axioms

Article

Fuzzy Testing Model Built on Confidence Interval of Process Capability Index C_{PMK}

Wei Lo [1], Tsun-Hung Huang [2], Kuen-Suan Chen [2,3,4,*], Chun-Min Yu [2,*] and Chun-Ming Yang [5]

[1] School of Business Administration, Guangxi University of Finance and Economics, Nanning 530007, China; 2018120010@gxufe.edu.cn
[2] Department of Industrial Engineering and Management, National Chin-Yi University of Technology, Taichung 411030, Taiwan; toby@ncut.edu.tw
[3] Department of Business Administration, Chaoyang University of Technology, Taichung 413310, Taiwan
[4] Department of Business Administration, Asia University, Taichung 413305, Taiwan
[5] School of Economics and Management, Dongguan University of Technology, Dongguan 523808, China; 2020812@dgut.edu.cn
* Correspondence: kschen@ncut.edu.tw (K.-S.C.); march@ncut.edu.tw (C.-M.Y.)

Abstract: A variety of process capability indices are applied to the quantitative measurement of the potential and performance of processes in manufacturing. As it is easy to understand the formulae of these indices, this method is easy to apply. Furthermore, a process capability index is frequently utilized by a manufacturer to gauge the quality of a process. This index can be utilized by not only an internal process engineer to assess the quality of the process but also as a communication tool for an external sales department. When the manufacturing process deviates from the target value T, the process capability index C_{PMK} can be quickly detected, which is conducive to the promotion of smart manufacturing. Therefore, this study applied the index C_{PMK} as an evaluation tool for process quality. As noted by some studies, process capability indices have unknown parameters and therefore must be estimated from sample data. Additionally, numerous studies have addressed that it is essential for companies to establish a rapid response mechanism, as they wish to make decisions quickly when using a small sample size. Considering the small sample size, this study proposed a 100 $(1-\alpha)\%$ confidence interval for the process capability index C_{PMK} based on suggestions from previous studies. Subsequently, this study built a fuzzy testing model on the 100 $(1-\alpha)\%$ confidence interval for the process capability index C_{PMK}. This fuzzy testing model can help enterprises make decisions rapidly with a small sample size, meeting their expectation of having a rapid response mechanism.

Keywords: process capability indices; unknown parameters; confidence interval; fuzzy testing model; mathematical programming method

MSC: 62C05; 62C86

check for updates

Citation: Lo, W.; Huang, T.-H.; Chen, K.-S.; Yu, C.-M.; Yang, C.-M. Fuzzy Testing Model Built on Confidence Interval of Process Capability Index C_{PMK}. *Axioms* **2024**, *13*, 379. https://doi.org/10.3390/axioms13060379

Academic Editors: Ta-Chung Chu, Wei-Chang Yeh and Salvatore Sessa

Received: 14 April 2024
Revised: 30 May 2024
Accepted: 2 June 2024
Published: 4 June 2024

1. Introduction

A number of process capability indices are unitless, measuring items produced in processes to determine whether they can achieve the quality level required by the product designer [1–3]. Indeed, process capability indices are common tools utilized by companies to gauge process quality. They can be offered to internal process engineers to assess process quality as well as viewed as communication tools for sales departments in external companies [4–7]. The two most widely used capability indices, C_P and C_{PK}, as Kane [8] suggested, are displayed below:

$$C_P = \frac{USL - LSL}{6\sigma} = \frac{d}{3\sigma} \quad (1)$$

and

$$C_{PK} = Min\left\{\frac{USL-\mu}{3\sigma}, \frac{\mu-LSL}{3\sigma}\right\} = \frac{d-|\mu-M|}{3\sigma}. \quad (2)$$

In the above equations, USL denotes the Upper Specification Limit, LSL denotes the Lower Specification Limit, μ refers to the process mean, σ refers to the process standard deviation, and $M = (LSL + USL)/2$ refers to the midpoint of the specification interval (LSL, USL). Boyles [9] noted that indices C_P and C_{PK} are based on yields and independent of the target T. Accordingly, they may fail to explain process centering, which refers to the capability of gathering data around the target. To tackle this problem, Chan, Cheng, and Spiring [10] came up with the Taguchi capability index C_{PM}, as presented below:

$$C_{PM} = \frac{USL-LSL}{6\sqrt{\sigma^2+(\mu-T)^2}} = \frac{d}{3\sqrt{\sigma^2+(\mu-T)^2}}, \quad (3)$$

where T represents the target value. $(\mu-T)^2+\sigma^2 = E(X-T)^2$ refers to the expected loss function of Taguchi. Considering the Taguchi capability index C_{PM}, Pearn, Kotz, and Johnson [11] took the following example with $T = \{3(USL) + (LSL)\}/4$ and $\sigma = d/3$. Process A, with $\mu_A = T - d/2 = m$, and process B, with $\mu_B = T + d/2 = USL$, both yielded the same result $- C_{PM} = 0.555$. Nonetheless, the expected non-conforming proportions were approximately 0.27% and 50%, respectively. We can tell, in this case, that the Taguchi capability index C_{PM} measures process capability inconsistently. To overcome the problem, the process capability index C_{PMK} proposed by Choi and Owen [12] is employed to handle the processes with asymmetric tolerances. For symmetrical tolerances, the index C_{PMK} is expressed as follows:

