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On Semi-Vector Spaces and Semi-Algebras with Applications in Fuzzy Automata

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Abstract: In this paper, we expand the theory of semi-vector spaces and semi-algebras, both over the semi-field of nonnegative real numbers $\mathbb{R}_0^+$. More precisely, we prove several new results concerning these theories. We introduce to the literature the concept of eigenvalues and eigenvectors of a semi-linear operator, describing how to compute them. The topological properties of semi-vector spaces, such as completeness and separability, are also investigated here. New families of semi-vector spaces derived from the semi-metric, semi-norm and semi-inner product, among others, are exhibited. Furthermore, we show several new results concerning semi-algebras. After this theoretical approach, we apply such a theory in fuzzy automata. More precisely, we describe the semi-algebra of A-fuzzy regular languages and we apply the theory of fuzzy automata for counting patterns in DNA sequences.

Keywords: semi-vector space; semi-algebras; semi-linear operators

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1. Introduction

The concept of semi-vector space was introduced by Prakash and Sertel in [1]. Roughly speaking, semi-vector spaces are "vector spaces" where the scalars are in a semi-field. Although the concept of semi-vector space was investigated over time, there exist few works available in the literature dealing with such spaces [1–7]. This fact occurs maybe due to the limitations that such a concept brings, i.e., the non-existence of a (additive) symmetric for some (for all) semi-vectors. A textbook on this topic of research is the book by Kandasamy [8].

Although the seminal paper on semi-vector spaces was the paper by Prakash and Sertel [1], the idea of such a concept is implicit in [7], where Radstrom showed that a semi-vector space over the semi-field of nonnegative real numbers can be extended to a real vector space (see [7], Theorem 1-B.). In [1], Prakash and Sertel investigated the structure of topological semi-vector spaces. The authors were concerned with the study of the existence of fixed points in compact convex sets and also with generating min–max theorems in topological semi-vector spaces. In [6], Prakash and Sertel investigated the properties of the topological semi-vector space consisting of nonempty compact subsets of a real Hausdorff topological vector space. In [5], Pap investigated and formulated the concept of integrals of functions having, as counter-domains, complete semi-vector spaces. W. Gahler and S. Gahler [2] showed that a (ordered) semi-vector space can be extended to a (ordered) vector space and a (ordered) semi-algebra can be extended to a (ordered) algebra. Moreover, they provided an extension of fuzzy numbers. Janyska et al. [3] developed such a theory (of semi-vector spaces) by proving useful results and defining the semi-tensor product of (semi-free) semi-vector spaces. They were also interested in proposing an algebraic model of physical scales. Canarutto [9] explored the concept of semi-vector spaces to express aspects and to exploit nonstandard mathematical notions of the basics of quantum particle physics on a curved Lorentzian background. Moreover, he dealt with the case of electroweak interactions. Additionally, in [10], Canarutto provided a suitable formulation of the fundamental mathematical concepts with respect to quantum field theory. Such a paper presents a natural application of the concept of semi-vector spaces and semi-algebras. Recently, Bedregal et al. [4] investigated (ordered) semi-vector spaces over a weak semi-field K (i.e., both $(K, +)$ and $(K, \bullet)$ are monoids) in the context of fuzzy sets and applying the results in multi-criteria group decision-making.

In this paper, we show new results on the theory of semi-vector spaces and semi-algebras. The semi-field of scalars considered here is the semi-field of nonnegative real numbers. We prove several results in the context of semi-vector spaces and semi-linear transformations. We introduce the concept of semi-eigenvalues and semi-eigenvectors of an operator and of a matrix, showing how to compute it in specific cases. We investigate topological properties such as completeness, compactness and separability of semi-vector spaces. Additionally, we present interesting new families of semi-vector spaces derived from semi-metric, semi-norm, semi-inner product and metric-preserving functions, among others. Furthermore, we show several new results concerning semi-algebras. To summarize, we provide new results on semi-vector spaces and semi-algebras, although such theories are very difficult to investigate due to the fact that vectors do not even have (additive) symmetry.

The main motivation of this research is to present an expansion of both theories: semi-vector spaces and semi-algebras. Since a semi-vector space (semi-algebra) is a natural extension of a vector space (algebra), this paper provides new useful tools that can be utilized in several areas of research. In particular, we apply some new results in order to count patterns in DNA sequences (see, Section 4.3). Moreover, due to the fact that the fuzzy theory is correlated with semi-vector spaces and semi-algebras, the new results presented here can be applied directly in the study of novel results on such a theory (fuzzy theory).

In fuzzy sets theory, introduced by Zadeh in [11], the "sets" can have uncertainty frontiers. To deal with this uncertainty one utilizes values in the interval $[0, 1]$ as membership degrees. From then, many extensions of this theory have been proposed; see [12], for instance. On the other hand, in [13], the author proposed the notion of fuzzy languages and fuzzy automata which can be useful to process natural languages instead of formal languages [14] as is the case of automata theory [15]. Many extensions of fuzzy automata have been proposed, ([16–19]). In this paper, we also introduce a new extension of fuzzy automata, where the membership degree takes values in a semi-algebra.

The paper is organized as follows. In Section 2, we recall some concepts on semi-vector spaces which will be utilized in this work. In Section 3, we present and prove several results concerning semi-vector spaces and semi-linear transformations. We introduce naturally the concepts of the eigenvalue and eigenvector of a semi-linear operator. Additionally, we exhibit and show interesting examples of semi-vector spaces derived from semi-metric, semi-norms and metric-preserving functions, among others. The results concerning semi-algebras are also presented. In Section 4, we show relationships between Fuzzy Set Theory and semi-algebras. More precisely, in Section 4.1, we show some relationships between semi-algebras and fuzzy automata; in Section 4.2, we present the semi-algebras of A-fuzzy regular languages; and, in Section 4.3, we apply the theory of fuzzy automata for counting patterns in DNA sequences. Finally, this paper's conclusion is presented in Section 5.

2. Preliminaries

The purpose of this section is to recall important facts about semi-vector spaces that are necessary for the development of this work. In order to define such a concept, it is necessary to define the concepts of semi-ring and semi-field.

Definition 1. *A semi-ring $(S, +, \bullet)$ is a set S endowed with two binary operations, $+ : S \times S \longrightarrow S$ (addition), $\bullet : S \times S \longrightarrow S$ (multiplication) such that: (1) $(S, +)$ is a commutative monoid; (2) $(S, \bullet)$ is a semigroup; (3) the multiplication $\bullet$ is distributive with respect to $+$: $\forall\, x, y, z \in S$, $(x + y) \bullet z = x \bullet z + y \bullet z$ and $x \bullet (y + z) = x \bullet y + x \bullet z$.*

We write S instead of writing $(S, +, \bullet)$ if there is no possibility of confusion. If the multiplication $\bullet$ is commutative, then S is a commutative semi-ring. If there exists $1 \in S$, such that $\forall\, x \in S$, one has $1 \bullet x = x \bullet 1 = x$, then S is a semi-ring with identity.

Definition 2 ([8] Definition 3.1.1)**.** *A semi-field is an ordered triple $(K, +, \bullet)$ which is a commutative semi-ring with a unit satisfying the following conditions: (1) $\forall\, x, y \in K$, if $x + y = 0$, then $x = y = 0$; (2) if $x, y \in K$ and $x \bullet y = 0$, then $x = 0$ or $y = 0$.*

Before proceeding further, it is interesting to observe that, in [2], the authors considered the additive cancellation law in the definition of a semi-vector space. In [3], the authors did not assume the existence of the zero (null) vector.

In this paper, we consider the definition of a semi-vector space in the context of that shown in Section 3.1 of [2].

Definition 3. *A semi-vector space over a semi-field K is an ordered triple $(V, +, \cdot)$, where V is a non-empty set endowed with the operations $+ : V \times V \longrightarrow V$ (vector addition) and $\cdot : K \times V \longrightarrow V$ (scalar multiplication) such that:*

(1) *$(V, +)$ is an abelian monoid equipped with the additive cancellation law: $\forall\, u, v, w \in V$, if $u + v = u + w$, then $v = w$;*
(2) *$\forall\, \alpha \in K$ and $\forall\, u, v \in V$, $\alpha(u + v) = \alpha u + \alpha v$;*
(3) *$\forall\, \alpha, \beta \in K$ and $\forall\, v \in V$, $(\alpha + \beta)v = \alpha v + \beta v$;*
(4) *$\forall\, \alpha, \beta \in K$ and $\forall\, v \in V$, $(\alpha\beta)v = \alpha(\beta v)$;*
(5) *$\forall\, v \in V$ and $1 \in K$, $1v = v$.*

Note that, from Item (1) of Definition 3, all semi-vector spaces considered in this paper are *regular*, that is, the additive cancellation law is satisfied. The zero (or null) vector of V, which is unique, will be denoted by 0_V. Let $v \in V$, $v \neq 0_V$. If there exists $u \in V$, such that $v + u = 0_V$, then v is said to be *symmetrizable*. A semi-vector space V is said to be *simple* if the unique symmetrizable element is the zero vector 0_V. In other words, V is simple if it has no nonzero symmetrizable elements.

Definition 4 ([3] Definition 1.4)**.** *Let V be a simple semi-vector space over $\mathbb{R}_0^+$. A subset $B \subset V$ is called a semi-basis of V if every $v \in V$, $v \neq 0_V$, can be written in a unique way as $v = \sum_{i \in I_v} v^{(i)} b_i$, where $v^{(i)} \in \mathbb{R}^+$, $b_i \in B$ and I_v is a finite family of indices uniquely determined by v. The finite subset $B_v \subset B$ defined by $B_v := \{b_i\}_{i \in I_v}$ is uniquely determined by v. If a semi-vector space V admits a semi-basis, then it is said to be semi-free.*

The concept of semi-dimension can be defined in an analogous way to semi-free semi-vector spaces due to the next results.

Corollary 1 ([3] Corollary 1.7)**.** *Let V be a semi-free semi-vector space. Then, all semi-bases of V have the same cardinality.*

Therefore, the semi-dimension of a semi-free semi-vector space is the cardinality of a semi-basis (consequently, of all semi-bases) of V. We next present some examples of semi-vector spaces.

Example 1. *All real vector spaces are semi-vector spaces, but they are not simple.*

Example 2. *The set* $[\mathbb{R}_0^+]^n = \underbrace{\mathbb{R}_0^+ \times \ldots \times \mathbb{R}_0^+}_{n \text{ times}}$ *endowed with the usual sum of coordinates and scalar multiplication is a semi-vector space over* $\mathbb{R}_0^+$*.*

Example 3. *The set* $\mathcal{M}_{n\times m}(\mathbb{R}_0^+)$ *of matrices* $n \times m$ *whose entries are nonnegative real numbers equipped with the sum of matrices and multiplication of a matrix by a scalar (in* $\mathbb{R}_0^+$*, of course) is a semi-vector space over* $\mathbb{R}_0^+$*.*

Example 4. *The set* $\mathcal{P}_n[x]$ *of polynomials with coefficients from* $\mathbb{R}_0^+$ *and degrees less than or equal to* n*, equipped with the usual sum of polynomials and the scalar multiplication of a scalar by a polynomial, is a semi-vector space.*

Definition 5. *Let* $(V, +, \cdot)$ *be a semi-vector space over* $\mathbb{R}_0^+$*. We say that a non-empty subset* W *of* V *is a semi-subspace of* V *if* W *is closed under both the addition and scalar multiplication of* V*, that is,*

(1) $\forall\, w_1, w_2 \in W \Longrightarrow w_1 + w_2 \in W$;
(2) $\forall\, \lambda \in \mathbb{R}_0^+$ *and* $\forall\, w \in W \Longrightarrow \lambda w \in W$.

The uniqueness of the zero vector implies that, for each $\lambda \in \mathbb{R}_0^+$, one has $\lambda 0_V = 0_V$. Moreover, if $v \in V$, it follows that $0v = 0v + 0v$; applying the regularity, one obtains $0v = 0_V$. Therefore, from Item (2), every semi-subspace contains the zero vector.

Example 5. *Let* $\mathbb{Q}_0^+$ *denote the set of nonnegative rational numbers. The semi-vector space* $\mathbb{Q}_0^+$ *considered as an* $\mathbb{Q}_0^+$ *space is a semi-subspace of* $\mathbb{R}_0^+$ *considered as an* $\mathbb{Q}_0^+$ *space.*

Example 6. *The set of diagonal matrices of order* n *with entries in* $\mathbb{R}_0^+$ *is a semi-subspace of* $\mathcal{M}_n(\mathbb{R}_0^+)$*, where the latter is the semi-vector space of square matrices with entries in* $\mathbb{R}_0^+$ *(according to Example 3).*

Definition 6 ([3] Definition 1.22)**.** *Let* V *and* W *be two semi-vector spaces over* $\mathbb{R}_0^+$ *and* $T : V \longrightarrow W$ *be a map. We say that* T *is a semi-linear transformation if: (1)* $\forall\, v_1, v_2 \in V$, $T(v_1 + v_2) = T(v_1) + T(v_2)$*; (2)* $\forall \lambda \in \mathbb{R}_0^+$ *and* $\forall\, v \in V$, $T(\lambda v) = \lambda T(v)$*.*

If U and V are semi-vector spaces, then the set $\mathrm{Hom}(U, V) = \{T : U \longrightarrow V; T \text{ is semi-linear}\}$ is also a semi-vector space.

3. The New Results

In this section, we present the contributions of this work. More precisely, we show new properties on semi-vector spaces and we introduce the concepts of the eigenvalue and eigenvector of a semi-linear operator. In Section 3.1, we investigate properties of complete semi-vector spaces. In Section 3.2, we provide examples of interesting semi-vector spaces, and, in Section 3.3, we prove several results with respect to semi-algebras.

We start with important remarks.

Remark 1.

