axioms

MDPI

Article

Covariance and Uncertainty Principle for Dispersive Pulse Propagation

Leon Cohen

Department of Physics, City University of New York, 695 Park Avenue, New York, NY 10065, USA; leon.cohen@hunter.cuny.edu

Abstract: We develop the concept of covariance for waves and show that it plays a fundamental role in understanding the evolution of a propagating pulse. The concept clarifies several issues regarding the spread of a pulse and the motion of the mean. Exact results are obtained for the time dependence of the covariance between position and wavenumber and the covariance between position and group velocity. We also derive relevant uncertainty principles for waves.

Keywords: dispersion; pulse; propagation; covariance; uncertainty principle

MSC: 35C07

Citation: Cohen, L. Covariance and Uncertainty Principle for Dispersive Pulse Propagation. *Axioms* **2024**, *13*, 242. https://doi.org/10.3390/axioms13040242

Academic Editors: Mourad Sini and Hans J. Haubold

Received: 15 February 2024
Revised: 27 March 2024
Accepted: 29 March 2024
Published: 8 April 2024

1. Introduction

For waves governed by a linear differential equation of the form

$$\sum_{n=0}^{N_x} b_n \frac{\partial^n u}{\partial x^n} - \sum_{n=0}^{N_t} a_n \frac{\partial^n u}{\partial t^n} = 0 \tag{1}$$

one substitutes $e^{ikx-i\omega t}$ into Equation (1) to obtain [1–4] the dispersion relation

$$\sum_{n=0}^{N_x} b_n (ik)^n - \sum_{n=0}^{N_t} a_n (-i\omega)^n = 0 \tag{2}$$

Generally, there is more than one solution to the dispersion relation, and the solutions may be complex. Each solution is called a mode. In this paper, we deal with the propagation of a single mode, and we write

$$\omega = \omega(k) \tag{3}$$

Furthermore, we assume that $\omega(k)$ is real, which implies that there is no damping. The solution to the wave equation for each mode is [1]

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int S(k,0)\, e^{ikx-i\omega(k)t}\, dk \tag{4}$$

where $S(k,0)$ is the initial spatial spectrum, which is obtained from the initial pulse, $u(x,0)$, by way of

$$S(k,0) = \frac{1}{\sqrt{2\pi}} \int u(x,0)\, e^{-ikx}\, dx \tag{5}$$

If one defines the time-dependent spectrum, $S(k,t)$, by [5–7]

$$S(k,t) = S(k,0) e^{-i\omega(k)t} \tag{6}$$

then $u(x,t)$ and $S(k,t)$ form Fourier transform pairs between x and k for all time

$$u(x,t) \;=\; \frac{1}{\sqrt{2\pi}}\int S(k,t)\,e^{ikx}\,dk \tag{7}$$

$$S(k,t) \;=\; \frac{1}{\sqrt{2\pi}}\int u(x,t)\,e^{-ikx}\,dx \tag{8}$$

Our aim is to study the dynamics of the modes and, in particular, to study the concept of covariance and uncertainty principles as they apply to wave propagation.

Because of the fact that $u(x,t)$ and $S(k,t)$ are Fourier transform pairs, one can calculate averages in either the position representation or the k-representation. For example, the moments of the pulse at time t are given by

$$\langle\, x^n \,\rangle_t = \int x^{\,n} |u(x,t)|^2\,dx \tag{9}$$

but may also be calculated in the k-representation by way of

$$\langle\, x^n \,\rangle_t = \int S^*(k,t)\,\mathcal{X}^n\,S(k,t)\,dk \tag{10}$$

where the position operator, $\mathcal{X}$, in the k-representation, is given by

$$\mathcal{X} = -\,\frac{1}{i}\frac{\partial}{\partial k} \tag{11}$$

Similarly, the k moments

$$\langle\, k^n \,\rangle_t = \int k^{\,n} |S(k,t)\,|^2\,dk \tag{12}$$

may be calculated by way of

$$\langle\, k^n \,\rangle_t = \int u^*(x,t)\,\mathcal{K}^n\,u(x,t)\,dx \tag{13}$$

where $\mathcal{K}$ is the wavenumber operator in the position representation

$$\mathcal{K} = \frac{1}{i}\frac{\partial}{\partial x} \tag{14}$$

Notation: Operators will be denoted by calligraphic letters. Commutators and anticommutators are denoted by the usual notation,

$$[\mathcal{A},\mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A} \tag{15}$$

and

$$[\mathcal{A},\mathcal{B}]_+ = \mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A} \tag{16}$$

respectively. Indefinite integrals imply integration from $-\infty$ to ∞, unless otherwise noted. *Time*-dependent moments, such as, for example, the mean, will be denoted as $\langle\, x \,\rangle_t$. For the case of time-dependent standard deviation for a quantity x, it will be denoted by $\sigma_x^2(t)$, and similarly for the covariance.

2. Motion of Mean and Spread

In Appendix A, we show that the mean of a pulse propagates with constant velocity

$$\langle\, x \,\rangle_t = \langle\, x \,\rangle_0 + \; Vt \tag{17}$$

where V is given by

$$V = \int \frac{\partial W(k)}{\partial k} \, |S(k,0)|^2 \, dk \tag{18}$$

We define, as standard, the group velocity, $v(k)$, by

$$v(k) = \frac{\partial W(k)}{\partial k} \tag{19}$$

The velocity, V, is the average of the group velocity at time zero,

$$V = \langle v(k) \rangle_0 \tag{20}$$

If we define the group velocity operator by $v(\mathcal{K})$, then V may also calculate the velocity in the position representation by way of

$$V = \int u^*(x,0) \; v(\mathcal{K}) \, u(x,0) \, dx$$

Comments:

1. It should be noted that the group velocity, $v(k)$, enters in a straightforward way, namely in the calculation of $\langle \, x \, \rangle_t$.

2. V not only depends on the group velocity, but also the initial spectrum. Thus, how a medium is disturbed affects the velocity of propagation of the mean and other quantities.

