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Edge DP-Coloring in K_4-Minor Free Graphs and Planar Graphs

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Abstract: The edge DP-chromatic number of G, denoted by $\chi'_{DP}(G)$, is the minimum k such that G is edge DP-k-colorable. In 1999, Juvan, Mohar, and Thomas proved that the edge list chromatic number of K_4-minor free graph G with $\Delta \geq 3$ is Δ. In this paper, we prove that if G is a K_4-minor free graph, then $\chi'_{DP}(G) \in \{\Delta, \Delta+1\}$, and equality $\chi'_{DP}(G) = \Delta + 1$ holds for some K_4-minor free graph G with $\Delta = 3$. Moreover, if G is a planar graph with $\Delta \geq 9$ and with no intersecting triangles, then $\chi'_{DP}(G) = \Delta$.

Keywords: edge coloring; edge DP-coloring; planar graph; K_4-minor free graph

MSC: 05C15; 05C10

1. Introduction

Graphs considered in this paper are finite and simple. For a graph G and $e, e' \in E(G)$, let $d(e, e')$ be the length of the shortest path between the endpoints of e and e'. The edge chromatic number of G, denoted by $\chi'(G)$, is the minimum number of colors needed to color the edges of G so that edges e and e' with $d(e, e') = 0$ are colored by distinct colors. An edge list assignment L assigns to each edge e a set $L(e)$ of permissible colors. We say G is edge L-colorable if there exists a function $\phi : E(G) \to \bigcup L(e)$ such that

1. $\phi(e) \in L(e), \forall e \in E(G)$;
2. $\phi(e) \neq \phi(e')$, if $d(e, e') = 0$.

The edge list chromatic number $\chi'_l(G)$ is the smallest k such that G is edge L-colorable for every edge list assignment L with $|L(e)| \geq k$.

DP-coloring is a generalization of list coloring introduced by Dvořák and Postle [1]. Then, Bernshteyn and Kostochka introduced edge DP-coloring of G as DP-coloring of the line graph of G [2]. To be precise, edge DP-coloring of a graph G is defined as follows:

Definition 1. *Assume G is a graph and $g \in \mathbb{N}^{E(G)}$ is a mapping that assigns to each edge e of G a positive integer $g(e)$. A* cover *of the line graph of G is a pair (L, M), where $L = \{L(e) : e \in E(G)\}$ is a family of pairwise disjoint sets, and $M = \{M_{ee'} : e \neq e', d(e, e') = 0\}$ is a family of bipartite graphs such that $M_{ee'}$ has the bipartite set $L(e)$ and $L(e')$ and $\Delta(M_{ee'}) \leq 1$. An* g-cover *is a cover (L, M) such that $|L(e)| \geq g(e)$ for each edge e. An (L, M)-edge coloring of G is a mapping ϕ such that $\phi(e) \in L(e)$ for each edge e, and for any pair of edges e, e' with $d(e, e') = 0$, $\{\phi(e), \phi(e')\} \notin M_{ee'}$ (for convenience, we write $M_{ee'}$ for $E(M_{ee'})$).*

Definition 2. *If G has an (L, M)-edge coloring for every g-cover (L, M), then we say G is* edge DP-g-colorable. *If $g(e) = k$ for each edge e, then edge DP-g-colorable is called edge DP-k-colorable. The* edge DP-chromatic number *of G, denoted by $\chi'_{DP}(G)$, is the minimum integer k such that G is edge DP-k-colorable.*

It is easy to see and well known that for any graph G, $\chi'(G) \leq \chi'_l(G) \leq \chi'_{DP}(G)$.

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A graph G is K_4-minor free if it has no subgraph isomorphic to a subdivision of K_4. There are many good results based on the K_4-minor free graphs. Beaudou, Foucaud, and Naserasr [3] studied the homomorphism of K_4-minor free graphs with odd girth. Combined with results from [4,5], it is obvious that every K_4-minor free graph is 2-degenerate, and its strong edge chromatic number is at most $6\Delta - 7$. Meanwhile, Juvan, Mohar, and Thomas proved the following theorem.

Theorem 1 ([6]). *Let G be a K_4-minor free graph with $\Delta \geq 3$. Then $\chi'_l(G) = \Delta$.*

The following conjecture is known as the edge list coloring conjecture, which was proposed by several researchers (see [7,8]).

Conjecture 1. *$\chi'_l(G) = \chi'(G)$ for a loopless multigraph G.*

The conjecture is verified for bipartite multigraphs [9], complete graphs of odd order [10], and complete graphs K_{p+1} where p is an odd prime [11], and remains largely open in general.

Vizing's theorem implies that for every simple graph, the edge chromatic number is at most $\Delta + 1$. The next conjecture is a combination of Conjecture 1 and Vizing's theorem.

Conjecture 2. *$\chi'_l(G) \leq \Delta + 1$ for a simple graph G.*

Borodin [12] proved Conjecture 2 for planar graphs G when $\Delta(G) \geq 9$, and Bonamy [13] improved this result to planar graphs with $\Delta \geq 8$.

Theorem 2 ([12]). *If G is a planar graph with $\Delta \geq 9$, then $\chi'_l(G) \leq \Delta + 1$.*

Theorem 3 ([13]). *If G is a planar graph with $\Delta \geq 8$, then $\chi'_l(G) \leq \Delta + 1$.*

By Theorem 3, we can see that if G is a planar graph with $\Delta \geq 8$, then $\chi'_l(G) \in \{\Delta, \Delta + 1\}$. Moreover, edge DP-coloring and DP-coloring of graphs are also studied in the literature [14–16]. In particular, the following two results were proved by Zhang et al. in [16].

Theorem 4. *Assume that G is a planar graph with maximum degree Δ such that G has no cycle of length k for $k \in \{3, 4\}$. Then, $\chi'_{DP}(G) = \Delta$ if either $\Delta \geq 7$ and $k = 4$, or $\Delta \geq 8$ and $k = 3$.*

Theorem 5. *If G is a planar graph with maximum degree $\Delta \geq 9$, then $\chi'_{DP}(G) \leq \Delta + 1$.*

In this paper, we first study edge DP-coloring of K_4-minor free graphs by following the theorem and constructing a K_4-minor free graph with $\Delta = 3$, which is not edge DP-3-colorable.

Theorem 6. *Let G be a K_4-minor free graph with maximum degree Δ. Then, $\chi'_{DP}(G) \leq \Delta + 1$.*

Then, consideration of the edge DP-chromatic number of planar graph G with $\Delta \geq 9$ as an improvement of Theorem 5 is given by following Corollary 1, which is implied from Theorem 7.

Theorem 7. *If G is a planar graph with maximum degree Δ, and with no intersecting triangles, then $\chi'_{DP}(G) = max\{\Delta, 9\}$.*

Corollary 1. *If G is a planar graph with $\Delta \geq 9$ and with no intersecting triangles, then $\chi'_{DP}(G) = \Delta$.*

Compared with Theorems 5 and 7, the discharging method, as a powerful tool for graph-coloring problems, is also used to consider reducible configurations. By our result, we give one sufficient condition for planar graph G with $\Delta \geq 9$ such that $\chi'_{DP}(G) = \Delta$.

Our paper is organized as follows. In Section 2, some lemmas that are used in the proof of Theorems 6 and 7 are listed. In Section 3, we complete the proof of Theorem 6, and a K_4-minor free graph with $\Delta = 3$ that is not edge DP-3-colorable is given, corresponding the matching assignment $\mathcal{M}_L$, which is described in this part. In Section 4, the proof of Theorem 7 is given. Finally, in the Section 5, the conclusions of this paper are stressed.

2. Some Preliminaries

In this section, we introduce some lemmas that are used in the proof of our results. It is well known [6] that a K_4-minor free graph has some special structure, as shown in the lemma below.

Lemma 1 ([6])**.** *There exists one of the following structures in every K_4-minor free graph G:*

(a) A vertex of degree at most one;
(b) Two distinct vertices of degree two with the same neighbors;
(c) Two distinct vertices u,v and not necessarily distinct vertices $w, z \in V(G)\backslash\{u, v\}$ such that the neighbors of v are u and w, and every neighbor of u is equal v, w, or z;
(d) Five distinct vertices v_1,v_2,u_1,u_2,w such that the neighbors of w are u_1,u_2,v_1,v_2, and the neighbors of v_i are w and u_i for $i = 1, 2$.

