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The New G-Double-Laplace Transforms and One-Dimensional Coupled Sine-Gordon Equations

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Abstract: This paper establishes a novel technique, which is called the G-double-Laplace transform. This technique is an extension of the generalized Laplace transform. We study its properties with examples and various theorems related to the G-double-Laplace transform that have been addressed and proven. Finally, we apply the G-double-Laplace transform decomposition method to solve the nonlinear sine-Gordon and coupled sine-Gordon equations. This method is a combination of the G-double-Laplace transform and decomposition method. In addition, some examples are examined to establish the accuracy and effectiveness of this technique.

1. Introduction

The nonlinear partial differential equation models in many areas of mathematical physics perform an essential role in theoretical sciences. The sine-Gordon equation is one of the widely important nonlinear equations that plays a vital role in physical science and engineering. This equation has many applications such as relativistic field theory, the Josephson junction, and mechanical transmission lines. The nonlinear sine-Gordon equation (SGE) is a class of hyperbolic partial differential equations, which is usually utilized to explain the physical phenomena in several areas of engineering and science, for instance, nonlinear problems in physics and the diffusion of fluxions; see [1]. The sine-Gordon equation was first introduced in the course of study of various problems of differential geometry [2]. The approximate solution of the one-dimensional sine-Gordon equation was obtained by utilizing the combined Laplace transform method and VIM [3]. The author in [4] studied the two-dimensional SGE on Cartesian grids by improving an energy-conserving local discontinuous Galerkin method. The numerical solutions of the one-dimensional coupled nonlinear sine-Gordon equations are obtained by using the geometric meshless method [5]. Many techniques have been applied to obtain the approximate analytical solutions of sine-Gordon equations, such as the reduced differential transform method [6] and the modified Adomian decomposition method applied to obtain approximate Jacobi elliptic function solutions of the (N + 1)-dimensional sine-Gordon equation [7], and the authors in [8] used an analytical method called the Natural Decomposition Method (NDM) for solving the nonlinear sine-Gordon equation. The nonlinear coupled sine-Gordon equations were first presented in [9]. The author in [10] studied the numerical solution of the coupled sine-Gordon equation by applying the modified decomposition method. Recently, many authors have utilized diverse methods to obtain the solution of nonlinear coupled sine-Gordon equations; for example, the authors in [11] introduced rational exponential techniques to find the exact solutions of a coupled sine-Gordon equation; the authors in [12] discussed the solution of sine-Gordon and coupled sine-Gordon equations by using the

homotopy perturbation method. The author [13] employed the double-Sumudu transform iterative method for solving the one-dimensional coupled nonlinear sine-Gordon equation. The authors [14] applied the triple-Laplace transform method to obtain the solution of the two-dimensional coupled nonlinear sine-Gordon equation. In recent years, various mathematicians have applied the double-Laplace transform to solve different types of partial differential equations (see [15–17]). The generalized Laplace transform was introduced first in [18]; further properties of this transform are given in [19] and were subsequently utilized to solve certain nonlinear dynamical models with non-integer order in [20]. This work aims to introduce a new integral transform called the G-double-Laplace transform. This new integral transform is derived from the generalized Laplace transform. It is proposed with the decomposition method to facilitate the procedure of solving sine-Gordon and coupled sine-Gordon equations and to help researchers solve several problems related to physics and engineering in the future.

Here, we recall the definition of the generalized Laplace transform as follows.

Definition 1 ([18]). *Let $f(t)$ be an integrable function, for all $t \geq 0$. The generalized integral transform G_α of the function $f(t)$ is given by*

$$F(s) = G_\alpha(f) = s^\alpha \int_0^\infty f(t) e^{-\frac{t}{s}} dv, \tag{1}$$

for $s \in \mathbb{C}$ and $\alpha \in Z$.

2. Main Results

2.1. The Idea of G-Double-Laplace Transform

Here, we introduce some necessary definitions and theorems of the existence of G-double-Laplace transform, which are useful in this paper.

Definition 2. *Let $f(x,t)$ be an integrable function on $[0, \infty) \times [0, \infty)$. The G-double-Laplace transform of*
$f(x,t)$ is given by

$$G_x G_t[f(x,t)] = F(p,s) = p^\alpha s^\alpha \int_0^\infty \int_0^\infty e^{-\left(\frac{x}{p} + \frac{t}{s}\right)} f(x,t) \, dx \, dt, \tag{2}$$

where $\alpha \in \mathbb{Z}$, $p, s \in \mathbb{C}$, the symbol $G_x G_t$ indicates the transform of x and t, respectively, and the function $F(p,s)$ is denoted as the G-double Laplace transform of $f(x,t)$.

A prominent advantage of the G-double-Laplace transform is that it is easy to generate the following transformations:

1. If we put $\alpha = 0$, $s = \frac{1}{s}$, and $p = \frac{1}{p}$, we obtain the double-Laplace transform:

$$L_x L_t(f(x,t)) = F(p,s) = \int_0^\infty \int_0^\infty f(x,t) e^{-(px+st)} dt \, dx, \tag{3}$$

2. If we put $\alpha = 0$, $q = \frac{1}{p}$ and replace s with ϖ, we obtain the Laplace–Yang transform:

$$L_x Y(f(x,t)) = F(p,\varpi) = \int_0^\infty \int_0^\infty f(x,t) e^{-\left(qx + \frac{t}{\varpi}\right)} dt \, dx, \tag{4}$$

3. With $\alpha = -1$ and replacing p, s with u, v, respectively, we obtain the double-Sumudu transform:

$$S_x S_t(f(x,t)) = F(u,v) = \frac{1}{uv} \int_0^\infty \int_0^\infty f(x,t) e^{-\left(\frac{x}{u} + \frac{t}{v}\right)} dt dx. \tag{5}$$

The inverse G-double-Laplace transform (IDGLT) is

$$G_p^{-1} G_s^{-1}(F(p,s)) = f(x,t) = \frac{1}{(2\pi i)^2} \int_{\tau - i\infty}^{\tau + i\infty} \int_{\varsigma - i\infty}^{\varsigma + i\infty} e^{\frac{1}{\rho}x + \frac{1}{s}t} F(p,s) ds \, d\rho,$$

where $G_p^{-1} G_s^{-1}$ indicates the IDGLT.

The existence conditions of the G-double-Laplace transform of the function $f(x,t)$ are defined as follows. Assume that $f(x,t)$ is piecewise continuous on $[0, \infty) \times [0, \infty)$ and has an exponential order at infinity with

$$|f(x,t)| \le K e^{m_1 x + m_2 t}, \tag{6}$$

for $x > X, t > T$, where $K \ge 0$, and m_1, m_2, X, T are constants.

We write

$$f(x,t) = O(e^{m_1 x + m_2 t}) \text{ as } x \to \infty, \, t \to \infty$$

or similarly,

$$\lim_{\substack{x \to \infty \\ t \to \infty}} e^{-\frac{1}{\mu}x - \frac{1}{\lambda}t} |f(x,t)| = K \lim_{\substack{x \to \infty \\ t \to \infty}} e^{-\left(\frac{1}{\mu} - m_1\right)x - \left(\frac{1}{\lambda} - m_2\right)t} = 0, \tag{7}$$

whenever $\frac{1}{\mu} > m_1$ and $\frac{1}{\lambda} > m_2$. Such a function $f(x,t)$ is simply called an exponential order as $x \to \infty, t \to \infty$, and clearly, it does not grow faster than $K e^{m_1 x + m_2 t}$ as $x \to \infty, t \to \infty$.