The spread of the pulse is defined in the usual way

$$\sigma_x^2(t) = \langle \, x^2 \, \rangle_t - \langle \, x \, \rangle_t^2 \tag{21}$$

is calculated to be

$$\sigma_x^2(t) = \sigma_x^2(0) + 2\, t \, \mathrm{Cov}_{xv}(0) + t^2 \sigma_v^2(0) \tag{22}$$

where

$$\mathrm{Cov}_{xv}(0) = \tfrac{1}{2} \langle \; v\mathcal{X} + \mathcal{X}\, v \, \rangle_0 - \langle v \, \rangle_0 \langle \, x \, \rangle_0 \tag{23}$$

and σ_v is the standard deviation of the group velocity

$$\sigma_v^2 = \langle \; v^2 \, \rangle - V^2 \tag{24}$$

The properties and meaning of $\mathrm{Cov}_{xv}(0)$ will be discussed in the next section.

3. Covariance for Waves

In standard probability theory, the covariance between two random variables a and b is given by

$$\mathrm{Cov}_{ab} = \langle ab \, \rangle - \langle a \rangle \langle \, b \, \rangle \tag{25}$$

where $\langle ab \, \rangle$ is the first mixed moment between a and b. An important property of covariance is that

$$\sigma_a^2 \, \sigma_b^2 \geq \mathrm{Cov}_{ab}^2 \tag{26}$$

where σ_a and σ_b are the standard deviations of a and b.

For the case of operators, we define the covariance by

$$\mathrm{Cov}_{ab} = \tfrac{1}{2} \langle \, \mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A} \, \rangle - \langle \, \mathcal{A} \, \rangle \langle \, \mathcal{B} \, \rangle \tag{27}$$

and, when the quantities are time-dependent, we shall generally use the notation

$$\mathrm{Cov}_{ab}(t) = \tfrac{1}{2} \langle \, \mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A} \, \rangle_t - \langle \mathcal{A} \rangle_t \langle \, \mathcal{B} \, \rangle_t \tag{28}$$

Also, we define the first mixed moment by

$$\langle\, ab\,\rangle_t = \tfrac{1}{2}\langle\, \mathcal{AB} + \mathcal{BA}\,\rangle_t = \tfrac{1}{2}\int S^*(k,t)(\,\mathcal{AB} + \mathcal{BA}\,)S(k,t)\,dk \tag{29}$$

To show that the operator definition of covariance, Equation (27), also satisfies Equation (26), we consider the operator

$$\mathcal{C} = \mathcal{A} - c\mathcal{B} \tag{30}$$

where c is a real number. The standard deviation of $\mathcal{C}$ is

$$\sigma_c^2 = \langle\, C^2\,\rangle - \langle\, C\,\rangle^2 \tag{31}$$

$$= \langle\, (\mathcal{A} - c\mathcal{B})^2\,\rangle - \langle\, \mathcal{A} - c\mathcal{B}\,\rangle^2 \tag{32}$$

which simplifies to

$$\sigma_c^2 = \sigma_a^2 + c^2\sigma_b^2 - c((\mathcal{AB} + \mathcal{AB})\rangle - 2c\langle\, \mathcal{A}\,\rangle\langle\, \mathcal{B}\,\rangle) \tag{33}$$

Using the operator definition of Cov, Equation (27), we have

$$\sigma_c^2 = \sigma_a^2 + c^2\sigma_b^2 - 2c\mathrm{Cov}_{ab} \tag{34}$$

Now, we take

$$c = \frac{\mathrm{Cov}_{ab}}{\sigma_b^2} \tag{35}$$

and substitute it into Equation (34) to obtain

$$\sigma_c^2 = \frac{\sigma_a^2\sigma_b^2 - \mathrm{Cov}_{ab}^2}{\sigma_b^2} \tag{36}$$

Since all the standard deviations are positive, we must have

$$\sigma_a^2\sigma_b^2 \geq \mathrm{Cov}_{ab}^2 \tag{37}$$

This shows that the operator definition of covariance, Equation (27), satisfies the standard condition for covariance, Equation (26).

3.1. Covariance between Position and Group Velocity

For the first mixed moment of position and group velocity, we have

$$\langle\, x\,v\,\rangle_t = \tfrac{1}{2}\langle\, v\mathcal{X} + \mathcal{X}v\,\rangle_t = \tfrac{1}{2}\int S^*(k,t)(v\mathcal{X} + \mathcal{X}v)S(k,t)\,dk \tag{38}$$

Writing $S(k,t)$ in terms of its amplitude and phase

$$S(k,t) \;= |S(k,t)|\, e^{i\psi(k,t)} \tag{39}$$

we obtain that

$$\langle\, xv\,\rangle_t = -\int v(k)\,\frac{\partial\psi(k,t)}{\partial k}|S(k,t)|^2\,dk \tag{40}$$

This is true in general. But, for pulse propagation, we have

$$S(k,t) = |S(k,0)|\, e^{i\psi(k,0) - iW(k)t} \tag{41}$$

where $\psi(k,0)$ is the phase of the spectrum at time zero. The phase at time t is, therefore, given by

$$\psi(k,t) = \psi(k,0) - W(k)t \tag{42}$$

Hence,

$$\frac{\partial\psi(k,t)}{\partial k} = \frac{\partial\psi(k,0)}{\partial k} - v(k)t \tag{43}$$

Substituting Equation (43) into Equation (40), we have

$$\langle xv \rangle_t = \langle x\, v \rangle_0 + \langle\, v^2 \,\rangle t \tag{44}$$

where

$$\langle\, v^2 \,\rangle = \int v^2(k)\, |S(k,0)|^2\, dk \tag{45}$$

The covariance between position and group velocity at time t is, then,

$$\mathrm{Cov}_{xv}(t) = \langle\, xv \,\rangle_t - V\langle\, x \,\rangle_t \tag{46}$$

$$= \langle x\, v \,\rangle_0 + \langle\, v^2 \,\rangle t - V(\langle\, x \,\rangle_0 + Vt) \tag{47}$$

which gives

$$\mathrm{Cov}_{xv}(t) = \mathrm{Cov}_{xv}(0) + t\,\sigma_v^2 \tag{48}$$

This shows that the covariance increases linearly in time. It also shows that, irrespective of the sign of the initial covariance, eventually, the covariance becomes positive and remains so.