Below, we provide a special edge list assignment L and show that the graph of (d) (as shown in Figure 1) in Lemma 1 is edge DP-L-colorable.

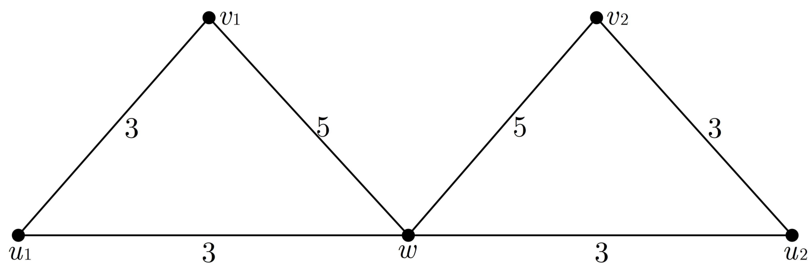


Figure 1. Graph and the value of its list assignment of (d) in Lemma 1.

Lemma 2. *Let H be the graph in Figure 1, and L be an edge list assignment with $|L(e)| \geq 3$ if $e \in \{v_1u_1, v_2u_2, u_1w, u_2w\}$, and $|L(e)| \geq 5$ if $e \in \{v_1w, v_2w\}$. Then, H has an (L, M)-edge coloring for any cover (L, M).*

Proof. The proof is trivial since H can be colored greedily by the order of u_1w, u_2w, u_1v_1, v_1w, v_2w, v_2u_2. □

Let G be a graph with a cover (L, M). Suppose that H is a subgraph of G and $G' = G - E(H)$ has an (L', M')-edge coloring with

$$L' = \{L'(e) = L(e) \in L : e \in E(G')\} \text{ and } M' = \{M'_{ee'} = M_{ee'} \in M : e \sim_{G'} e'\}.$$

There is a mapping ϕ' such that $\phi'(e) \in L'(e)$ for each edge e in G', and for any pair of edges e, e' with $d_{G'}(e, e') = 0$, $\{\phi'(e), \phi'(e')\} \notin M'_{ee'}$.

For $e \in E(H)$, we define a new list assignment $L^*(e)$ and a new family of bipartite graphs M^* as below:

$$L^*(e) = L(e) \backslash \bigcup_{e' \sim e} \{c \in L(e) : \exists \phi'(e') \in L'(e') \text{ s.t. } \{c, \phi'(e')\} \in M_{ee'} \in M\}$$

and

$$M^*_{ee'} = \{\{c, c'\} \in M_{ee'} \in M : c \in L^*(e), c' \in L^*(e')\}, \forall e \sim_H e'.$$

If G' has an (L', M')-edge coloring ϕ' and H has an (L^*, M^*)-edge coloring ϕ^*, then $\phi' \cup \phi^*$ is an (L, M)-edge coloring of G. Hence, G has an (L, M)-edge coloring. This gives the following lemma, proved in [16].

Lemma 3 ([16])**.** *Let G be a graph with a cover (L, M), and H be a subgraph of G. If $G - E(H)$ has an (L', M')-edge coloring and H has an (L^*, M^*)-edge coloring, then G has an (L, M)-edge coloring, where L', M', L^*, and M^* are defined as above.*

Besides, Zhang et al. [16] provided the following lemmas as powerful tools to study the edge DP-coloring of planar graphs. Let G be a graph; a vertex v of G is a pendant vertex if v has degree 1. Similarly, an edge of a graph is said to be pendant if one of its endpoints is a pendant vertex.

Lemma 4 ([16])**.** *Let G be a cycle with a pendant edge, and L be an edge list assignment of G satisfying $|L(uv)| \geq d(u) + d(v) - 2$ for every $uv \in E(G)$. Then, G has an (L, M)-edge coloring for any cover (L, M), where M is a family of bipartite graphs over L.*

Lemma 5 ([16])**.** *Let $G = C + \{v_1v_{2i} : i \in [2, t-1]\} + v_1u(t \geq 3)$, where $C = v_1v_2 \ldots v_{2t}v_1$ is a cycle and v_1u is a pendant edge. If L is an edge list assignment of G satisfying $|L(v_1u)| \geq t$, $|L(v_1v_{2i})| \geq t + 1$ for $i \in [1, t]$, $|L(e)| \geq 2$ for other edges e of G, then G has an (L, M)-edge coloring for any cover (L, M), where M is a family of bipartite graphs over L.*

Before the next section, some notations need to be introduced in advance. Let G be a graph and f be a k-face; all vertices of f will be ordered as $v_1, v_2 \ldots, v_k$ clockwise, and denoted by $[v_1v_2 \ldots v_k]$. The definitions of all symbols used are shown clearly by the following table.

The Definitions of All Symbols

Symbol	**Definition**
$V(G)$	The set of all vertices in G
$F(G)$	The set of all faces in G
$k(k^+; k^-)$-vertex	A vertex of degree k (at least k; at most k)
$k(k^+; k^-)$-face	A face of length k (at least k; at most k)
$(d_1, d_2, \ldots, d_k)$-face	All vertices of face will be ordered as $v_1, v_2 \ldots, v_k$ clockwise and for each $i \in [k]$, v_i is d_i-vertex
(d_1, d_2)-edge	v_1v_2 is a (d_1, d_2)-edge if v_i has degree d_i for $i \in [2]$

3. Edge DP-Coloring of K_4-Minor Free Graph

In this section, we provide a proof of Theorem 6 and construct a K_4-minor free graph with $\Delta = 3$ that is not edge DP-3-colorable.

Theorem 8. *Let G be a K_4-minor free graph with maximum degree Δ; then, $\chi'_{DP}(G) \leq \Delta + 1$.*

Proof. Assume that G is a counterexample of Theorem 6 with $|V(G)| + |E(G)|$ minimal. Let (L, M) be a cover with $|L(e)| = \Delta + 1$ for all $e \in E(G)$ such that G does not have an (L, M)-edge coloring. As G is a K_4-minor free graph, G contains one of the structures (a–d) in Lemma 1.

(a) Assume that $u \in V(G)$ with $e = uv \in E(G)$ and $d(u) = 1$. Then, $G' = G - e$ has an (L', M')-edge coloring by minimality, where L' and M' are defined in Lemma 3. Note that e has at most $\Delta - 1$ incident edges in G'. Thus, we have $|L^*(e)| \geq 2$, and so e can be colored properly. Then, G has an (L, M)-edge coloring by Lemma 3, a contradiction.

(b) Let $u, v \in V(G)$ such that $N(u) = N(v) = \{w, y\}$ (it may happen that $uv \in E(G)$ and $N(u) = \{v, w\}, N(v) = \{u, w\}$). Set $G' = G - E(C)$, where $C = uwvyu$ (or $C = uvw$). By minimality, G' has an (L', M')-edge coloring. And C has an (L^*, M^*)-edge coloring because of $|L^*(e)| \geq 3, \forall e \in E(C)$. Similarly, G has an (L, M)-edge coloring by Lemma 3, a contradiction.

(c) If $w = z$, then this situation is included in (a) or (b). Assume that $w \neq z$. Then, $G - uv$ has an (L', M')-edge coloring by minimality. Note that $d(v) = 2$ and $v \in N(u) \subseteq \{v, w, z\}$. Thus,

$$\begin{aligned} |L^*(uv)| &= \Delta + 1 - (d(u) - 1) - (d(v) - 1) \geq \max\{d(u), d(v)\} - d(u) - d(v) + 3 \\ &\geq \max\{d(u), d(v)\} - d(u) + 1 \geq 1. \end{aligned}$$

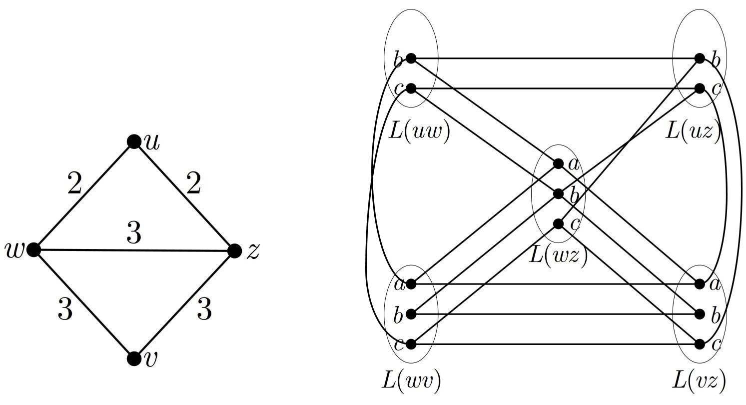
So uv could be colored directly. By Lemma 3, G could have an (L, M)-edge coloring, a contradiction.