Theorem 1. *If the function $f(x,t)$ is a continuous function defined on $(0, X)$ and $(0, T)$ with exponential order $e^{m_1 x + m_2 t}$, then the G-double-Laplace transform of $f(x,t)$ exists for all $\mathrm{Re}\left(\frac{1}{p}\right) > \frac{1}{\mu}$ and $\mathrm{Re}\left(\frac{1}{s}\right) > \frac{1}{\lambda}$.*

Proof. By placing Equation (2) into Equation (6), we obtain

$$
\begin{aligned}
|F(p,s)| &= \left| p^\alpha s^\alpha \int_0^\infty \int_0^\infty e^{-\left(\frac{x}{p} + \frac{t}{s}\right)} f(x,t) dx dt \right| \le K \left| e^{\left(m_1 - \frac{1}{p}\right)x + \left(m_2 - \frac{1}{s}\right)t} \right| \\
&= \frac{R p^{\alpha+1} s^{\alpha+1}}{(1 - m_1 p)(1 - m_2 s)}.
\end{aligned}
$$

Utilizing the condition $\mathrm{Re}\left(\frac{1}{p}\right) > \frac{1}{\mu}$ and $\mathrm{Re}\left(\frac{1}{s}\right) > \frac{1}{\lambda}$, we obtain

$$\lim_{\substack{p \to \infty \\ s \to \infty}} |F(p,s)| = 0 \text{ or } \lim_{\substack{p \to \infty \\ s \to \infty}} F(p,s) = 0,$$

$\square$

2.2. Some Properties of G-Double-Laplace Transform

In this part, several properties of the G-double-Laplace transform are explained and proven, such as the linearity property, the change of scale property, and the convolution theorem.

Linearity property: Consider $f(x,t)$ and $g(x,t)$ to be functions of the variables x and t, then

$$G_x G_t[af(x,t) + bg(x,t)] = aF(p,s) + bG(p,s).$$

Theorem 2. *Let $f(x,t)$ be a periodic function of periods a and b, where $f(x+a,t+b) = f(x,t)$ for all x and t, and if $F(p,s)$ exists, then*

$$F(p,s) = \frac{p^\alpha s^\alpha \int_0^a \int_0^b e^{-\frac{x}{p}-\frac{t}{s}} f(x,t) dt\, dx}{\left(1 - e^{-\frac{a}{p}-\frac{b}{s}}\right)}.$$

Proof. Using Definition 1, we obtain

$$G_x G_t[f(x,t)] = p^\alpha s^\alpha \int_0^a \int_0^b e^{-\frac{x}{p}-\frac{t}{s}} f(x,t) dt\, dx + p^\alpha s^\alpha \int_a^\infty \int_b^\infty e^{-\frac{x}{p}-\frac{t}{s}} f(x,t) dt\, dx.$$

Putting $x = u+a, t = v+b$ in the last double-integral in the above equation, we obtain

$$
\begin{aligned}
F(p,s) &= p^\alpha s^\alpha \int_0^a \int_0^b e^{-\frac{x}{p}-\frac{t}{s}} f(x,t) dt\, dx + p^\alpha s^\alpha \int_0^\infty \int_0^\infty e^{-\frac{u}{p}-\frac{v}{s}} f(u+a, v+b) dv\, du \\
&= p^\alpha s^\alpha \int_0^a \int_0^b e^{-\frac{x}{p}-\frac{t}{s}} f(x,t) dt\, dx + p^\alpha s^\alpha e^{-\frac{a}{p}-\frac{b}{s}} \int_0^\infty \int_0^\infty e^{-\frac{u}{p}-\frac{v}{s}} f(u,v) dv\, du \\
F(p,s) &= p^\alpha s^\alpha \int_0^a \int_0^b e^{-\frac{x}{p}-\frac{t}{s}} f(x,t) dt\, dx + e^{-\frac{a}{p}-\frac{b}{s}} F(p,s),
\end{aligned}
$$

therefore

$$F(p,s) = \frac{p^\alpha s^\alpha \int_0^a \int_0^b e^{-\frac{x}{p}-\frac{t}{s}} f(x,t) dt\, dx}{\left(1 - e^{-\frac{a}{p}-\frac{b}{s}}\right)}.$$

$\square$

Theorem 3. *If $G_x G_t[f(x,t)] = F(p,s)$, then*

$$G_x G_t[f(x-\rho, t-\sigma) H(x-\rho, t-\sigma)] = e^{-\frac{\rho}{p}-\frac{\sigma}{s}} F(p,s)$$

where $H(x,t)$ is the Heaviside unit step function.

Proof. Using Equation (38), we obtain

$$
\begin{aligned}
&G_x G_t[f(x-\rho, t-\sigma) H(x-\rho, t-\sigma)] \\
&= p^\alpha s^\alpha \int_0^\infty \int_0^\infty e^{-\left(\frac{x}{p}+\frac{t}{s}\right)} f(x-\rho, t-\sigma) H(x-\rho, t-\sigma) dx\, dt \\
&= p^\alpha s^\alpha \int_\rho^\infty \int_\sigma^\infty e^{-\left(\frac{x}{p}+\frac{t}{s}\right)} f(x-\rho, t-\sigma) dx\, dt
\end{aligned}
$$

where $H(x,t)$ is the Heaviside unit step function defined by $H(x-\rho, t-\sigma) = 1$, at $x > \rho$ and $t > \rho$, and $H(x-\rho, t-\sigma) = 0$ when $x < \rho$ and $t < \sigma$. Putting $\zeta = x-\rho$ and $\eta = t-\sigma$, we obtain

$$G_x G_t[f(x-\rho, t-\sigma)H(x-\rho, t-\sigma)] = p^\alpha s^\alpha e^{-\frac{\rho}{p}-\frac{\sigma}{s}} \int\limits_0^\infty \int\limits_0^\infty e^{-(\frac{x}{p}+\frac{t}{s})} f(\zeta, \eta) d\zeta d\eta$$

$$= e^{-\frac{\rho}{p}-\frac{\sigma}{s}} F(p,s)$$

□

Theorem 4. *(Convolution theorem) Let $G_x G_t[f(x,t)]$ and $G_x G_t[y(x,t)]$ exist and $G_x G_t[f(x,t)] = F(p,s)$, $G_x G_t[y(x,t)] = Y(p,s)$, then*

$$p^\alpha s^\alpha G_x G_t[f(x,t)**y(x,t)] = F(p,s)Y(p,s), \tag{8}$$

where

$$f(x,t)**y(x,t) = \int\limits_0^x \int\limits_0^t f(x-\zeta, t-\eta)y(\zeta, \eta) d\zeta d\eta,$$

and the symbol $**$ indicates the double-convolution with respect to x and t.

Proof. Using the definition of the G-double-Laplace transform, we obtain

$$
\begin{aligned}
G_x G_t[f(x,t)**y(x,t)] &= p^\alpha s^\alpha \int\limits_0^\infty \int\limits_0^\infty e^{-\frac{x}{p}-\frac{t}{s}} f(x,t)**y(x,t) dx dt \\
&= p^\alpha s^\alpha \int\limits_0^\infty \int\limits_0^\infty e^{-\frac{x}{p}-\frac{t}{s}} \left(\int\limits_0^x \int\limits_0^t f(x-\zeta, t-\eta)y(\zeta, \eta) d\zeta d\eta \right) dx dt.
\end{aligned} \tag{9}
$$

Set $\rho = x - \zeta$ and $\sigma = t - \eta$, and apply the upper bound of integrals to $x \to \infty$ and $t \to \infty$; Equation (42) can be written as

$$
\begin{aligned}
&G_x G_t[f(x,t)**y(x,t)] \\
&= p^\alpha s^\alpha \int\limits_0^\infty \int\limits_0^\infty e^{-\frac{\zeta}{p}-\frac{\eta}{s}} f(x-\rho, t-\sigma) d\zeta d\eta \int\limits_{-\zeta}^\infty \int\limits_{-\eta}^\infty e^{-\frac{\rho}{p}-\frac{\sigma}{s}} y(\rho, \sigma) d\rho d\sigma;
\end{aligned} \tag{10}
$$

the functions $f(x,t)$ and $g(x,t)$ equal zero at $x < 0$, $t < 0$; therefore, it follows with respect to the lower limit of integrations that

$$
\begin{aligned}
&G_x G_t[f(x,t)**y(x,t)] \\
&= \frac{1}{p^\alpha s^\alpha} \left(p^\alpha s^\alpha \int\limits_0^\infty \int\limits_0^\infty e^{-\zeta \mu - \frac{\eta}{s}} f(\zeta, \eta) d\zeta d\eta \right) \left(p^\alpha s^\alpha \int\limits_0^\infty \int\limits_0^\infty e^{-p\rho-\frac{\sigma}{s}} y(\rho, \sigma) d\rho d\sigma \right),
\end{aligned}
$$

and it is easy to see that

$$p^\alpha s^\alpha G_x G_t[f(x,t)**g(x,t)] = F(p,s)Y(p,s).$$

□

Example 1 ([21]). *The G-double-Laplace transform of $f(x,t) = (xt)^n$ is determined by*

$$G_x G_t\left[(xt)^n\right] = (n!)^2 s^{\alpha+n+1} p^{\alpha+n+1},$$

where n is a non-negative integer. If $\mu(>-1)$ and $\nu(>-1) \in \mathbb{R}$, then

$$G_x G_t[x^\mu t^\nu] = \Gamma(\mu+1)\Gamma(\nu+1)p^{\alpha+\mu+1}s^{\alpha+\nu+1},$$

can be derived from the definition of the G-double-Laplace transform, and we have

$$
\begin{aligned}
G_x G_t [x^\mu t^\nu] &= p^\alpha s^\alpha \int_0^\infty \int_0^\infty x^\mu t^\nu e^{-\frac{x}{p} - \frac{t}{s}} \, dt \, dx \\
&= \left(p^\alpha \int_0^\infty x^\mu e^{-\frac{x}{p}} \, dx \right) \left(s^\alpha \int_0^\infty t^\nu e^{-\frac{t}{s}} \, dt \right).
\end{aligned}
\tag{11}
$$