3.2. Covariance between Position and Wave Number

Let us first consider the evolution of the first mixed moment

$$\langle\, xk \,\rangle_t = \tfrac{1}{2}\langle\, k\mathcal{X} + \mathcal{X}k \,\rangle_t \tag{49}$$

$$= \tfrac{1}{2}\int S^*(k,t)(k\mathcal{X} + \mathcal{X}k)S(k,t)\, dk \tag{50}$$

Using Equation (39), direct calculation gives

$$\langle\, xk \,\rangle_t = -\int k\, \frac{\partial\psi(k,t)}{\partial k} |S(k,t)|^2\, dk \tag{51}$$

This is true in general. But, for Equation (42),

$$\frac{\partial\psi(k,t)}{\partial k} = \frac{\partial\psi(k,0)}{\partial k} - \frac{\partial W(k)}{\partial k}t \tag{52}$$

and, therefore,

$$\langle\, xk \,\rangle_t = -\int k\left(\frac{\partial\psi(k,0)}{\partial k} - \frac{\partial W(k)}{\partial k}t\right)|S(k,t)|^2\, dk \tag{53}$$

which gives

$$\langle x\, k \,\rangle_t = \langle x\, k \,\rangle_0 + \langle k\, v(k) \,\rangle_0 t \tag{54}$$

The covariance between x and k at time t is

$$\mathrm{Cov}_{xk}(t) = \langle x\, k \,\rangle_t - \langle x \,\rangle_t \langle\, k \,\rangle_t \tag{55}$$

Putting Equation (54) into Equation (55), we obtain

$$\mathrm{Cov}_{xk}(t) = \mathrm{Cov}_{xk}(0) + t\,\mathrm{Cov}_{kv}(0) \tag{56}$$

4. Uncertainty Principle for Waves

For any two operators, $\mathcal{A}$ and $\mathcal{B}$, the usual textbook result for the uncertainty product is that [8,9]

$$\sigma_a^2\,\sigma_b^2 \geq \tfrac{1}{4}|\,\langle\,[\mathcal{A},\mathcal{B}\,]\,\rangle\,|^2 \tag{57}$$

However, there is a stronger version [10,11]

$$\sigma_a^2\,\sigma_b^2 \geq \frac{1}{4}|\,\langle\,[\mathcal{A},\mathcal{B}\,]\,\rangle\,|^2 + \mathrm{Cov}_{ab}^2 \tag{58}$$

where

$$\mathrm{Cov}_{ab} = \frac{1}{2}\langle\,\mathcal{AB}+\mathcal{BA}\,\rangle - \langle\,\mathcal{A}\,\rangle\langle\,\mathcal{B}\,\rangle \tag{59}$$

For completeness, a proof of Equation (58) is given in Appendix B.

4.1. Uncertainty Principle for Position and Wavenumber

For position and wavenumber, we have that

$$[\,\mathcal{X},\mathcal{K}\,] = i \tag{60}$$

Therefore, the uncertainty product inequality is

$$\sigma_x^2(t)\sigma_k^2(t) \geq \frac{1}{4} + \mathrm{Cov}_{xk}^2(t) \tag{61}$$

Using Equation (56), we may write

$$\sigma_x^2(t)\sigma_k^2(t) \geq \frac{1}{4} + \mathrm{Cov}_{xk}^2(0) + 2t\mathrm{Cov}_{kv}(0)\,\mathrm{Cov}_{xk}(0) + t^2\,\mathrm{Cov}_{kv}^2(0) \tag{62}$$

Since we know the exact standard deviations, given by Equations (22) and (24), we have that

$$\sigma_x^2(t)\sigma_k^2(t) = \sigma_x^2(0)\sigma_k^2(0) + 2t\mathrm{Cov}_{xv}(0)\sigma_k^2 + t^2\sigma_v^2\sigma_k^2 \qquad \text{(exact)} \tag{63}$$

Using Equation (37), we have that

$$\sigma_x^2(t)\sigma_k^2(t) \geq \frac{1}{4} + \mathrm{Cov}_{xk}^2(0) + 2t\mathrm{Cov}_{kv}\,(0)\mathrm{Cov}_{xk}(0) + t^2\,\sigma_v^2\sigma_k^2 \tag{64}$$

which shows that the uncertainty principle is an approximate estimation of the uncertainty product, but gives the exact answer for time going to infinity.

4.2. Uncertainty Principle for Position and Group Velocity

For position $\mathcal{X}$ and group velocity $v(\mathcal{K})$, the commutator is

$$[\,\mathcal{X},v(\mathcal{K})\,] = iv'(\mathcal{K}) \tag{65}$$

and, therefore, the uncertainty product for position and group velocity is given by

$$\sigma_x^2(t)\sigma_v^2(t) \geq \frac{1}{4}|\,\langle\,v'(\mathcal{K})\,\rangle\,|^2 + \mathrm{Cov}_{xv}^2(t) \tag{66}$$

Using Equation (48), we have

$$\sigma_x^2(t)\sigma_v^2(t) \geq \frac{1}{4}|\,\langle\,v'(\mathcal{K})\,\rangle\,|^2 + \mathrm{Cov}_{xv}^2(0) + 2t\sigma_v^2\mathrm{Cov}_{xv}(0) + t^2\sigma_v^4 \tag{67}$$

The exact uncertainty product is given by

$$\sigma_x^2(t)\sigma_v^2(t) = \sigma_x^2(0)\sigma_v^2(0) + 2\,t\,\mathrm{Cov}_{xv}(0)\sigma_v^2 + t^2\sigma_v^4 \tag{68}$$

In comparing the exact answer, Equation (68) with Equation (67), we see that the time-dependent terms are identical

5. Meaning of Group Velocity and Covariance

What is generally called the group velocity, $v(k)$, is a misnomer as it is not the velocity of the group of waves. The velocity of the group is V. The group velocity indicates what frequencies exist and their individual speeds. To ascertain the contribution of a particular k-frequency, it is weighted by the energy density spectrum $|S(k,0)|^2$. The average of all frequencies is the velocity, V, as given by the integral in Equation (18).