(1) *Throughout this section, we always consider that the semi-field* K *is the set of nonnegative real numbers, i.e.,* $K = \mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$*.*

(2) *In the whole section (except Subsection 3.2), we assume that the semi-vector spaces are simple, i.e., the unique symmetrizable element is the zero vector* 0_V.
(3) *It is well-known that a semi-vector space* $(V, +, \cdot)$ *can be always extended to a vector space according to the equivalence relation on* $V \times V$ *defined by :* $(u_1, v_1) \sim (u_2, v_2)$ *if and only if* $u_1 + v_2 = v_1 + u_2$ *(see [7]; see also [2] (Section 3.4)). However, our results were obtained without utilizing such a natural embedding. In other words, if we want to compute, for instance, the eigenvalues of a matrix defined over* $\mathbb{R}_0^+$, *we cannot solve the problem in the associated vector spaces and then discard the negative ones. Put differently, all computations performed here are restricted to nonnegative real numbers and also to the fact that a none vector (with the exception of* 0_V*) is (additive and) symmetrical. However, we will show that, even in this case, several results can be obtained.*

Proposition 1. *Let* V *be a semi-vector space over* $\mathbb{R}_0^+$. *Then, the following hold:*
(1) *Let* $v \in V$, $v \neq 0_V$, *and* $\lambda \in \mathbb{R}_0^+$; *if* $\lambda v = 0_V$, *then* $\lambda = 0$.
(2) *If* $\alpha, \beta \in \mathbb{R}_0^+$, $v \in V$ *and* $v \neq 0_V$, *then the equality* $\alpha v = \beta v$ *implies that* $\alpha = \beta$.

Proof. (1) If $\lambda \neq 0$, then there exists its multiplicative inverse λ^{-1}, hence $1v = \lambda^{-1} 0_V = 0_V$, i.e., $v = 0_V$, a contradiction.
(2) If $\alpha \neq \beta$, assume w.l.o.g. that $\alpha > \beta$, i.e., there exists a positive real number c such that $\alpha = \beta + c$. Thus, $\alpha v = \beta v$ implies $\beta v + cv = \beta v$. From the cancellation law, we have $cv = 0_V$, and from Item (1) it follows that $c = 0$, i.e., a contradiction. □

We next introduce in the literature the concepts of the eigenvalue and eigenvector of a semi-linear operator.

Definition 7. *Let* V *be a semi-vector space and* $T : V \longrightarrow V$ *be a semi-linear operator. If there exists a non-zero vector* $v \in V$ *and a nonnegative real number* λ, *such that* $T(v) = \lambda v$, *then* λ *is an eigenvalue of* T *and* v *is an eigenvector of* T *associated with* λ.

As is natural, the zero vector joined to the set of the eigenvectors associated with a given eigenvalue has a semi-subspace structure.

Proposition 2. *Let* V *be a semi-vector space over* $\mathbb{R}_0^+$ *and* $T : V \longrightarrow V$ *be a semi-linear operator. Then, the set* $V_\lambda = \{v \in V; T(v) = \lambda v\}$ *is a semi-subspace of* V.

Proof. From the hypotheses, V_λ is non-empty. Let $u, v \in V_\lambda$, i.e., $T(u) = \lambda u$ and $T(v) = \lambda v$. Hence, $T(u + v) = T(u) + T(v) = \lambda(u + v)$, i.e., $u + v \in V_\lambda$. Further, if $\alpha \in \mathbb{R}_0^+$ and $u \in V$, it follows that $T(\alpha u) = \alpha T(u) = \lambda(\alpha u)$, that is, $\alpha u \in V_\lambda$. Therefore, V_λ is a semi-subspace of V. □

The next natural step would be to introduce the characteristic polynomial of a matrix, according to the standard linear algebra. However, how does one compute $\det(A - \lambda I)$ if $-\lambda$ can be a negative real number? Based on this fact, we must be careful to compute the eigenvectors of a matrix. In fact, the main tools to be utilized in computing the eigenvalues/eigenvectors of a square matrix whose entries are nonnegative real numbers is the additive cancellation law in $\mathbb{R}_0^+$ and also the fact that positive real numbers have multiplicative inverses. However, in many cases, such tools are insufficient to solve the problem. Let us see some cases where it is possible to compute the eigenvalues/eigenvectors of a matrix.

Example 7. *Let us see how to obtain (if there exists) an eigenvalue/eigenvector of a diagonal matrix* $A \in \mathcal{M}_2(\mathbb{R}_0^+)$,

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},$$

where $a \neq b$ are both not zero. We obtain $\lambda = a$ with associated eigenvector $x(1,0)$ and $\lambda = b$ with associated eigenvector $y(0,1)$.

If $a \neq 0$ and $b = 0$, then $\lambda = a$ with eigenvectors $x(1,0)$.

If $a = 0$ and $b \neq 0$, then $\lambda = b$ with eigenvectors $y(0,1)$.

Example 8. *Let $A \in \mathcal{M}_2(\mathbb{R}_0^+)$ be a matrix of the form*

$$A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix},$$

where $a \neq b$ are positive real numbers.

From direct computations, it follows that $\lambda = a$ with eigenvectors $(x,0)$.

If V and W are semi-free semi-vector spaces, then it is possible to define the matrix of a semi-linear transformation $T : V \longrightarrow W$ as in the usual case (vector spaces).

Definition 8. *Let $T : V \longrightarrow W$ be a semi-liner transformation between semi-free semi-vector spaces with semi-basis B_1 and B_2, respectively. Then, the matrix $[T]_{B_1}^{B_2}$ is the matrix of the transformation T.*

Theorem 1. *Let V be a semi-free semi-vector space over $\mathbb{R}_0^+$ and let $T : V \longrightarrow V$ be a semi-linear operator. Then, T admits a semi-basis $B = \{v_1, v_2, \ldots, v_n\}$ such that $[T]_B^B$ is diagonal if and only if B consists of eigenvectors of T.*

Proof. The proof is analogous to the case of vector spaces. Let $B = \{v_1, v_2, \ldots, v_n\}$ be a semi-basis of V whose elements are eigenvectors of T. We then have the following:

$$\begin{aligned} T(v_1) &= \lambda_1 v_1 + 0v_2 + \ldots + 0v_n, \\ T(v_2) &= 0v_1 + \lambda_2 v_2 + \ldots + 0v_n, \\ &\vdots \\ T(v_n) &= 0v_1 + 0v_2 + \ldots + \lambda_n v_n, \end{aligned}$$

which implies that $[T]_B^B$ is of the form

$$[T]_B^B = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

On the other hand, let $B^* = \{w_1, w_2, \ldots, w_n\}$ be a semi-basis of V, such that $[T]_{B^*}^{B^*}$ is diagonal:

$$[T]_{B^*}^{B^*} = \begin{bmatrix} \alpha_1 & 0 & 0 & \ldots & 0 \\ 0 & \alpha_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & \alpha_n \end{bmatrix}.$$

Thus,

$$\begin{aligned} T(w_1) &= \alpha_1 w_1 + 0w_2 + \ldots + 0w_n = \alpha_1 w_1, \\ T(w_2) &= 0w_1 + \alpha_2 w_2 + \ldots + 0w_n = \alpha_2 w_2, \\ &\vdots \\ T(w_n) &= 0w_1 + 0w_2 + \ldots + \alpha_n w_n = \alpha_2 w_n. \end{aligned}$$

This means that w_i are eigenvectors of T with corresponding eigenvalues α_i, for all $i = 1, 2, \ldots, n$. □

Definition 9. *Let $T : V \longrightarrow W$ be a semi-linear transformation. The set* $\mathrm{Ker}(T) = \{v \in V; T(v) = 0_W\}$ *is called kernel of T.*

Proposition 3. *Let $T : V \longrightarrow W$ be a semi-linear transformation. Then, the following hold:*
(1) $\mathrm{Ker}(T)$ *is a semi-subspace of V;*
(2) *If T is injective then* $\mathrm{Ker}(T) = \{0_V\}$*;*
(3) *If V has semi-dimension* 1*, then* $\mathrm{Ker}(T) = \{0_V\}$ *implies that T is injective.*

Proof. (1) We have $T(0_V) = T(0_V) + T(0_V)$. Since W is regular, it follows that $T(0_V) = 0_W$, which implies $\mathrm{Ker}(T) \neq \emptyset$. If $u, v \in \mathrm{Ker}(T)$ and $\lambda \in \mathbb{R}_0^+$, then $u + v \in \mathrm{Ker}(T)$ and $\lambda v \in \mathrm{Ker}(T)$, which implies that $\mathrm{Ker}(T)$ is a semi-subspace of V.
(2) Since $T(0_V) = 0_W$, it follows that $\{0_V\} \subseteq \mathrm{Ker}(T)$. On the other hand, let $u \in \mathrm{Ker}(T)$, that is, $T(u) = 0_W$. Since T is injective, one has $u = 0_V$. Hence, $\mathrm{Ker}(T) = \{0_V\}$.
(3) Let $B = \{v_0\}$ be a semi-basis of V. Assume that $T(u) = T(v)$, where $u, v \in V$ are such that $u = \alpha v_0$ and $v = \beta v_0$. Hence, $\alpha T(v_0) = \beta T(v_0)$. Since $\mathrm{Ker}(T) = \{0_V\}$ and $v_0 \neq 0$, it follows that $T(v_0) \neq 0$. From Item (2) of Proposition 1, one has $\alpha = \beta$, i.e., $u = v$. □

Definition 10. *Let $T : V \longrightarrow W$ be a semi-linear transformation. The image of T is the set of all vectors $w \in W$ such that there exists $v \in V$ with $T(v) = w$, that is,* $\mathrm{Im}(T) = \{w \in W; \exists\, v \in V \text{ with } T(v) = w\}$.

Proposition 4. *Let $T : V \longrightarrow W$ be a semi-linear transformation. Then, the image of T is a semi-subspace of W.*

Proof. The set $\mathrm{Im}(T)$ is non-empty because $T(0_V) = 0_W$. It is easy to see that, if $w_1, w_2 \in \mathrm{Im}(T)$ and $\lambda \in \mathbb{R}_0^+$, then $w_1 + w_2 \in \mathrm{Im}(T)$ and $\lambda w_1 \in \mathrm{Im}(T)$. □

Recall that two semi-vector spaces V and W over a semi-field K are isomorphic; there exists a bijective semi-linear transformation from V to W.

Theorem 2. *Let V be a n-dimensional semi-free semi-vector space over $\mathbb{R}_0^+$. Then, V is isomorphic to $(\mathbb{R}_0^+)^n$.*

Proof. Let $B = \{v_1, v_2, \ldots, v_n\}$ be a semi-basis of V and consider the canonical semi-basis $e_i = (0, 0, \ldots, 0, \underbrace{1}_{i}, 0, \ldots, 0)$ of $(\mathbb{R}_0^+)^n$, where $i = 1, 2, \ldots, n$. Define the map $T : V \longrightarrow (\mathbb{R}_0^+)^n$ as follows: for each $v = \sum_{i=1}^{n} a_i v_i \in V$, put $T(v) = \sum_{i=1}^{n} a_i e_i$. It is easy to see that T is bijective semi-linear transformation, i.e., V is isomorphic to $(\mathbb{R}_0^+)^n$, as required. □

3.1. Complete Semi-Vector Spaces

Here, we define and study complete semi-vector spaces, i.e., semi-vector spaces whose norm (inner product) induces a metric under which the space is complete.

Definition 11. *Let V be a semi-vector space over $\mathbb{R}_0^+$. If there exists a norm $\| \, \| : V \longrightarrow \mathbb{R}_0^+$ on V, we say that V is a normed semi-vector space (or normed semi-space, for short). If the norm defines a metric on V under which V is complete then V is said to be Banach semi-vector space.*

Definition 12. *Let V be a semi-vector space over $\mathbb{R}_0^+$. If there exists an inner product $\langle \, , \, \rangle : V \times V \longrightarrow \mathbb{R}_0^+$ on V, then V is an inner product semi-vector space (or inner product semi-space). If the inner product defines a metric on V under which V is complete, then V is said to be Hilbert semi-vector space.*

The well-known norms on $\mathbb{R}^n$ are also norms on $[\mathbb{R}_0^+]^n$, as we show in the next propositions.

Proposition 5. *Let $V = [\mathbb{R}_0^+]^n$ be the Euclidean semi-vector space (over $\mathbb{R}_0^+$) of semi-dimension n . Define the function $\| \, \| : V \longrightarrow \mathbb{R}_0^+$ as follows: if $x = (x_1, x_2, \ldots, x_n) \in V$, put $\|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$. Then, $\| \, \|$ is a norm on V, called the Euclidean norm on V.*

Proof. It is clear that $\|x\| = 0$ if and only if $x = (0, \ldots, 0)$ and for all $\alpha \in \mathbb{R}_0^+$ and $x \in V$, $\|\alpha x\| = |\alpha| \|x\|$. To show the triangle inequality, it is sufficient to apply the Cauchy–Schwarz inequality in $\mathbb{R}_0^+$: if $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are semi-vectors in V, then $\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} \cdot \left(\sum_{i=1}^{n} y_i^2 \right)^{1/2}$. □

In the next results, we show that the Euclidean norm on $[\mathbb{R}_0^+]^n$ generates the Euclidean metric on it.

Proposition 6. *Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ be semi-vectors in $V = [\mathbb{R}_0^+]^n$. Define the function $d : V \times V \longrightarrow \mathbb{R}_0^+$ as follows: for every fixed i, if $x_i = y_i$ put $c_i = 0$; if $x_i \neq y_i$, put $\varphi_i = \psi_i + c_i$, where $\varphi_i = \max\{x_i, y_i\}$ and $\psi_i = \min\{x_i, y_i\}$ (in this case, $c_i > 0$); then consider $d(x, y) = \sqrt{c_1^2 + \ldots + c_n^2}$. The function d is a metric on V.*

Remark 2. *Note that, in Proposition 6, we could have defined c_i simply by the nonnegative real number satisfying $\max\{x_i, y_i\} = \min\{x_i, y_i\} + c_i$. However, we prefer to separate the cases when $c_i = 0$ and $c_i > 0$ in order to improve the readability of this paper.*

Proof. It is easy to see that $d(x, y) = 0$ if and only if $x = y$ and $d(x, y) = d(y, x)$.