(d) Let H be the graph in Figure 1. Then $G' = G - E(H)$ has an (L', M')-edge coloring by minimality. It is easy to check that for all $e \in E(H)$, $|L^*(e)|$ satisfies the condition in Lemma 2. Thus, H has an (L^*, M^*)-edge coloring, and so G has an (L, M)-edge coloring, a contradiction. □

In the following, we will introduce a K_4-minor free graph G with a maximum degree of 3, which is not edge DP-3-colorable. In particular, we will define a K_4-minor free graph G, a cover (L, M) with $|L(e)| = 3, \forall e \in E(G)$ such that G has no (L, M)-edge coloring.

Definition 3. *Let (L, M) be a cover of G and (e_i, e_j) be an adjacent pair. We call the cover M* straight *over (e_i, e_j) if every $\{(e_i, c_1)(e_j, c_2)\} \in M_{e_i e_j}$ satisfies $c_1 = c_2$. Especially, for edge set E, we call M straight over E if M is straight over every adjacent pair (e_i, e_j), $\forall e_i, e_j \in E$.*

Lemma 6. *Let H be the graph in Figure 2a with an edge list assignment L, where $L(e) = \{b, c\}$ for $e \in \{uw, uz\}$, and $L(e) = \{a, b, c\}$ otherwise. Then, there exists a family of bipartite graphs M' over L such that H does not have any (L, M')-edge coloring.*



(**a**) H (**b**) A family of bipartite graphs M' over L

Figure 2. Graph H and a cover (M', L) of H.

Proof. Let M' be the family of bipartite graphs over L, as shown in Figure 2b, which is straight over $(uw, uz), (wv, wz), (wv, vz)$ and (vz, wz). Assume that H has an (L, M')-edge coloring ϕ. Without loss of generality, we may assume that $\phi(uw) = b$. Then, the only choice for edge uz, wz, vz is c, c, b, respectively. Now we cannot find available colors for wv. Similarly, we can get a contradiction if $\phi(uw) = c$. Therefore, H does not have an (L, M')-edge coloring. □

Lemma 7. *Let G be a K_4-minor free graph as shown in Figure 3. Then, G is not edge DP-3-colorable.*

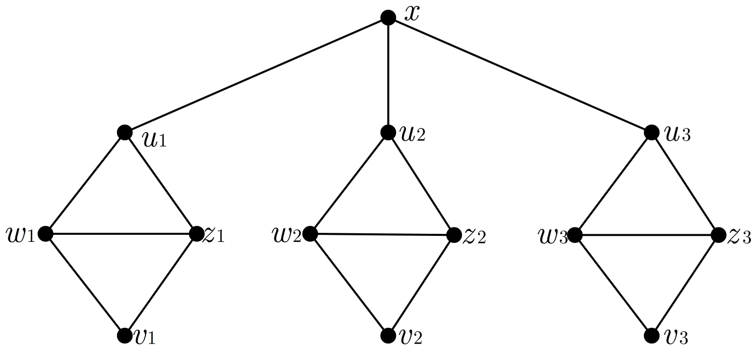


Figure 3. K_4-minor free graph G.

Proof. Let $L(e) = \{a,b,c\}$ for each $e \in E(G)$. Set $E_0 = \bigcup_{i=1,2,3} \{xu_i, u_iw_i, u_iz_i\}$, $E_i = \{u_iw_i, u_iz_i, w_iz_i, w_iv_i, v_iz_i\}$ and $H_i = G[E_i]$ for $i \in [3]$. Note that H_i is a copy of H in Lemma 6. Now, we will define a family of bipartite graphs M over L such that G has no (L,M)-edge coloring. Let M be a family of bipartite graphs over L such that

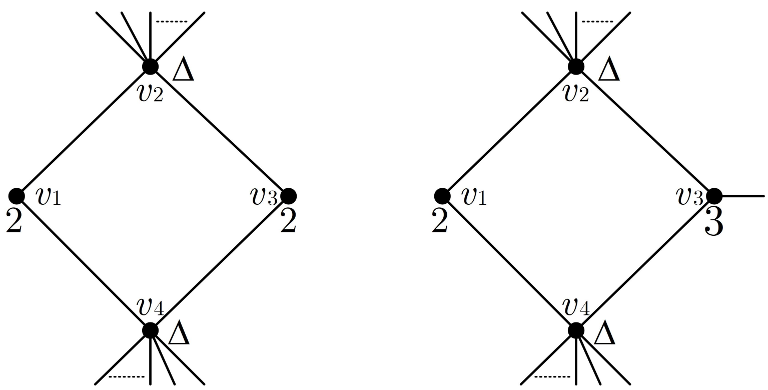
- M is straight over E_0;
- For $i \in [3]$, $M_{ee'} = M'_{ee'}$, $\forall e, e' \in E_i = \{u_iw_i, u_iz_i, w_iz_i, w_iv_i, v_iz_i\}$(M' is shown in Figure 2b).

Assume that G has an (L,M)-edge coloring ϕ. As M is straight over $\{xu_1, xu_2, xu_3\}$, there must exist exactly one edge $xu_j, j \in \{1,2,3\}$, say $j = 1$ s.t. $\phi(xu_1) = a$. Then a is not available for u_1w_1 and u_1z_1. Thus, for graph H_1, the remaining list assignment L' satisfies $L'(e) = \{b,c\}$ for $e \in \{u_1w_1, u_1z_1\}$ and $L'(e) = \{a,b,c\}$ otherwise. By Lemma 6, H_1 does not have (L',M')-edge coloring, and so G does not have an (L,M)-edge coloring. Therefore, G is not edge DP-3-colorable. □

4. Proof of Theorem 7

Assume G is a counterexample to Theorem 7 such that $|E(G)|$ is minimal. Then, there exists a cover (L,M) with $|L(e)| = \Delta$ for $e \in E(G)$ such that G has no (L,M)-edge coloring.

The lemma below shows some properties of the minimal counterexample G. We say f is a special 4-face with facial cycle $[v_1v_2v_3v_4]$ if either (Type I) $d(v_1) = d(v_3) = 2$, $d(v_2) = d(v_4) = \Delta$ or (Type II) $d(v_1) = 2, d(v_3) = 3, d(v_2) = d(v_4) = \Delta$. Let $\mathcal{F}_1$ ($\mathcal{F}_2$) be the set of special faces of Type I (Type II), respectively. Figure 4 describes these two kinds of special faces.



(**a**) Special face of Type I (**b**) Special face of Type II

Figure 4. Two kinds of special 4-faces.

Lemma 8. *If G is a counterexample to Theorem 7 with $|E(G)|$ minimal, then all of following hold:*

(a) *G is connected.*
(b) *Each $v \in V(G)$ is incident to at most one triangle.*
(c) *$d(u) + d(v) \geq \Delta + 2$ for any edge $uv \in E(G)$.*
(d) *If $d(v) = \Delta$ and v is incident to some $f \in \mathcal{F}_1$, then all vertices in $N_G(v) \setminus V(f)$ are 3^+-vertex.*
(e) *G does not contain H_0 (shown in Figure 5a) as its subgraph, where Δ-vertex v_1 is incident to a 2-vertex u and two special faces of Type II.*
(f) *G does not contain H_1(shown in Figure 5b) as its subgraph, where Δ-vertex v is incident to a 2-vertex u_0 and three 4-faces. Moreover, $[u_1u_3vu_2]$ and $[w_2vw_3w_1]$ are two special faces of Type II and $[vu_3xw_3]$ shares a common $(\Delta, 3)$-edge with two special faces. Here, u_1 and w_1 are not necessarily different.*

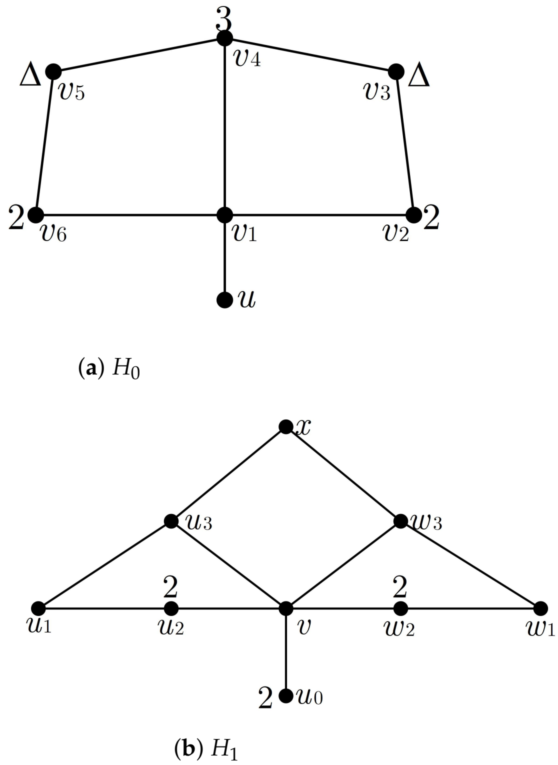


Figure 5. Two configurations of (e) and (f) in Lemma 8.