Now. by substituting $k = \frac{x}{p}$, $x = kp$, $r = \frac{t}{s}$, $t = rs$ into Equation (11), we obtain

$$
\begin{aligned}
G_x G_t [x^\mu t^\nu] &= \left(p^{\alpha+\mu+1} \int_0^\infty k^\mu e^{-k} \, dk \right) \left(s^{\alpha+\nu+1} \int_0^\infty r^\nu e^{-r} \, dr \right) \\
&= \Gamma(\mu+1)\Gamma(\nu+1) p^{\alpha+\mu+1} s^{\alpha+\nu+1},
\end{aligned}
$$

where the Gamma functions of $\mu + 1$ and $\Gamma(\nu + 1)$ are defined by the convergent integrals:

$$
\Gamma(\mu+1) = \int_0^\infty k^\mu e^{-k} \, dk, \ \mu > 0
$$

$$
\Gamma(\nu+1) = \int_0^\infty r^\nu e^{-r} \, dk, \ \nu > 0
$$

Example 2. *The G-double-Laplace transform for the following function:*

$$
f(x,t) = H(x) \otimes H(t) \ln x \ln t,
$$

is given by

$$
G_x G_t [H(x) \otimes H(t) \ln x \ln t] = s^\alpha p^\alpha \int_0^\infty \int_0^\infty \ln x \ln t \, e^{-\frac{x}{p} - \frac{v}{s}} \, dt \, dx,
$$

where $H(x,t) = H(x) \otimes H(t)$ is the dimensional Heaviside function and $\otimes$ is a tensor product; see [22].

Let $\zeta = \frac{1}{p} x$ and $\eta = \frac{1}{s} t$; the integral becomes

$$
\begin{aligned}
G_x G_t [H(x) \otimes H(t) \ln x \ln t] &= p^{\alpha+1} s^{\alpha+1} \int_0^\infty e^{-\eta} \ln(s\eta) \left(\int_0^\infty e^{-\zeta} \ln(p\zeta) d\zeta \right) d\eta \\
&= p^{\alpha+1} s^{\alpha+1} (-\gamma + \ln p)(-\gamma + \ln s),
\end{aligned}
\tag{12}
$$

where $\gamma = \int_0^\infty e^{-\eta} \ln(\eta) d\eta = 0.5772\ldots$ is Euler's constant.

Theorem 5. *If the G-double-Laplace transform of the function $f(x,t)$ is given by $G_x G_t [f(x,t)] = F(p,s)$, then (GDLT) of $\frac{\partial^n f(x,t)}{\partial x^n}$ and $\frac{\partial^n f(x,t)}{\partial t^n}$ are given:*

$$
G_x G_t \left[\frac{\partial^n f(x,t)}{\partial x^n} \right] = \frac{F(p,s)}{p^n} - p^\alpha \sum_{k=1}^n \frac{1}{p^{n-k}} G_t \left[\frac{\partial^{k-1} f(0,t)}{\partial x^{k-1}} \right],
\tag{13}
$$

and

$$
G_x G_t \left[\frac{\partial^n f(x,t)}{\partial t^n} \right] = \frac{F(p,s)}{s^n} - s^\alpha \sum_{k=1}^n \frac{1}{s^{n-k}} G_x \left[\frac{\partial^{k-1} f(x,0)}{\partial t^{k-1}} \right].
\tag{14}
$$

Proof. Now, by substituting $n = 1$ in Equation (13),

$$G_x G_t \left[\frac{\partial f(x,t)}{\partial x} \right] = s^\alpha \int_0^\infty e^{-\frac{t}{s}} \left[p^\alpha \int_0^\infty e^{-\frac{x}{p}} \frac{\partial f(x,t)}{\partial x} dx \right] dt, \tag{15}$$

first, we calculate the integral inside the bracket for Equation (15), and we obtain

$$
\begin{aligned}
p^\alpha \int_0^\infty e^{-\frac{x}{p}} \frac{\partial f(x,t)}{\partial x} dx &= p^\alpha \left[e^{-\frac{x}{p}} f(x,t) \right]_0^\infty + \frac{p^\alpha}{p} \int_0^\infty e^{-\frac{x}{p}} f(x,t) dx \\
&= \frac{1}{p} F(p,t) - p^\alpha f(0,t),
\end{aligned}
\tag{16}
$$

then, by substituting Equation (16) into Equation (15), we have

$$
\begin{aligned}
G_x G_t \left[\frac{\partial f(x,t)}{\partial x} \right] &= s^\alpha \int_0^\infty e^{-\frac{t}{s}} \left[\frac{1}{p} F_\alpha(p,t) - p^\alpha f(0,t) \right] dt \\
&= s^\alpha \int_0^\infty e^{-\frac{t}{s}} \frac{1}{p} F_\alpha(p,t) dt - s^\alpha \int_0^\infty e^{-\frac{t}{s}} p^\alpha f(0,t) dt;
\end{aligned}
$$

we conclude that

$$G_x G_t \left[\frac{\partial f(x,t)}{\partial x} \right] = \frac{1}{p} F(p,s) - p^\alpha G_t[f(0,t)]. \tag{17}$$

Let us consider the case of $n = 2$. In a similar manner,

$$G_x G_t \left[\frac{\partial^2 f(x,t)}{\partial x^2} \right] = \frac{1}{p^2} F_{2\alpha}(p,s) - p^{\alpha-1} G_t[f(0,t)] - p^\alpha G_t \left[\frac{\partial f(0,t)}{\partial x} \right].$$

Let us suppose that $n = m$ is valid for some m. Thus,

$$
\begin{aligned}
G_x G_t \left[\frac{\partial^m f(x,t)}{\partial x^m} \right] = &\frac{1}{p^m} F_{2\alpha}(p,s) - \frac{p^\alpha}{p^{m-1}} G_t[f(0,t)] - \frac{p^\alpha}{p^{m-2}} G_t \left[\frac{\partial f(0,t)}{\partial x} \right] \\
&- \cdots - p^{\alpha-1} G_t \left[\frac{\partial^{m-2} f(0,t)}{\partial \mu^{m-2}} \right] - p^\alpha G_t \left[\frac{\partial^{m-1} f(0,t)}{\partial \mu^{m-1}} \right],
\end{aligned}
\tag{18}
$$

holds for $\frac{\partial^0}{\partial} = 1$; see [19]; now we show that

$$
\begin{aligned}
G_x G_t \left[\frac{\partial^{m+1} f(x,t)}{\partial x^{m+1}} \right] = &\frac{1}{p^{m+1}} F(p,s) - \frac{p^\alpha}{p^m} G_t[f(0,\nu)] - \frac{p^\alpha}{p^{m-1}} G_t \left[\frac{\partial f(0,\nu)}{\partial \mu} \right] \\
&- \cdots - p^{\alpha-2} G_t \left[\frac{\partial^{m-1} f(0,\nu)}{\partial \mu^{m-1}} \right] - p^{\alpha-1} G_t \left[\frac{\partial^m f(0,\nu)}{\partial \mu^m} \right].
\end{aligned}
\tag{19}
$$