$$C_{PMK} = \frac{d-|\mu-T|}{3\sqrt{\sigma^2+(\mu-T)^2}}. \quad (4)$$

As noted by Vännman [13], the rankings of indices C_P, C_{PK}, C_{PM}, and C_{PMK} are in the following order: (1) C_{PMK}, (2) C_{PM}, (3) C_{PK}, and (4) C_P, based on their sensitivity to the departure of the process mean μ from the target value T. These four process capability indices are favorable for processes with quality characteristics of the nominal-the-better (NTB) type. Among them, C_{PMK} integrates the numerator of index C_{PK} with the denominator of index C_{PM}. The deviation can thus be detected quickly as the manufacturing process deviates from the target value T, which helps promote smart manufacturing. Therefore, this paper utilizes the process capability index C_{PMK} as an evaluation tool for process quality. As noted by some studies, process capability indices have unknown parameters and therefore must be estimated from sample data [14]. In addition, as highlighted by many studies, companies typically seek a rapid response mechanism, enabling them to make decisions quickly while utilizing a small sample size [15,16]. If decisions, however, are made based on a small number of samples, there will be a risk of misjudgment due to sampling error. Given the case of small sample size, this paper follows some suggestions from previous studies and derives a 100 $(1-\alpha)$% confidence interval for the process capability index C_{PMK}. Next, building upon the 100 $(1-\alpha)$% confidence interval for the process capability index C_{PMK}, this paper develops a fuzzy testing model. This model, on the basis of the confidence interval, helps enterprises make quick decisions with a small sample size, fulfilling their need for a rapid response mechanism.

As noted by various studies, machine tools made in Taiwan won first place worldwide in terms of output value and sales volume. They are mainly sold to emerging markets in Southeast Asia and Eastern Europe. The central region of Taiwan is an industrial center for precision machinery and machine tools. It combines machine tool parts factories, aerospace, and medical industries, and connects parts processing and maintenance industries, forming a large cluster of machine tools and machinery industries [17,18]. Additionally, several studies have indicated that the high clustering effect of the machine tool industry in Taiwan has enabled central Taiwan to develop a robust industry chain for machine tools;

therefore, Taiwan plays a vital role in the world machine tool industry [19]. In view of this, we demonstrate how to implement the proposed fuzzy evaluation model using an axis produced by a machining factory in the central region of Taiwan.

In this paper, we organize the remaining sections as follows. In the Section 2, we demonstrate how to derive the Maximum Likelihood Estimator (*MLE*) as well as 100 $(1 - \alpha)\%$ confidence regions for the process mean and the process standard deviation, respectively. This study utilizes the process capability index C_{PMK} as the object function and adopts the 100 $(1 - \alpha)\%$ confidence regions for the process mean and regions for the process standard deviation as the feasible solution areas. Subsequently, we apply mathematical programming to find a 100 $(1 - \alpha)\%$ confidence interval for the process capability index C_{PMK}. In the Section 3, we develop a fuzzy testing model using the 100 $(1 - \alpha)\%$ confidence interval of the process capability index C_{PMK} to measure the process quality, so as to learn whether it reaches the required quality level. In this model, we first derive a triangular fuzzy number and then obtain its membership function. Next, based on fuzzy testing rules, we can determine whether the process quality satisfies the requirement, which can serve as a reference for other industries. As mentioned before, central Taiwan is an industrial center for machine tools. Therefore, in the Section 4, an axis manufactured by a machining factory in the central region of Taiwan is used as an empirical example to illustrate how to apply the proposed fuzzy testing model. In the Section 5, conclusions are presented.

2. Confidence Interval for Process Capability Index C_{PMK}

A random variable, denoted with X, has a normal distribution with the mean (μ) and the standard deviation (σ). Let $(X_1, X_2, \ldots, Xn)$ be a random sample received from a normal process. Then the Maximum Likelihood Estimators (*MLEs*) of the process mean (μ) and the process standard deviation (σ) are written in Equation (5) and Equation (6), respectively:

$$\mu^* = \frac{1}{n}\sum_{i=1}^{n} X_i, \tag{5}$$

and

$$\sigma^* = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}. \tag{6}$$

Furthermore, the estimator of the process capability index C_{PMK} is denoted by

$$C^*_{PMK} = \frac{d - |\mu^* - T|}{3\sqrt{\sigma^{*2} + (\mu^* - T)^2}}. \tag{7}$$

Let random variables be $Z = \sqrt{n}(\mu^* - \mu)/\sigma$ and $K = n\sigma^{*2}/\sigma^2$. As normality is assumed, μ^* and σ^{*2} are mutually independent, and so are random variables Z and K [20]. The random variable Z is denoted as $Z \sim N(0,1)$, following a normal distribution, while the random variable K, denoted with χ^2_{n-1}, represents a chi-squared distribution including $n - 1$ degrees of freedom. Therefore, we have

$$p\left(-Z_{0.5-\sqrt{1-\alpha}/2} \le \sqrt{n}(\mu^* - \mu)/\sigma \le Z_{0.5-\sqrt{1-\alpha}/2}\right) = \sqrt{1-\alpha} \tag{8}$$

and

$$p\left(\chi^2_{0.5-\sqrt{1-\alpha}/2;n-1} \le n\sigma^{*2}/\sigma^2 \le \chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}\right) = \sqrt{1-\alpha}, \tag{9}$$

where $Z_{0.5-\sqrt{1-\alpha}/2}$ is the upper $0.5 - \sqrt{1-\alpha}/2$ quintile of the standard normal distribution, $\chi^2_{0.5-\sqrt{1-\alpha}/2;n-1}$ is the lower $0.5 - \sqrt{1-\alpha}/2$ quintile of χ^2_{n-1}, and $\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}$ is the lower $0.5 + \sqrt{1-\alpha}/2$ quintile of χ^2_{n-1}. Thus, we can further obtain $p(A) = 1 - \alpha$, where

$$A = p\left\{\mu^* - e\times\sigma \le \mu \le \overline{X} + e\times\sigma,\ \sqrt{\frac{n}{\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}}}\sigma^* \le \sigma \le \sqrt{\frac{n}{\chi^2_{0.5-\sqrt{1-\alpha}/2;n-1}}}\sigma^*\right\}. \tag{10}$$

where we have $e = Z_{0.5-\sqrt{1-\alpha}/2}/\sqrt{n}$. In the random sample $(X_1, X_2, \ldots, X_n)$, the observed values are written as $(x_1, x_2, \ldots, x_n)$. Let the observed values of μ^* and σ^* be μ_0^* and σ_0^*, expressed as follows:

$$\mu_0^* = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{11}$$

and

$$\sigma_0^* = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(x_i - \mu_0^*\right)^2}. \tag{12}$$