We will next prove the triangle inequality. To do this, let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ and $z = (z_1, z_2, \ldots, z_n)$ be semi-vectors in $V = [\mathbb{R}_0^+]^n$. We look first at a fixed i. If $x_i = y_i = z_i$ or if two of them are equal, then $d(x_i, z_i) \leq d(x_i, y_i) + d(y_i, z_i)$. Let us then assume that x_i, y_i and z_i are pairwise distinct. We have to analyze the six cases: (1) $x_i < y_i < z_i$; (2) $x_i < z_i < y_i$; (3) $y_i < x_i < z_i$; (4) $y_i < z_i < x_i$; (5) $z_i < x_i < y_i$; (6) $z_i < y_i < x_i$. In order to verify the triangle inequality, we will see what occurs in the worst cases. More precisely, we assume that for all $i = 1, 2, \ldots, n$ we have $x_i < y_i < z_i$ or, equivalently, $z_i < y_i < x_i$. Since both cases are analogous, we only verify the (first) case $x_i < y_i < z_i$, for all i. In such cases, there exist positive real numbers a_i, b_i, for all $i = 1, 2, \ldots, n$, such that $y_i = x_i + a_i$ and $z_i = y_i + b_i$, which implies $z_i = x_i + a_i + b_i$. We need to show that $d(x, z) \leq d(x, y) + d(y, z)$, i.e., $\left(\sum_{i=1}^{n} (a_i + b_i)^2 \right)^{1/2} \leq \left(\sum_{i=1}^{n} a_i^2 \right)^{1/2} + \left(\sum_{i=1}^{n} b_i^2 \right)^{1/2}$. The last inequality is equivalent to the inequality $\sum_{i=1}^{n} (a_i + b_i)^2 \leq \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2 \left(\sum_{i=1}^{n} a_i^2 \right)^{1/2} \cdot \left(\sum_{i=1}^{n} b_i^2 \right)^{1/2}$. Developing the first member of the previous inequality and deleting the corresponding terms with the first two terms in the second member following the multiplication by $1/2$, we have $\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \cdot \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}$, which is the Cauchy–Schwarz inequality in $\mathbb{R}_0^+$. Therefore, d satisfies the triangle inequality and, hence, is a metric on V. □

Remark 3. *Note that Proposition 6 means that the Euclidean norm on $[\mathbb{R}_0^+]^n$ (see Proposition 5) generates the Euclidean metric on $[\mathbb{R}_0^+]^n$. This result is analogous to the fact that every norm defined on vector spaces generates a metric on it. Further, a semi-vector space V is Banach (see Definition 11) if the norm generates a metric under which every Cauchy sequence in V converges to an element of V.*

Proposition 7. *Let $V = [\mathbb{R}_0^+]^n$ and define the function $\langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{R}_0^+$ as follows: if $u = (x_1, x_2, \ldots, x_n)$ and $v = (y_1, y_2, \ldots, y_n)$ are semi-vectors in V, put $\langle u, v\rangle = \sum_{i=1}^{n} x_i y_i$. Then, $\langle\ ,\ \rangle$ is an inner product on V, called the dot product.*

Proof. The proof is immediate. □

Proposition 8. *The dot product on $V = [\mathbb{R}_0^+]^n$ generates the Euclidean norm on V.*

Proof. If $x = (x_1, x_2, \ldots, x_n) \in V$, define the norm of x by $\|x\| = \sqrt{\langle x, x\rangle}$. Note that the norm is exactly the Euclidean norm given in Proposition 5. □

Remark 4. *We observe that, if an inner product on a semi-vector space V generates a norm $\|\ \|$ and such a norm generates a metric d on V, then V is a Hilbert space (according to Definition 12) if every Cauchy sequence in V converges with respect to d to an element of V.*

Proposition 9. *Let $V = [\mathbb{R}_0^+]^n$ and define the function $\|\ \|_1 : V \longrightarrow \mathbb{R}_0^+$ as follows: if $x = (x_1, x_2, \ldots, x_n) \in V$, $\|x\|_1 = \sum_{i=1}^{n} x_i$. Then, $\|x\|_1$ is a norm on V.*

Proof. The proof is direct. □

Proposition 10. *Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ be semi-vectors in $V = [\mathbb{R}_0^+]^n$. Define the function $d_1 : V \times V \longrightarrow \mathbb{R}_0^+$ in the following way. For every fixed i, if $x_i = y_i$, put $c_i = 0$; if $x_i \neq y_i$, put $\varphi_i = \psi_i + c_i$, where $\varphi_i = \max\{x_i, y_i\}$ and $\psi_i = \min\{x_i, y_i\}$. Let us consider that $d_1(x, y) = \sum_{i=1}^{n} c_i$. Then, the function d_1 is a metric on V derived from the norm $\|\ \|_1$ shown in Proposition 9.*

Proof. We only prove the triangle inequality. To avoid the stress of notation, we make the same considerations as in the proof of Proposition 6. We then fix i and only investigate the worst case $x_i < y_i < z_i$. In this case, there exist positive real numbers a_i, b_i for all $i = 1, 2, \ldots, n$, such that $y_i = x_i + a_i$ and $z_i = y_i + b_i$, which implies $z_i = x_i + a_i + b_i$. Then, for all i, $d_1(x_i, z_i) \leq d_1(x_i, y_i) + d_1(y_i, z_i)$; hence, $d_1(x, z) = \sum_{i=1}^{n} d_1(x_i, z_i) = \sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} d_1(x_i, y_i) + \sum_{i=1}^{n} d_1(y_i, z_i) = d_1(x, y) + d_1(y, z)$. Therefore, d_1 is a metric on V. □

Proposition 11. *Let $V = [\mathbb{R}_0^+]^n$ be the Euclidean semi-vector space of semi-dimension n. Define the function $\|\ \|_2 : V \longrightarrow \mathbb{R}_0^+$ as follows: if $x = (x_1, x_2, \ldots, x_n) \in V$, take $\|x\|_2 = \max_i\{x_i\}$. Then, $\|x\|_2$ is a norm on V.*

Proposition 12. *Keeping the notation of Proposition 6, define the function $d_2 : V \times V \longrightarrow \mathbb{R}_0^+$ such that $d_2(x, y) = \max_i\{c_i\}$. Then, d_2 is a metric on V. Moreover, d_2 is obtained from the norm $\| \|_2$ exhibited in Proposition 11.*

Proposition 13. *The norms $\| \|$, $\| \|_1$ and $\| \|_2$ shown in Propositions 5, 9 and 11 are equivalent.*

Proof. It can immediately be seen that $\| \|_2 \leq \| \| \leq \| \|_1 \leq n\| \|_2$. □

In a natural way, we can define the norm of a bounded semi-linear transformation.

Definition 13. *Let V and W be two normed semi-vector spaces and let $T : V \longrightarrow W$ be a semi-linear transformation. We say that T is bounded if there exists a real number $c > 0$, such that $\|T(v)\| \leq c\|v\|$.*

If $T : V \longrightarrow W$ is bounded and $v \neq 0_V$, we can consider the quotient $\frac{\|T(v)\|}{\|v\|}$. Since such a quotient is upper bounded by c, the supremum $\sup_{v\in V, v\neq 0_V} \dfrac{\|T(v)\|}{\|v\|}$ exists and it is, at most, c. We then define

$$\|T\| = \sup_{v\in V, v\neq 0_V} \frac{\|T(v)\|}{\|v\|}.$$

Proposition 14. *Let $T : V \longrightarrow W$ be a bounded semi-linear transformation. Then, the following hold:*
(1) *T sends bounded sets in bounded sets;*
(2) *$\|T\|$ is a norm, called norm of T;*
(3) *$\|T\|$ can be written in the form $\|T\| = \sup_{v\in V, \|v\|=1} \|T(v)\|$.*

Proof. Items (1) and (2) are immediate. The proof of Item (3) is analogous to the standard proof but we present it here to guarantee that our mathematical tools are sufficient to perform it. Let $v \neq 0_V$ be a semi-vector with norm $\|v\| = a \neq 0$ and set $u = (1/a)v$. Thus, $\|u\| = 1$ and since T is semi-linear, one has

$$\|T\| = \sup_{v\in V, v\neq 0_V} \frac{1}{a}\|T(v)\| = \sup_{v\in V, v\neq 0_V} \|T((1/a)v)\| = \sup_{u\in V, \|u\|=1} \|T(u)\| = \sup_{v\in V, \|v\|=1} \|T(v)\|.$$

□

Semi-Spaces l_+^∞, l_+^p and $C_+[a, b]$

In this subsection, we investigate the topological aspects of some semi-vector spaces over $\mathbb{R}_0^+$, such as completeness and separability. We investigate the sequence spaces l_+^∞, l_+^p, $C_+[a, b]$, which will be defined in the sequence.

We first study the space l_+^∞, the set of all bounded sequences of nonnegative real numbers. Before studying such a space, we must define a metric on it, since the metric in l^∞, which is defined as $d(x, y) = \sup_{i\in\mathbb{N}} |x_i - y_i|$, where $x = (x_i)$ and $y = (y_i)$ are sequences in l^∞, has no meaning to us, because there is no sense in considering $-y_i$ if $y_i > 0$. Based on this fact, we circumvent this problem by utilizing the total order of $\mathbb{R}$ according to Proposition 6. Let $x = (\mu_i)$ and $y = (\nu_i)$ be sequences in l_+^∞. We then fix i, and define c_i as was carried out in Proposition 6: if $\mu_i = \nu_i$, then we put $c_i = 0$; if $\mu_i \neq \nu_i$, let $\gamma_i = \max\{\mu_i, \nu_i\}$ and $\psi_i = \min\{\mu_i, \nu_i\}$; then, there exists a positive real number c_i such that $\gamma_i = \psi_i + c_i$ and, in place of $|\mu_i - \nu_i|$, we put c_i. Thus, our metric becomes

$$d(x, y) = \sup_{i\in\mathbb{N}}\{c_i\}. \quad (1)$$

It is clear that $d(x, y)$, as shown in Equation (1), defines a metric. However, we must show that the tools that we have are sufficient to prove this fact, once we are working on $\mathbb{R}_0^+$.

Proposition 15. *The function d shown in Equation (1) is a metric on* l_+^∞.

Proof. It is clear that $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$. Let $x = (\mu_i)$ and $y = (\nu_i)$ be two sequences in l_+^∞. Then, for every fixed $i \in \mathbb{N}$, if $c_i = d(\mu_i, \nu_i) = 0$ then $\mu_i = \nu_i$, i.e., $d(\mu_i, \nu_i) = d(\nu_i, \mu_i)$. If $c_i > 0$ then $c_i = d(\mu_i, \nu_i)$ is computed by $\gamma_i = \psi_i + c_i$, where $\gamma_i = \max\{\mu_i, \nu_i\}$ and $\psi_i = \min\{\mu_i, \nu_i\}$. Hence, $d(\nu_i, \mu_i) = c_i^*$ is computed by $\gamma_i^* = \psi_i^* + c_i^*$, where $\gamma_i^* = \max\{\nu_i, \mu_i\}$ and $\psi_i^* = \min\{\nu_i, \mu_i\}$, which implies $d(\mu_i, \nu_i) = d(\nu_i, \mu_i)$. Taking the supremum over all i's we have $d(x, y) = \sup_{i \in \mathbb{N}}\{c_i\} = \sup_{i \in \mathbb{N}}\{c_i^*\} = d(y, x)$.

To show the triangle inequality, let $x = (\mu_i)$, $y = (\nu_i)$ and $z = (\eta_i)$ be sequences in l_+^∞. For every fixed i, we will prove that $d(\mu_i, \eta_i) \leq d(\mu_i, \nu_i) + d(\nu_i, \eta_i)$. If $\nu_i = \mu_i = \eta_i$, the result is trivial. If two of them are equal, the result is also trivial. Assume that μ_i, ν_i and η_i are pairwise distinct. As in the proof of Proposition 6, we must investigate the six cases: (1) $\mu_i < \nu_i < \eta_i$; (2) $\mu_i < \eta_i < \nu_i$; (3) $\nu_i < \mu_i < \eta_i$; (4) $\nu_i < \eta_i < \mu_i$; (5) $\eta_i < \mu_i < \nu_i$; (6) $\eta_i < \nu_i < \mu_i$. We only show (1) and (2).

To show (1), note that there exist positive real numbers c_i and $c_i^{'}$, such that $\nu_i = \mu_i + c_i$ and $\eta_i = \nu_i + c_i^{'}$, which implies $\eta_i = \mu_i + c_i + c_i^{'}$. Hence, $d(\mu_i, \eta_i) = c_i + c_i^{'} = d(\mu_i, \nu_i) + d(\nu_i, \eta_i)$.

Let us show (2). There exist positive real numbers b_i and $b_i^{'}$, such that $\eta_i = \mu_i + b_i$ and $\nu_i = \eta_i + b_i^{'}$, so $\nu_i = \mu_i + b_i + b_i^{'}$. Therefore, $d(\mu_i, \eta_i) = b_i < d(\mu_i, \nu_i) + d(\nu_i, \eta_i) = b_i + 2b_i^{'}$.

Taking the supremum over all i's, we have

$$\sup_{i \in \mathbb{N}}\{d(\mu_i, \eta_i)\} \leq \sup_{i \in \mathbb{N}}\{d(\mu_i, \nu_i)\} + \sup_{i \in \mathbb{N}}\{d(\nu_i, \eta_i)\},$$

i.e., $d(x, z) \leq d(x, y) + d(y, z)$. Therefore, d is a metric on l_+^∞. □

Definition 14. *The metric space* l_+^∞ *is the set of all bounded sequences of nonnegative real numbers equipped with the metric* $d(x, y) = \sup_{i \in \mathbb{N}}\{c_i\}$ *given previously.*

We prove that l_+^∞ equipped with the previous metric is complete.