Proof. (a,b) It is trivial, since G is a minimal counterexample and does not have intersecting triangles.

(c) Assume that there is an edge $e = uv$ of G satisfying $d(u) + d(v) \leq \Delta + 1$. Let $G' = G - \{uv\}$. By minimality, G' has an (L', M')-edge coloring for cover (L', M'), the definition of L' and M' are given in Section 2. Note that $|L^*(e)| \geq \Delta - (d(u) + d(v) - 2) \geq 1$. Thus, e could be colored. By Lemma 3, G has an (L, M)-edge coloring, a contradiction.

(d) Let $v \in V(G)$ with $d(v) = \Delta$ and $f \in \mathcal{F}_1$ with $f = [v_1vv_3v_4]$. Note that $d(v_1) = d(v_3) = 2$ and $d(v_4) = \Delta$. Assume there exists $u \in N_G(v) \setminus V(f)$ with $d(u) \leq 2$. Let H be a subgraph of G with $V(H) = V(f) + \{u\}$ and $E(H) = E(f) + \{vu\}$. Then, by the minimality of G, $G - E(H)$ has an (L', M')-edge coloring ϕ'. Now we consider $|L^*(e)|$ for $e \in E(H)$. It is easy to check that $|L^*(vu)| \geq \Delta - (1 + \Delta - 3) \geq 2$, $|L^*(e)| \geq \Delta - (\Delta - 2) = 2$ for $e \in \{v_3v_4, v_4v_1\}$, $|L^*(e)| \geq \Delta - (\Delta - 3) = 3$ for $e \in \{v_1v, vv_3\}$. By Lemma 4, H has an (L^*, M^*)-edge coloring ϕ^*. Since $G - E(H)$ has an (L', M')-edge coloring ϕ', G has an (L, M)-edge coloring $\phi' \cup \phi^*$.

(e) Assume $H_0 \subseteq G$. Then, $G - E(H_0)$ has an (L', M')-edge coloring. Note that H_0 is a structure described in Lemma 5 with $t = 3$. Based on its degree condition, it is easy to check that $|L^*(v_1u)| \geq \Delta - (\Delta - 4 + 1) = 3$, $|L^*(v_1v_2)| = |L^*(v_1v_6)| = |L^*(v_1v_4)| \geq \Delta - (\Delta - 4) = 4$, and $|L^*(e)| \geq \Delta - (\Delta - 2) = 2$ for $e \in E(H_0)$ otherwise. By Lemma 5, H_0 has an (L^*, M^*)-edge coloring. Thus, G has an (L, M)-edge coloring by Lemma 3, which is a contradiction.

(f) Observe that H_1 is the structure of Lemma 5 with $t = 4$ and $d(x) \leq \Delta$. Thus, we have $|L^*(e)| \geq 2$ for $e \in \{u_1u_2, u_1u_3, w_1w_2, w_1w_3, xu_3, xw_3\}$, $|L^*(e)| \geq 5$ for $e \in \{vu_2, vw_2, vu_3, vw_3\}$ and $|L^*(vu_0)| \geq 4$. Thus, H_1 has an (L^*, M^*)-edge coloring, and so G has an (L, M)-edge coloring by Lemma 3, which is a contradiction. □

To drive a contradiction by discharging analysis, we first define an initial charge ch as $ch(v) = 2d(v) - 6$ for $v \in V(G)$ and $ch(f) = d(f) - 6$ for $f \in F(G)$.

By Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$, the total sum of charges of vertices and faces satisfies the following identity:

$$\sum_{x \in V(G) \cup F(G)} ch(x) = \sum_{v \in V(G)} (2d(v) - 6) + \sum_{x \in F(G)} (d(f) - 6) = -12.$$

Next, we design appropriate discharging rules and redistribute charges accordingly. Once the discharging is finished, a new charge ch^* is produced. Note that the discharging process preserves the total sum of charges of G. However, we will show that $ch^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$, which leads to an obvious contradiction, and subsequently the proof is complete.

Our discharging rules are defined as follows. (For $3, 4, 5$-face f, we always assume $v_1 \in V(f)$ has the minimal degree).

(R1) Every 2-vertex v receives 1 from every incident face.
(R2) If a Δ-vertex v is incident to a 6^+-face f; and

- (R2.1) v is adjacent to two 2-vertices which are incident to f, then v gives 1 to f.
- (R2.2) v is adjacent to exactly one 2-vertex which is incident to f, then v gives $\frac{1}{2}$ to f.

(R3) Let $f = [v_1v_2v_3]$ be a 3-face.

- (R3.1) If $d(v_1) = 2$, then each of v_2 and v_3 gives 2 to f, respectively.
- (R3.2) If $d(v_1) = 3$, then each of v_2 and v_3 gives $\frac{3}{2}$ to f, respectively.
- (R3.3) If $d(v_1) = 4$, then each of v_2 and v_3 gives $\frac{5}{4}$ to f; v_1 gives $\frac{1}{2}$ to f.
- (R3.4) If $d(v_1) = 5$,
  - and $\exists i \in \{2, 3\}$ with $d(v_i) = \Delta - 3$, then each of v_1, v_2 and v_3 gives 1 to f.
  - otherwise, v_1 gives $\frac{1}{2}$ to f, each of v_2 and v_3 gives $\frac{5}{4}$ to f.
- (R3.5) If $d(v_1) \geq 6$, then each of v_1, v_2 and v_3 gives 1 to f.

(R4) Let $f = [v_1v_2v_3v_4]$ be a 4-face.

- (R4.1) If $d(v_1) = 2$ (note that $d(v_2) = d(v_4) = \Delta$), and
  - $d(v_3) = 2$, then each of v_2 and v_4 gives 2 to f ($f \in \mathcal{F}_1$ due to (c) in Lemma 8).
  - $d(v_3) = 3$, then each of v_2 and v_4 gives $\frac{3}{2}$ to f ($f \in \mathcal{F}_2$).
  - $d(v_3) = 4$, then each of v_2 and v_4 gives $\frac{5}{4}$ to f and v_3 gives $\frac{1}{2}$ to f.
  - $d(v_3) = 5$, then each of v_2 and v_4 gives $\frac{11}{10}$ to f and v_3 gives $\frac{4}{5}$ to f.
  - $d(v_3) \geq 6$, then each of v_2 and v_4 gives 1 to f, v_3 gives 1 to f.
- (R4.2) If $d(v_1) = 3$, and
  - $d(v_3) = 3$, then each of v_2 and v_4 gives 1 to f.
  - $d(v_3) \geq 4$, then each of v_2 and v_4 gives $\frac{3}{4}$ to f and v_3 gives $\frac{1}{2}$ to f.
- (R4.3) If $d(v_1) \geq 4$, then each of v_1, v_2, v_3 and v_4 gives $\frac{1}{2}$ to f.

(R5) Let $f = [v_1v_2v_3v_4v_5]$ be a 5-face. By (c) in Lemma 8 and symmetry, we could assume v_3 has the second smallest degree.

(R5.1) If $d(v_1) = 2$,

- and $d(v_3) = 2$, then each of v_2, v_4 and v_5 gives 1 to f.
- and $d(v_3) = 3$, then each of v_2 and v_4 gives $\frac{1}{2}$ to f, v_5 gives 1 to f.
- otherwise, each of v_2, v_3, v_4 and v_5 gives $\frac{1}{2}$ to f.

(R5.2) If $d(v_1) = 3$,

- and $d(v_3) = 3$, then each of v_2, v_4 and v_5 gives $\frac{1}{3}$ to f.
- otherwise, each of v_2, v_3, v_4 and v_5 gives $\frac{1}{4}$ to f.