Then the observed value for the estimator C^*_{PMK} is denoted by

$$C^*_{PMK0} = \frac{d - \left|\mu_0^* - T\right|}{3\sqrt{\sigma_0^{*2} + (\mu_0^* - T)^2}}. \tag{13}$$

Furthermore, the 100 $(1-\alpha)$% confidence region of (μ,σ), denoted with $CR(\mu,\sigma)$, is written as follows:

$$CR(\mu,\sigma) = \{(\mu,\sigma)|\mu_0^* - e\times\sigma \le \mu \le \mu_0^* + e\times\sigma,\ \sigma_L \le \sigma \le \sigma_U\}, \tag{14}$$

where we have $\sigma_L = \sigma_0^*\sqrt{n/\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}}$ and $\sigma_U = \sigma_0^*\sqrt{n/\chi^2_{0.5-\sqrt{1-\alpha}/2;n-1}}$. Chen [14] thinks that since the index C_{PMK} is a function of (μ,σ), then the probability of (μ,σ) belonging to $CR(\mu,\sigma)$ is as high as $1-\alpha$. Thus, in this paper, the process capability index C_{PMK} was employed as an object function while the $1-\alpha$ confidence region of (μ,σ) was used as a feasible solution area. Therefore, when $p\{(\mu,\sigma) \in CR(\mu,\sigma)\} \ge 1-\alpha$, then we have $p\{LC_{PMK} \le C_{PMK} \le UC_{PMK}\} \ge 1-\alpha$. Accordingly, the upper confidence limit for the process capability index C_{PMK} is defined in the model of mathematical programming as

$$\begin{cases} UC_{PMK} = Max\ C_{PMK}(\mu,\sigma) \\ subject\ to \\ \mu_0^* - e\times\sigma \le \mu \le \mu_0^* + e\times\sigma \\ \sigma_L \le \sigma \le \sigma_U \end{cases}. \tag{15}$$

For any process standard deviation, when σ is bigger than or equal to σ_L $(\sigma \ge \sigma_L)$, then we have $C_{PMK}(\mu,\sigma) \le C_{PMK}(\mu,\sigma_L)$. Therefore, the mathematical programming model of Equation (15) can be rewritten as below:

$$\begin{cases} UC_{PMK} = Max\ C_{PMK}(\mu,\sigma_L) \\ subject\ to \\ \mu_0^* - e\times\sigma_L \le \mu \le \mu_0^* + e\times\sigma_L \end{cases}. \tag{16}$$

Similarly, for any process standard deviation, when σ is smaller than or equal to σ_U $(\sigma \le \sigma_U)$, then we have $C_{PMK}(\mu,\sigma)\ C_{PMK}(\mu,\sigma_U)$. Therefore, the lower confidence limit for the process capability index C_{PMK} in the mathematical programming model is displayed as follows:

$$\begin{cases} LC_{PMK} = Min\ C_{PMK}(\mu,\sigma_U) \\ subject\ to \\ \mu_0^* - e\times\sigma_U \le \mu \le \mu_0^* + e\times\sigma_U \end{cases}. \tag{17}$$

Based on the above, this article proposes a process to explain how to use a mathematical programming method to solve the 100 $(1-\alpha)$% upper confidence limit and lower confidence limit of the index C_{PMK}. Then, a confidence interval-based fuzzy testing method was developed using Chen's method [14]. The development process of this fuzzy test is as follows:

Step 1: Expressing the index as a function of μ mean and standard deviation σ is as follows:

$$C_{PMK}(\mu,\sigma) = \frac{d-|\mu - T|}{3\sqrt{\sigma^2 + (\mu - T)^2}}. \tag{18}$$

Step 2: Derive the 100 $(1-\alpha)$% confidence region of (μ,σ) as shown in Equation (14).

Step 3: Taking $C_{PMK}(\mu,\sigma)$ as the objective function and $CR(\mu,\sigma)$ as the feasible solution area, the maximum value UC_{PMK} and the minimum value LC_{PMK} are respectively obtained as shown in Equations (16) and (17).

Step 4: According to the confidence interval of index C_{PMK}, the resemble triangular fuzzy number and its membership function are constructed as shown in Equation (38). Then, develop a confidence interval-based fuzzy test method.

Next, we derive the 100 $(1-\alpha)$% confidence interval for the process capability index C_{PMK} based on Case 1$\mu_0^* - e \times \sigma_U \le T \le \mu_0^* + e \times \sigma_U$, Case 2 $T < \mu_0^* - e \times \sigma_U$, and Case 3 $\mu_0^* + e \times \sigma_U < T$, respectively, as follows:

Case 1: $\mu_0^* - e \times \sigma_U \le T \le \mu_0^* + e \times \sigma_U$

In this case, we find $\mu = T$ and process capability index $C_{PMK} = d/(3\sigma)$. According to Equations (16) and (17), the 100 $(1-\alpha)$% confidence interval for the process capability index C_{PMK} is $[LC_{PMK}, UC_{PMK}]$, where

$$LC_{PMK} = \frac{d}{3\sigma_U} = C_{PMK0}^* \times \sqrt{\frac{\chi^2_{0.5-\sqrt{1-\alpha}/2;n-1}}{n}}; \tag{19}$$

$$UC_{PMK} = \frac{d}{3\sigma_L} = C_{PMK0}^* \times \sqrt{\frac{\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}}{n}}. \tag{20}$$

