axioms

Article

Characterization of Isoclinic, Transversally Geodesic and Grassmannizable Webs

Jihad Saab [1] and Rafik Absi [2,*]

[1] Laboratoire de Recherche en Éco-Innovation Industrielle et Énergétique LR2E, École Supérieure d'Ingénieurs ECAM-EPMI, 95092 Cergy-Pontoise, France; j.saab@ecam-epmi.com
[2] Laboratoire Quartz, ECAM-EPMI, 95092 Cergy-Pontoise, France
* Correspondence: r.absi@ecam-epmi.com

Abstract: One of the most relevant topics in web theory is linearization. A particular class of linearizable webs is the Grassmannizable web. Akivis gave a characterization of such a web, showing that Grassmannizable webs are equivalent to isoclinic and transversally geodesic webs. The obstructions given by Akivis that characterize isoclinic and transversally geodesic webs are computed locally, and it is difficult to give them an interpretation in relation to torsion or curvature of the unique Chern connection associated with a web. In this paper, using Nagy's web formalism, Frölisher—Nejenhuis theory for derivation associated with vector differential forms, and Grifone's connection theory for tensorial algebra on the tangent bundle, we find invariants associated with almost-Grassmann structures expressed in terms of torsion, curvature, and Nagy's tensors, and we provide an interpretation in terms of these invariants for the isoclinic, transversally geodesic, Grassmannizable, and parallelizable webs.

Keywords: webs; affine structures; linearization; gronwall conjecture; loop; net

MSC: Primary 53A60; Secondary 53C36

check for updates

Citation: Saab, J.; Absi, R. Characterization of Isoclinic, Transversally Geodesic and Grassmannizable Webs. *Axioms* **2024**, *13*, 386. https://doi.org/10.3390/axioms13060386

Academic Editors: Zhigang Wang, Yanlin Li and Juan De Dios Pérez

Received: 3 May 2024
Revised: 31 May 2024
Accepted: 3 June 2024
Published: 6 June 2024

1. Introduction

Webs theory is a theory that dates back to the beginning of the last century. It was seen as a new topic in differential geometry. Webs theory started with Gronwall [1], Blashke [2], and his students and then with the Russian school led by Akivis [3]. During the last century, it has been studied only in a local approach, and all the results were in terms of local coordinates, which made the computation and interpretation a little complicated. More precisely, a differential 3- web on a manifold M is a triplet of three foliations such that the tangent spaces to the leaves of any two foliations through a point of M are complementary subspaces of TM. We can immediately see that if a web is defined on M, then the dimension of M is necessarily even, $dimM = 2r$ and the dimension of the leaves of the three webs is r. Two webs are said to be equivalent at a point $p \in M$ if there is a germ of local diffeomorphisms at p that exchanges them. A 3- web on $\mathbb{R}^{2r}$ is said to be linear (respectively parallel) if its leaves are pieces of $r-$planes (respectively parallel $r-$planes). A linearizable (respectively parallelizable) web on M is a web equivalent to a linear web (respectively parallel web). Towards the end of the last century, Nagy [4] introduced an intrinsic formalism of web theory, which consists in seeing a web as the given of two endomorphism fields $\{h,\ j\}$ of TM, verifying some relations. The most important result in the linearisability problem is due to Akivis [3,5], who gave a characterization of a particular class of linearizable webs, the grassmanianizable webs, showing that these webs are at the same time isoclinic and transversally geodesic. Linearizable webs are an old, open problem. It was treated by [6–9], but in the case of 3 webs on a 2 dimensional manifold, Grifone, Muzsnay and Saab, [10] elaborate an elegant characterization. In their work, they show that for any given 3 web, one can decide if it's linearizable or not. This work was contested by [11,12] for the famous example of 3 web on $\mathbb{R}^2$:

$$\begin{cases} x & = & cte \\ y & = & cte \\ (x+y)e^{-x} & = & cte \end{cases}$$

which is linearizable by the authors of [10]. In [11,12], they claimed that it is not linearizable. Three years later, the work in [13] shows again that this web is linearizable, and finally the end of the controversy was given in 2018 in the work of [14] confirming that the work in [10] was correct and this web is linearizable.

Later in 2011, in his work [15], Wang continued a part of the work given by [10] concerning the Granwall conjecture and showed that for a planar 3-web $\mathcal{W}$ on a connected surface M, which admits two distinct linearizations: if $x_0 \in M$ is a reference point and if the web curvature of $\mathcal{W}$ vanishes to order three at x_0, then the web curvature vanishes identically, and $\mathcal{W}$ is locally equivalent to an algebraic web.

A very important work was carried out by Agafonov [16,17], who showed that the Gronwall conjecture is true for 3- webs whose two foliations are pencils of lines. The proof is based on two facts:

(1) The conjecture is true for webs with infinitesimal symmetry.
(2) A web with a degenerated signature set admits an infinitesimal symmetry.