To understand the meaning of covariance for waves, we consider a pulse that, for the moment, is traveling to the right, and has a spatial extension. Let us suppose that the k-frequencies are distributed within the spatial extension of the pulse in a random manner. In such a case, the covariance between the position x and $v(k)$ would be zero. On the other hand, let us suppose that the high k-frequencies are concentrated on the right-hand side of the pulse, and the low k-frequencies on the left side of the pulse. In such a case, the covariance between $v(k)$ and x would be positive and would indicate the relative relation between high k-frequency velocities and the front of the pulse. Conversely, the covariance would be negative if the high-frequency values are on the left side of the pulse. Hence, the reason that the covariance is fundamental is that it indicates whether the high k- frequencies are on the right or left side of the pulse. If the covariance is positive, the high frequencies are to the right of the low frequencies, but, if it is negative, it means that they are arranged the opposite way. Now, according to Equation (48),

$$\mathrm{Cov}_{xv}(t) = \mathrm{Cov}_{xv}(0) + t\,\sigma_v^2 \tag{69}$$

the evolution of the covariance is such that for large times, it is always positive, irrespective of the sign of the initial covariance

$$\mathrm{Cov}_{xv}(t) \to \infty \qquad\qquad \text{for } t \to \infty \tag{70}$$

That is, the pulse evolves so that the high velocity frequencies are at the front of the pulse. The reason is that, if the high frequencies are initially on the left side of the pulse, they will eventually catch up to the front and remain so.

6. Arbitrary Waves

In the above discussion, we have assumed a special type of motion, namely dispersive motion governed by Equation (4). It is of interest to consider what aspects of the previous results apply to an arbitrary pulse where we do not specify a particular equation of motion. Let us suppose the wave $u(x,t)$ and its Fourier transform $S(k,t)$ are written in terms of their respective amplitudes and phases,

$$u(x,t) = |u(x,t)|\, e^{i\varphi(x,t)} = A(x,t)\, e^{i\varphi(x,t)} \tag{71}$$

$$S(k,t) = |S(k,t)|\, e^{i\psi(k,t)} = B(k,t)\, e^{i\psi(k,t)} \tag{72}$$

If one calculates the first two moments and standard deviations, one obtains

$$\langle\, x \,\rangle_t = -\int \frac{\partial \psi}{\partial k} B^2(k,t) dk \tag{73}$$

$$\langle\, x^2 \,\rangle_t = \int \left(\left(\frac{1}{B} \frac{\partial B}{\partial k} \right)^2 + \left(\frac{\partial \psi}{\partial k} \right)^2 \right) B^2(k,t) dk \tag{74}$$

$$\sigma^2_{x|t} = \int \left(\left(\frac{1}{B} \frac{\partial B}{\partial k} \right)^2 + \left(\frac{\partial \psi}{\partial k} + \langle\, x \,\rangle \right)^2 \right) B^2(k,t) dk \tag{75}$$

and

$$\langle\, k \,\rangle_t = \int \frac{\partial \varphi}{\partial x} A^2(x,t) dx \tag{76}$$

$$\langle\, k^2 \,\rangle_t = \int \left[\left(\frac{1}{A} \frac{\partial A}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial x} \right)^2 \right] A^2(x,t) dx \tag{77}$$

$$\sigma^2_{k|t} = \int \left[\left(\frac{1}{A} \frac{\partial A}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial x} - \langle\, k \,\rangle \right)^2 \right] A^2(x,t) dx \tag{78}$$

Now, let us consider the first mixed moment of x and k:

$$\langle\, xk \,\rangle = \frac{1}{2} \langle\, [\, \mathcal{X}, \mathcal{K} \,]_+ \,\rangle \tag{79}$$

We calculate $\langle\, xk \,\rangle$ in the position representation by way of

$$\langle\, xk \,\rangle_t = \tfrac{1}{2} \int u^*(x,t)(x\mathcal{K} + \mathcal{K}x) u(x,t)\, dx \tag{80}$$

resulting in

$$\langle\, x\, k \,\rangle_t \;=\; \int x \frac{\partial \varphi}{\partial x} \,|\, u(x,t) \,|^2 dx \tag{81}$$

However, if one calculates it in the k representation using

$$\langle\, xk \,\rangle_t = \tfrac{1}{2} \int S^*(k,t)(k\mathcal{X} + \mathcal{X}k) S(k,t)\, dk \tag{82}$$

then

$$\langle\, x\, k \,\rangle_t \;=\; -\int k \frac{\partial \psi}{\partial k} \,|\, S(k,t) \,|^2 dk \tag{83}$$

is obtained. Equating the two expressions, we have

$$\int x \frac{\partial \varphi}{\partial x} \,|\, u(x,t) \,|^2 dx \;=\; -\int k \frac{\partial \psi}{\partial k} \,|\, S(k,t) \,|^2 dk \tag{84}$$

This is an interesting relation because it shows a connection between amplitudes and phase of the pulse and its spectrum.

The covariance, therefore, may be written as

$$\mathrm{Cov}_{xk} \;=\; \left\langle x \frac{\partial \varphi}{\partial x} \right\rangle - \langle x \,\rangle \langle\, k \,\rangle \tag{85}$$

or

$$\mathrm{Cov}_{xk} \;=\; -\left\langle k \,\frac{\partial \psi}{\partial k} \right\rangle - \langle x \,\rangle \langle\, k \,\rangle \tag{86}$$

This shows that, in the x representation, the wavenumber, k, acts as $\frac{\partial \varphi}{\partial x}$ and that, in the wavenumber representation, x acts as $-\frac{\partial \psi}{\partial k}$.