Theorem 3. *The space* l_+^∞ *with the metric* $d(x, y) = \sup_{i \in \mathbb{N}}\{c_i\}$ *shown above is complete.*

Proof. The proof follows the same line as the standard proof of completeness of l^∞; however, it is necessary to adapt it to the metric (written above) in terms of nonnegative real numbers. Let (x_n) be a Cauchy sequence in l_+^∞, where $x_i = (\eta_1^{(i)}, \eta_2^{(i)}, \ldots)$. We must show that (x_n) converges to an element of l_+^∞. As (x_n) is Cauchy, given $\epsilon > 0$, there exists a positive integer, k such that, for all $n, m > k$,

$$d(x_n, x_m) = \sup_{j \in \mathbb{N}}\{c_j^{(n,m)}\} < \epsilon,$$

where $c_j^{(n,m)}$ is a nonnegative real number, such that, if $\eta_j^{(n)} = \eta_j^{(m)}$ then $c_j^{(n,m)} = 0$, and if $\eta_j^{(n)} \neq \eta_j^{(m)}$ then $c_j^{(n,m)}$ is given by $\max\{\eta_j^{(n)}, \eta_j^{(m)}\} = \min\{\eta_j^{(n)}, \eta_j^{(m)}\} + c_j^{(n,m)}$. This implies that, for each fixed j, one has

$$c_j^{(n,m)} < \epsilon, \tag{2}$$

where $n, m > k$. Thus, for each fixed j, it follows that $(\eta_j^{(1)}, \eta_j^{(2)}, \ldots)$ is a Cauchy sequence in $\mathbb{R}_0^+$. Since $\mathbb{R}_0^+$ is a complete metric space, the sequence $(\eta_j^{(1)}, \eta_j^{(2)}, \ldots)$ converges to an element η_j in $\mathbb{R}_0^+$. Hence, for each j, we form the sequence x whose coordinates are the limits η_j, i.e., $x = (\eta_1, \eta_2, \eta_3, \ldots)$. We must show that $x \in l_+^\infty$ and $x_n \longrightarrow x$.

To show that x is a bounded sequence, let us consider the number $c_j^{(n,\infty)}$ defined as follows: if $\eta_j = \eta_j^{(n)}$ then $c_j^{(n,\infty)} = 0$, and if $\eta_j \neq \eta_j^{(n)}$, define $c_j^{(n,\infty)}$ as being the positive real number satisfying $\max\{\eta_j, \eta_j^{(n)}\} = \min\{\eta_j, \eta_j^{(n)}\} + c_j^{(n,\infty)}$. From the inequality (2), one has

$$c_j^{(n,\infty)} \leq \epsilon. \tag{3}$$

Because $\eta_j \leq \eta_j^{(n)} + c_j^{(n,\infty)}$ and since $\eta_j^{(n)} \in l_+^\infty$, it follows that η_j is a bounded sequence for every j. Hence, $x = (\eta_1, \eta_2, \eta_3, \ldots) \in l_+^\infty$. From (3), we have

$$\sup_{j \in \mathbb{N}} \{c_j^{(n,\infty)}\} \leq \epsilon,$$

which implies that $x_n \longrightarrow x$. Therefore, l_+^∞ is complete. □

Although l_+^∞ is a complete metric space, it is not separable.

Theorem 4. *The space l_+^∞ with the metric $d(x, y) = \sup_{i \in \mathbb{N}} \{c_i\}$ is not separable.*

Proof. The proof is the same as shown in ([20] 1.3-9), so it is omitted. □

Let us define the space analogous to the space l^p.

Definition 15. *Let $p \geq 1$ be a fixed real number. The set l_+^p consists of all sequences $x = (\eta_1, \eta_2, \eta_3, \ldots)$ of nonnegative real numbers, such that $\sum_{i=1}^{\infty} (\eta_i)^p < \infty$, whose metric is defined by $d(x, y) = \left[\sum_{i=1}^{\infty} [c_i]^p\right]^{1/p}$, where $y = (\mu_1, \mu_2, \mu_3, \ldots)$ and c_i is defined as follows: $c_i = 0$ if $\mu_i = \eta_i$, and if $\mu_i > \eta_i$ (respect. $\eta_i > \mu_i$) then $c_i > 0$ is such that $\mu_i = \eta_i + c_i$.*

Theorem 5. *The space l_+^p with the metric $d(x, y) = \left[\sum_{i=1}^{\infty} [c_i]^p\right]^{1/p}$ exhibited above is complete.*

Proof. Recall that the given two sequences (μ_i) and (η_i) in l_+^p the Minkowski inequality for sums reads as

$$\left[\sum_{i=1}^{\infty} |\mu_i + \eta_i|^p\right]^{1/p} \leq \left[\sum_{j=1}^{\infty} |\mu_j|^p\right]^{1/p} + \left[\sum_{k=1}^{\infty} |\eta_k|^p\right]^{1/p}.$$

Applying the Minkowski inequality as per ([20] 1.5-4) with some adaptations, it follows that $d(x, y)$ is, in fact, a metric. In order to prove the completeness of l_+^p, we proceed similarly as in the proof of Theorem 3 with some adaptations. The main adaptation is performed according to the proof of completeness of l^p in ([20] 1.5-4) replacing the last equality $x = x_m + (x - x_m) \in l^p$ (after Equation (5)) by two equalities in order to avoid negative real numbers.

(1) If the i-th coordinate $x^{(i)} - x_m^{(i)}$ of the sequence $x - x_m$ is positive, then define $c_m^{(i)} = x^{(i)} - x_m^{(i)}$ and write $x^{(i)} = x_m^{(i)} + c_m^{(i)}$. From the Minkowski inequality, it follows that the sequence $(x^{(i)})_i$ is in l_+^p.

(2) If $x^{(j)} - x_m^{(j)}$ is negative, then define $c_m^{(j)} = x_m^{(j)} - x^{(j)}$ and write $x_m^{(j)} = x^{(j)} + c_m^{(j)}$. Since $x_m \in l_+^p$, from the comparison criterion for positive series it follows that the sequence $(x^{(j)})_j$ is also in l_+^p.

□

Theorem 6. *The space l_+^p is separable.*

Proof. The proof follows the same line of ([20] 1.3-10). □

Definition 16. *Let $I = [a, b]$ be a closed interval in $\mathbb{R}_0^+$, where $a \geq 0$ and $a < b$. Then, $\mathrm{C}_+[a, b]$ is the set of all continuous nonnegative real valued functions on $I = [a, b]$, whose metric is defined by $d(f(t), g(t)) = \max_{t \in I}\{c(t)\}$, where $c(t)$ is given by $\max\{f(t), g(t)\} = \min\{f(t), g(t)\} + c(t)$.*

Theorem 7. *The metric space $(\mathrm{C}_+[a, b], d)$, where d is given in Definition 16, is complete.*

Proof. The proof follows the same lines as the standard one with some modifications. Let (f_m) be a Cauchy sequence in $\mathrm{C}_+[a, b]$. Given, $\epsilon > 0$ there exists a positive integer N such that, for all $m, n > N$, it follows that

$$d(f_m, f_n) = \max_{t \in I}\{c_{m,n}(t)\} < \epsilon, \tag{4}$$

where $\max\{f_m(t), f_n(t)\} = \min\{f_m(t), f_n(t)\} + c_{m,n}(t)$. Thus, for any fixed $t_0 \in I$, we have $c_{m,n}(t_0) < \epsilon$, for all $m, n > N$. This means that $(f_1(t_0), f_2(t_0), \ldots)$ is a Cauchy sequence in $\mathbb{R}_0^+$, which converges to $f(t_0)$ when $m \longrightarrow \infty$ since $\mathbb{R}_0^+$ is complete. We then define a function $f : [a, b] \longrightarrow \mathbb{R}_0^+$ such that, for each $t \in [a, b]$, we put $f(t)$. Taking $n \longrightarrow \infty$ in (4), we obtain $\max_{t \in I}\{c_m(t)\} \leq \epsilon$ for all $m > N$, where $\max\{f_m(t), f(t)\} = \min\{f_m(t), f(t)\} + c_m(t)$, which implies $c_m(t) \leq \epsilon$ for all $t \in I$. This fact means that $(f_m(t))$ converges to $f(t)$ uniformly on I, i.e., $f \in \mathrm{C}_+[a, b]$ because the functions f_m's are continuous on I. Therefore, $\mathrm{C}_+[a, b]$ is complete, as desired. □

3.2. Interesting Semi-Vector Spaces

In this section, we exhibit semi-vector spaces over $K = \mathbb{R}_0^+$ derived from semi-metrics, semi-metric-preserving functions, semi-norms, semi-inner products and sub-linear functionals. Recall that a semi-metric is a metric without the condition that $d(x, y) = 0$ if and only if $x = y$.

Theorem 8. *Let X be a semi-metric space and $\mathcal{M}_X = \{d : X \times X \longrightarrow \mathbb{R}; d$ is a semi-metric on $X\}$. Then, $(\mathcal{M}_X, +, \cdot)$ is a semi-vector space over $\mathbb{R}_0^+$, where $+$ and $\cdot$ are the pointwise addition and the scalar multiplication (in $\mathbb{R}_0^+$), respectively.*

Proof. We first show that $\mathcal{M}_X$ is closed under addition. Let $d_1, d_2 \in \mathcal{M}_X$ and set $d := d_1 + d_2$. It is clear that d is a nonnegative real-valued function. Moreover, for all $x, y \in X$, $d(x, y) = d(y, x)$. Let $x \in X$; $d(x, x) = d_1(x, x) + d_2(x, x) = 0$. For all $x, y, z \in X$, $d(x, z) = d_1(x, z) + d_2(x, z) \leq [d_1(x, y) + d_2(x, y)] + [d_1(y, z) + d_2(y, z)] = d(x, y) + d(y, z)$.

Let us show that $\mathcal{M}_X$ is closed under scalar multiplication. Let $d_1 \in \mathcal{M}_X$ and define $d = \lambda d_1$, where $\lambda \in \mathbb{R}_0^+$. It is clear that d is real-valued nonnegative and for all $x, y \in X$, $d(x, y) = d(y, x)$. Moreover, if $x \in X$, $d(x, x) = 0$. For all $x, y, z \in X$, $d(x, z) = \lambda d_1(x, z) \leq \lambda[d_1(x, y) + d_1(y, z)] = d(x, y) + d(y, z)$. This means that $\mathcal{M}_X$ is closed under scalar multiplication.

It is easy to see that $(\mathcal{M}_X, +, \cdot)$ satisfies the other conditions of Definition 3. □

Let (X, d) be a metric space. In [21], Corazza investigated interesting functions $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$, such that the composite of f with d, i.e., $X \times X \xrightarrow{d} \mathbb{R}_0^+ \xrightarrow{f} \mathbb{R}_0^+$, also generates a metric on X. Let us put this concept formally.

Definition 17. *Let $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ be a function. We say that f is metric-preserving if, for all metric spaces (X, d), the composite $f \circ d$ is a metric.*

For our purpose, we will consider semi-metric preserving functions as follows.

Definition 18. *Let $f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ be a function. We say that f is semi-metric-preserving if, for all semi-metric spaces (X, d), the composite $f \circ d$ is a semi-metric.*

We next show that the set of semi-metric preserving functions has a semi-vector space structure.

Theorem 9. *Let $\mathcal{F}_{pres} = \{f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+; f \text{ is semi-metric preserving}\}$. Then, $(\mathcal{F}_{pres}, +, \cdot)$ is a semi-vector space over $\mathbb{R}_0^+$, where $+$ and $\cdot$ are the pointwise addition and the scalar multiplication (in $\mathbb{R}_0^+$), respectively.*

Proof. We begin by showing that $\mathcal{F}_{pres}$ is closed under pointwise addition and scalar multiplication.

Let $f, g \in \mathcal{F}_{pres}$. Given a semi-metric space (X, d), we must prove that $(f + g) \circ d$ is also semi-metric preserving. We know that $[(f + g) \circ d](x, y) \geq 0$ for all $x, y \in X$. Let $x \in X$; then $[(f + g) \circ d](x, x) = f(d(x, x)) + g(d(x, x)) = 0$. It is clear that $[(f + g) \circ d](x, y) = [(f + g) \circ d](y, x)$. Let $x, y, z \in X$. One has $[(f + g) \circ d](x, y) = f(d(x, y)) + g(d(x, y)) \leq [f(d(x, z)) + g(d(x, z))] + [f(d(z, y)) + g(d(z, y))] = (f + g)(d(x, z)) + (f + g)(d(z, y)) = [(f + g) \circ d](x, z) + [(f + g) \circ d](z, y)$.

Here, we show that, for each $f \in \mathcal{F}_{pres}$ and $\alpha \in \mathbb{R}_0^+$, it follows that $\alpha f \in \mathcal{F}_{pres}$. We show only the triangular inequality since the other conditions are immediate. Let us calculate the following: $[\alpha f \circ d](x, y) = \alpha f(d(x, y)) \leq \alpha f(d(x, z)) + \alpha f(d(z, y)) = [\alpha f \circ d](x, z) + [\alpha f \circ d](z, y)$.

The null vector is the null function $0_f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$. The other conditions are easy to verify. □

Theorem 10. *Let V be a semi-normed real vector space and $\mathcal{N}_V = \{\| \ \| : V \longrightarrow \mathbb{R}; \| \ \| \text{ is a semi-norm on } V\}$. Then, $(\mathcal{N}_V, +, \cdot)$ is a semi-vector space over $\mathbb{R}_0^+$, where $+$ and $\cdot$ are pointwise addition and scalar multiplication (in $\mathbb{R}_0^+$), respectively.*

Proof. From the hypotheses, $\mathcal{N}_V$ is non-empty. Let $\| \ \|_1, \| \ \|_2 \in \mathcal{N}_V$ and set $\| \ \| := \| \ \|_1 + \| \ \|_2$. For all $v \in V$, $\|v\| \geq 0$. If $v \in V$ and $\alpha \in \mathbb{R}$, then $\|\alpha v\| = |\alpha| \|v\|$. For every $u, v \in V$, it follows that $\|u + v\| := \|u + v\|_1 + \|u + v\|_2 \leq (\|u\|_1 + \|u\|_2) + (\|v\|_1 + \|v\|_2) = \|u\| + \|v\|$. Hence, $\mathcal{N}_V$ is closed under addition.