By the notion of $n = 1$, we have

$$
\begin{aligned}
G_x G_t \left[\frac{\partial^{m+1} f(x,t)}{\partial x^{m+1}} \right] = &\frac{1}{p} \left[\begin{array}{l} \frac{1}{p^m} F(p,s) - \frac{p^\alpha}{p^{m-1}} G_t[f(0,t)] - \frac{p^\alpha}{p^{m-2}} G_t \left[\frac{\partial f(0,t)}{\partial x} \right] \\ - \cdots - p^{\alpha-1} G_t \left[\frac{\partial^{m-2} f(0,t)}{\partial x^{m-2}} \right] - p^\alpha G_t \left[\frac{\partial^{m-1} f(0,t)}{\partial x^{m-1}} \right] \end{array} \right] \\
&- p^\alpha G_t \left[\frac{\partial^m f(0,t)}{\partial x^m} \right],
\end{aligned}
$$

From the above equation, the formula inside the bracket can be written in the following form:

$$
\begin{aligned}
G_x G_t \left[\frac{\partial^m f(x,t)}{\partial x^m} \right] = &\frac{1}{p^m} F(p,s) - \frac{p^\alpha}{p^{m-1}} G_t[f(0,t)] - \frac{p^\alpha}{p^{m-2}} G_t \left[\frac{\partial f(0,t)}{\partial x} \right] \\
&\cdots - p^\alpha G_t \left[\frac{\partial^{m-1} f(0,t)}{\partial x^{m-1}} \right];
\end{aligned}
$$

therefore,

$$G_x G_t \left[\frac{\partial^{m+1} f(x,t)}{\partial x^{m+1}} \right] = \frac{1}{p} G_x G_t \left[\frac{\partial^m f(x,t)}{\partial x^m} \right] - p^\alpha G_x G_t \left[\frac{\partial^m f(0,t)}{\partial x^m} \right];$$

hence, Equation (13) can be written as follows:

$$G_x G_t \left[\frac{\partial^n f(x,t)}{\partial x^n} \right] = \frac{F(p,s)}{p^n} - p^\alpha \sum_{k=1}^{n} \frac{1}{p^{n-k}} G_t \left[\frac{\partial^{k-1} f(0,t)}{\partial x^{k-1}} \right].$$

Similarly, we can prove Equation (14). $\square$

In the following example, we apply the G-double-Laplace transform to solve the one-dimensional telegraphic equation.

Example 3. *Consider the homogeneous telegraphic equation denoted by*

$$\psi_{xx} - \psi_{tt} - \psi_t - \psi = 0, \tag{20}$$

with boundary conditions:

$$\psi(0,t) = e^{-t}, \quad \psi_x(0,t) = e^{-t}, \tag{21}$$

and initial conditions:

$$\psi(x,0) = e^x, \quad \psi_t(x,0) = -e^x. \tag{22}$$

Using the G-double-Laplace transform on Equation (20), we have

$$\frac{1}{p^2} \Psi(p,s) - p^{\alpha-1} G_t[\psi(0,t)] - p^\alpha G_t[\psi_x(0,t)]$$

$$- \frac{1}{s^2} \Psi(p,s) - s^{\alpha-1} G_x[\psi(x,0)] - s^\alpha G_x[\psi_t(x,0)]$$

$$- \frac{1}{s} \Psi(p,s) - s^\alpha G_x[\psi(x,0)] - \Psi(p,s)$$

$$= 0. \tag{23}$$

Taking the G-Laplace transform for Equations (21) and (22), we obtain

$$G_t[\psi(0,t)] = \frac{s^{\alpha+1}}{(1+s)}, \; G_t[\psi_x(0,t)] = \frac{s^{\alpha+1}}{(1+s)}$$

$$G_x[\psi(x,0)] = \frac{p^{\alpha+1}}{(1-p)}, \; G_x[\psi_t(x,0)] = -\frac{p^{\alpha+1}}{(1-p)}, \tag{24}$$

substituting Equation (24) into Equation (23), we obtain

$$\frac{1}{p^2} \Psi(p,s) - p^{\alpha-1} \frac{s^{\alpha+1}}{(1+s)} - p^\alpha \frac{s^{\alpha+1}}{(1+s)}$$

$$- \left[\frac{1}{s^2} \Psi(p,s) - s^{\alpha-1} \frac{p^{\alpha+1}}{(1-p)} + s^\alpha \frac{p^{\alpha+1}}{(1-p)} \right]$$

$$- \left[\frac{1}{s} \Psi(p,s) - s^\alpha \frac{p^{\alpha+1}}{(1-p)} \right] - \Psi(p,s)$$

$$= 0; \tag{25}$$

through arranging Equation (25), we obtain

$$\left[\frac{1}{p^2} - \frac{1}{s^2} - \frac{1}{s} - 1\right]\Psi(p,s)$$

$$= p^{\alpha-1}\frac{s^{\alpha+1}}{(1+s)} + p^\alpha\frac{s^{\alpha+1}}{(1+s)} - s^{\alpha-1}\frac{p^{\alpha+1}}{(1-p)}; \tag{26}$$

by simplifying Equation (26), we have

$$\Psi(p,s) = \frac{s^{\alpha+1}p^{\alpha+1}}{(1+s)(1-p)}. \tag{27}$$

Taking the inverse G-double-Laplace transform for Equation (27), we obtain the solution of Equation (20) as follows:

$$\psi(x,t) = e^{x-t}.$$

The solution is in agreement with the result obtained by [16].

3. G-Double-Laplace Transform Decomposition Method and One-Dimensional Coupled Sine-Gordon Equation

This section discusses the nonlinear initial-boundary-value problem (IBVP) of the one-dimensional sine-Gordon and coupled sine-Gordon equations by applying the G-double-Laplace transform decomposition methods. Here, we offer two problems.

Problem 1: In this part, the essential notion of the G-double-Laplace decomposition method is demonstrated in the following nonlinear sine-Gordon equation:

$$\psi_{tt} - \psi_{xx} = \sin(\psi(x,t)) \tag{28}$$

subject to the initial condition:

$$\psi(x,0) = f(x), \quad \psi_t(x,0) = g(x). \tag{29}$$

By using the G-double-Laplace transform for Equation (28) and the G-Laplace transform for Equation (29), we obtain

$$\begin{aligned}\Psi(p,s) &= s^{\alpha+1}\Psi(p,0) + s^{\alpha+2}\Psi_t(p,0)\\ &\quad + s^2 G_x G_t[\psi_{xx} + \sin(\psi(x,t))],\end{aligned} \tag{30}$$

and

$$\Psi(p,0) = F(p) \text{ and } \Psi_t(p,0) = G(p), \tag{31}$$

respectively. Then, substituting Equation (31) into Equation (30), we obtain

$$\begin{aligned}\Psi(p,s) &= s^{\alpha+1}F(p) + s^{\alpha+2}G(p)\\ &\quad + s^2 G_x G_t[\psi_{xx} + \sin(\psi(x,t))].\end{aligned} \tag{32}$$

Applying the inverse G-double-Laplace transform of Equation (32), we achieve

$$\psi(x,t) = f(x) + tg(x) + G_p^{-1}G_s^{-1}\left[s^2 G_x G_t[\psi_{xx} + \sin(\psi)]\right], \tag{33}$$

where the symbol $G_p^{-1}G_s^{-1}$ indicates the G-double-Laplace transform, and we assume here the inverse exists. Now, we suppose a series solution is given by

$$\psi(x,t) = \sum_{m=0}^{\infty}\psi_m(x,t), \tag{34}$$

By substituting Equation (34) into Equation (33), we obtain

$$\sum_{m=0}^{\infty} \psi_m = f(x) + tg(x) + G_p^{-1} G_s^{-1} \left[s^2 G_x G_t \left[\sum_{m=0}^{\infty} \psi_{mxx} + \sum_{m=0}^{\infty} A_m \right] \right], \tag{35}$$

where A_m is a nonlinear term given by

$$\sin(\psi(x,t)) = \sum_{m=0}^{\infty} A_m, \tag{36}$$

where the first few terms of the Adomian polynomials A_m are denoted by

$$
\begin{aligned}
A_0 &= \sin(\psi_0(x,t)), \quad A_1 = \psi_1(x,t)\cos(\psi_0(x,t)) \\
A_2 &= \psi_2(x,t)\cos(\psi_0(x,t)) - \frac{1}{2!}\psi_1^2(x,t)\sin(\psi_0(x,t)) \\
A_3 &= \psi_2(x,t)\cos(\psi_0(x,t)) - \psi_1(x,t)\psi_2(x,t)\sin(\psi_0(x,t)) \\
&\quad - \frac{1}{3!}\psi_1^3(x,t)\cos(\psi_0(x,t)).
\end{aligned} \tag{37}
$$

For more details, see [23]. Then, by matching both sides of Equation (35) above, we can easily create the recursive relation as follows:

$$\psi_0(x,t) = f(x) + tg(x),$$

and the rest of the components are given by

$$\psi_{m+1} = G_p^{-1} G_s^{-1} \left[s^2 G_x G_t [\psi_{mxx} + A_m] \right],$$

for all $m \geq 0$.