In addition, we speculate that the minimum uncertainty relation takes the form

$$\sigma_{x|t}^2 \sigma_{k|t}^2 \geq \left\{ \int \left(\frac{1}{B} \frac{\partial B}{\partial k} \right)^2 B^2(k,t) dk \right\} \left\{ \int \left(\frac{1}{A} \frac{\partial A}{\partial x} \right)^2 A^2(x,t) dx \right\} \tag{87}$$

7. Quasi-Probability Approach

The concept of covariance implies the existence of a joint distribution of the two variables. In the past sections, we have defined it in terms of operators without consideration of a possible joint distribution. In the quantum mechanical case, Cartwright [12] was the first to define covariance in terms of a joint distribution of position and momentum. She used the Wigner distribution [13–15] for the joint distribution. In the case of position and wavenumber, the Wigner distribution is defined by

$$W(x,k,t) = \frac{1}{2\pi} \int u^*(x - \tfrac{1}{2}\tau, t)\, u(x + \tfrac{1}{2}\tau, t)\, e^{-i\tau k} d\tau \tag{88}$$

which, in terms of the spectrum, may be written as

$$W(x,k,t) = \frac{1}{2\pi} \int S^*(k + \tfrac{1}{2}\theta, t)\, S(k - \tfrac{1}{2}\theta, t)\, e^{-i\theta x} d\theta \tag{89}$$

Because the Wigner and other similar distributions are not manifestly positive, they are often called quasi-distributions. The Wigner distribution satisfies the marginals

$$\int W(x,k,t) dk = |u(x,t)|^2 \tag{90}$$

$$\int W(x,k,t) dx = |S(k,t)|^2 \tag{91}$$

Therefore, all functions of position or wavenumber are given correctly by the Wigner distribution. In particular, the moments $\langle x^n \rangle$ and $\langle k^n \rangle$ are correctly given by

$$\langle x^n \rangle = \iint x^n W(x,k,t) dk dx \tag{92}$$

$$\langle k^n \rangle = \iint k^n W(x,k,t) dk dx \tag{93}$$

Following Cartwright [12], we define the first mixed moment by

$$\langle xk \rangle = \iint xk W(x,k,t) dx dk \tag{94}$$

Straightforward calculation leads to Equation (81) and/or Equation (83).
We now specialize to the case of pulse propagation, where

$$S(k,t) = S(k,0)\, e^{-i\omega(k)t} \tag{95}$$

We have

$$W(x,k,t) = \frac{1}{2\pi} \int S^*(k + \tfrac{1}{2}\theta, 0)\, S(k - \tfrac{1}{2}\theta, 0) e^{-i\theta x} e^{i[\omega(k+\theta/2) - \omega(k-\theta/2)]t} d\theta \tag{96}$$

At $t = 0$,

$$W(x,k,0) = \frac{1}{2\pi} \int S^*(k + \tfrac{1}{2}\theta, 0)\, S(k - \tfrac{1}{2}\theta, 0) e^{-i\theta x} d\theta \tag{97}$$

and inverting yields

$$S^*(k + \tfrac{1}{2}\theta, 0)\, S(k - \tfrac{1}{2}\theta, 0) = \int W(x,k,0)\, e^{i\theta x} dx \tag{98}$$

Inserting Equation (98) into Equation (96), we have

$$W(x,k,t) = \frac{1}{2\pi}\iint W(x',k,0)\, e^{-i\theta(x'-x)} e^{i[\omega(k+\theta/2)-\omega(k-\theta/2)]t} d\theta dx' \tag{99}$$

which expresses the Wigner distribution at time t in terms of the Wigner distribution at time zero.

Now, we consider the first mixed moment at time t,

$$\langle xk\rangle_t = \frac{1}{2\pi}\iint xkW(x',k,0)\, e^{i\theta(x'-x)} e^{i[\omega(k+\theta/2)-\omega(k-\theta/2)]t} d\theta dx' dx dk \tag{100}$$

Simplification leads to

$$\langle xk\rangle_t = \iint xkW(x,k,0)\, dxdk + t\iint kv(k)W(x,k,0)\, dxdk \tag{101}$$

and, therefore, we have

$$\langle xk\rangle_t = \langle xk\rangle_0 + t\langle kv(k)\rangle_0 \tag{102}$$

which is the answer obtained by the operator method, namely Equation (54).

For the case of position and group velocity, the mixed moment is

$$\langle xv(k)\rangle_t = \iint xv(k)W(x,k,t)dxdk \tag{103}$$

which, for pulse propagation, is given by

$$\langle xv(k)\rangle_t = \frac{1}{2\pi}\iint xv(k)W(x',k,0)\, e^{i\theta(x'-x)} e^{i[\omega(k+\theta/2)-\omega(k-\theta/2)]t} d\theta dx' dx dk \tag{104}$$

Simplification leads to

$$\langle xv(k)\rangle_t = \iint xv(k)W(x,k,0)\, dxdk + t\iint v^2(k)W(x,k,0)\, dxdk \tag{105}$$

and, therefore,

$$\langle xv(k)\rangle_t = \langle xv(k)\rangle_0 + t\left\langle v^2(k)\right\rangle_0 \tag{106}$$

which is the same as Equation (44).

