suppose that $b_m = 0, c_m = 0, t_m = \beta_m(1 - l)$. Thus, we have

$$t_m \in [0,1], \sum_{m=1}^{\infty} t_m = \infty, b_m = o(t_m), \sum_{m=1}^{\infty} c_m < \infty.$$

From Lemma 5, we have the result. □

7. Conclusions

The theory of singular integral operators has a wide range of applications in almost all areas of physics and engineering such as, for instance, in electromagnetism, optic, elasticity, fluid dynamics, geophysics, theory of orthogonal polynomials, and in asymptotic analysis up to modern quantum field theory. In this paper, we use the theory of singular integral operators to study Riemann-type and Dirichlet-type problems in Clifford analysis. One may further extrapolate these findings to the related boundary value problems in Clifford analysis. Boundary value problems in Clifford analysis have remarkable applications in mathematical physics, the mechanics of deformable bodies, electromagnetism, and relativistic quantum mechanics.

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