Example 4. *Consider the following nonlinear sine-Gordon equation of the form:*

$$\psi_{tt} - \psi_{xx} = \sin(\psi(x,t)), \tag{38}$$

with the initial conditions:

$$\psi(x,0) = \frac{\pi}{2}, \quad \psi_t(x,0) = 1. \tag{39}$$

Using the G-double-Laplace transform of Equation (38) with the initial conditions, we obtain

$$
\begin{aligned}
\Psi(p,s) &= s^{\alpha+1}\Psi(p,0) + s^{\alpha+2}\Psi_t(p,0) \\
&\quad + s^2 G_x G_t [\psi_{xx} + \sin(\psi(x,t))],
\end{aligned} \tag{40}
$$

where $\Psi(p,0)$ and $\Psi_t(p,0)$ are given by

$$\Psi(p,0) = \frac{\pi p^{\alpha+1}}{2} \text{ and } \Psi_t(p,0) = p^{\alpha+1}; \tag{41}$$

substituting Equation (41) into Equation (40), we obtain

$$\Psi(p,s) = \frac{\pi p^{\alpha+1} s^{\alpha+1}}{2} + p^{\alpha+1} s^{\alpha+2} + s^2 G_x G_t [\psi_{xx} + \sin(\psi(x,t))]. \tag{42}$$

Applying the inverse G-double-Laplace transform of Equation (42), we achieve

$$\psi(x,t) = \frac{\pi}{2} + t + G_p^{-1} G_s^{-1} \left[s^2 G_x G_t [\psi_{xx} + \sin(\psi)] \right]. \tag{43}$$

The solution of Equation ([38](#)) is given by infinite series as follows:

$$\psi(x,t) = \sum_{m=0}^{\infty} \psi_m(x,t), \tag{44}$$

By substituting Equation ([44](#)) into Equation ([43](#)), we obtain

$$\sum_{m=0}^{\infty} \psi_m = \frac{\pi}{2} + t + G_p^{-1}G_s^{-1}\left[s^2 G_x G_t\left[\sum_{m=0}^{\infty} \psi_{mxx} + \sum_{m=0}^{\infty} A_m\right]\right], \tag{45}$$

where A_m is the Adomian polynomial that is given by Equations ([36](#)) and ([37](#)). Then, by matching both sides of Equation ([45](#)), we can easily create the iterative relation as follows:

$$\begin{aligned}
\psi_0 &= \frac{\pi}{2} + t \\
\psi_{m+1} &= G_p^{-1}G_s^{-1}\left[s^2 G_x G_t[\psi_{mxx} + A_m]\right],
\end{aligned} \tag{46}$$

where $m \geq 0$; at $m = 0$, we have

$$\begin{aligned}
\psi_1 &= G_p^{-1}G_s^{-1}\left[s^2 G_x G_t[\psi_{0xx} + A_0]\right] \\
&= G_p^{-1}G_s^{-1}\left[s^2 G_x G_t\left[\sin\left(\frac{\pi}{2} + t\right)\right]\right] \\
&= G_p^{-1}G_s^{-1}\left[s^2 G_x G_t[\cos(t)]\right] \\
&= G_p^{-1}G_s^{-1}\left[s^2 G_x G_t\left[1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots\right]\right] \\
&= G_p^{-1}G_s^{-1}\left[s^{\alpha+3} - s^{\alpha+5} + s^{\alpha+7} - \cdots\right] \\
&= \frac{t^2}{2!} - \frac{t^4}{4!} + \frac{t^6}{6!} - \cdots \\
\psi_1 &= 1 - \cos t;
\end{aligned}$$

at $m = 1$,

$$\begin{aligned}
\psi_2 &= G_p^{-1}G_s^{-1}\left[s^2 G_x G_t[\psi_{1xx} + A_1]\right] \\
&= G_p^{-1}G_s^{-1}\left[s^2 G_x G_t\left[(1 - \cos t)\cos\left(\frac{\pi}{2} + t\right)\right]\right] \\
&= G_p^{-1}G_s^{-1}\left[s^2 G_x G_t[\sin t \cos t - \sin t]\right] \\
&= G_p^{-1}G_s^{-1}\left[s^2 G_x G_t\left[\frac{1}{2}\sin 2t - \sin t\right]\right] \\
&= G_p^{-1}G_s^{-1}\left[s^2 G_x G_t\left[\begin{array}{c}\frac{1}{2}\left(2t - \frac{(2t)^3}{3!} + \frac{(2t)^5}{5!} - \cdots\right) \\ -\left(t - \frac{(t)^3}{3!} + \frac{(t)^5}{5!} - \cdots\right)\end{array}\right]\right] \\
&= G_p^{-1}G_s^{-1}\left[\begin{array}{c}\frac{1}{8}\left(8s^{\alpha+4} - 32s^{\alpha+6} + 128s^{\alpha+8} - \cdots\right) - \\ -\left(s^{\alpha+4} - 3s^{\alpha+6} + s^{\alpha+8} - \cdots\right)\end{array}\right] \\
&= -\frac{3}{4}t - \frac{1}{8}\left(2t - \frac{(2t)^3}{3!} + \frac{(2t)^5}{5!} - \cdots\right) \\
&\quad + \left(t - \frac{(t)^3}{3!} + \frac{(t)^5}{5!} - \cdots\right) \\
&= \sin t - \frac{\sin 2t}{8} - \frac{3}{4}t,
\end{aligned}$$

and so on. Then, the approximation solution of Equations (38) and (39) is given by

$$\psi = \psi_0 + \psi_1 + \psi_2 + \ldots.$$

Therefore,

$$\psi = \frac{\pi}{2} + t + 1 - \cos t + \sin t - \frac{\sin 2t}{8} - \frac{3}{4}t + \ldots;$$

hence,

$$\psi = \frac{\pi}{2} + t + \frac{t^2}{2!} - \frac{t^4}{4!} + \ldots.$$

The approximate solution is in agreement with the result obtained by [8,23].

Problem 2: Consider the following general form of the one-dimensional coupled sine-Gordon equation:

$$\begin{aligned} \psi_{tt} - \psi_{xx} &= -\sin(\psi - \phi) + f(x,t) \\ \phi_{tt} - \phi_{xx} &= \sin(\psi - \phi) + g(x,t), \end{aligned} \tag{47}$$

with initial conditions:

$$\begin{aligned} \psi(x,0) &= f_1(x), \ \psi_t(x,0) = f_2(x) \\ \phi(x,0) &= g_1(x), \ \phi_t(x,0) = g_2(x), \end{aligned} \tag{48}$$

and boundary conditions:

$$\begin{aligned} \psi(0,t) &= f_3(t), \ \psi_x(0,t) = f_4(t) \\ \phi(0,t) &= g_3(t), \ \phi_x(0,t) = g_4(t). \end{aligned} \tag{49}$$