Other Quasi-Distributions

Cartwright [12] pointed out that there are an infinite number of distributions that give the same result as the Winger distribution. All quasi-distributions may be expressed as [10,12,16]

$$C(x,k,t) = \frac{1}{4\pi^2}\iiint u^*(q-\tfrac{1}{2}\tau,t)u(q+\tfrac{1}{2}\tau,t)\Phi(\theta,\tau)\, e^{-i\theta x - i\tau k + i\theta q}\, d\theta\, d\tau\, dq \tag{107}$$

where $\Phi(\theta,\tau)$ is the kernel function that characterizes the distribution. If we want the covariance calculated with $C(x,k,t)$ to be the same as that with Wigner, that is,

$$\iint xkC(x,k,t)dxdk = \iint xkW(x,k,t)dxdk \tag{108}$$

then the kernel must satisfy

$$\left.\frac{\partial^2}{\partial\theta\partial\tau}\Phi(\theta,\tau)\right|_{\theta,\tau=0} \tag{109}$$

There are an infinite number of distributions that satisfy Equation (109). Among them are the Choi–Williams [17,18], the Margenau–Hill [19], and Rihackek [20] distributions, among others [10].

8. Dual Case

In the previous sections, we considered the physical case where we have a spatial wave at a given time and allow it to evolve. An example is plucking a string and letting go at time zero. The dual case is where, at a fixed position, we create a pulse as a function of time, and examine its evolution as a function of position. Examples are radar, sonar, and a fiber optics pulse. For the fundamental solution, we take $e^{-ikx+i\omega t}$ instead of $e^{ikx-i\omega t}$ because that will make the results mathematically and physically similar to the previous case. We sbstitute $e^{-ikx+i\omega t}$ into Equation (1) to obtain

$$\sum_{n=0}^{N_x} b_n(-ik)^n - \sum_{n=0}^{N_t} a_n(i\omega)^n = 0 \tag{110}$$

In this case, we solve for k as a function of ω

$$k = k(\omega) \tag{111}$$

For the case where we generate a signal at position $x = 0$, the general solution is then

$$u(x,t) \;=\; \frac{1}{\sqrt{2\pi}} \int F(\omega,0)\; e^{-ik(\omega)x+i\omega t} d\omega \tag{112}$$

where $F(\omega,0)$ is the spectrum at $x = 0$, given by

$$F(\omega,0) = \frac{1}{\sqrt{2\pi}} \int u(0,t)\; e^{-i\omega t} dt \tag{113}$$

Analogous to Equation (6), we define

$$F(\omega,x) = F(\omega,0)\, e^{-ik(\omega)x} \tag{114}$$

in which case,

$$u(x,t) \;=\; \frac{1}{\sqrt{2\pi}} \int F(\omega,x)\, e^{i\omega t} d\omega \tag{115}$$

$$F(\omega,x) \;=\; \frac{1}{\sqrt{2\pi}} \int u(x,t)\, e^{-i\omega t} dt \tag{116}$$

which shows that $u(x,t)$ and $F(\omega,x)$ form Fourier transform pairs between frequency and time at position x.

Analogous to group velocity, one defines the "group slowness" or the "unit transit time" by

$$\tau(\omega) = \frac{d}{d\omega} k(\omega) \tag{117}$$

All the equations in the previous sections may be written down by analogy. Here, we give just the main ones relevant to the concept of covariance.

The time moments at position x are given by

$$\langle\, t^n \,\rangle_x = \int t^n\, |u(x,t)|^2\, dt = \int F^*(\omega,x) \mathcal{T}^n F(\omega,x)\, d\omega \tag{118}$$

where $\mathcal{T}$ is the time operator in the frequency domain

$$\mathcal{T} = i\,\frac{\partial}{\partial\omega} \tag{119}$$

If we write

$$F(\omega,x) = |F(\omega,x)|e^{i\eta(\omega,x)} \tag{120}$$

then, in general,

$$\langle\, \tau t \,\rangle_x = \tfrac{1}{2}\langle\, \tau\mathcal{T} + \mathcal{T}\tau \,\rangle_x = -\int \tau(\omega)\frac{\partial\eta}{\partial\omega}|F(\omega,t)|^2\, d\omega \tag{121}$$

The covariance between time and transit time at position x is given by

$$\mathrm{Cov}_{t\tau}(x) = \langle\, t\tau \,\rangle_x - \langle\, \tau \,\rangle_x \langle\, t \,\rangle_x \tag{122}$$

If we write the spectrum in terms of amplitude and phase

$$F(\omega,x) = |F(\omega,x)|\, e^{i\eta(\omega,x)} = |F(\omega,0)|\, e^{i\eta(\omega,0)}e^{-ik(\omega)x} \tag{123}$$

then

$$\eta(\omega,x) = \eta(\omega,0) - k(\omega)x \tag{124}$$

and

$$\frac{\partial\eta(\omega,x)}{\partial\omega} = \frac{\partial\eta(\omega,0)}{\partial\omega} - \tau x \tag{125}$$

Substituting Equation (125) into Equation (121), the mixed moment is

$$\langle t\,\tau \,\rangle_x = \langle t\,\tau \,\rangle_0 + \langle\, \tau^2 \,\rangle x \tag{126}$$

from which it follows that

$$\mathrm{Cov}_{t\tau}(x) = \mathrm{Cov}_{t\tau}(0) \;\; + x\,\sigma_\tau^2 \tag{127}$$

For the covariance between time and frequency at position x, defined by

$$\mathrm{Cov}_{t\omega}(x) = \langle t\,\omega \,\rangle_x - \langle t \,\rangle_x \langle\, \omega \,\rangle_x \tag{128}$$

we have

$$\langle\, t\omega \,\rangle_x = \tfrac{1}{2}\langle\, \omega\mathcal{T} + \mathcal{T}\omega \,\rangle_x = -\int \omega\,\frac{\partial\eta(\omega,x)}{\partial\omega}|F(\omega,x)|^2\, d\omega \tag{129}$$

and

$$\langle t\,\omega \,\rangle_x = \langle t\,\omega \,\rangle_0 + \langle \omega\,\tau \,\rangle x \tag{130}$$

One obtains

$$\mathrm{Cov}_{t\omega}(x) = \mathrm{Cov}_{t\omega}(0) + x\,\mathrm{Cov}_{\omega\tau}(0) \tag{131}$$

Funding: This research received no external funding.