(R5.3) If v_1 is a 4^+-vertex, then each v_i gives $\frac{1}{5}$ to f for $i \in [5]$.

Note that each face f only gives a charge to incident 2-vertex. Assume that $|f| \geq 6$. If there is a 2-vertex $v \in V(f)$, then there exist two Δ-vertices as its neighbors on f by (c) in Lemma 8. Thus, by (R1) and (R2), each Δ-vertices gives at least $1/2$ to f, and so $ch^*(f) \geq ch(f) \geq 0$. If f is a d-face, $d \in \{3, 4, 5\}$, then $ch^*(f) = 0$ by rules (R3), (R4) and (R5). After all, the final charge of every face is nonnegative.

As $d(u) + d(v) \geq \Delta + 2$ for any edge $uv \in E(G)$ (by (c) of Lemma 8), each v of G has degree at least 2. Now, we will consider the final charge of d-vertex, where $d \in [2, \Delta]$. If $d(v) = 2$, then $ch(v) = -2$. As v does not give out any charge, $ch^*(v) = -2 + 1 \times 2 = 0$ by rule (R1). Now it suffices to consider the vertex v with $d(v) \in [3, \Delta]$.

Claim 1. *If v is d-vertex with $d \in [3, \Delta - 1]$, then $ch^*(v) \geq 0$.*

Proof. If $d(v) = 3$, then v does not give out or get any charge by all discharging rules. So, $ch^*(v) = ch(v) = 0$.

If $d(v) = 4$, then v gives at most $\frac{1}{2}$ to every incident face by rules (R3), (R4) and (R5). So, $ch^*(v) \geq 2 - 4 \times \frac{1}{2} = 0$.

Next considering any 5-vertex v. If v is not incident to any 3-face, then by rules (R4) and (R5), we obtain $ch^*(v) \geq 4 - 5 \times \frac{4}{5} = 0$. Otherwise, v is incident to exactly one 3-face by (b) in Lemma 8. For $i \in [5]$, let $u_i \in N(v)$ in cyclic order and for $i \in [5]$, let f_i be the face incident to v with vu_i, vu_{i+1}, where indices are taken module 5. Assume that the unique 3-face is f_1. If v gives at most $\frac{4}{5}$ to f_1, then $ch^*(v) \geq 0$ obviously. So, considering v gives 1 to f_1 by rule (R3.4). Then, $\exists i \in \{1, 2\}$ such that $d(u_i) = \Delta - 3$, and so, v gives at most $\frac{1}{2}$ to the face $f \in \{f_2, f_5\}$ with $u_i \in V(f)$ by (R4.3) and (R5). Thus, $ch^*(v) \geq 4 - 1 - \frac{1}{2} - 3 \times \frac{4}{5} \geq 0$.

If $d(v) = 6$, then v sends at most 1 to every incident face by (R3), (R4), and (R5). So, $ch^*(v) \geq 6 - 1 \times 6 = 0$.

If $d(v) \in [7, \Delta - 1]$, by (R3), (R4) and (R5), then v sends at most $\frac{3}{2}$ to 3-face, and v sends at most 1 to every incident 4^+-face. Therefore, we obtain $ch^*(v) \geq 2d - 6 - \frac{3}{2} \times 1 - (d - 1) \times 1 = d - 6.5 > 0$. □

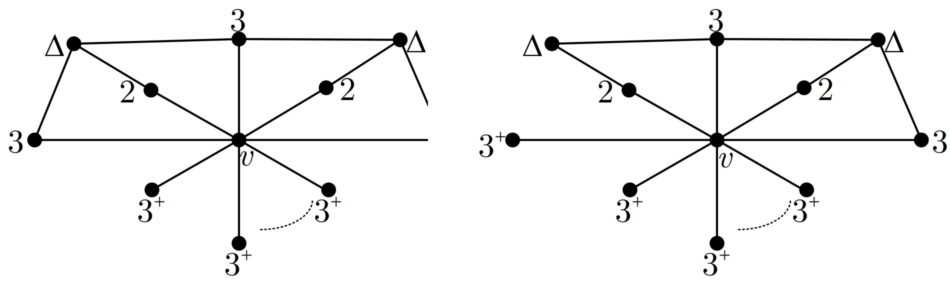
Claim 2. *If v is a Δ-vertex, then $ch^*(v) \geq 0$.*

Proof. For $i \in [\Delta]$, let $u_i \in N(v)$ in cyclic order and f_i be the face incident to v with vu_i, vu_{i+1}, where indices are taken module Δ. Let $F = \{f_i : i \in [\Delta]\}$. We call $P \subseteq F$ is a *petal* of v if P is formed by some consecutive adjacent $f \in \mathcal{F}_2$ and set $l = \max\{|P| : P \text{ is a petal of } v\}$.

If $l \geq 5$, then there is some subgraph H_0 of G, which contradicts (e) of Lemma 8. Thus, we can have the following observation.

Observation 1. *Let w be the number of petals around v. If $\mathcal{F}_2 \cap F \neq \emptyset$, i.e., $w \geq 1$, then $l \leq 4$ and one of the following situations appears.*

(s1) $l = 4$, $\omega = 1$ (as shown in Figure 6a);
(s2) $l = 3$, $\omega = 1$ (as shown in Figure 6b);
(s3) $l = 2$ and two consecutive adjacent special faces share common $(\Delta, 3)$-edge, $\omega = 1$;
(s4) $l = 2$ and two consecutive adjacent special faces share common $(\Delta, 2)$-edge, $\omega \leq \left\lfloor \frac{\Delta}{2} \right\rfloor$;
(s5) $l = 1$, $\omega \leq \left\lfloor \frac{\Delta}{2} \right\rfloor$.



(**a**) (s1) (**b**) (s2)

Figure 6. (s1) and (s2) in Observation 1.

Note that if there is some 3-face in F, there is exactly one 3-face in F since there are no intersecting triangles in G. Now we will consider three cases based on the existence of special faces in F. Assume that $\mathcal{F}_i \cap F = \emptyset$ for $i \in [2]$. By the (R2), (R3), (R4) and (R5), v sends at most 2 to the possible unique 3-face and at most $\frac{5}{4}$ to 4^+-faces. Thus, $ch^*(v) \geq 2\Delta - 6 - 2 - \frac{5}{4}(\Delta - 1) \geq 0$ since $\Delta \geq 9$. Assume that there is some special 4-face of Type I in F, say $f_1 \in \mathcal{F}_1 \cap F$. By (d) in Lemma 8, the degree of u_i is at least 3, for $i \in \{3, 4, \ldots, \Delta\}$. If one of f_2 and f_Δ is 3-face, say f_2, then v sends 2 to f_1, at most 2 to f_2, at most $\frac{3}{2}$ to f_Δ, and at most 1 to $f_i, i \in [3, 4, \ldots, \Delta - 1]$ by rules (R3.5). So, $ch^*(v) \geq 2\Delta - 6 - 2 - 2 - \frac{3}{2} - 1 \times (\Delta - 3) > 0$ since $\Delta \geq 9$. Otherwise, v sends 2 to f_1, at most $\frac{3}{2}$ to f_2, f_Δ and possible 3-face, and at most 1 to other $f_i \in F$. So, $ch^*(v) \geq 2\Delta - 6 - 2 - 3 \times \frac{3}{2} - 1 \times (\Delta - 4) \geq 0$.

In the following, we will assume that $F \cap \mathcal{F}_1 = \emptyset$ and $F \cap \mathcal{F}_2 \neq \emptyset$, i.e., there is some $f_i \in \mathcal{F}_2$ for $i \in [\Delta]$. It suffices to consider all situations in Observation 1. Note that v sends $\frac{3}{2}$ to each $f \in \mathcal{F}_2 \cap F$.

(s1) (see in Figure 6a): As every vertex u_i that is not incident to faces in $\mathcal{F}_2 \cap F$ is 3^+-vertex, v sends at most $\frac{3}{2}$ to the possible 3-face and at most 1 to every 4^+-faces not contained in the petal. Thus, $ch^*(v) \geq 2\Delta - 6 - (4+1) \times \frac{3}{2} - 1 \times (\Delta - 5) \geq 0$ since $\Delta \geq 9$.