Case 2: $T < \mu_0^* - e \times \sigma_U$

In this case, for any $\mu \le \mu_0^* + e \times \sigma_U$, we have $C_{PMK}(\mu,\sigma_U) \ge C_{PMK}(\mu_0^* + e \times \sigma_U, \sigma_U)$. Based on Equation (17), the lower confidence limit for the process capability index C_{PMK} is depicted below:

$$LC_{PMK} = \frac{d - (\mu_0^* + e \times \sigma_U - T)}{3\sqrt{\sigma_U^2 + (\mu_0^* + e \times \sigma_U - T)^2}}. \tag{21}$$

Similarly, for any $\mu_0^* - e \times \sigma_L \le \mu$, we have $C_{PMK}(\mu,\sigma_L) \le C_{PMK}(\mu_0^* - e \times \sigma_L, \sigma_L)$. Based on Equation (16), the upper confidence limit for the process capability index C_{PMK} is shown as follows:

$$UC_{PMK} = \frac{d - (\mu_0^* - e \times \sigma_L - T)}{3\sqrt{\sigma_L^2 + (\mu_0^* - e \times \sigma_L - T)^2}}. \tag{22}$$

Case 3: $\mu_0^* + e \times \sigma_U < T$

In this case, for any $\mu_0^* - e \times \sigma_U \le \mu$, we have $C_{PMK}(\mu,\sigma_U) \ge C_{PMK}(\mu_0^* - e \times \sigma_U, \sigma_U)$. Based on Equation (17), the lower confidence limit for the process capability index C_{PMK} is displayed below:

$$LC_{PMK} = \frac{d - (T - (\mu_0^* - e \times \sigma_U))}{3\sqrt{\sigma_U^2 + \left(T - (\mu_0^* - e \times \sigma_U)\right)^2}}. \tag{23}$$

Similarly, for any $\mu \le \mu_0^* + e \times \sigma_L$, we have $C_{PMK}(\mu,\sigma_L) \le C_{PMK}(\mu_0^* + e \times \sigma_L, \sigma_L)$. Based on Equation (16), the upper confidence limit of the process capability index C_{PMK} can be depicted as follows:

$$UC_{PMK} = \frac{d - (T - (\mu_0^* + e \times \sigma_L))}{3\sqrt{\sigma_L^2 + (T - (\mu_0^* + e \times \sigma_L))^2}}. \quad (24)$$

Based on the above three cases, this paper builds a method for fuzzy testing upon the confidence interval for the process capability index C_{PMK}.

3. Fuzzy Testing Model Based on Confidence Interval of Process Capability Index C_{PMK}

As mentioned above, in pursuit of a rapid response mechanism, companies usually operate with a small sample size. Following several suggestions from previous studies, in this paper, we constructed a fuzzy testing method on the basis of the confidence interval for the process capability index C_{PMK}, given a small sample size. Pearn and Chen [21] defined the levels required by the process capability indices in the following table.

To identify whether the process capability index C_{PMK} is greater than or equal to C, the null hypothesis, denoted with H_0, and the alternative hypothesis, denoted with H_1, for fuzzy testing are stated below:

H_0: $C_{PMK} \geq C$ (indicating the process capability has achieved the desired level);

H_1: $C_{PMK} < C$ (indicating the process capability has not achieved the desired level).

Customers or process engineers can propose the required value C corresponding to the process capability index C_{PMK} with reference to Table 1. Based on the statistical testing rules mentioned above and Chen's method [14], this paper builds the fuzzy testing model upon the observed values for the estimator and the 100 $(1 - \alpha)$% confidence interval of the process capability index C_{PMK}. According to Chen and Lin [22], the α—cuts of the triangular fuzzy number $\widetilde{C}_{PMK}$ can be written as follows:

$$\widetilde{C}_{PMK}[\alpha] = \begin{cases} [C_{PMK1}(\alpha), C_{PMK2}(\alpha)], \ for\ 0.01 \leq \alpha \leq 1 \\ [C_{PMK1}(0.01), C_{PMK2}(0.01)], \ for\ 0 \leq \alpha \leq 0.01 \end{cases}. \quad (25)$$

Table 1. The levels required by the process capability indices.

Required Level	Capability Index Value
Inadequate	$C_{PMK} < 1.00$
Capable	$1.00 \leq C_{PMK} < 1.33$
Satisfactory	$1.33 \leq C_{PMK} < 1.50$
Excellent	$1.50 \leq C_{PMK} < 2.00$
Superb	$2.00 \leq C_{PMK}$

As mentioned earlier, this paper derived the 100 $(1 - \alpha)$% confidence interval for the process capability index C_{PMK} from Case 1 $\mu_0^* - e \times \sigma_U \leq T \leq \mu_0^* + e \times \sigma_U$, Case 2 $T < \mu_0^* - e \times \sigma_U$, and Case 3 $\mu_0^* + e \times \sigma_U < T$. According to these three cases, C_{PMK1} (α) and C_{PMK2} (α) can be depicted separately as follows.