We next show that $\mathcal{N}_V$ is closed under scalar multiplication. Let $\| \ \|_1 \in \mathcal{N}_V$ and define $\| \ \| := \lambda \| \ \|_1$, where $\lambda \in \mathbb{R}_0^+$. For all $v \in V$, $\|v\| \geq 0$. If $\alpha \in \mathbb{R}$ and $v \in V$, $\|\alpha v\| = |\alpha| (\lambda \|v\|_1) = |\alpha| \|v\|$. Let $u, v \in V$. Then, $\|u + v\| \leq \lambda \|u\|_1 + \lambda \|v\|_1 = \|u\| + \|v\|$. Therefore, $\mathcal{N}_V$ is closed under addition and scalar multiplication over $\mathbb{R}_0^+$.

The zero vector is the null function $\mathbf{0} : V \longrightarrow \mathbb{R}$. The other conditions of Definition 3 are straightforward. □

Remark 5. *Note that $\mathcal{N}_V^{\diamond} = \{\| \ \| : V \longrightarrow \mathbb{R}; \| \ \| \text{ is a norm on } V\}$ is also closed under both pointwise function addition and scalar multiplication.*

Lemma 1. *Let $T : V \longrightarrow W$ be a linear transformation.*

(1) *If $\| \ \| : W \longrightarrow \mathbb{R}$ is a semi-norm on W, then $\| \ \| \circ T : V \longrightarrow \mathbb{R}$ is a semi-norm on V.*

(2) *If T is injective linear and $\| \, \| : W \longrightarrow \mathbb{R}$ is a norm on W, then $\| \, \| \circ T$ is a norm on V.*

Proof. We only show Item (1). It is clear that $[\| \, \| \circ T](v) \geq 0$ for all $v \in V$. For all $\alpha \in \mathbb{R}$ and $v \in V$, $[\| \, \| \circ T](\alpha v) = |\alpha| \|T(v)\| = |\alpha| [\| \, \| \circ T](v)$. Moreover, $\forall \, v_1, v_2 \in V$, $[\| \, \| \circ T](v_1 + v_2) \leq [\| \, \| \circ T](v_1) + [\| \, \| \circ T](v_2)$. Therefore, $\| \, \| \circ T$ is a semi-norm on V. □

Theorem 11. *Let V and W be two semi-normed vector spaces and $T : V \longrightarrow W$ be a linear transformation. Then*

$$\mathcal{N}_{V_T} = \{\| \, \| \circ T : V \longrightarrow \mathbb{R}; \| \, \| \text{ is a semi-norm on } W\}$$

is a semi-subspace of $(\mathcal{N}_V, +, \cdot)$.

Proof. From the hypotheses, it follows that $\mathcal{N}_{V_T}$ is non-empty. From Item (1) of Lemma 1, it follows that $\| \, \| \circ T$ is a semi-norm on V. Let $f, g \in \mathcal{N}_{V_T}$, i.e., $f = \| \, \|_1 \circ T$ and $g = \| \, \|_2 \circ T$, where $\| \, \|_1$ and $\| \, \|_2$ are semi-norms on W. Then, $f + g = [\| \, \|_1 + \| \, \|_2] \circ T \in \mathcal{N}_{V_T}$. For every nonnegative real number λ and $f \in \mathcal{N}_{V_T}$, $\lambda f = \lambda[\| \, \| \circ T] = (\lambda \| \, \|) \circ T \in \mathcal{N}_{V_T}$. □

Theorem 12. *Let $\mathcal{N}$ be the class whose members are $\{\mathcal{N}_V\}$, where $\mathcal{N}_V$'s are given in Theorem 10. Let $\mathrm{Hom}(\mathcal{N})$ be the class whose members are the sets*

$$\mathrm{hom}(\mathcal{N}_V, \mathcal{N}_W) = \{F_T : \mathcal{N}_V \longrightarrow \mathcal{N}_W; F_T(\| \, \|_V) = \| \, \|_V \circ T\},$$

where $T : W \longrightarrow V$ is a linear transformation and $\| \, \|_V$ is a semi-norm on V. Then, $(\mathcal{N}, \mathrm{Hom}(\mathcal{N}), Id, \circ)$ is a category.

Proof. The sets $\mathrm{hom}(\mathcal{N}_V, \mathcal{N}_W)$ are pairwise disjointed. For each $\mathcal{N}_V$, there exists $Id_{(\mathcal{N}_V)}$ given by $Id_{(\mathcal{N}_V)}(\| \, \|_V) = \| \, \|_V = \| \, \|_V \circ Id_{(V)}$. It is clear that, if $F_T : \mathcal{N}_V \longrightarrow \mathcal{N}_W$, then $F_T \circ Id_{(\mathcal{N}_V)} = F_T$ and $Id_{(\mathcal{N}_W)} \circ F_T = F_T$.

It is easy to see that, for every $T : W \longrightarrow V$ linear transformation, the map F_T is semi-linear, i.e., $F_T(\| \, \|_V^{(1)} + \| \, \|_V^{(2)}) = F_T(\| \, \|_V^{(1)}) + F_T(\| \, \|_V^{(2)})$ and $F_T(\lambda \| \, \|_V) = \lambda F_T(\| \, \|_V)$, for every $\| \, \|_V, \| \, \|_V^{(1)}, \| \, \|_V^{(2)} \in \mathcal{N}_V$ and $\lambda \in \mathbb{R}_0^+$.

Let $\mathcal{N}_U, \mathcal{N}_V, \mathcal{N}_W, \mathcal{N}_X \in \mathcal{N}$ and $F_{T_1} \in \mathrm{hom}(\mathcal{N}_U, \mathcal{N}_V)$, $F_{T_2} \in \mathrm{hom}(\mathcal{N}_V, \mathcal{N}_W)$, $F_{T_3} \in \mathrm{hom}(\mathcal{N}_W, \mathcal{N}_X)$, i.e.,

$$\mathcal{N}_U \xrightarrow{F_{T_1}} \mathcal{N}_V \xrightarrow{F_{T_2}} \mathcal{N}_W \xrightarrow{F_{T_3}} \mathcal{N}_X.$$

The linear transformations are of the forms

$$X \xrightarrow{T_3} W \xrightarrow{T_2} V \xrightarrow{T_1} U \xrightarrow{\| \, \|_U} \mathbb{R}.$$

The associativity $(F_{T_3} \circ F_{T_2}) \circ F_{T_1} = F_{T_3} \circ (F_{T_2} \circ F_{T_1})$ follows from the associativity of composition of the maps. Moreover, the map $F_{T_3} \circ F_{T_2} \circ F_{T_1} \in \mathrm{Hom}(\mathcal{N})$ because $F_{T_3} \circ F_{T_2} \circ F_{T_1} = (\| \, \|_U) \circ (T_1 \circ T_2 \circ T_3)$ and $T_1 \circ T_2 \circ T_3$ is a linear transformation. Therefore, $(\mathcal{N}, \mathrm{Hom}(\mathcal{N}), Id, \circ)$ is a category, as required. □

Theorem 13. *Let V be a real vector space endowed with a semi-inner product and let $\mathcal{P}_V = \{\langle \, , \, \rangle : V \times V \longrightarrow \mathbb{R}; \langle \, , \, \rangle \text{ is a semi-inner product on } V\}$. Then, $(\mathcal{P}_V, +, \cdot)$ is a semi-vector space over $\mathbb{R}_0^+$, where $+$ and $\cdot$ are pointwise addition and scalar multiplication (in $\mathbb{R}_0^+$), respectively.*

Proof. The proof is analogous to that of Theorems 8 and 10. □

Proposition 16. *Let V, W be two vector spaces and $T_1, T_2 : V \longrightarrow W$ be two linear transformations. Let us consider the map $T_1 \times T_2 : V \times V \longrightarrow W \times W$ given by $T_1 \times T_2(u, v) =$*

$(T_1(u), T_2(v))$. *If* $\langle\ ,\ \rangle$ *is a semi-inner product on* W*, then* $\langle\ ,\ \rangle \circ T_1 \times T_2$ *is a semi-inner product on* V*.*

Proof. The proof is immediate, so it is omitted. □

Let V be a real vector space. Recall that a sub-linear functional on V is a functional $t : V \longrightarrow \mathbb{R}$ which is sub-additive: $\forall\, u, v \in V$, $t(u + v) \le t(u) + t(v)$; and positive-homogeneous: $\forall\, \alpha \in \mathbb{R}_0^+$ and $\forall\, v \in V$, $t(\alpha v) = \alpha t(v)$.

Theorem 14. *Let* V *be a real vector space. Let us consider* $\mathcal{S}_V = \{S : V \longrightarrow \mathbb{R}; S \text{ is sub-linear on } V\}$*. Then,* $(\mathcal{S}_V, +, \cdot)$ *is a semi-vector space on* $\mathbb{R}_0^+$*, where* $+$ *and* $\cdot$ *are pointwise addition and scalar multiplication (in* $\mathbb{R}_0^+$*), respectively.*

Proof. The proof follows the same line of that of Theorems 8 and 10 and 13. □

3.3. Semi-Algebras

We start this section by recalling the definition of semi-algebra and semi-sub-algebra. For more details, the reader can consult [2]. In [22], Olivier and Serrato investigated relation semi-algebras, i.e., a semi-algebra being both a Boolean algebra and an involutive semi-monoid, satisfying some conditions (see page 2 in Ref. [22] for more details). Roy [23] studied the semi-algebras of continuous and monotone functions on compact ordered spaces.

Definition 19. *A semi-algebra* A *over a semi-field* K *(or a* K*-semi-algebra) is a semi-vector space* A *over* K *endowed with a binary operation called the multiplication of semi-vectors* $\bullet : A \times A \longrightarrow A$ *such that* $\forall\, u, v, w \in A$ *and* $\lambda \in K$*:*

(1a) $u \bullet (v + w) = (u \bullet v) + (u \bullet w)$ *(left-distributivity);*
(1b) $(u + v) \bullet w = (u \bullet w) + (v \bullet w)$ *(right-distributivity);*
(2) $\lambda(u \bullet v) = (\lambda u) \bullet v = u \bullet (\lambda v)$*.*

A semi-algebra A is *associative* if $(u \bullet v) \bullet w = u \bullet (v \bullet w)$ for all $u, v, w \in A$; A is said to be *commutative* (or abelian) if the multiplication is commutative, that is, $\forall\, u, v \in A$, $u \bullet v = v \bullet u$; A is called a semi-algebra with identity if there exists an element $1_A \in A$ such that $\forall\, u \in A$, $1_A \bullet u = u \bullet 1_A = u$; the element 1_A is called the identity of A. The identity element of a semi-algebra A is unique (if exists). If A is a semi-free semi-vector space, then the dimension of A is its dimension regarded as a semi-vector space. A semi-algebra is *simple* if it is simple as a semi-vector space.

Example 9. *The set* $\mathbb{R}_0^+$ *is a commutative semi-algebra with identity* $e = 1$*.*

Example 10. *The set of square matrices of order* n *whose entries are in* $\mathbb{R}_0^+$*, are equipped with the sum of matrices, the multiplication of a matrix by a scalar (in* $\mathbb{R}_0^+$*, of course) and by the multiplication of matrices, constituting an associative and non-commutative semi-algebra with identity* $e = I_n$ *(the identity matrix of order* n*), over* $\mathbb{R}_0^+$*.*

Example 11. *Let* V *be a semi-vector space over a semi-field* K*. Then, the set* $\mathcal{L}(V, V) = \{T : V \longrightarrow V; T \text{ is a semi-linear operator}\}$ *is a semi-vector space. If we define a vector multiplication as the composite of semi-linear operators (which is also semi-linear), then we have a semi-algebra over* K*.*

Definition 20. *Let* A *be a semi-algebra over* K*. We say that a non-empty set* $S \subseteq A$ *is a semi-subalgebra if* S *is closed under the operations of* A*, that is,*

(1) $\forall\, u, v \in A$, $u + v \in A$;
(2) $\forall\, u, v \in A$, $u \bullet v \in A$;
(3) $\forall\, \lambda \in K$ *and* $\forall u \in A$, $\lambda u \in A$.

Definition 21. *Let A and B two semi-algebras over K. We say that a map $T : A \longrightarrow B$ is an K-semi-algebra homomorphism if, $\forall\, u, v \in A$ and $\lambda \in K$, the following conditions hold:*

(1) $T(u + v) = T(u) + T(v)$;
(2) $T(u \bullet v) = T(u) \bullet T(v)$;
(3) $T(\lambda v) = \lambda T(v)$.

Definition 21 means that T is both a semi-ring homomorphism and also semi-linear (as a semi-vector space).

Definition 22. *Let A and B be two K-semi-algebras. A K-semi-algebra isomorphism $T : A \longrightarrow B$ is a bijective K-semi-algebra homomorphism. If there exists such an isomorphism, we say that A is isomorphic to B, written $A \cong B$.*

The following results seems to be new, because semi-algebras over $\mathbb{R}_0^+$ have not been investigated much in the literature.

Proposition 17. *Assume that A and B are two K-semi-algebras, where $K = \mathbb{R}_0^+$ and A has identity 1_A. Let $T : A \longrightarrow B$ be a K-semi-algebra homomorphism. Then, the following properties hold:*

(1) $T(0_A) = 0_B$;
(2) *If $u \in A$ is invertible, then its inverse is unique and $(u^{-1})^{-1} = u$;*
(3) *If T is surjective, then $T(1_A) = 1_B$, i.e., B also has identity; furthermore, $T(u^{-1}) = [T(u)]^{-1}$;*
(4) *If $u, v \in A$ are invertible, then $(u \bullet v)^{-1} = v^{-1} \bullet u^{-1}$;*
(5) *The composite of K-semi-algebra homomorphisms is also a K-semi-algebra homomorphism;*
(6) *If T is a K-semi-algebra isomorphism, then $T^{-1} : B \longrightarrow A$ is also;*
(7) *The relation $A \sim B$, if and only if A is isomorphic to B, is an equivalence relation.*

Proof. Note that Item (1) holds because the additive cancelation law holds in the definition of semi-vector spaces (see Definition 3). We only show Item (3) since the remaining items are direct. Let $v \in B$; then, there exists $u \in A$ such that $T(u) = v$. It then follows that $v \bullet T(1_A) = T(u \bullet 1_A) = v$ and $T(1_A) \bullet v = T(1_A \bullet u) = v$, which means that $T(1_A)$ is the identity of B, i.e., $T(1_A) = 1_B$.