(s2) Assume v has a petal $P = \{f_1, f_2, f_3\}$ (as shown in Figure 6b). By (e) in Lemma 8, the degree of u_i is at least 3 for $i \in [4, \Delta]$. If $|f_\Delta| = 3$, then f_1 has a chord vu_Δ. So that is impossible and f_Δ must be a 4^+-face. Therefore, v sends at most $\frac{3}{2}$ to the possible 3-face, at most $\frac{5}{4}$ to f_Δ, at most 1 to other 4^+-faces, and so $ch^*(v) \geq 2\Delta - 6 - (3+1) \times \frac{3}{2} - \frac{5}{4} - 1 \times (\Delta - 5) > 0$ since $\Delta \geq 9$.

(s3) Assume v has exactly one petal $P = \{f_1, f_2\}$. Similarly, v sends $\frac{3}{2}$ to f_i for $i \in [2]$ and the degree of u_i is at least 3, for $i \in [4, \Delta]$. If one of f_3 and f_Δ is a 3-face, say f_3, then f_2 as a face has a chord vu_4, a contradiction. So f_3 and f_Δ are 4^+-face. Therefore, v sends at most $\frac{5}{4}$ to f_3, f_Δ, sends at most $\frac{3}{2}$ to the possible 3-face and sends at most 1 to other 4^+-faces. So, $ch^*(v) \geq 2\Delta - 6 - 2 \times \frac{3}{2} - 2 \times \frac{5}{4} - \frac{3}{2} - 1 \times (\Delta - 5) \geq 0$ since $\Delta \geq 9$.

Before the analysis of (s4) and (s5), let $\mathcal{P}_i$ be the set of all petals of v with size i and $|\mathcal{P}_i| = \omega_i$ for $i \in [2]$. Set $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then, $|\mathcal{P}| = \omega_1 + \omega_2 = \omega$. Let $F' = \{f_i \in F : f_i \notin P \in \mathcal{P}$ and f_i shares an $(\Delta, 3)$-edge with a face in $F \cap \mathcal{F}_2\}$ and set $|F'| = \alpha$. Note that for any $f \in F$:

- If $f \in F'$ and $|f| \geq 4$, then v sends at most 1 to f.
- If $f \in F'$ and $|f| = 3$, then v sends $\frac{3}{2}$ to f.

- If $f \in P$ for some $P \in \mathcal{P}$, then v sends $\frac{3}{2}$ to f.
- Otherwise, v sends at most $\frac{5}{4}$ to every 4^+-face f.

Now we consider two cases based on the existence of a 3-face.

Case 1. *For any $i \in [\Delta]$, $|f_i| \geq 4$.*

(s4) Note that $\omega_2 \neq 0$ and each $P \in \mathcal{P}_2$ is sharing $(\Delta, 3)$-edges with two faces in F'. When $\alpha \leq \omega_2$, the only possibility is that $\alpha = \omega_2$ and $\omega_1 = 0$. In this situation, every face in F' shares $(\Delta, 3)$-edge with two different petals in $\mathcal{P}_2$ respectively. Hence, $\alpha = \omega_2 = \frac{\Delta}{3}$ implies that $ch^*(v) = 2\Delta - 6 - 3\omega_2 - \frac{\Delta}{3} = \frac{2}{3}\Delta - 6 \geq 0$ by $\Delta \geq 9$.

Thus, α is at least $\omega_2 + 1$. So,

$$\begin{aligned} ch^*(v) &\geq 2\Delta - 6 - \frac{3}{2}\omega_1 - 3\omega_2 - (\omega_2 + 1) - \frac{5}{4}(\Delta - \omega_1 - 2\omega_2 - (\omega_2 + 1)) \\ &= \frac{3}{4}\Delta - \frac{23}{4} - \frac{1}{4}\omega_2 - \frac{1}{4}\omega_1 \geq 0, \end{aligned}$$

since $\Delta \geq 9$ and $\omega_1 + \omega_2 = \omega \leq \left\lfloor \frac{\Delta}{2} \right\rfloor$.

(s5) Note that $\omega_1 \neq 0, \omega_2 = 0$. As each $P \in \mathcal{P}$ is sharing an $(\Delta, 3)$-edge with some face in F', $\alpha \geq \left\lceil \frac{\omega_1}{2} \right\rceil$. Thus, we have

$$\begin{aligned} ch^*(v) &\geq 2\Delta - 6 - \frac{3}{2}\omega_1 - \left\lceil \frac{\omega_1}{2} \right\rceil - \frac{5}{4}(\Delta - \omega_1 - \left\lceil \frac{\omega_1}{2} \right\rceil) \\ &= \frac{3}{4}\Delta - 6 - \frac{1}{4}\omega_1 + \frac{1}{4}\left\lceil \frac{\omega_1}{2} \right\rceil \geq 0, \end{aligned}$$

since $\Delta \geq 9$ and $\omega_1 = \omega \leq \left\lfloor \frac{\Delta}{2} \right\rfloor$.

Case 2. *There is some $i \in [\Delta]$ such that f_i is a 3-face. We may assume that $|f_1| = 3$ and $d(u_1) \leq d(u_2)$.*

(s4) Similarly, we have α is at least $\omega_2 + 1$ and $\omega_2 \neq 0$. Note that v sends at most $\frac{3}{2}$ to each petal in $\mathcal{P}_1$ and sends at most 3 to each petal in $\mathcal{P}_2$. Assume that f_1 is not a $(2, \Delta, \Delta)$-face. If $f_1 \notin F'$, then v sends at most $\frac{3}{2}$ to f_1, sends at most 1 to $w_2 + 1$ faces in F', and sends at most $\frac{5}{4}$ to other 4^+-face. Thus, we have

$$\begin{aligned} ch^*(v) &\geq 2\Delta - 6 - \frac{3}{2} - \frac{3}{2}\omega_1 - 3\omega_2 - (\omega_2 + 1) - \frac{5}{4}(\Delta - 1 - \omega_1 - 2\omega_2 - \omega_2 - 1) \\ &= \frac{3}{4}\Delta - \frac{24}{4} - \frac{1}{4}\omega_1 - \frac{1}{4}\omega_2 \end{aligned} \quad (1)$$

If $f_1 \in F'$, then f_1 contains a $(\Delta, 3)$-edge and one of $f_2, f_\Delta \in \mathcal{F}_2$, say $f_\Delta \in \mathcal{F}_2$. Thus, f_2 is a 4^+-face containing a $(\Delta, (\Delta - 1)^+)$-edge. If $f_2 \notin F'$, then v sends at most $\frac{3}{2}$ to f_1, sends at most 1 to f_2 and w_2 faces in F'. Thus, we have

$$\begin{aligned} ch^*(v) &\geq 2\Delta - 6 - \frac{3}{2} - \frac{3}{2}\omega_1 - 3\omega_2 - \omega_2 - 1 - \frac{5}{4}(\Delta - 1 - \omega_1 - 2\omega_2 - \omega_2 - 1) \\ &= \frac{3}{4}\Delta - \frac{24}{4} - \frac{1}{4}\omega_1 - \frac{1}{4}\omega_2. \end{aligned} \quad (2)$$

Otherwise, $f_2 \in F'$ and f_2 is a 4^+-face containing a $(\Delta, 3)$-edge. Thus, f_2 receives at most $\frac{3}{4}$ from v by (R2), (R4) and (R5). And so,

$$ch^*(v) \geq 2\Delta - 6 - \frac{3}{2} - \frac{3}{2}\omega_1 - 3\omega_2 - (\omega_2 - 1) - \frac{3}{4} - \frac{5}{4}(\Delta - 1 - \omega_1 - 2\omega_2 - (\omega_2 - 1) - 1)$$
$$= \frac{3}{4}\Delta - \frac{24}{4} - \frac{1}{4}\omega_1 - \frac{1}{4}\omega_2. \quad (3)$$

As $\Delta \geq 9$ and $\omega_1 + \omega_2 = \omega \leq \left\lfloor \frac{\Delta}{2} \right\rfloor$, Equations (1)–(3) are not negative except when $\Delta = 9$ and $\omega_2 = 1, \omega_1 = 3$. Then, f_1 is adjacent to some $P \in \mathcal{P}_1$ and some $P \in \mathcal{P}_2$, and so there is a $(2,3)$-edge or $(3,3)$-edge in f_1. It is a contradiction since $d(u) + d(v) \geq \Delta + 2$ for any $uv \in E(G)$.