Taking the G-double-Laplace transform on both sides of the system of Equation (47) and the single G-Laplace transform for the initial, boundary conditions and simplifying the output, we obtain

$$\begin{aligned} \left(\frac{1}{s^2} - \frac{1}{p^2}\right)\Psi(p,s) &= \frac{p^2 s^2}{p^2 - s^2}\left(s^{\alpha-1}F_1(p) + s^\alpha F_2(p) - p^{\alpha-1}F_3(s)\right) \\ &\quad - p^\alpha F_4(s) + F(p,s) + G_x G_t[\psi_{xx}] \\ &\quad - G_x G_t[\sin(\psi - \phi)], \end{aligned} \tag{50}$$

and

$$\begin{aligned} \left(\frac{1}{s^2} - \frac{1}{p^2}\right)\Phi(p,s) &= \frac{p^2 s^2}{p^2 - s^2}\left(s^{\alpha-1}G_1(p) + s^\alpha G_2(p) - p^{\alpha-1}G_3(s)\right) \\ &\quad - p^\alpha G_4(s) + G(p,s) + G_x G_t[\phi_{xx}] \\ &\quad + G_x G_t[\sin(\psi - \phi)]; \end{aligned} \tag{51}$$

by arranging Equations (50) and (51), we obtain

$$\begin{aligned} \Psi(p,s) &= \frac{p^2 s^2}{p^2 - s^2}\left(s^{\alpha-1}F_1(p) + s^\alpha F_2(p) - p^{\alpha-1}F_3(s)\right) \\ &\quad - \frac{p^2 s^2}{p^2 - s^2}\left(p^\alpha F_4(s) - F(p,s)\right) \\ &\quad + \frac{p^2 s^2}{p^2 - s^2}G_x G_t[\psi_{xx} - \sin(\psi - \phi)], \end{aligned} \tag{52}$$

and

$$
\begin{aligned}
\Phi(p,s) \;=\; & \frac{p^2 s^2}{p^2 - s^2}\left(s^{\alpha-1}G_1(p) + s^\alpha G_2(p) - p^{\alpha-1}G_3(s)\right) \\
& - \frac{p^2 s^2}{p^2 - s^2}\left(p^\alpha G_4(s) + G(p,s)\right) \\
& + \frac{p^2 s^2}{p^2 - s^2}G_x G_t[\phi_{xx} + \sin(\psi - \phi)].
\end{aligned}
\tag{53}
$$

Using the inverse G-double-Laplace transform in Equations (52) and (53),

$$
\begin{aligned}
\psi(x,t) \;=\; & G_p^{-1}G_s^{-1}\left[\frac{p^2 s^2}{p^2 - s^2}\left(s^{\alpha-1}F_1(p) + s^\alpha F_2(p) - p^{\alpha-1}F_3(s)\right)\right] \\
& - G_p^{-1}G_s^{-1}\left[\frac{p^2 s^2}{p^2 - s^2}\left(p^\alpha F_4(s) - F(p,s)\right)\right] \\
& + G_p^{-1}G_s^{-1}\left[\frac{p^2 s^2}{p^2 - s^2}G_x G_t[\psi_{xx} - \sin(\psi - \phi)]\right],
\end{aligned}
\tag{54}
$$

and

$$
\begin{aligned}
\phi(x,t) \;=\; & G_p^{-1}G_s^{-1}\left[\frac{p^2 s^2}{p^2 - s^2}\left(s^{\alpha-1}G_1(p) + s^\alpha G_2(p) - p^{\alpha-1}G_3(s)\right)\right] \\
& - G_p^{-1}G_s^{-1}\left[\frac{p^2 s^2}{p^2 - s^2}\left(p^\alpha G_4(s) + G(p,s)\right)\right] \\
& + G_p^{-1}G_s^{-1}\left[\frac{p^2 s^2}{p^2 - s^2}G_x G_t[\phi_{xx} + \sin(\psi - \phi)]\right],
\end{aligned}
\tag{55}
$$

The solution of Equation (47) is given by an infinite series as follows:

$$
\psi(x,t) = \sum_{m=0}^{\infty}\psi_m(x,t), \; \phi(x,t) = \sum_{m=0}^{\infty}\phi_m(x,t);
\tag{56}
$$

by substituting Equation (56) into Equations (54) and (55), we obtain

$$
\begin{aligned}
\sum_{m=0}^{\infty}\psi_m \;=\; & G_p^{-1}G_s^{-1}\left[\frac{p^2 s^2}{p^2 - s^2}\left(s^{\alpha-1}F_1(p) + s^\alpha F_2(p) - p^{\alpha-1}F_3(s)\right)\right] \\
& - G_p^{-1}G_s^{-1}\left[\frac{p^2 s^2}{p^2 - s^2}\left(p^\alpha F_4(s) - F(p,s)\right)\right] \\
& + G_p^{-1}G_s^{-1}\left[\frac{p^2 s^2}{p^2 - s^2}G_x G_t\left[\sum_{m=0}^{\infty}\psi_{mxx} - \sin\sum_{m=0}^{\infty}(\psi_m - \phi_m)\right]\right],
\end{aligned}
\tag{57}
$$

and

$$
\begin{aligned}
\sum_{m=0}^{\infty}\phi_m \;=\; & G_p^{-1}G_s^{-1}\left[\frac{p^2 s^2}{p^2 - s^2}\left(s^{\alpha-1}G_1(p) + s^\alpha G_2(p) - p^{\alpha-1}G_3(s)\right)\right] \\
& - G_p^{-1}G_s^{-1}\left[\frac{p^2 s^2}{p^2 - s^2}\left(p^\alpha G_4(s) + G(p,s)\right)\right] \\
& + G_p^{-1}G_s^{-1}\left[\frac{p^2 s^2}{p^2 - s^2}G_x G_t\left[\sum_{m=0}^{\infty}\phi_{mxx} + \sin\sum_{m=0}^{\infty}(\psi_m - \phi_m)\right]\right];
\end{aligned}
\tag{58}
$$

the iterative relation of Equations (57) and (58) is defined by

$$
\begin{aligned}
\psi_0 \;=\; & G_p^{-1}G_s^{-1}\left[\frac{p^2 s^2}{p^2 - s^2}\left(s^{\alpha-1}F_1(p) + s^\alpha F_2(p) - p^{\alpha-1}F_3(s)\right)\right] \\
& - G_p^{-1}G_s^{-1}\left[\frac{p^2 s^2}{p^2 - s^2}\left(p^\alpha F_4(s) - F(p,s)\right)\right],
\end{aligned}
\tag{58}
$$

$$\phi_0 = G_p^{-1}G_s^{-1}\left[\frac{p^2s^2}{p^2-s^2}\left(s^{\alpha-1}G_1(p)+s^\alpha G_2(p)-p^{\alpha-1}G_3(s)\right)\right]$$
$$-G_p^{-1}G_s^{-1}\left[\frac{p^2s^2}{p^2-s^2}\left(p^\alpha G_4(s)+G(p,s)\right)\right], \tag{59}$$

and

$$\psi_{m+1} = G_p^{-1}G_s^{-1}\left[\frac{p^2s^2}{p^2-s^2}G_xG_t[\psi_{mxx}-\sin(\psi_m-\phi_m)]\right]$$
$$\phi_{m+1} = G_p^{-1}G_s^{-1}\left[\frac{p^2s^2}{p^2-s^2}G_xG_t[\phi_{mxx}+\sin(\psi_m-\phi_m)]\right], \tag{60}$$

where $m \geq 0$; we suppose that the inverse of Equations (58)–(60) exist.

In the following examples, the one-dimensional coupled sine-Gordon equations are studied by applying the G-double-Laplace transform decomposition method to show the effectiveness of this method.

Example 5. *Consider the following IBVP for a one-dimensional coupled sine-Gordon equation:*

$$\psi_{tt}-\psi_{xx} = -\sin(\psi-\phi)+\sin(t\sin(x)-e^{-t}\sin(x))$$
$$+t\sin(x)$$
$$\phi_{tt}-\phi_{xx} = \sin(\psi-\phi)-\sin(t\sin(x)-e^{-t}\sin(x))$$
$$+2e^{-t}\sin(x), \tag{61}$$

with IC:

$$\psi(x,0) = 0,\ \psi_t(x,0)=\sin(x)$$
$$\phi(x,0) = \sin(x),\ \phi_t(x,0)=-\sin(x), \tag{62}$$

and BC:

$$\psi(0,t) = 0,\ \psi_x(0,t)=t$$
$$\phi(0,t) = 0,\ \phi_x(0,t)=e^{-t}. \tag{63}$$