We have $T(u) \bullet T(u^{-1}) = T(u \bullet u^{-1}) = T(1_A) = 1_B$ and $T(u^{-1}) \bullet T(u) = T(u^{-1} \bullet u) = T(1_A) = 1_B$, which implies $T(u^{-1}) = [T(u)]^{-1}$. □

Proposition 18. *If A is a K-semi-algebra with identity 1_A, then A can be embedded in $\mathcal{L}(A, A)$, the semi-algebra of semi-linear operators on A.*

Proof. For every fixed $v \in A$, define $v^* : A \longrightarrow A$ as $v^*(x) = v \bullet x$. It is easy to see that v^* is a semi-linear operator on A. Define $h : A \longrightarrow \mathcal{L}(A, A)$ by $h(v) = v^*$. We must show that h is a injective K-semi-algebra homomorphism where the product in $\mathcal{L}(A, A)$ is the composite of maps from A into A. Fixing $u, v \in A$, we have the following: $[h(u + v)](x) = (u + v)^*(x) = (u + v) \bullet x = u \bullet x + v \bullet x = u^*(x) + v^*(x) = [h(u)](x) + [h(v)](x)$, hence $h(u + v) = h(u) + h(v)$. For $\lambda \in K$ and $v \in A$, it follows that $[h(\lambda v)](x) = (\lambda v)^*(x) = (\lambda v)x = \lambda(vx) = [\lambda h(v)](x)$, i.e., $h(\lambda v) = \lambda h(v)$. For fixed $u, v \in A$, $[h(u \bullet v)](x) = (u \bullet v)^*(x) = (u \bullet v) \bullet x = u \bullet (v \bullet x) = u \bullet v^*(x) = u^*(v^*(x)) = [h(u) \circ h(v)](x)$, i.e., $h(u \bullet v) = h(u) \circ h(v)$. Assume that $h(u) = h(v)$, that is, $u^* = v^*$; hence, for every $x \in A$, $u^*(x) = v^*(x)$, i.e., $u \bullet x = v \bullet x$. Taking, in particular, $x = 1_A$, it follows that $u = v$, which implies that h is injective. Therefore, A is isomorphic to $h(A)$, where $h(A) \subseteq \mathcal{L}(A, A)$. □

Definition 23. *Let A be a semi-vector space over a semi-field K. Then, A is said to be a Lie semi-algebra if A is equipped with a product $[\ ,\ ] : A \times A \longrightarrow A$ such that the following conditions hold:*

(1) *$[\ ,\ ]$ is semi-bilinear, i.e., fixing the first (second) variable, $[\ ,\ ]$ is semi-linear with respect to the second (first) one;*
(2) *$[\ ,\ ]$ is anti-symmetric, i.e., $[v, v] = 0_V\ \forall\, v \in A$;*

(3) $[\ ,\ ]$ *satisfies the Jacobi identity:* $\forall\, u, v, w \in A$, $[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0_V$.

From Definition 23, we can see that a Lie semi-algebra can be non-associative, i.e., the product $[\ ,\ ]$ is not always associative.

Let us now consider the semi-algebra $\mathcal{M}_n(\mathbb{R}_0^+)$ of matrices of order n with entries in $\mathbb{R}_0^+$ (see Example 10). We know that $\mathcal{M}_n(\mathbb{R}_0^+)$ is simple, i.e., with the exception of the zero matrix (zero vector), no matrix is (additive) symmetric. Therefore, the product of such matrices can be nonzero. However, in the case of a Lie semi-algebra A, if A is simple, then the unique product $[\ ,\ ]$, that can be defined over A, is the zero product, as is shown in the next result.

Proposition 19. *If A is a simple Lie semi-algebra over a semi-field K, then the semi-algebra is abelian, i.e., $[u, v] = 0_V$ for all $u, v \in A$.*

Proof. Assume that $u, v \in A$ and $[u, v] \neq 0_V$. From Items (1) and (2) of Definition 23, it follows that $[u+v, u+v] = [u, u] + [u, v] + [v, u] + [v, v] = 0_V$, i.e., $[u, v] + [v, u] = 0_V$. This means that $[u, v]$ has symmetric $[v, u] \neq 0_V$, a contradiction. □

Definition 24. *Let A be a Lie semi-algebra over a semi-field K. A Lie semi-subalgebra $B \subseteq A$ is a semi-subspace of A which is closed under $[u, v]$, i.e., for all $u, v \in B$, $[u, v] \in B$.*

Corollary 2. *All semi-subspaces of A are semi-subalgebras of A.*

Proof. Apply Proposition 19. □

4. Fuzzy Set Theory and Semi-Algebras

The theory of semi-vector spaces and semi-algebras is a natural generalization of the corresponding theories of vector spaces and algebras. Since the scalars are in semi-fields (weak semi-fields), some standard properties do not hold in this new context. However, as we have shown in Section 3, even in the case of the nonexistence of symmetrizable elements, several results are still true. An application of the theory of semi-vector spaces is in the investigation of Fuzzy Set Theory, which was introduced by Lotfali Askar-Zadeh [11]. In fact, such a theory fits in the investigation/extension of results concerning fuzzy sets and their corresponding theory. Let us see an example.

Let L be a linearly ordered complete lattice with distinct smallest and largest elements 0 and 1. Recall that a L-fuzzy number is a function $x : \mathbb{R} \longrightarrow L$ on the field of real numbers satisfying the following items (see [2] Sect. 1.1): (1) for each $\alpha \in L_0$ the set $x_\alpha = \{\varphi \in \mathbb{R}; \alpha \leq x(\varphi)\}$ is a closed interval $[x_{\alpha l}, x_{\alpha r}]$, where $L_0 = \{\alpha \in L; \alpha > 0\}$; (2) $\{\varphi \in \mathbb{R}; 0 < x(\varphi)\}$ is bounded.

The addition of two fuzzy numbers x and y is the fuzzy number $x + y$ defined for each $r \in \mathbb{R}$ by

$$(x + y)(r) = \sup\{x(s) \wedge_L y(t); s + t = r\}.$$

Analogously, the product of x and y is the fuzzy number $x \cdot y$, for each $r \in \mathbb{R}$ given by

$$(x \cdot y)(r) = \sup\{x(s) \wedge_L y(t); st = r\}.$$

The scalar product of $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}_L$ is the fuzzy number αx such that $\alpha x = \tilde{\alpha} \cdot x$, where

$$\widetilde{\alpha}(r) = \begin{cases} 1 & \text{if } r = \alpha \\ 0 & \text{if } r \neq \alpha. \end{cases}$$

We denote the set $\mathbb{R}_L$ to be the set of all fuzzy numbers; $\mathbb{R}_L$ can be equipped with a partial order in the following manner: $x \leq y$ if and only if $x_{\alpha l} \leq y_{\alpha l}$ and $x_{\alpha r} \leq y_{\alpha r}$ for all $\alpha \in L_0$. Additionally, in [24], the authors investigated linear orders on fuzzy numbers which refine this partial order. In this scenario, Gahler et al. showed that the concepts of semi-algebras can be utilized to extend the concept of fuzzy numbers, according to the following proposition:

Proposition 20 ([2] Proposition 19)**.** *The set $\mathbb{R}_L$ is an ordered commutative semi-algebra.*

Thus, a direct utilization of the investigation of the structures of semi-vector spaces and semi-algebras is the possibility to generate new interesting results on the Fuzzy Set Theory.

Let $L_n([0,1]) = \{(x_1, x_2, \ldots, x_n) \in [0,1]^n | x_1 \leq x_2 \leq \ldots \leq x_n\}$. Shang et al., in [25], introduced a new type of fuzzy sets where the membership are elements of $L_n([0,1])$. The product order $\leq_n^p$ on $L_n([0,1])$ is given as follows: for all $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ vectors in $L_n([0,1])$, define $x \leq_n^p y \Longleftrightarrow \pi_i(x) \leq \pi_i(x)$ for each $i \in \{1, 2, \ldots, n\}$, where $\pi_i : L_n([0,1]) \longrightarrow [0,1]$ is the i-th projection $\pi_i(x_1, x_2, \ldots, x_n) = x_i$ [26]. Another work relating semi-vector spaces and Fuzzy Set Theory is the study by Bedregal et al. [4]. In order to study the aggregation functions (geometric mean, weighted average and ordered weighted averaging, among others) with respect to an admissible order (a total order $\preceq$ on $L_n([0,1])$ such that, for all $x, y \in L_n([0,1])$, $x \leq_n^p y \Longrightarrow x \preceq y$), the authors worked with semi-vector spaces over the weak semi-field $U = ([0,1], \oplus, \cdot)$ where, for all $x, y \in [0,1]$, $x \oplus y = \min\{1, x+y\}$ and $\cdot$ is the usual multiplication. With these concepts in mind, the authors showed two important results:

Theorem 15 ([4] Theorem 1)**.** *$\mathcal{L}_n([0,1]) = (L_n([0,1], \dot{+}, \odot)$ is a semi-vector space over U, where $r \odot v = (rx_1, \ldots, rx_n)$ and $u \dot{+} v = (x_1 \oplus y_1, \ldots, x_n \oplus y_n)$. Moreover, $(\mathcal{L}_n([0,1]), \leq_n^p)$ is an ordered semi-vector space over U, where $\leq_n^p$ is the product order.*

Proposition 21 ([4] Propostion 2)**.** *For any bijection $f : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$, the pair $(\mathcal{L}_n([0,1]), \preceq_f)$ is an ordered semi-vector space over U, where $\preceq_f$, defined in ([4] Example 1), is an admissible order.*

Clearly, except for the additive cancellation law, $\mathcal{L}_n([0,1])$ jointly with the operation $u \bullet v = (x_1y_1, \ldots, x_ny_n)$, for all $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ vectors in $L_n([0,1])$, is a commutative semi-algebra over U. As a consequence of the investigation conducted, the authors propose an algorithm to perform a multi-criteria and multi-expert decision-making method.

Summarizing the ideas: The theory of semi-vector spaces in [4] can be related to our theory of semi-algebras, increasing the connection between fuzzy set theory and semi-algebras. Therefore, it is important to understand deeply which are the algebraic and geometry structures of semi-vector spaces, providing, in this way, support for the development of our own theory as well as other interesting theories such as, for example, the Fuzzy Set Theory. In the next subsection, we provide a connection between K-semi-algebra and fuzzy formal language [14].

4.1. K-Fuzzy Automata

Let A be a K-semi-algebra and U a set. Then, a K-fuzzy set F over U is a function $F : U \to A$. The support of F is the set $Supp(F) = \{u \in U | F(u) \neq 0_A\}$.

Let X^* be the free monoid generated by a set of input symbols X with concatenation as a binary operation. We will denote this by ε the identity element of X^*, i.e., the empty string. An A-fuzzy language over a set of input symbols X has any function $L : X^* \to A$.

Definition 25. *Let A be a K-semi-algebra with identity 1_A. Then, the system $M = \langle Q, X, \varrho, \iota, \tau \rangle$ is a K-Fuzzy Finite Automaton, K-FFA for short, if Q and X are nonempty finite disjoint sets, $\varrho : Q \times (X \cup \{\varepsilon\}) \times Q \to A$ is such that $\varrho(q, \varepsilon, q) = 1_A$ for each $q \in Q$, $\iota : Q \to A$ and $\tau : Q \to A$. The elements of Q are the states and elements of X of input symbols. The mapping ϱ, ι and τ are the K-fuzzy transition function, K-fuzzy initial state set and K-fuzzy final state set, respectively.*

Example 12. *Let A be a commutative K-semi-algebra with identity 1_A and three different elements $a, b, c \in A - \{0_A\}$. Then, $M = \langle Q, X, \varrho, \iota, \tau\rangle$ where $Q = \{q_0, q_1, q_2, q_3\}$, $X = \{0, 1\}$, and for each $i, j \in \{0, 1, 2, 3\}$*

$$\varrho(q_i, x, q_j) = \begin{cases} a & \text{if } x = 0 \text{ and } i = j \in \{0,3\} \\ b & \text{if } x = 1 \text{ and, either } j = i+1 \text{ or } j = i \in \{0,3\} \\ 1_A & \text{if } x = \varepsilon \text{ and } i = j \\ 0_A & \text{otherwise} \end{cases}$$

$$\iota(q_i) = \begin{cases} c & \text{if } i = 0 \\ 0_A & \text{otherwise,} \end{cases} \qquad \tau(q_i) = \begin{cases} c & \text{if } i = 3 \\ 0_A & \text{otherwise.} \end{cases}$$

is a K-FFA. Analogously, as occurs in automata theory, finite automata are graphically represented. In particular, the graphical representation of this K-FFA is presented in Figure 1.

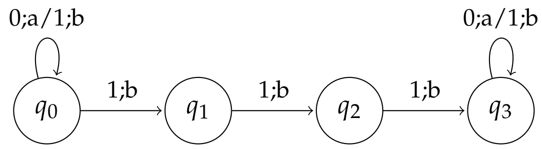

Figure 1. Graphical representation of a *K*-FFA.

Notice that, if A is the K-semi-algebra of fuzzy numbers, then $1_A = \widetilde{1}$, $0_A = \widetilde{0}$ and a, b and c are arbitrary fuzzy numbers (different from $\widetilde{0}$).

Definition 26. *Let A be a K-semi-algebra with identity 1_A and $M = \langle Q, X, \varrho, \iota, \tau\rangle$ be a K-FFA. Then, the extension of ϱ is the mapping $\varrho^* : Q \times X^* \times Q \to A$ recursively defined for each $q, p \in Q$, by*

$$\varrho^*(q, \varepsilon, p) = \varrho(q, \varepsilon, p) \text{ and } \varrho^*(q, wa, p) = \sum_{q' \in Q} \varrho^*(q, w, q') \bullet \varrho(q', a, p),$$

whenever $w \in X^*$ and $a \in X$, where the sum is with respect to the addition of the K-semi-algebra.