Assume that f_1 is a $(2,\Delta,\Delta)$-face. Clearly, we have $f_1 \notin F'$ and one of f_2, f_Δ contains a (Δ,Δ)-edge. Without loss of generality, we may assume that f_2 is a 4^+-face with a (Δ,Δ)-edge. Similarly, if $f_2 \notin F'$, then f_2 receives at most 1 from v by (R2), (R4) and (R5). Thus,

$$ch^*(v) \geq 2\Delta - 6 - 2 - \frac{3}{2}\omega_1 - 3\omega_2 - (\omega_2 + 1) - 1 - \frac{5}{4}(\Delta - 1 - \omega_1 - 2\omega_2 - (\omega_2 + 1) - 1)$$
$$= \frac{3}{4}\Delta - \frac{25}{4} - \frac{1}{4}\omega_1 - \frac{1}{4}\omega_2. \quad (4)$$

Otherwise, $f_2 \in F'$ and f_2 is a 4^+-face containing a $(\Delta,3)$-edge. Thus, each $f \in F' - f_2$ receives at most 1 from v and f_2 receives at most $\frac{3}{4}$ from v by (R2), (R4) and (R5). So,

$$ch^*(v) \geq 2\Delta - 6 - 2 - \frac{3}{2}\omega_1 - 3\omega_2 - \omega_2 - \frac{3}{4} - \frac{5}{4}(\Delta - 1 - \omega_1 - 2\omega_2 - \omega_2 - 1)$$
$$= \frac{3}{4}\Delta - \frac{25}{4} - \frac{1}{4}\omega_1 - \frac{1}{4}\omega_2. \quad (5)$$

As $\Delta \geq 9$, $\omega_1 + \omega_2 = \omega \leq \left\lfloor \frac{\Delta}{2} \right\rfloor$ and the existence of the 3-face, Equations (4) and (5) are not negative unless when $\Delta = 9$ and $\omega_2 = 2, \omega_1 = 1$, or $\omega_2 = 1, \omega_1 = 2$. Note that v sends 2 to face f_1, sends 3 to each petal in $\mathcal{P}_2$ and sends $\frac{3}{2}$ to each petal in $\mathcal{P}_1$.

When $\Delta = 9$ and $\omega_2 = 2, \omega_1 = 1$, it suffices to consider the structure in Figure 7a by symmetry and the existence of $(2,\Delta,\Delta)$-face. Note that f_5 and f_8 are not 4-faces due to (f) in Lemma 8. So, f_5 and f_8 receive at most $\frac{1}{3}$ from v by (R5.2) respectively. Thus, we have $ch^*(v) \geq 12 - 6 - \frac{3}{2} - 2 - 1 - \frac{1}{3} \times 2 \geq 0$.

Now, we consider the situation when $\Delta = 9$ and $\omega_2 = 1, \omega_1 = 2$. Note that for any 4^+-face $f \in F'$, f receives at most 1 from v. By all degree conditions and the absence of chord in any face, the unique $(2,\Delta,\Delta)$-face does not share any edge with any petals. All situations are shown in Figure 7b–h. Based on $F \cap \mathcal{F}_1 = \emptyset$ and (R4.1, R5.1). Thus, we have $ch^*(v) \geq 12 - 6 - 2 - 4 * 1 = 0$.

(s5) Note that $\omega_1 = \omega \leq \left\lfloor \frac{\Delta}{2} \right\rfloor$ and $\alpha \geq \lceil \frac{\omega_1}{2} \rceil$.

Assume f_1 is not a $(2,\Delta,\Delta)$-face. If $f_1 \notin F'$, then

$$ch^*(v) \geq 2\Delta - 6 - \frac{3}{2} - \frac{3}{2}\omega_1 - \left\lceil \frac{\omega_1}{2} \right\rceil - \frac{5}{4}(\Delta - 1 - \omega_1 - \left\lceil \frac{\omega_1}{2} \right\rceil)$$
$$= \frac{3}{4}\Delta - \frac{25}{4} - \frac{1}{4}\omega_1 + \frac{1}{4}\left\lceil \frac{\omega_1}{2} \right\rceil \geq 0,$$

since $\Delta \geq 9$.

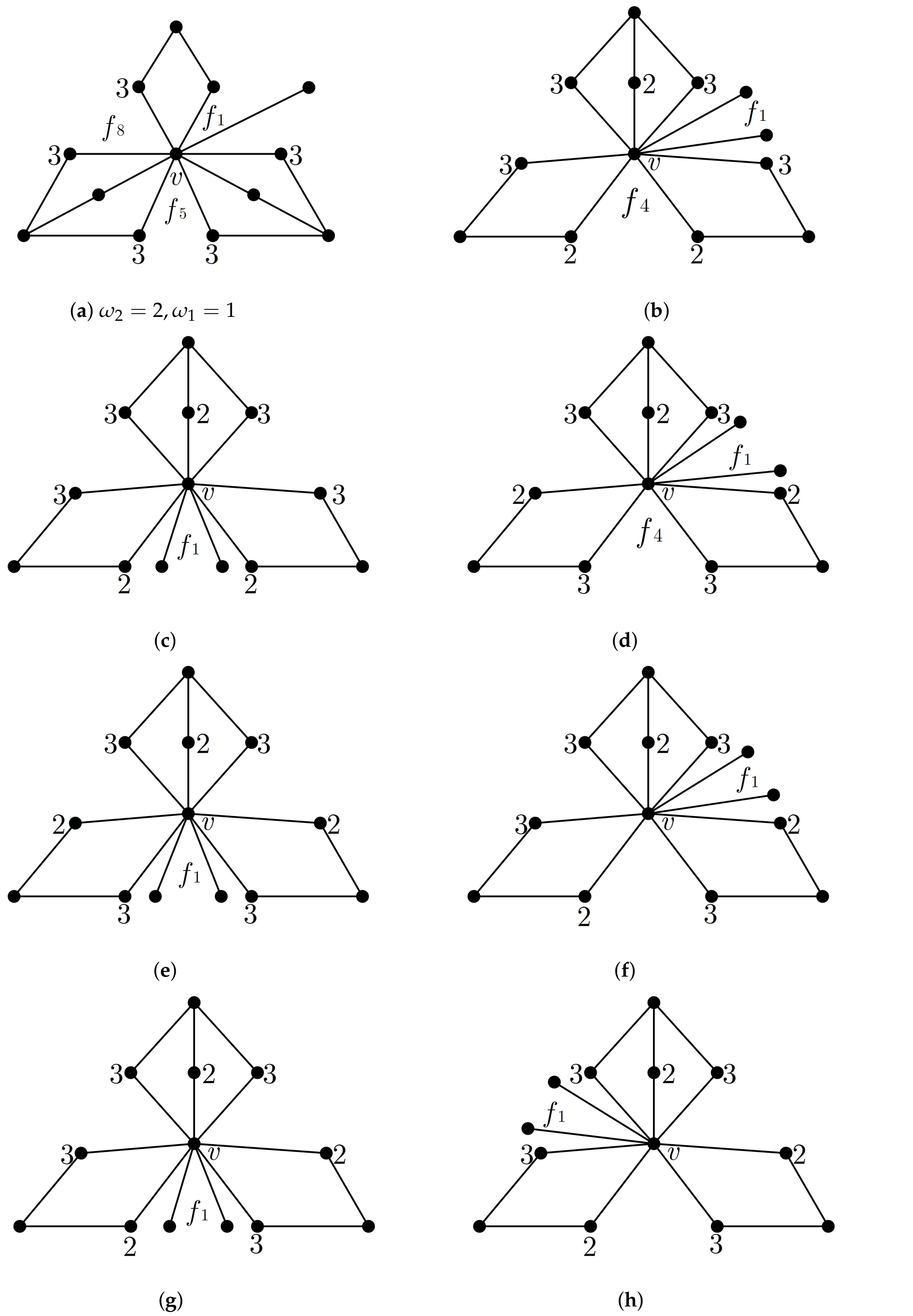


Figure 7. When $\Delta = 9$ and $\omega_2 = 2, \omega_1 = 1$, or $\omega_2 = 1, \omega_1 = 2$.

If $f_1 \in F'$, then we can assume that $f_\Delta \in \mathcal{F}_2$, and so f_2 is a 4^+-face containing a $(\Delta, (\Delta-1)^+)$-edge. Note that $|F' - f_1| \geq \left\lceil \frac{\omega_1 - 1}{2} \right\rceil$. If $f_2 \notin F'$, then f_2 receives at most 1 from v by (R2), (R4) and (R5). Thus,

$$
\begin{aligned}
ch^*(v) &\geq 2\Delta - 6 - \frac{3}{2} - \frac{3}{2}\omega_1 - \left\lceil \frac{\omega_1 - 1}{2} \right\rceil - 1 - \frac{5}{4}(\Delta - 1 - \omega_1 - \left\lceil \frac{\omega_1 - 1}{2} \right\rceil - 1) \\
&= \frac{3}{4}\Delta - \frac{24}{4} - \frac{1}{4}\omega_1 + \frac{1}{4}\left\lceil \frac{\omega_1 - 1}{2} \right\rceil \geq 0,
\end{aligned}
$$

since $\Delta \geq 9$.