Applying the G-double-Laplace transform of Equation (61) and the single G-Laplace transform for initial, boundary conditions and simplifying the output, we acquire

$$\Psi(p,s) = \frac{p^{\alpha+4}s^{\alpha+2}}{(p^2-s^2)(1+p^2)}-\frac{p^{\alpha+2}s^{\alpha+4}}{(p^2-s^2)}+\frac{p^{\alpha+4}s^{\alpha+4}}{(p^2-s^2)(1+p^2)}$$
$$+\frac{p^2s^2}{p^2-s^2}G_xG_t\left[\psi_{xx}-\sin(\psi-\phi)+\sin(t\sin(x)-e^{-t}\sin(x))\right]$$
$$, \tag{64}$$

and

$$\Phi(p,s) = \frac{p^{\alpha+4}s^{\alpha+1}(1+s)-p^{\alpha+4}s^{\alpha+2}(1+s)}{(p^2-s^2)(1+p^2)(1+s)}$$
$$\frac{-p^{\alpha+2}s^{\alpha+3}(1+p^2)+2p^{\alpha+4}s^{\alpha+3}}{(p^2-s^2)(1+p^2)(1+s)}$$
$$+\frac{p^2s^2}{p^2-s^2}G_xG_t\left[\phi_{xx}+\sin(\psi-\phi)-\sin(t\sin(x)-e^{-t}\sin(x))\right]. \tag{65}$$

Arranging Equations (64) and (65), one can obtain

$$\Psi(p,s) = \frac{p^{\alpha+2}s^{\alpha+2}}{(1+p^2)}+\frac{p^2s^2}{p^2-s^2}G_xG_t[\Delta], \tag{66}$$

and

$$\Phi(p,s) = \frac{p^{\alpha+2} s^{\alpha+1}}{(1+p^2)(1+s)} + \frac{p^2 s^2}{p^2 - s^2} G_x G_t [\Lambda], \tag{67}$$

where

$$\begin{aligned}
\Delta &= \psi_{xx} - \sin(\psi - \phi) + \sin(t \sin(x) - e^{-t} \sin(x)) \\
\Lambda &= \phi_{xx} + \sin(\psi - \phi) - \sin(t \sin(x) - e^{-t} \sin(x)).
\end{aligned}$$

Using the inverse G-double-Laplace transform in Equations (52) and (53),

$$\psi(x,t) = t \sin x + G_p^{-1} G_s^{-1} \left[\frac{p^2 s^2}{p^2 - s^2} G_x G_t [\Delta] \right], \tag{68}$$

and

$$\phi(x,t) = e^{-t} \sin x + G_p^{-1} G_s^{-1} \left[\frac{p^2 s^2}{p^2 - s^2} G_x G_t [\Lambda] \right]; \tag{69}$$

by substituting Equation (56) into Equations (68) and (69), we obtain

$$\sum_{m=0}^{\infty} \psi_m = t \sin x + G_p^{-1} G_s^{-1} \left[\frac{p^2 s^2}{p^2 - s^2} G_x G_t \left[\sum_{m=0}^{\infty} \Delta_m \right] \right], \tag{70}$$

and

$$\sum_{m=0}^{\infty} \phi_m = e^{-t} \sin x + G_p^{-1} G_s^{-1} \left[\frac{p^2 s^2}{p^2 - s^2} G_x G_t \left[\sum_{m=0}^{\infty} \Lambda_m \right] \right]; \tag{71}$$

we obtain the following approximations:

$$\psi_0 = t \sin x, \ \phi_0 = e^{-t} \sin x,$$

and

$$\begin{aligned}
\psi_{m+1} &= G_p^{-1} G_s^{-1} \left[\frac{p^2 s^2}{p^2 - s^2} G_x G_t [\Delta_m] \right] \\
\phi_{m+1} &= G_p^{-1} G_s^{-1} \left[\frac{p^2 s^2}{p^2 - s^2} G_x G_t [\Lambda_m] \right],
\end{aligned}$$

where $m \geq 0$; at $m = 0$

$$\begin{aligned}
\psi_1 &= G_p^{-1} G_s^{-1} \left[\frac{p^2 s^2}{p^2 - s^2} G_x G_t \left[\psi_{0xx} - \sin(\psi_0 - \phi_0) + \sin(t \sin(x) - e^{-t} \sin(x)) \right] \right] \\
&= G_p^{-1} G_s^{-1} \left[\frac{p^2 s^2}{p^2 - s^2} G_x G_t [0] \right] = 0 \\
\phi_1 &= G_p^{-1} G_s^{-1} \left[\frac{p^2 s^2}{p^2 - s^2} G_x G_t \left[\phi_{0xx} + \sin(\psi_0 - \phi_0) - \sin(t \sin(x) - e^{-t} \sin(x)) \right] \right] \\
&= G_p^{-1} G_s^{-1} \left[\frac{p^2 s^2}{p^2 - s^2} G_x G_t [0] \right] = 0,
\end{aligned}$$

and in a similar way, we obtain $\psi_2 = \psi_3 = \cdots = \psi_m = 0$ and $\phi_2 = \phi_3 = \cdots = \phi_m = 0$, which means that method's convergence is very fast. Therefore, the series solution is given by

$$\begin{aligned}
\psi(x,t) &= \psi_0 + \psi_1 + \psi_2 + \ldots \\
\phi(x,t) &= \phi_0 + \phi_1 + \phi_2 + \ldots;
\end{aligned}$$

hence,

$$\begin{aligned} \psi(x,t) &= t\sin x \\ \phi(x,t) &= e^{-t}\sin x. \end{aligned}$$

The approximate solution is in agreement with the result obtained by [13].

Example 6. *Consider the following IBVP for the one-dimensional coupled sine-Gordon equation on the domain $\Omega = [0,\pi]$, $t > 0$:*

$$\begin{aligned} \psi_{tt} - \psi_{xx} &= -\sin(\psi - \phi) + \sin(\cos(x+2t) - \sin(x)\sinh(t)) \\ &\quad -3\cos(x+2t) \\ \phi_{tt} - \phi_{xx} &= \sin(\psi - \phi) - \sin(\cos(x+2t) - \sin(x)\sinh(t)) \\ &\quad +2\sin(x)\sinh(t), \end{aligned} \tag{72}$$

with IC:

$$\begin{aligned} \psi(x,0) &= \cos x, \ \psi_t(x,0) = -2\sin(x) \\ \phi(x,0) &= 0, \ \phi_t(x,0) = \sin(x) \end{aligned} \tag{73}$$

and BC:

$$\begin{aligned} \psi(0,t) &= \cos 2t, \ \psi_x(0,t) = -\sin 2t \\ \phi(0,t) &= 0, \ \phi_x(0,t) = \sinh t. \end{aligned} \tag{74}$$

By employing the proposed method for Equation (72), we obtain

$$\Psi(p,s) = \frac{p^{\alpha+1}s^{\alpha+1}(1-2ps)}{(1+p^2)(1+4s^2)} + \frac{p^2 s^2}{p^2 - s^2}G_x G_t[\Pi], \tag{75}$$

and

$$\Phi(p,s) = \frac{p^{\alpha+2}s^{\alpha+2}}{(1+p^2)(1-s^2)} + \frac{p^2 s^2}{p^2 - s^2}G_x G_t[\Theta], \tag{76}$$

where

$$\begin{aligned} \Pi &= \psi_{xx} - \sin(\psi - \phi) + \sin(\cos(x+2t) - \sin(x)\sinh(t)) \\ \Theta &= \phi_{xx} + \sin(\psi - \phi) + \sin(\cos(x+2t) - \sin(x)\sinh(t)), \end{aligned}$$

Taking the inverse G-double-Laplace transform into Equations (75) and (76),

$$\psi(x,t) = \cos(x+2t) + G_p^{-1}G_s^{-1}\left[\frac{p^2 s^2}{p^2 - s^2}G_x G_t[\Pi]\right], \tag{77}$$

and

$$\phi(x,t) = \sin x \sinh t + G_p^{-1}G_s^{-1}\left[\frac{p^2 s^2}{p^2 - s^2}G_x G_t[\Theta]\right]; \tag{78}$$

by substituting Equation (56) into Equations (77) and (78), one can obtain

$$\sum_{m=0}^{\infty}\psi_m = \sin(x+2t) + G_p^{-1}G_s^{-1}\left[\frac{p^2 s^2}{p^2 - s^2}G_x G_t\left[\sum_{m=0}^{\infty}\Pi_m\right]\right], \tag{79}$$

and

$$\sum_{m=0}^{\infty}\phi_m = \sin x \sinh t + G_p^{-1}G_s^{-1}\left[\frac{p^2 s^2}{p^2 - s^2}G_x G_t\left[\sum_{m=0}^{\infty}\Theta_m\right]\right], \tag{80}$$