Case 1: $\mu_0^* - e \times \sigma_U \leq T \leq \mu_0^* + e \times \sigma_U$

In this case, the process capability index C_{PMK} is represented as

$$C_{PMK} = \frac{d}{3\sigma}. \quad (26)$$

The observed values of the estimator C_{PMK}^* is represented as

$$C_{PMK0}^* = \frac{d}{3\sigma_0^*}. \quad (27)$$

Based on the above equations, Equations (19) and (20), $C_{PMK1}(\alpha)$ and $C_{PMK2}(\alpha)$ are expressed as follows:

$$C_{PMK1}(\alpha) = C^*_{PMK0} \times \sqrt{\frac{\chi^2_{0.5-\sqrt{1-\alpha}/2;n-1}}{n}}, \tag{28}$$

$$C_{PMK2}(\alpha) = C^*_{PMK0} \times \sqrt{\frac{\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}}{n}}. \tag{29}$$

Case 2: $T < \mu_0^* - e \times \sigma_U$

The process capability index C_{PMK} is represented as

$$C_{PMK} = \frac{d - (\mu - T)}{3\sqrt{\sigma^2 + (\mu - T)^2}}. \tag{30}$$

The observed value of the estimator C^*_{PMK} is represented as

$$C^*_{PMK0} = \frac{d - (\mu_0^* - T)}{3\sqrt{\sigma_0^{*2} + (\mu_0^* - T)^2}}. \tag{31}$$

Based on the above equations, Equations (21) and (22), $C_{PMK1}(\alpha)$, and $C_{PMK2}(\alpha)$ are expressed as follows:

$$e = Z_{0.5-\sqrt{1-\alpha}/2}/\sqrt{n},\ \sigma_L = \sigma_0^* \sqrt{n/\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}},\ \text{and } \sigma_U = \frac{n\sigma_0^{*2}}{\chi^2_{0.5-\sqrt{1-\alpha}/2;n-1}}$$

$$C_{PMK1}(\alpha) = \frac{d - \left(\mu_0^* + \sigma_0^* \sqrt{\frac{Z_{0.5-\sqrt{1-\alpha}/2}}{\chi^2_{0.5-\sqrt{1-\alpha}/2;n-1}}} - T\right)}{3\sqrt{\frac{n\sigma_0^{*2}}{\chi^2_{0.5-\sqrt{1-\alpha}/2;n-1}} + \left(\mu_0^* + \sigma_0^* \sqrt{\frac{Z_{0.5-\sqrt{1-\alpha}/2}}{\chi^2_{0.5-\sqrt{1-\alpha}/2;n-1}}} - T\right)^2}} \tag{32}$$

$$C_{PMK2}(\alpha) = \frac{d - \left(\mu_0^* - \sigma_0^* \sqrt{\frac{Z_{0.5-\sqrt{1-\alpha}/2}}{\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}}} - T\right)}{3\sqrt{\frac{n\sigma_0^{*2}}{\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}} + \left(\mu_0^* - \sigma_0^* \sqrt{\frac{Z_{0.5-\sqrt{1-\alpha}/2}}{\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}}} - T\right)^2}} \tag{33}$$

Case 3: $\mu_0^* + e \times \sigma_U < T$

In this case, the process capability index C_{PMK} is defined as:

$$C_{PMK} = \frac{d - (\mu - T)}{3\sqrt{\sigma^2 + (\mu - T)^2}}. \tag{34}$$

The observed values of the estimator C^*_{PMK} is defined as:

$$C^*_{PMK0} = \frac{d - (\mu_0^* - T)}{3\sqrt{\sigma_0^{*2} + (\mu_0^* - T)^2}}. \tag{35}$$

Based on the above equations, Equations (23) and (24), $C_{PMK1}(\alpha)$ and $C_{PMK2}(\alpha)$ are derived as follows:

$$C_{PMK1}(\alpha) = \frac{d - \left(T - \left(\mu_0^* - \sigma_0^* \sqrt{\frac{Z_{0.5-\sqrt{1-\alpha}/2}}{\chi^2_{0.5-\sqrt{1-\alpha}/2;n-1}}}\right)\right)}{3\sqrt{\frac{n\sigma_0^{*2}}{\chi^2_{0.5-\sqrt{1-\alpha}/2;n-1}} + \left(T - \left(\mu_0^* - \sigma_0^* \sqrt{\frac{Z_{0.5-\sqrt{1-\alpha}/2}}{\chi^2_{0.5-\sqrt{1-\alpha}/2;n-1}}}\right)\right)^2}} \tag{36}$$

and

$$C_{PMK2}(\alpha) = \frac{d - \left(T - \left(\mu_0^* + \sigma_0^* \sqrt{\frac{Z_{0.5-\sqrt{1-\alpha}/2}}{\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}}}\right)\right)}{3\sqrt{\frac{n\sigma_0^{*2}}{\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}} + \left(T - \left(\mu_0^* + \sigma_0^* \sqrt{\frac{Z_{0.5-\sqrt{1-\alpha}/2}}{\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}}}\right)\right)^2}}. \quad (37)$$

Therefore, we have the resemble triangular fuzzy number, $\widetilde{C}_{PMK} = \Delta\,(K_L, K_M, K_R)$, where $K_L = C_{PMK1}(0.01)$, $K_M = C_{PMK1}(1) = C_{PMK2}(1)$ and $K_R = C_{PMK2}(0.01)$. Its membership function is expressed as follows:

$$h(x) = \begin{cases} 0 & if\ x < K_L \\ \alpha_1 & if\ K_L \le x < K_M \\ 1 & if\ x = K_M \\ \alpha_2 & if\ K_M < x \le K_R \\ 0 & if\ x > K_R \end{cases}, \quad (38)$$

where α_1 is determined by $C_{PMK1}(\alpha_1) = x$ and α_2 is determined by $C_{PMK2}(\alpha_2) = x$. Before proposing the fuzzy testing method for the process capability index C_{PMK}, we reviewed the following statistical testing rules:

(1) When the upper confidence limit of the process capability index C_{PMK} exceeds or equals C ($UC_{PMK} \ge C$), do not reject H_0 and conclude that $C_{PMK} \ge C$.
(2) When the upper confidence limit of the process capability index C_{PMK} is smaller than C ($UC_{PMK} < C$), reject H_0 and conclude that $C_{PMK} < C$.

Then, this paper developed a fuzzy testing method, considering the confidence interval for the process capability index C_{PMK} based on statistical testing rules. According to Equation (36), for the fuzzy number $\widetilde{C}_{PMK}$, its membership function $h(x)$ is presented with the vertical line $x = C$ in Figure 1.