Data Availability Statement: Data is contained within the article.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A. Dynamics of Modes

We first obtain the differential equation for a mode, $u(x,t)$, characterized by $\omega(k)$. We differentiate Equation (4) with respect to time to have

$$i\frac{\partial}{\partial t}u(x,t) = \frac{1}{\sqrt{2\pi}}\int \omega(k)S(k,0)\, e^{ikx-iW(k)t}\; dk \tag{A1}$$

$$= \frac{1}{\sqrt{2\pi}}\int \omega\left(\frac{1}{i}\frac{\partial}{\partial x}\right)S(k,0)\, e^{ikx-i\omega(k)t}\; dk \tag{A2}$$

which we write as

$$i\frac{\partial}{\partial t}u(x,t) \;= \omega(\mathcal{K})u(x,t) \tag{A3}$$

This equation is of the form of the Schrödinger equation with $W(\mathcal{K})$ acting as the Hamiltonian. Since we are taking W to be real, $W(\mathcal{K})$ is Hermitian, and, further $e^{-i\,W(\mathcal{K})\,t}$ is unitary. The symbolic solution to Equation (A3) is

$$u(x,t) \;=\; e^{-i\,\omega(\mathcal{K})\,t}\,u(x,0) \tag{A4}$$

The equation of motion for $S(k,t)$ is obtained from Equation (6),

$$i\frac{\partial}{\partial t}S(k,t) = \omega(k)S(k,t) \tag{A5}$$

Appendix A.1. Dynamical Variable

Starting with

$$\langle\mathcal{A}\rangle_t = \int u^*(x,t)\mathcal{A}(0)\;u(x,t)dx \tag{A6}$$

The usual derivation of the Heisenberg equation of motion leads to [8,9]

$$\mathcal{A}(t) \;=\; e^{i\,\omega(\mathcal{K})\,t}\mathcal{A}(0)\,e^{-i\,\omega(\mathcal{K})\,t} \tag{A7}$$

and

$$\frac{d\mathcal{A}}{dt} \;=\; \frac{1}{i}\,[\mathcal{A},\omega(\mathcal{K})] \tag{A8}$$

Appendix A.2. Equations of Motion for Position and Wave Number

In Equation (A8), we take $A = \mathcal{K}$, and since $\mathcal{K}$ and $W(\mathcal{K})$ commute, we have

$$\frac{d\mathcal{K}}{dt} \;=\; \frac{1}{i}[\mathcal{K},\omega(\mathcal{K})] = 0 \tag{A9}$$

and, therefore,

$$\mathcal{K}(t) = \mathcal{K}(0) \tag{A10}$$

Also, any function of $\mathcal{K}(t)$ will not change in time. In particular, all the moments are time-independent:

$$\langle k^n\,\rangle_t = \langle k^{\,n}\rangle_0. \tag{A11}$$

Now, we consider the position operator $\mathcal{X}$. Using Equation (A8), we have

$$\frac{d\mathcal{X}}{dt} \;=\; \frac{1}{i}[\mathcal{X},\omega(\mathcal{K})] \tag{A12}$$

The commutator works out to be

$$[\mathcal{X},\omega(\mathcal{K})] = -\frac{1}{i}\frac{d}{dk}\omega(\mathcal{K}) = -\,\frac{1}{i}v(\mathcal{K}) \tag{A13}$$

Hence,

$$\frac{d\mathcal{X}}{dt} = v(\mathcal{K}) \tag{A14}$$

But

$$v(\mathcal{K}(t)\,) = v(\mathcal{K}(0)\,) \tag{A15}$$

and, therefore,

$$\mathcal{X}(t) = \mathcal{X}\,(0) + v(\mathcal{K}(0))t \tag{A16}$$

Taking the expectation values of both sides, we have

$$\langle\, x\,\rangle_t = \langle\, x\,\rangle_0 + \ Vt \tag{A17}$$

where

$$V = \int v(k)\, |S(k,0)|^2\, dk \tag{A18}$$

For the second moment, we square Equation (A16) to obtain

$$\mathcal{X}^2(t) = \mathcal{X}^{\,2}(0) + [\mathcal{X}\,(0)v(\mathcal{K}) + v(\mathcal{K})X\,(0)]t{+}v^2(\mathcal{K})t^2 \tag{A19}$$

Taking expectation values gives

$$\langle\, x^2\,\rangle_t = \langle\, x^2\,\rangle_0 + t\,\langle\, v\mathcal{X} + \mathcal{X}v\,\rangle_0 + t^2\,\langle\, v^2\,\rangle \tag{A20}$$

where

$$\langle\, v^2\,\rangle = \int v^2(k)\, |S(k,0)|^2\, dk \tag{A21}$$

and

$$\langle\, v\mathcal{X} + \mathcal{X}v\,\rangle_0 = \int S^*(k,0)(\, v\mathcal{X} + \mathcal{X}v)\, |S(k,0)\, dk \tag{A22}$$

For the standard deviation, one obtains

$$\sigma^2_{x|t} = \sigma^2_{x|0} + 2\,t\,\mathrm{Cov}_{xv}(0) + t^2\sigma^2_v \tag{A23}$$

where

$$\mathrm{Cov}_{xv}(0) = \tfrac{1}{2}\langle\, v\mathcal{X} + \mathcal{X}v\,\rangle_0 - \langle\, v\,\rangle_0\langle\, x\,\rangle_0 \tag{A24}$$

and where σ_v^2 is the dispersion of the group velocity

$$\sigma^2_v = \int (v(k) - V)^2\, |S(k,0)|^2\, dk \tag{A25}$$

Appendix B. Proof of the Uncertainty Principle with Covariance

We consider two Hermitian operators $\mathcal{A}$ and $\mathcal{B}$ and, without loss of generality, we can assume that their means are zero. The uncertainty product in a state $\psi(x)$ is

$$\sigma^2_a\,\sigma^2_b = \int\ \psi^*(x)\mathcal{A}^2\,\psi(x)\ dx\ \ \times\ \int \psi^*(x)\,\mathcal{B}^2\,\psi(x)\,dx \tag{A26}$$