Definition 27. *Let A be a K-semi-algebra with identity and $M = \langle Q, X, \varrho, \iota, \tau\rangle$ be a K-FFA. M is deterministic if*

1. *there is $q_0 \in Q$ such that, for each $q \in Q$, $\iota(q) \neq 0_A \Leftrightarrow q = q_0$;*
2. *for each $q, p \in Q$, such that $p \neq q$, $\varrho(q, \varepsilon, p) = 0_A$;*
3. *for each $q, p, p' \in Q$ and $a \in X$, $\varrho(q, a, p) \neq 0_A$ and $\varrho(q, a, p') \neq 0_A$, then $p = p'$.*

Proposition 22. *Let A be an associative K-semi-algebra with identity 1_A and $M = \langle Q, X, \varrho, \iota, \tau\rangle$ be a K-FFA such that $\varrho(q, \varepsilon, p) = 0_A$ whenever $q, p \in Q$ and $q \neq p$. If 0_A is a right annihilator element, i.e., $x \bullet 0_A = 0_A$ for each $x \in A$, then for each $v, w \in X^*$ and $q, p \in Q$, we have*

$$\varrho^*(q, vw, p) = \sum_{q' \in Q} \varrho^*(q, v, q') \bullet \varrho^*(q', w, p).$$

Proof. The proof is confirmed by induction on $n = |w|$. If $n = 0$, then $w = \varepsilon$. Hence, for each $v \in X^*$, since $\varrho(q, \varepsilon, p) = 0_A$ whenever $q, p \in Q$ and $q \neq p$, it follows that

$$\begin{aligned} \varrho^*(q, vw, p) &= \varrho^*(q, v, p) \\ &= \varrho^*(q, v, p) \bullet 1_A + \sum_{q' \in Q, q' \neq q} \varrho(q, v, q') \bullet 0_A \\ &= \varrho^*(q, v, p) \bullet \varrho(p, \varepsilon, p) + \sum_{q' \in Q, q' \neq q} \varrho^*(q, v, q') \bullet \varrho(q', \varepsilon, p) \end{aligned}$$

$$= \sum_{q' \in Q} \varrho^*(q, v, q') \bullet \varrho(q', \varepsilon, p)$$
$$= \sum_{q' \in Q} \varrho^*(q, v, q') \bullet \varrho^*(q', w, p).$$

Suppose now that $\varrho^*(q, vw, p) = \varrho^*(q, v, p) \bullet \varrho^*(q, w, p)$ for any $v, w \in X^*$ such that $|w| = n$. Thus, if $u \in X^*$ is such that $|u| = n + 1$, then there are $w \in X^*$ and $a \in X$ such that $|w| = n$ and $u = wa$. Therefore,

$$\begin{aligned}
\varrho^*(q, vu, p) &= \sum_{q'' \in Q} \varrho^*(q, vw, q'') \bullet \varrho(q'', a, p) \\
&= \sum_{q'' \in Q} \left(\sum_{q' \in Q} \varrho^*(q, v, q') \bullet \varrho^*(q', w, q'') \right) \bullet \varrho(q'', a, p) \text{ by inductive hyp.} \\
&= \sum_{q'' \in Q} \left(\sum_{q' \in Q} ((\varrho^*(q, v, q') \bullet \varrho^*(q', w, q'')) \bullet \varrho(q'', a, p) \right) \text{ by right-distributivity} \\
&= \sum_{q'' \in Q} \left(\sum_{q' \in Q} \varrho^*(q, v, q') \bullet (\varrho^*(q', w, q'') \bullet \varrho(q'', a, p)) \right) \text{ by associativity} \\
&= \sum_{q' \in Q} \left(\sum_{q'' \in Q} \varrho^*(q, v, q') \bullet (\varrho^*(q', w, q'') \bullet \varrho(q'', a, p)) \right) \\
&= \sum_{q' \in Q} \left(\varrho^*(q, v, q') \bullet \sum_{q'' \in Q} (\varrho^*(q', w, q'') \bullet \varrho(q'', a, p)) \right) \text{ by left-distributivity} \\
&= \sum_{q' \in Q} (\varrho^*(q, v, q') \bullet \varrho^*(q', u, p)).
\end{aligned}$$

□

Definition 28. *Let A be a K-semi-algebra with identity and $M = \langle Q, X, \varrho, \iota, \tau \rangle$ be a K-FFA. Then, the A-fuzzy language accepted by M is $L_M : X^* \to A$ where, for each $w \in X^*$,*

$$L_M(w) = \sum_{q, p \in Q} \iota(q) \bullet (\varrho^*(q, w, p) \bullet \tau(p)).$$

A-fuzzy languages accepted by a K-FFA on a nonempty set of input symbols X will be called A-fuzzy regular languages on X and the set of all them will be denoted by $\mathcal{FRL}_A^X$.

Example 13. *The A-fuzzy language accepted by the K-FFA of Example 12, for each $w \in X^*$, is*

$$L_M(w) = \begin{cases} c^2 \bullet (a^{m(w)} \bullet b^{n(w)}) & \text{if } 111 \text{ is a substring of } w \\ 0_A & \text{otherwise,} \end{cases}$$

where $m(\varepsilon) = 0$, $m(wx) = m(w) + 1 - x$, $n(\varepsilon) = 0$, $n(wx) = n(w) + x$, $x^0 = 1_A$, $x^{k+1} = x^k \bullet x$.

4.2. The Semi-Algebras of A-Fuzzy Regular Languages

In the following, for each $\alpha \in K$, $\alpha^0 = 1$ and $\alpha^{n+1} = \alpha \cdot \alpha^n$ for each positive integer n.

Definition 29. *Let L, L_1 and L_2 be A-fuzzy languages over a set of input symbols X, respectively. We then define the following:*

Scalar product of an A-fuzzy language: *given $\alpha \in K$, the scalar product of α with L is the A-fuzzy language $\alpha \odot L : X^* \to A$, where $\alpha \odot L(w) = \alpha^{|w|} \cdot L(w)$ for each $w \in X^*$;*

Addition of A-fuzzy languages: *the addition of L_1 and L_2 is the A-fuzzy language $L_1 \oplus L_2 : X^* \to A$, where $L_1 \oplus L_2(w) = L_1(w) + L_2(w)$ for each $w \in X^*$;*

Multiplication of A-fuzzy languages: *the multiplication of L_1 and L_2 is the A-fuzzy language $L_1 \circledbullet L_2 : X^* \to A$, where $L_1 \circledbullet L_2(w) = L_1(w) \bullet L_2(w)$ for each $w \in X^*$.*

Lemma 2. *Let X be a non-empty set of input symbols and A be a K-semi-algebra with identity. Then, for each family $a_1, \ldots, a_m, b_1, \ldots, b_n \in A$, we have that*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_i \bullet b_j = \left(\sum_{i=1}^{m} a_i\right) \bullet \left(\sum_{j=1}^{n} b_j\right). \tag{5}$$

Proof. We prove by induction on n. If $n = 1$, then

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=1}^{n} a_i \bullet b_j &= \sum_{i=1}^{m} a_i \bullet b_1 \\ &= a_1 \bullet b_1 + (a_2 \bullet b_1 + \ldots + (a_{m-2} \bullet b_1 + ((a_{m-1} + a_m) \bullet b_1))\ldots) \text{ by Def. 19(1b)} \\ &= a_1 \bullet b_1 + (a_2 \bullet b_1 + \ldots ((a_{m-2} + (a_{m-1} + a_m)) \bullet b_1)\ldots) \text{ by Definition 19(1b)} \\ &\vdots \\ &= a_1 \bullet b_1 + \left(\left(\sum_{i=2}^{m} a_i\right) \bullet b_1\right) \text{ by Definition 19(1b)} \\ &= \left(\sum_{i=1}^{m} a_i\right) \bullet b_1 \text{ by Definition 19(1b).}\end{aligned}$$

Assume that Equation (5) holds for each $k \leq n$. Then,

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=1}^{n+1} a_i \bullet b_j &= \sum_{i=1}^{m}\left(a_i \bullet b_1 + \left(\sum_{j=2}^{n+1} a_i \bullet b_j\right)\right) \\ &= \left(\sum_{i=1}^{m} a_i \bullet b_1\right) + \left(\sum_{i=1}^{m}\sum_{j=2}^{n+1} a_i \bullet b_j\right) \text{ by associativity and commutativity of } + \\ &= \left(\sum_{i=1}^{m} a_i\right) \bullet b_1 + \left(\sum_{i=1}^{m} a_i\right) \bullet \left(\sum_{j=1}^{n} b_j\right) \text{ by Induction hypotheses} \\ &= \left(\sum_{i=1}^{m} a_i\right) \bullet \left(\sum_{j=1}^{n+1} b_j\right) \text{ by Definition 19(1a).}\end{aligned}$$

□

Theorem 16. *Let X be a non-empty set of input symbols and A be a K-semi-algebra with identity. The K-fuzzy regular languages on X are closed under the scalar product. Moreover, if 0_A is the left annihilator element of the $\bullet$, i.e., $0_A \bullet x = 0_A$ for each $x \in A$, then the K-fuzzy regular languages on X are closed under addition and multiplication operations in Definition 29.*

Proof. Let L be K-fuzzy regular language on X and $\alpha \in K$. Then, there exists a K-FFA $M = \langle Q, X, \varrho, \iota, \tau\rangle$ such that $L_M = L$. We now define $M^\alpha = \langle Q, X, \varrho_\alpha, \iota, \tau\rangle$ where, for each $q, p \in Q$ and $a \in X$, $\varrho_\alpha(q, a, p) = \alpha \cdot \varrho(q, a, p)$ and $\varrho_\alpha(q, \varepsilon, p) = \varrho(q, \varepsilon, p)$. Clearly, M^α is a K-FFA.

We first prove by induction on $w \in X^*$ that

$$\varrho^*_\alpha(q, w, p) = \alpha^{|w|} \cdot (\varrho^*(q, w, p)). \tag{6}$$

If $w = \varepsilon$, from Definition 3(5), one has $\varrho^*_\alpha(q, w, p) = \varrho_\alpha(q, \varepsilon, p) = \varrho(q, \varepsilon, p) = \alpha^{|w|} \cdot (\varrho(q, w, p))$.

Suppose that $\varrho^*_\alpha(q,w,p) = \alpha^{|w|} \cdot \varrho^*(q,w,p)$ when (n before ∣w∣ was deleted) $|w| = n$. Then,

$$\begin{aligned}
\varrho^*_\alpha(q,wa,p) &= \sum_{q'\in Q} \varrho^*_\alpha(q,w,q') \bullet \varrho_\alpha(q',a,p) \\
&= \sum_{q'\in Q} (\alpha^{|w|} \cdot \varrho^*(q,w,q')) \bullet (\alpha \cdot \varrho(q',a,p)) \text{ by inductive Hyp.} \\
&= \sum_{q'\in Q} \alpha^{|w|+1} \cdot (\varrho^*(q,w,q') \bullet \varrho(q',a,p)) \text{ by Definition 19(2)} \\
&= \alpha^{|wa|} \cdot \varrho^*(q,wa,p) \text{ by Definition 3(2).}
\end{aligned}$$

Therefore, for each $w \in X^*$, we have

$$\begin{aligned}
L_{M^\alpha}(w) &= \sum_{q,p\in Q} \iota(q) \bullet (\varrho^*_\alpha(q,w,p) \bullet \tau(p)) \\
&= \sum_{q,p\in Q} \iota(q) \bullet ((\alpha^{|w|} \cdot \varrho^*(q,w,p)) \bullet \tau(p)) \text{ by Equation (6)} \\
&= \sum_{q,p\in Q} \alpha^{|w|} \cdot (\iota(q) \bullet (\varrho^*(q,w,p) \bullet \tau(p))) \text{ by Definition 19(2)} \\
&= \alpha^{|w|} \cdot \left(\sum_{q,p\in Q} \iota(q) \bullet (\varrho^*(q,w,p) \bullet \tau(p)) \right) \text{ by Definition 3(2)} \\
&= \alpha^{|w|} \cdot (L_M(w)) \\
&= \alpha \odot L_M(w)
\end{aligned}$$

Therefore, $\alpha \odot L \in \mathcal{FRL}^X_A$, i.e., the K-fuzzy regular languages on X, are closed under the scalar product operator.

Next, if $L_1, L_2 \in \mathcal{FRL}^X_A$, then there exist K-FFAs $M_1 = \langle Q_1, X, \varrho_1, \iota_1, \tau_1 \rangle$ and $M_2 = \langle Q_2, X, \varrho_2, \iota_2, \tau_2 \rangle$ such that $Q_1 \cap Q_2 = \emptyset$, $L_{M_1} = L_1$ and $L_{M_2} = L_2$. Then, $M_{1\oplus 2} = \langle Q, X, \varrho, \iota, \tau \rangle$, where $Q = Q_1 \cup Q_2$, and for each $q, p \in Q$ and $a \in X \cup \{\varepsilon\}$,

$$\varrho(q,a,p) = \begin{cases} \varrho_1(q,a,p) & \text{if } q,p \in Q_1 \\ \varrho_2(q,a,p) & \text{if } q,p \in Q_2 \\ 0_A & \text{otherwise} \end{cases}$$

$$\iota(q) = \begin{cases} \iota_1(q) & \text{if } q \in Q_1 \\ \iota_2(q) & \text{if } q \in Q_2, \end{cases} \quad \tau(q) = \begin{cases} \tau_1(q) & \text{if } q \in Q_1 \\ \tau_2(q) & \text{if } q \in Q_2 \end{cases}$$

is clearly a K-FFA. We will prove that $L_{M_{1\oplus 2}} = L_{M_1} \oplus L_{M_2}$. Before this, note that

$$\varrho^*(q,w,p) = \begin{cases} \varrho^*_1(q,w,p) & \text{if } q,p \in Q_1 \\ \varrho^*_2(q,w,p) & \text{if } q,p \in Q_2 \\ 0_A & \text{otherwise.} \end{cases} \tag{7}$$

Since 0_A is the left annihilator element of the $\bullet$ and neutral element of $+$, it follows that

$$\begin{aligned}
L_{M_{1\oplus 2}}(w) &= \sum_{q,p\in Q} \iota(q) \bullet (\varrho^*(q,w,p) \bullet \tau(p)) \\
&= \left(\sum_{q,p\in Q_1} \iota(q) \bullet (\varrho^*(q,w,p) \bullet \tau(p)) \right) + \left(\sum_{q,p\in Q_2} \iota(q) \bullet (\varrho^*(q,w,p) \bullet \tau(p)) \right) + \\
&\quad \left(\sum_{q\in Q_1, p\in Q_2} \iota(q) \bullet (\varrho^*(q,w,p) \bullet \tau(p)) \right) + \left(\sum_{q\in Q_2, p\in Q_1} \iota(q) \bullet (\varrho^*(q,w,p) \bullet \tau(p)) \right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{q,p\in Q_1} \iota_1(q)\bullet(\varrho_1^*(q,w,p)\bullet\tau_1(p))\right) + \left(\sum_{q,p\in Q_2} \iota_2(q)\bullet(\varrho_2^*(q,w,p)\bullet\tau_2(p))\right) + \\
&\quad \left(\sum_{q\in Q_1,p\in Q_2} \iota_1(q)\bullet(0_A\bullet\tau_2(p))\right) + \left(\sum_{q\in Q_2,p\in Q_1} \iota_2(q)\bullet(0_A\bullet\tau_1(p))\right) \text{ by Equation (7)} \\
&= \left(\sum_{q,p\in Q_1} \iota_1(q)\bullet(\varrho_1^*(q,w,p)\bullet\tau_1(p))\right) + \left(\sum_{q,p\in Q_2} \iota_2(q)\bullet(\varrho_2^*(q,w,p)\bullet\tau_2(p))\right) \\
&\quad + 0_A + 0_A \\
&= L_{M_1}(w) + L_{M_2}(w) = L_{M_1}\oplus L_{M_2}(w).
\end{aligned}
$$

Therefore, $L_1 \oplus L_2 \in \mathcal{FRL}_A^X$, i.e., the K-fuzzy regular languages on X, are closed under the addition operator.