By comparing both sides of Equations (79) and (80), we have

$$\psi_0 = \sin(x + 2t), \quad \phi_0 = \sin x \sinh t,$$

and in general, the iteration relation is given by

$$\psi_{m+1} = G_p^{-1} G_s^{-1} \left[\frac{p^2 s^2}{p^2 - s^2} G_x G_t \left[\begin{array}{c} \psi_{mxx} - \sin(\psi_m - \phi_m) \\ + \sin(\cos(x + 2t) - \sin(x) \sinh(t)) \end{array} \right] \right]$$

$$\phi_{m+1} = G_p^{-1} G_s^{-1} \left[\frac{p^2 s^2}{p^2 - s^2} G_x G_t \left[\begin{array}{c} \phi_{mxx} + \sin(\psi_m - \phi_m) \\ - \sin(\cos(x + 2t) - \sin(x) \sinh(t)) \end{array} \right] \right],$$

where $m \geq 0$; at $m = 0$,

$$\psi_1 = G_p^{-1} G_s^{-1} \left[\frac{p^2 s^2}{p^2 - s^2} G_x G_t \left[\begin{array}{c} \psi_{xx} - \sin(\psi_0 - \phi_0) \\ + \sin(\cos(x + 2t) - \sin(x) \sinh(t)) \end{array} \right] \right]$$

$$= G_p^{-1} G_s^{-1} \left[\frac{p^2 s^2}{p^2 - s^2} G_x G_t [0] \right] = 0,$$

and

$$\phi_1 = G_p^{-1} G_s^{-1} \left[\frac{p^2 s^2}{p^2 - s^2} G_x G_t \left[\begin{array}{c} \phi_{xx} + \sin(\psi_0 - \phi_0) \\ - \sin(\cos(x + 2t) - \sin(x) \sinh(t)) \end{array} \right] \right]$$

$$= G_p^{-1} G_s^{-1} \left[\frac{p^2 s^2}{p^2 - s^2} G_x G_t [0] \right] = 0;$$

in the same manner, we obtain $\psi_2 = \psi_3 = \cdots = \psi_m = 0$ and $\phi_2 = \phi_3 = \cdots = \phi_m = 0$.
Hence, the series solution is given by

$$\psi(x, t) = \psi_0 + \psi_1 + \psi_2 + \ldots$$
$$\phi(x, t) = \phi_0 + \phi_1 + \phi_2 + \ldots;$$

hence.

$$\psi(x, t) = \cos(x + 2t)$$
$$\phi(x, t) = \sin x \sinh t.$$

The approximate solution is in agreement with the result obtained by [13].

4. Conclusions

In the course of this work, we introduced a new technique called the G-double-Laplace transform, which accurately describes its various definitions, theorems, existence conditions, partial derivatives, and the double-convolution theorems. Taking advantage of these new insights, we successfully obtained the exact solution of the one-dimensional coupled nonlinear sine-Gordon equation. To check the applicability of our technique, we provided some demonstrative examples. From the results we obtained, we strongly advise applying this method in the future so it can be extended to solve different nonlinear fractional partial differential equations related to physics and engineering problems.

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References

1. Guo, P.A.; Boldbaatar, D.; Yi, D.; Dai, P. Numerical Solution of Sine-Gordon Equation with the Local Kriging Meshless Method. *Math. Probl. Eng.* **2020**, *2020*, 9057387. [CrossRef]
2. Barone, A.; Esposito, F.; Magee, C.J.; Scott, A.C. Theory and Applications of the Sine-Gordon Equation. *Riv. Nuovo C.* **1971**, *1*, 227–267. [CrossRef]
3. Fayadh, A.A.; Faraj, D.S. Combined Laplace Transform-Variational Iteration Method for Sine-Gordon Equation. *Dep. Math. Comput. Appl.* **2019**, *31*, 61–64.
4. Baccouch, M. Optimal Error Estimates of the Local Discontinuous Galerkin Method the Two Dimensional Sine-Gordon Equation on Cartesian Grids. *Int. J. Numer. Anal. Model.* **2019**, *16*, 436–462.
5. Hashemi, M.S. Numerical Study of the One-Dimensional Coupled Nonlinear Sine-Gordon Equations by a Novel Geometric Meshless Method. *Eng. Comput.* **2021**, *37*, 3397–3407. [CrossRef]
6. Keskin, R.S.; Koc, B. Numerical Solution of Sine-Gordon Equation by Reduced Differential Transform Method. In Proceedings of the World Congress on Engineering, London, UK, 6–8 July 2011; Volume 1, pp. 6–8.
7. Wang, Q. An Application of the Modified Adomian Decomposition Method for (n + 1)-Dimensional Sine-Gordon Field. *Appl. Math. Comput.* **2006**, *181*, 147–152. [CrossRef]
8. Maitama, S.; Hamza, Y.F. An Analytical Method for Solving Nonlinear Sine-Gordon Equation. *Sohag J. Math.* **2020**, *7*, 5–10.
9. Khusnutdinova, K.R.; Pelinovsky, D.E. On the Exchange of Energy in Coupled Klein-Gordon Equations. *Wave Motion* **2003**, *38*, 1–10. [CrossRef]
10. Ray, S.S. A Numerical Solution of the Coupled Sine-Gordon Equation Using the Modified Decomposition Method. *Appl. Math. Comput.* **2006**, *175*, 1046–1054. [CrossRef]
11. Salas, A.H. Exact Solutions of Coupled Sine-Gordon Equations. *Nonlinear Anal. Real World Appl.* **2010**, *11*, 3930–3935. [CrossRef]
12. Sadighi, A.; Ganji, D.D.; Ganjavi, B. Traveling Wave Solutions of the Sine-Gordon and the Coupled Sine-Gordon Equations Using the Homotopy-Perturbation Method. *Trans. B Mech. Eng.* **2009**, *16*, 189–195.
13. Deresse, A.T. Double Sumudu Transform Iterative Method for One-Dimensional Nonlinear Coupled Sine-Gordon Equation. *Adv. Math. Phys.* **2022**, *2022*, 6977692. [CrossRef]
14. Deresse, A.T.; Mussa, Y.O.; Gizaw, A.K. Solutions of Two-Dimensional Nonlinear Sine-Gordon Equation via Triple Laplace Transform Coupled with Iterative Method. *J. Appl. Math.* **2021**, *2021*, 9279022. [CrossRef]
15. Eltayeb, H.; Kılıçman, A. A Note on Solutions of Wave, Laplace's and Heat Equations with Convolution Terms by Using a Double Laplace Transform. *Appl. Math. Lett.* **2008**, *21*, 1324–1329. [CrossRef]
16. Eltayeb, H.; Kılıçman, A. A Note on Double Laplace Transform and Telegraphic Equations. *Abstr. Appl. Anal.* **2013**, *2013*, 932578. [CrossRef]
17. Debnath, L. The Double Laplace Transforms and Their Properties with Applications to Functional, Integral and Partial Differential Equations. *Int. J. Appl. Comput. Math.* **2016**, *2*, 223–241. [CrossRef]
18. Kim, H. The Intrinsic Structure and Properties of Laplace-Typed Integral Transforms. *Math. Probl. Eng.* **2017**, *2017*, 1762729. [CrossRef]
19. Supaknaree, S.; Nonlapon, K.; Kim, H. Further Properties of Laplace-Type Integral Transform. *Dyn. Sys. Appl.* **2019**, *28*, 195–215. [CrossRef]
20. Ibrahim Nuruddeen, R.; Akbar, Y.; Kim, H. On the Application of G_α Integral Transform to Nonlinear Dynamical Models with Non-Integer Order Derivatives. *AIMS Math.* **2022**, *7*, 17859–17878. [CrossRef]
21. Alhefthi, R.K.; Eltayeb, H. The Solution of Coupled Burgers' Equation by G-Laplace Transform. *Symmetry* **2023**, *15*, 1764. [CrossRef]
22. Kilicman, A.; Eltayeb, H. A Note on Defining Singular Integral as Distribution and Partial Differential Equations with Convolution Term. *Math. Comput. Model.* **2009**, *49*, 327–336. [CrossRef]
23. Wazwaz, A.M. *Partial Differential Equations and Solitary Waves Theory*; Springer: Heidelberg, Germany, 2009.