Since the operators are Hermitian, we have that

$$\sigma^2_a\,\sigma^2_b = \int |\,\mathcal{A}\,\psi(x)\,|^2\, dx\ \ \times\ \int |\,\mathcal{B}\,\psi(x)\,|^2\, dx \tag{A27}$$

Since the means are zero, the covariance is given by

$$\mathrm{Cov}_{ab} = \frac{1}{2}\langle\, [\,\mathcal{A},\mathcal{B}\,]_+\,\rangle \tag{A28}$$

Applying the Schwartz inequality to Equation (A27), we have

$$\sigma^2_a\,\sigma^2_b \geq \left| \int \{\mathcal{A}\,\psi(x)\,\}^*\{\mathcal{B}\psi(x)\,\}\,dx \right|^2 \tag{A29}$$

$$= \left| \int \psi^*(x)\mathcal{A}\,\mathcal{B}\psi(x)\,dx \right|^2 \tag{A30}$$

giving that

$$\sigma^2_a\,\sigma^2_b \geq |\,\langle\mathcal{A}\,\mathcal{B}\rangle\,|^2 \tag{A31}$$

The product of two operators may be written as

$$\mathcal{A}\,\mathcal{B} = \frac{1}{2}\,[\,\mathcal{A},\mathcal{B}]_+ + \frac{1}{2}i\,[\,\mathcal{A},\mathcal{B}\,]/i \tag{A32}$$

where both $[\,\mathcal{A},\mathcal{B}]_+$ and $[\,\mathcal{A},\mathcal{B}\,]/i$ are Hermitian. Taking the expectation value of both sides, we have

$$\langle\,\mathcal{A}\,\mathcal{B}\,\rangle = \frac{1}{2}\langle\,[\,\mathcal{A},\mathcal{B}]_+\,\rangle + \frac{1}{2}i\langle\,[\,\mathcal{A},\mathcal{B}\,]/i\,\rangle \tag{A33}$$

$$= \mathrm{Cov}_{ab} + \frac{1}{2}i\langle\,[\,\mathcal{A},\mathcal{B}\,]/i\,\rangle \tag{A34}$$

Since $[\,\mathcal{A},\mathcal{B}]_+$ and $[\,\mathcal{A},\mathcal{B}\,]/i$ are Hermitian, the expectation values on the right-hand side are real and, therefore,

$$|\,\langle\mathcal{A}\,\mathcal{B}\rangle\,|^2 = \frac{1}{4}|\,\langle\,[\,\mathcal{A},\mathcal{B}\,]\,\rangle\,|^2 + \mathrm{Cov}_{ab}^2 \tag{A35}$$

Therefore, Equation (A31) gives that

$$\sigma_a^2\,\sigma_b^2 \geq \frac{1}{4}|\,\langle\,[\,\mathcal{A},\mathcal{B}\,]\,\rangle\,|^2 + \mathrm{Cov}_{ab}^2 \tag{A36}$$

References

1. Whitham, G. *Linear and Nonlinear Waves*; John Wiley and Sons: New York, NY, USA, 1974.
2. Graff, K. *Wave Motion in Elastic Solids*; Oxford University Press: Oxford, UK, 1975.
3. Jackson, J.D. *Classical Electrodynamics*; Wiley: Hoboken, NJ, USA, 1992.
4. Morse, P.H.; Ingard, K.U. *Theoretical Acoustics*; McGraw-Hill: New York, NY, USA, 1968.
5. Cohen, L. Why do wave packets sometimes contract? *J. Mod. Opt.* **2003**, *49*, 2365–2382. [CrossRef]
6. Loughlin, P.; Cohen, L. Local properties of dispersive pulses. *J. Mod. Opt.* **2002**, *49*, 2645–2655. [CrossRef]
7. Ben-Benjamin, J.S.; Cohen, L. Modes and Noise Propagation in Phase Space. *Coherent Opt. Phenom.* **2015**, *2*, 2299. [CrossRef]
8. Bohm, D. *Quantum Theory*; Prentice-Hall: New York, NY, USA, 1951.
9. Merzbacher, E. *Quantum Mechanics*; John Wiley & Sons, Inc.: Hoboken, NJ, USA, 1998.
10. Cohen, L. *Time-Frequency Analysis*; Prentice-Hall: Upper Saddle River, NJ, USA, 1995.
11. Cohen, L. *The Weyl Operator and Its Generalization*; Birkhäuser: Basel, Switzerland; Springer: New York, NY, USA, 2013.
12. Cartwright, N. Correlations without joint distributions in quantum mechanics. *Found. Phys.* **1974**, *4*, 127–136. [CrossRef]
13. Wigner, E.P. On the quantum correction for thermodynamic equilibrium. *Phys. Rev.* **1932**, *40*, 749–759. [CrossRef]
14. Hillery, M.; O'Connell, R.F.; Scully, M.O.; Wigner, E.P. Distribution functions in physics: Fundamentals. *Phys. Rep.* **1984**, *106*, 121. [CrossRef]
15. Schleich, W.P. *Quantum Optics in Phase Space*; Wiley-VCH: Weinheim, Germany, 2001.
16. Lee, H.W. Theory and application of the quantum phase-space distribution functions. *Phys. Rep.* **1995**, *259*, 147–211. [CrossRef]
17. Choi, H.; Williams, W. Improved time-frequency representation of multicomponent signals using exponential kernels. *IEEE Trans. ASSP* **1989**, *37*, 862–871. [CrossRef]
18. Jeong, J.; Williams, W. Kernel design for reduced interference distributions. *IEEE Trans. Signal Process.* **1992**, *40*, 402–412. [CrossRef]
19. Margenau, H.; Hill, R.N. Correlation between Measurements in Quantum Theory. *Prog. Theor. Phys.* **1961**, *26*, 722–738. [CrossRef]
20. Rihaczek, W. Signal Energy Distribution in Time and Frequency. *IEEE Trans. Inf. Theory* **1968**, *4*, 369–374. [CrossRef]