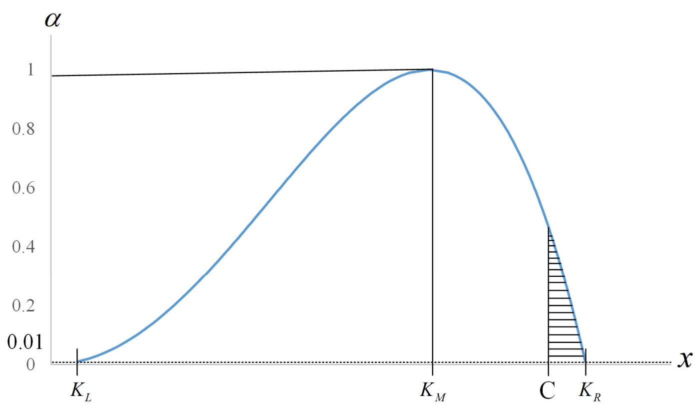

Figure 1. Membership function *h*(*x*) with vertical line *x* = *C*.

Based on Chen and Lin [22], let A_T denote the area in the diagram of membership function $h(x)$, and let A_R denote the area in the same graph but to the right of membership function $h(x)$ from the vertical line $x = C$, such that

$$A_T = \{(x,\ \alpha)|C_{PMK1}(\alpha) \le x \le C_{PMK2}(\alpha)\}. \quad (39)$$

and

$$A_R = \{(x,\ \alpha)|C \le x \le C_{PMK2}(\alpha)\}. \quad (40)$$

According to Yu et al. [23] and based on Equations (39) and (40), we let $d_T = K_R - K_L$ and $d_R = K_R - C$. Then, we have $d_R/d_T = (K_R - C)/(K_R - K_L)$. Also, we denote Case 1 as $\mu_0^* - e \times \sigma_U \le T \le \mu_0^* + e \times \sigma_U$, Case 2 as $T < \mu_0^* - e \times \sigma_U$, and Case 3 as $\mu_0^* + e \times \sigma_U < T$. More detailed explanations are listed below:

Case 1: $\mu_0^* - e \times \sigma_U \leq T \leq \mu_0^* + e \times \sigma_U$

$$d_R/d_T = \frac{K_R - C}{2(K_R - K_M)}, \tag{41}$$

where

$$K_R = C_{PMK0}^* \times \sqrt{\frac{\chi^2_{0.9975;n-1}}{n}} \tag{42}$$

and

$$K_M = C_{PMK0}^* \times \sqrt{\frac{\chi^2_{0.5;n-1}}{n}}. \tag{43}$$

Case 2: $T < \mu_0^* - e \times \sigma_U$

$$d_R/d_T = \frac{K_R - C}{2(K_R - K_M)}, \tag{44}$$

where

$$K_R = \frac{d - \left(\mu_0^* - \sigma_0^* \sqrt{\frac{Z_{0.0025}}{\chi^2_{0.9975;n-1}}} - T\right)}{3\sqrt{\frac{n\sigma_0^{*2}}{\chi^2_{0.9975;n-1}} + \left(\mu_0^* - \sigma_0^* \sqrt{\frac{Z_{0.0025}}{\chi^2_{0.9975;n-1}}} - T\right)^2}} \tag{45}$$

and

$$K_M = \frac{d - \left(\mu_0^* + \sigma_0^* \sqrt{\frac{Z_{0.0025}}{\chi^2_{0.5;n-1}}} - T\right)}{3\sqrt{\frac{n\sigma_0^{*2}}{\chi^2_{0.5;n-1}} + \left(\mu_0^* + \sigma_0^* \sqrt{\frac{Z_{0.0025}}{\chi^2_{0.5;n-1}}} - T\right)^2}}. \tag{46}$$

Case 3: $\mu_0^* + e \times \sigma_U < T$

$$d_R/d_T = \frac{K_R - C}{2(K_R - K_M)}, \tag{47}$$

where

$$K_R = \frac{d - \left(T - \left(\mu_0^* + \sigma_0^* \sqrt{\frac{Z_{0.0025}}{\chi^2_{0.9975;n-1}}}\right)\right)}{3\sqrt{\frac{n\sigma_0^{*2}}{\chi^2_{0.9975;n-1}} + \left(T - \left(\mu_0^* + \sigma_0^* \sqrt{\frac{Z_{0.0025}}{\chi^2_{0.9975;n-1}}}\right)\right)^2}} \tag{48}$$

and

$$K_M = \frac{d - \left(T - \left(\mu_0^* - \sigma_0^* \sqrt{\frac{Z_{0.0025}}{\chi^2_{0.5;n-1}}}\right)\right)}{3\sqrt{\frac{n\sigma_0^{*2}}{\chi^2_{0.5;n-1}} + \left(T - \left(\mu_0^* - \sigma_0^* \sqrt{\frac{Z_{0.0025}}{\chi^2_{0.5;n-1}}}\right)\right)^2}}. \tag{49}$$

Based on Yu et al. [23], we let $0 < \phi \leq 0.5$, then the fuzzy testing rules of the process capability index C_{PMK} are shown as follows:

(1) If $d_R/d_T \leq \phi$, then H_0 is rejected, and we can conclude that $C_{PMK} < C$.
(2) If $d_R/d_T > \phi$, then H_0 is not rejected, and we can conclude that $C_{PMK} \geq C$.