Otherwise, $f_2 \in F'$ and f_2 is a 4^+-face containing a $(\Delta, 3)$-edge. Then, $|F' - \{f_1, f_2\}| \geq \left\lceil \frac{\omega_1 - 2}{2} \right\rceil$ and each $f \in F' - \{f_1, f_2\}$ receives at most 1 from v and f_2 receives at most $\frac{3}{4}$ from v by (R2), (R4), and (R5). So,

$$
\begin{aligned}
ch^*(v) &\geq 2\Delta - 6 - \frac{3}{2} - \frac{3}{2}\omega_1 - \left\lceil \frac{\omega_1 - 2}{2} \right\rceil - \frac{3}{4} - \frac{5}{4}(\Delta - 1 - \omega_1 - \left\lceil \frac{\omega_1 - 2}{2} \right\rceil - 1) \\
&= \frac{3}{4}\Delta - \frac{23}{4} - \frac{1}{4}\omega_1 + \frac{1}{4}\left\lceil \frac{\omega_1 - 2}{2} \right\rceil \geq 0,
\end{aligned}
$$

since $\Delta \geq 9$.

Assume that f_1 is a $(2, \Delta, \Delta)$-face. Clearly, we have $f_1 \notin F'$. Similarly, if $f_2 \notin F'$, then f_2 receives at most 1 from v by (R4) and (R5). Thus,

$$
\begin{aligned}
ch^*(v) &\geq 2\Delta - 6 - 2 - \frac{3}{2}\omega_1 - \left\lceil \frac{\omega_1}{2} \right\rceil - 1 - \frac{5}{4}(\Delta - 1 - \omega_1 - \left\lceil \frac{\omega_1}{2} \right\rceil - 1) \\
&= \frac{3}{4}\Delta - \frac{26}{4} - \frac{1}{4}\omega_1 + \frac{1}{4}\left\lceil \frac{\omega_1}{2} \right\rceil.
\end{aligned} \tag{6}
$$

Otherwise, $f_2 \in F'$ and f_2 is a 4^+-face containing a $(\Delta, 3)$-edge. Thus, each $f \in F' - f_2$ receives at most 1 from v and f_2 receives at most $\frac{3}{4}$ from v by (R2), (R4) and (R5). Thus,

$$
\begin{aligned}
ch^*(v) &\geq 2\Delta - 6 - 2 - \frac{3}{2}\omega_1 - \left\lceil \frac{\omega_1 - 1}{2} \right\rceil - \frac{3}{4} - \frac{5}{4}(\Delta - 1 - \omega_1 - \left\lceil \frac{\omega_1 - 1}{2} \right\rceil - 1) \\
&= \frac{3}{4}\Delta - \frac{25}{4} - \frac{1}{4}\omega_1 + \frac{1}{4}\left\lceil \frac{\omega_1 - 1}{2} \right\rceil.
\end{aligned} \tag{7}
$$

Combined with $\Delta \geq 9$ and $\omega_1 \leq \left\lfloor \frac{\Delta}{2} \right\rfloor$, Equations (6) and (7) are not negative unless $\Delta = 9$, $\omega_1 = 4$ and $\alpha = 2$ (shown in Figure 8). Similarly, by all degree conditions and the absence of chord of any face, the unique 3-face f_1 does not appear in any position. □

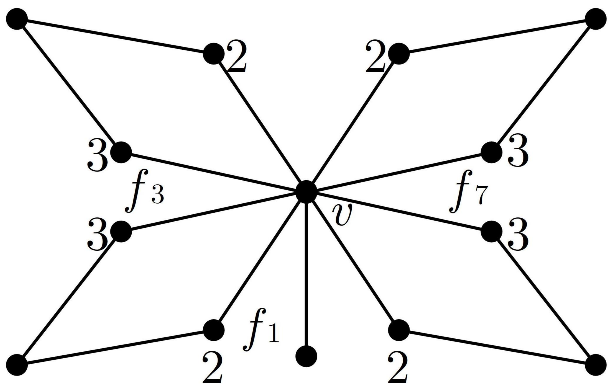


Figure 8. $\Delta = 9$, $\omega_1 = 4$ and $\alpha = 2$.

By Claims 1 and 2, for $d \in [3, \Delta]$, the final charge of d-vertex is nonnegative. The proof of Theorem 7 is completed.

5. Conclusions

Theorems 6 and 7 are two main results in our paper, we study the edge DP-chromatic number of K_4-minor free graphs and planar graph with $\Delta \geq 9$, respectively. Moreover, we also prove that the upper bound in Theorem 6 is sharp by one example.

For K_4-minor free graphs. On the one hand, by Theorem 6, the corresponding edge DP-chromatic number is Δ or $\Delta + 1$. On the other hand, we give one example to demonstrate that there exists some K_4-minor free graph satisfying with the edge DP-chromatic number is not Δ. It reflects that the upper bound in Theorem 6 is sharp.

For a planar graph G with $\Delta \geq 9$, some researchers proved $\chi'_{DP}(G) \leq \Delta + 1$. One sufficient condition is given for G such that $\chi'_{DP}(G) = \Delta$ in Theorem 7.

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References

1. Dv ořák, Z.; Postle, L. Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8. *J. Combin. Theory Ser. B* **2018**, *129*, 38–54. [CrossRef]
2. Bernshteyn, A.; Kostochka, A. On the differences between DP-coloring and list coloring. *Sib. Adv. Math.* **2019**, *29*, 183–189. [CrossRef]
3. Beaudou, L.; Foucaud, F.; Naserasr, R. Homomorphism bounds and edge-colourings of K_4-minor-free graphs. *J. Combin. Theory Ser. B* **2017**, *124*, 128–164. [CrossRef]
4. Duffin, R.J. Topology of series-parallel networks. *J. Math. Anal. Appl.* **1965**, *10*, 303–318. [CrossRef]
5. Wang, T. Strong chromatic index of *k*-degenerate graphs. *Discret. Math.* **2014**, *330*, 17–19. [CrossRef]
6. Juvan, M.; Mohar, B.; Thomas, R. List edge colorings of series-parallel graphs. *Electron. J. Combin.* **1999**, *6*, R42. [CrossRef] [PubMed]
7. Häggkvist, R.; Chetwynd, A. Some upper bounds on the total and list chromatic numbers of multigraphs. *J. Graph Theory* **1992**, *16*, 503–516. [CrossRef]
8. Jensen, T.R.; Toft, B. *Graph Coloring Problems*; Wiley-Interscience: New York, NY, USA, 1995.
9. Galvin, F. The list chromatic index of a bipartite multigraph. *J. Combin. Theory Ser. B* **1995**, *63*, 153–158. [CrossRef]
10. Häggkvist, R.; Janssen, J. New bounds on the list-chromatic index of complete graph and other simple graphs. *Combin. Probab. Comput.* **1997**, *6*, 295–313. [CrossRef]
11. Schauz, U. Proof of the list edge coloring conjecture for complete graphs of prime degree. *Electron. J. Combin.* **2014**, *21*, 3–43. [CrossRef] [PubMed]
12. Borodin, O.V. A generalization of Kotzig's theorem and prescribed edge coloring of planar graphs. *Mat. Zametki* **1990**, *48*, 22–28.
13. Bonamy, M. Planar graphs with $\Delta \geq 8$ are $(\Delta + 1)$-edge-choosable. *SIAM J. Discret. Math.* **2015**, *29*, 1735–1763. [CrossRef]
14. Kim, S.; Ozeki, K. A sufficient condition for DP-4-colorability. *Discret. Math.* **2018**, *341*, 1983–1986. [CrossRef]
15. Li, X.; Lv, J.; Zhang, M. DP-4-colorability of planar graphs without intersecting 5-cycles. *Discret. Math.* **2022**, *345*, 112790. [CrossRef]
16. Zhang, L.; Lu, Y.; Zhang, S. Edge DP-coloring in planar graphs. *Discret. Math.* **2021**, *344*, 112314. [CrossRef]