Finally, if $L_1, L_2 \in \mathcal{FRL}_A^X$, then there are K-FFAs $M_1 = \langle Q_1, X, \varrho_1, \iota_1, \tau_1\rangle$ and $M_2 = \langle Q_2, X, \varrho_2, \iota_2, \tau_2\rangle$ such that $Q_1 \cap Q_2 = \emptyset$, $L_{M_1} = L_1$ and $L_{M_2} = L_2$. Then, $M_1 \circledbullet M_2 = \langle Q, X, \varrho, \iota, \tau\rangle$ where $Q = Q_1 \times Q_2$, and for each $q, p \in Q$ and $a \in X$, $\varrho((q_1, q_2), a, (p_1, p_2)) = \varrho_1(q_1, a, p_1) \bullet \varrho_2(q_2, a, p_2)$ and

$$\iota((q_1,q_2)) = \iota_1(q_1)\bullet\iota_2(q_2) \text{ and } \tau((q_1,q_2)) = \tau_1(q_1)\bullet\tau_2(q_2) \tag{8}$$

is clearly a K-FFA. We will now prove that $L_{M_1 \circledbullet M_2} = L_{M_1} \circledbullet L_{M_2}$.

Since $\bullet$ is associative and commutative, then

$$\varrho^*((q_1,q_2),w,(p_1,p_2)) = \varrho_1^*(q_1,w,p_1)\bullet\varrho_2^*(q_2,w,p_2) \tag{9}$$

Since 0_A is an annihilator element of the $\bullet$ and neutral element of $+$, we have

$$
\begin{aligned}
L_{M_1\circledbullet M_2}(w) &= \sum_{(q_1,q_2),(p_1,p_2)\in Q} \iota((q_1,q_2))\bullet(\varrho^*((q_1,q_2),w,(p_1,p_2))\bullet\tau((p_1,p_2))) \\
&= \sum_{q_1,p_1\in Q_1,q_2,p_2\in Q_2} (\iota_1(q_1)\bullet\iota_2(q_2))\bullet((\varrho_1^*(q_1,w,p_1) \\
&\qquad \bullet\varrho_2^*(q_2,w,p_2))\bullet(\tau_1(q_1)\bullet\tau_2(q_2))) \text{ by Equations (8) and (9)} \\
&= \sum_{q_1,p_1\in Q_1,q_2,p_2\in Q_2} (\iota_1(q_1)\bullet(\varrho_1^*(q_1,w,p_1) \\
&\qquad \bullet\tau_1(q_1)))\bullet(\iota_2(q_2)\bullet(\varrho_2^*(q_2,w,p_2)\bullet\tau_2(q_2))) \text{ since } \bullet \text{ is comm. and assoc.} \\
&= \left(\sum_{q_1,p_1\in Q_1} (\iota_1(q_1)\bullet(\varrho_1^*(q_1,w,p_1)\bullet\tau_1(q_1)))\right) \\
&\quad \bullet\left(\sum_{q_2,p_2\in Q_2} (\iota_2(q_2)\bullet(\varrho_2^*(q_2,w,p_2)\bullet\tau_2(q_2)))\right) \text{ by Lemma 2} \\
&= L_{M_1}(w)\bullet L_{M_2}(w) = L_{M_1}\circledbullet L_{M_2}(w).
\end{aligned}
$$

Therefore, $L_1 \circledbullet L_2 \in \mathcal{FRL}_A^X$, i.e., the K-fuzzy regular languages on X, are closed under the product operator. □

Theorem 17. *Let K be a semi-field, A be a K-semi-algebra with identity and X a non-empty set of input symbols such that 0_A is a left annihilator element of the $\bullet$. Then, $(\mathcal{FRL}_A^X, \oplus, \odot, \circledbullet)$ is a K-semi-algebra with identity.*

Proof. This is straightforward from Definition 29 and from the fact that A is a K-semi-algebra. For example, to prove that $(\mathcal{FRL}_A^X, \oplus, \odot, \circledbullet)$ satisfies the left-distributivity, take $L_1, L_2, L_3 \in \mathcal{FRL}_A^X$. Then, for each $w \in X^*$,

$$
\begin{aligned}
(L_1 \circledbullet (L_2 \oplus L_3))(w) &= L_1(w) \bullet (L_2(w) + L_3(w)) \text{ by Definition 29} \\
&= ((L_1(w) \bullet L_2(w)) + (L_1(w) \bullet L_3(w)) \text{ by the left-distributivity of } A \\
&= ((L_1 \circledbullet L_2) \oplus (L_1 \circledbullet L_3))(w) \text{ by Definition 29}
\end{aligned}
$$

□

4.3. Counting Pattern in DNA Sequences

DNA sequences can contain many repetitions of some DNA sequences, called DNA patterns. In other words, a pattern is a contiguous sub-sequence of a DNA sequence. In most situations, the quantity of occurrences of a DNA pattern have important roles in determining if a DNA pattern is interesting or not [27] or to detect some mutational anomalies such as tandem duplication [28].

Let $X = \{\text{A,C,G,T}\}$ be the set of DNA characters or bases, and consider the patterns $u_1 = gata$ and $u_2 = cat$ and a semi-algebra $(A, \oplus, \otimes, \bullet)$ with identity 1_A over a semi-field K and annihilator $\mathbf{0}$ for $\bullet$.

Let us consider the K-FFA $M = \langle Q, X, \varrho, \iota, \tau \rangle$ such that $Q = \{q_1, \dots, q_8\}$, where ϱ is defined in Figure 2 with $a \in A - \{1_A, 0_A, \mathbf{0}\}$,

$$
\iota(q_i) = \begin{cases} 1_A & \text{if } i = 1 \\ \mathbf{0} & \text{otherwise} \end{cases} \quad \text{and } \tau(q_i) = \begin{cases} 1_A & \text{if } i = 2 \text{ or } i = 3 \\ \mathbf{0} & \text{otherwise.} \end{cases}
$$

Then, $L_M(w) = a^m \oplus a^n$ where m and n are, respectively, the number of occurrences of u_1 and u_2.

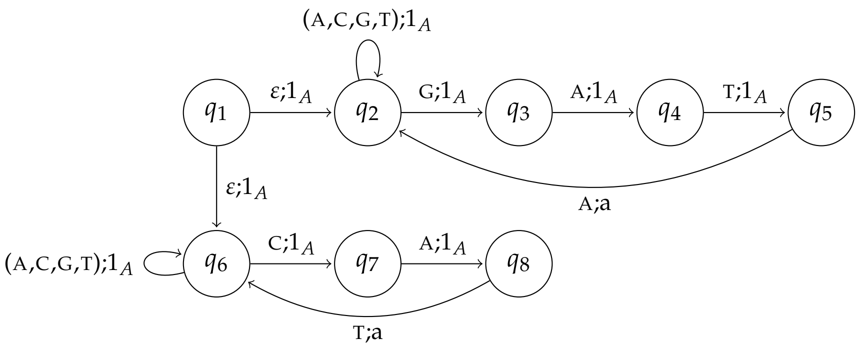

Figure 2. K-FFA for counting two DNA patterns.

If A is the set of nonnegative integers in unary (in unary, each nonnegative integer n is represented by a string of n symbols 1, denoted by 1^n and, therefore, 1^0 is the empty string) endowed with an extra element, denoted by $\mathbf{0}$, and the operations $1^m \oplus 1^n = 1^{m+n}$, $\mathbf{0} \oplus 1^m = 1^m \oplus \mathbf{0} = 1^m$, $1^m \bullet 1^n = 1^m 1^n = 1^{m+n}$, $\mathbf{0} \bullet 1^m = 1^m \bullet \mathbf{0} = \mathbf{0}$; then, for each $w \in X^*$, $L_M(w)$ is the sum (in unary) of the number of occurrences of u_1 and u_2 in w. For example, $L_M(\text{GACATTGCATGGATACATGTGATACb}) = 1^2 \oplus 1^3 = 1^5$.

It is worth noting that such a counting cannot be carried out either with nondeterministic finite automata or with fuzzy automata. In fact, nondeterministic finite automata just decide if a string is in the regular language or not and the Mealy or Moore machines are essentially deterministic. The case of fuzzy automata is similar, only deciding the membership degree of a string to a fuzzy language, i.e., a real value in $[0, 1]$. Of course, we can consider L-fuzzy automata, where L is a complete lattice, as in [29,30]. In particular, this complete lattice can be the set of nonnegative integers in unary A extended with infinitum, denoted here by A^∞.

A tentative of L-fuzzy automata for this purposes is shown in Figure 3. In this case, using the notation of [29], $rec_A(\text{GACATTGCATGGATACATGTGATAC}) = \bigvee_{p_1,\dots,p_{25} \in Q^{25}} I(p_1) \wedge \delta(p_1, \text{G} p_2) \wedge \dots \wedge \delta(p_{24}, \text{C} p_{25}) \wedge F(p_{25}) = I(o_1) \wedge \delta(o_1, \text{G}, o_2) \wedge \dots \wedge \delta(o_{24}, \text{C}, o_{25}) \wedge F(o_{25})$

where $o_1 = o_2 = o_5 = o_6 = o_7 = o_{10} = o_{11} = o_{15} = o_{18} = o_{19} = o_{20} = o_{24} = o_{25} = q_1$, $o_{12} = o_{21} = q_2$, $o_{13} = o_{22} = q_3$, $o_{14} = o_{23} = q_4$, $o_3 = o_8 = o_{16} = q_5$, and $o_4 = o_9 = o_{17} = q_6$.

Therefore, $rec_{\mathcal{A}}(\text{GACATTGCATGGATACATGTGATAC}) = 1_a \wedge \ldots \wedge 1_a \wedge a \wedge 1_A \wedge \ldots \wedge 1_A \wedge a \wedge 1_A \wedge \ldots \wedge 1_A \wedge a \wedge 1_A \wedge \ldots \wedge 1_A \wedge a \wedge 1_A \wedge \ldots \wedge 1_A \wedge a \wedge 1_A \wedge 1_A = a$, that is, these L-fuzzy automata do not perform a counting of u_1 and u_2. The unique way to achieve this counting is to enrich the lattice with operations like addition or concatenation if we consider A^∞. But, in this case, these operations must satisfy some properties, resulting in fuzzy automata valued in the algebra as the K-FFA proposal in this manuscript.

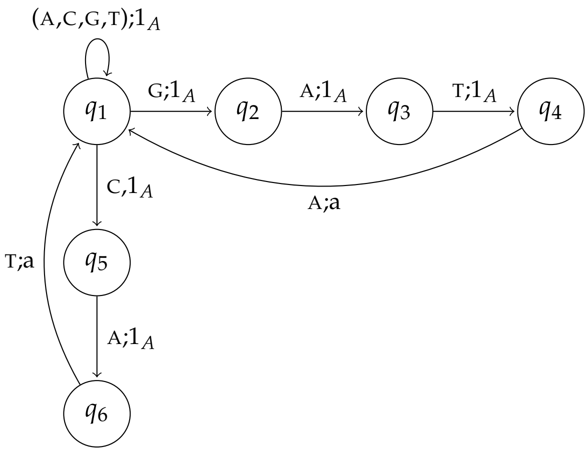

Figure 3. Tentative L-fuzzy automata for counting two DNA patterns.

5. Conclusions

In this paper, we have expanded the theory of semi-vector spaces as well as the theory of semi-algebras, both over the semi-field of nonnegative real numbers. Among these results, we introduced the concept of eigenvalues and eigenvectors of a semi-linear operator. The properties of completeness and separability were also investigated. Since semi-vector spaces and semi-algebras are correlated with fuzzy theory, we described the semi-algebra of A-fuzzy regular languages, after applying the theory of fuzzy automata for counting patterns in DNA sequences. In addition, we provided evidence that the counting of patterns, in general, cannot be achieved either with nondeterministic finite automata or L-fuzzy automata in the sense of [29]. Fuzzy automata have been applied in several areas of research, such as neural networks, learning machines, pattern recognition, control engineering, decision-making, robot control, clinical monitoring, image processing, etc. [31,32]. In particular, applications of fuzzy automata in syntactic pattern recognition can deal with pattern variability by defining inaccurate models [33]. In terms of future work, we intend to apply the K-Fuzzy Automata in Hand Gesture Recognition based on the approaches of [34,35].

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