4. An Application Example

It is known that the central region of Taiwan is a machine tool center. Therefore, this section of this paper demonstrates how to apply the proposed fuzzy testing model through an empirical example involving the axis machined by a manufacturer in central Taiwan. The fuzzy testing model built on the confidence interval of the process capability index C_{PMK} is an effective approach for deciding whether the process capability is acceptable or requires improvement. The target value T of the axis machined by the factory is 1.80 mm (T = 1.80), and the tolerance is 0.05 mm. Accordingly, the lower specification limit (LSL) is 1.75 mm (LSL = 1.75) and the upper specification limit (USL) is 1.85 mm (USL = 1.85). Thus, $T = (USL + LSL)/2$ and $d = (USL - LSL)/2$. According to customer requirements, the process engineer sets the required value for the process capability index C_{PMK} as 1.00. Aiming to gauge whether the value of the process capability index C_{PMK} exceeds or equals 1.00, the null hypothesis (H_0) and the alternative hypothesis (H_1) for fuzzy testing are listed as follows:

H_0: $C_{PMK} \geq 1.00$ (showing that the process capability is sufficient);

H_1: $C_{PMK} < 1.00$ (showing that the process capability is insufficient).

As mentioned earlier, in pursuit of a quick response mechanism, companies often opt for a small sample size. Let (x_1, x_2, . . ., x_{16}) be the observed values for a random sample (X_1, X_2, . . ., X_{16}). Then the observed values of μ^* and σ^* are μ_0^* and σ_0^*, respectively, as shown below:

$$\mu_0^* = \frac{1}{16}\sum_{i=1}^{16} x_i = 1.083 \tag{50}$$

and

$$\sigma_0^* = \sqrt{\frac{1}{16}\sum_{i=1}^{16}(x_i - \mu_0^*)^2} = 0.022. \tag{51}$$

Thus,

$$\mu_0^* - e \times \sigma_U = 1.083 - 0.702 \times 0.044 = 1.793$$

and

$$\mu_0^* + e \times \sigma_U = 1.083 + 0.702 \times 0.044 = 1.834.$$

The target value T belongs to the interval (1.793, 1.834). Thus, the observed value of the estimator C^*_{PMK} is calculated as follows:

$$C^*_{PMK0} = \frac{d}{3\sigma_0^*} = 0.758. \tag{52}$$

Based on Equation (50), we obtain the following values of K_R and K_M:

$$K_R = C^*_{PMK0} \times \sqrt{\frac{\chi^2_{0.9975;n-1}}{n}} = 0.758 \times \sqrt{\frac{34.950}{16}} = 1.120 \tag{53}$$

and

$$K_M = C^*_{PMK0} \times \sqrt{\frac{\chi^2_{0.5;n-1}}{n}} = 0.758 \times \sqrt{\frac{14.399}{16}} = 0.717. \tag{54}$$

Thus, the value of d_R/d_T is calculated as follows:

$$d_R/d_T = \frac{K_R - C}{2(K_R - K_M)} = \frac{1.120 - 1.00}{2 \times (1.120 - 0.717)} = 0.15 \tag{55}$$

Based on Yu et al. [23], we let $0 < \phi \leq 0.5$ and reviewed the fuzzy testing rules of the process capability index C_{PMK}. We obtained the following result:

(1) If $d_R/d_T \leq \phi$, then H_0 is rejected, and we can conclude that $C_{PMK} < C$.

The process engineer, drawing from past professional experience, analyzed and set the value of ϕ to 0.2; that is, ϕ = 0.2. According to the above fuzzy testing rule, since d_R/d_T is less than 0.2, the null hypothesis H_0 is rejected, and the conclusion $C_{PMK} < 1.00$ is drawn. In fact, the observed value of the estimator C^*_{PMK} is 0.758 ($C^*_{PMK0} = 0.758$), the upper confidence limit of the process capability index C_{PMK} is 1.120 ($UC_{PMK} = 1.120$) with $\alpha = 0.01$. If the result of the statistical inference shows that $C_{PMK} \geq 1.00$, it indicates that the proposed fuzzy testing model in this paper demonstrates greater practicality compared to the conventional statistical testing model.

5. Conclusions

Various process capability indices are applied to the quantitative measurement of the potential and performance of a process in the manufacturing industry. Not only can an internal process engineer use them to assess process quality, but an external sales department can also utilize them as a communication tool. The process capability index C_{PMK} can quickly detect process deviations from the target value, which is conducive to the promotion of smart manufacturing. Therefore, in this paper, we utilized the process capability index as a tool to evaluate process quality. Process capability indices, as noted by some studies, have unknown parameters and therefore must be estimated from sample data. In addition, as highlighted by many studies, companies typically pursue a rapid response mechanism, so they need to make decisions using a small sample size. Consequently, this study, based on some suggestions from previous studies for the case of small sample size, proposed the process capability index C_{PMK} with a 100 $(1 - \alpha)$% confidence interval. In the normal process condition where the sample mean and the sample variation are mutually independent, this study derived the 100 $(1 - \alpha)$% confidence region of (μ, σ). Then, this study adopted the process capability index C_{PMK} as an object function as well as the 100 $(1 - \alpha)$% confidence region of (μ, σ) as a feasible solution area, aiming to acquire the 100 $(1 - \alpha)$% confidence interval of the process capability index C_{PMK}. Immediately afterward, the 100 $(1 - \alpha)$% confidence interval of the process capability index C_{PMK} was utilized to establish a fuzzy testing model to evaluate process quality and see if it can achieve the required quality level. In this model, we first derived the triangular fuzzy number $\widetilde{C}_{PMK}$ and then obtained its membership function $h(x)$. According to the membership function $h(x)$, this study established fuzzy testing rules. Through these rules, we can tell if the process quality attains the required level, which can serve as a reference for other industries. As mentioned earlier, central Taiwan is an industrial center for machine tools. Accordingly, this study illustrated the use of the proposed fuzzy testing model with an example of the axis machined by a factory located in the central region of Taiwan. It is evident from this example that the proposed fuzzy testing model can exhibit greater practicality compared to the conventional statistical testing model.

Author Contributions: Conceptualization, W.L. and K.-S.C.; methodology, W.L., K.-S.C. and C.-M.Y. (Chun-Min Yu); software, T.-H.H.; validation, T.-H.H. and C.-M.Y. (Chun-Ming Yang); formal analysis, W.L., K.-S.C. and C.-M.Y. (Chun-Min Yu); resources, W.L.; data curation, C.-M.Y. (Chun-Ming Yang); writing—original draft preparation, W.L., T.-H.H., K.-S.C., C.-M.Y. (Chun-Min Yu) and C.-M.Y. (Chun-Ming Yang); writing—review and editing, W.L., K.-S.C. and C.-M.Y. (Chun-Min Yu); visualization, C.-M.Y. (Chun-Min Yu); supervision, K.-S.C.; project administration, W.L.; funding acquisition, W.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research was partially funded by the Guangxi Philosophy and Social Sciences Research Project under grant No. 23AGL001 and the National Natural Science Foundation of China under grant No. 72361002.

Data Availability Statement: The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Anderson, N.C.; Kovach, J.V. Reducing welding defects in turnaround projects: A lean six sigma case study. *Qual. Eng.* **2014**, *26*, 168–181. [CrossRef]
2. Gijo, E.V.; Scaria, J. Process improvement through Six Sigma with Beta correction: A case study of manufacturing company. *Int. J. Adv. Manuf. Technol.* **2014**, *71*, 717–730. [CrossRef]
3. Shafer, S.M.; Moeller, S.B. The effects of Six Sigma on corporate performance: An empirical investigation. *J. Oper. Manag.* **2012**, *30*, 521–532. [CrossRef]
4. Liao, M.Y.; Pearn, W.L. Modified weighted standard deviation index for adequately interpreting a supplier's lognormal process capability. *Proc. Inst. Mech. Eng. Part B-J. Eng. Manuf.* **2019**, *233*, 999–1008. [CrossRef]
5. Li, W.; Liu, G. Dynamic failure mode analysis approach based on an improved Taguchi process capability index. *Reliab. Eng. Syst. Saf.* **2022**, *218*, 108152. [CrossRef]
6. Shu, M.H.; Wang, T.C.; Hsu, B.M. Generalized quick-switch sampling systems indexed by Taguchi capability with record traceability. *Comput. Ind. Eng.* **2022**, *172*, 108577. [CrossRef]
7. Sanchez-Marquez, R.; Jabaloyes Vivas, J. Building a cpk control chart—A novel and practical method for practitioners. *Comput. Ind. Eng.* **2021**, *158*, 107428. [CrossRef]
8. Kane, V.E. Process capability indices. *J. Qual. Technol.* **1986**, *18*, 41–52. [CrossRef]
9. Boyles, R.A. The Taguchi Capability Index. *J. Qual. Technol.* **1991**, *23*, 17–26. [CrossRef]
10. Chan, L.K.; Cheng, S.W.; Spiring, F.A. A New Measure of Process Capability: C_{pm}. *J. Qual. Technol.* **1988**, *20*, 162–175. [CrossRef]
11. Pearn, W.L.; Kotz, S.; Johnson, N.L. Distributional and Inferential Properties of Process Capability Indices. *J. Qual. Technol.* **1992**, *24*, 216–231. [CrossRef]
12. Choi, B.C.; Owen, D.B. A study of a New Capability Index. *Commun. Stat.-Theory Methods* **1990**, *19*, 1231–1245. [CrossRef]
13. Vännman, K. A unified approach to capability indices. *Stat. Sin.* **1995**, *5*, 805–820.
14. Chen, K.S. Fuzzy testing decision-making model for intelligent manufacturing process with Taguchi capability index. *J. Intell. Fuzzy Syst.* **2020**, *38*, 2129–2139. [CrossRef]
15. Wu, C.H.; Hsu, Y.C.; Pearn, W.L. An improved measure of quality loss for notching processes. *Qual. Reliab. Eng. Int.* **2021**, *37*, 108–122. [CrossRef]
16. Arif, O.H.; Aslam, M.; Jun, C.H. Acceptance sampling plan for multiple manufacturing lines using EWMA process capability index. *J. Adv. Mech. Des. Syst. Manuf.* **2017**, *11*, JAMDSM0004. [CrossRef]
17. Chien, C.F.; Hong, T.Y.; Guo, H.Z. An empirical study for smart production for TFT-LCD to empower Industry 3.5. *J. Chin. Inst. Eng.* **2017**, *40*, 552–561. [CrossRef]
18. Wu, M.F.; Chen, H.Y.; Chang, T.C.; Wu, C.F. Quality evaluation of internal cylindrical grinding process with multiple quality characteristics for gear products. *Int. J. Prod. Res.* **2019**, *57*, 6687–6701. [CrossRef]
19. Lo, W.; Yang, C.M.; Lai, K.K.; Li, S.Y.; Chen, C.H. Developing a novel fuzzy evaluation model by one-sided specification capability indices. *Mathematics* **2021**, *9*, 1076. [CrossRef]
20. Chen, K.S.; Yu, C.M.; Huang, M.L. Fuzzy selection model for quality-based IC packaging process outsourcers. *IEEE Trans. Semicond. Manuf.* **2022**, *35*, 102–109. [CrossRef]
21. Pearn, W.L.; Chen, K.S. One-sided capability indices C_{pu} and C_{pl}: Decision making with sample information. *Int. J. Qual. Reliab. Manag.* **2002**, *19*, 221–245. [CrossRef]
22. Chen, H.Y.; Lin, K.P. Fuzzy supplier selection model based on lifetime performance index. *Expert Syst. Appl.* **2022**, *208*, 118135. [CrossRef]
23. Yu, C.M.; Lai, K.K.; Chen, K.S.; Chang, T.C. Process-quality evaluation for wire bonding with multiple gold wires. *IEEE Access* **2020**, *8*, 106075–106082. [CrossRef]