The characterizations given by Akivis [3] for Grassmannizable web are expressed locally, and their interpretations remain difficult to understand. In this paper, we adopt Nagy's modern formalism [4,18], and we use Frölicher-Nijenhuis' theory on the derivatives associated with differential vector forms [19], and Grifone's connections theory [20]. By a method similar to the one used to find Weil's projective tensor of a connection, we find 3 invariants, $\Im, \Re, \Re^1$, of Hangan's tensorial structures of type $(2, r)$, (cf. [21,22]). These invariants are intrinsically expressed in terms of torsion, curvature, and their derivatives, as well as Nagy's tensors h and j. We show that if $\mathcal{W} = (T^h, T^v, T^t)$ is an r- dimensional web on a manifold of dimension $2r$, then:

(1) $\mathcal{W}$ is isoclinic if and only if $\Im = 0$.
(2) $\mathcal{W}$ is transversally geodesic if and only if $\Re^1 = 0$.
(3) $\mathcal{W}$ is Grassmannizable if and only if $\Im = 0$ and $\Re^1 = 0$.
(4) If $\Im = 0$ and $\Re = 0$, then $\mathcal{W}$ is parallelizable.

Comparing this work to that of Akivis and his school, it can be said that, for the first time, this work provides intrinsically defined conditions in terms of torsion, curvature, and Nagy's tensors that characterize an isoclinic, transversally geodesic, or Grassmannizable web.

2. Almost-Grassmann Structures

Let M be a $2r-$ dimensional manifold, with $r \geq 1$.

Definition 1. *A differential* 3*-web on a manifold M is a triplet of foliations* $\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$ *such that the tangent spaces to the leaves of any two different foliations* $\{\mathcal{F}_\alpha, \mathcal{F}_\beta\}$ $(\alpha \neq \beta)$ *through a point of M are complementary subspaces of TM.*

We see immediately that the distributions tangent to the foliations have the same dimension $\frac{1}{2}\dim M = r$. We call the leaves of the foliations $\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$ horizontal, vertical, and transversal. Likewise, we call their tangent spaces *horizontal, vertical*, and *transversal* tangent spaces, and denote them by T^h, T^v and T^t.

A 3-web on M is said to be linear if its leaves are r-plans. A 3-web is said to be parallel if its leaves are r-parallel plans. Two webs are said to be equivalent at a point p of M if there exists a germ of local diffeomorphisms at p that exchanges them.

A web is said to be linearizable (respectively parallelizable) at a point p if it is equivalent at p to a linear web (respectively to a parallel web).

On another hand, we may need to introduce the case of families of non-integrable distributions:

Definition 2. *We call almost 3-web on M every three r-dimensional distributions (i.e., every three subbundles of dimension r of TM), which are two by two complementary subspaces of T_pM at any point p in M.*

The following theorem proved by Nagy [4] provides an elegant infinitesimal characterization of 3-web and their Chern connection.

Theorem 1 (*Nagy*)**.**

1. *To give an almost 3-web is equivalent to giving two (1,1)-tensor fields, $\{h, j\}$, satisfying the following conditions:*
   - (a) $h^2 = h,\ j^2 = id$
   - (b) $jh = vj$ *(where $v = id - h$)*
     The tensors $\{h,\ j\}$ define a web if, in addition, we have:
   - (c) *$Kerh$, Imh, and $Ker(j + id)$ are integrable distributions.*
2. *For all 3-web, there exists a unique covariant derivation ∇ on M satisfying:*
   - (a) $\nabla h = 0$
   - (b) $\nabla j = 0$
   - (c) $T(hX, vY) = 0$, *for any $X, Y \in TM$,*
     where T is the torsion of ∇.

   ∇ is called the Chern connection.
   The almost web given by $\{h,\ j\}$ is a web if and only if

$$[h, h] = 0 \quad \text{and} \quad j\,[j, j] = -\,[j, j]$$

or in terms of the torsion:

$$\begin{cases} v\boldsymbol{T}(hX, hY) = 0, \\ h\boldsymbol{T}(vX, vY) = 0, \\ j\boldsymbol{T}(X, Y) = -\boldsymbol{T}(jX, jY) \end{cases}$$

Indications for the proof:

(1) Let $\mathcal{W}$ be a web. We take for h the projection on T^h in the splitting $TM = T^h \oplus T^v$, j is the (1,1)-tensor field on M defined in the following way:

$$j\,|_{T^h} = \text{projection onto } T^v \text{ in the splitting } TM = T^v \oplus T^t$$
$$j\,|_{T^v} = \text{projection onto } T^h \text{ in the splitting } TM = T^h \oplus T^t.$$

Conversely, if $\{h, j\}$ satisfies the conditions of the theorem, we can define a web setting $T^h = \text{Im}h$, $T^v = \text{Ker}h$ and $T^t = \text{Ker}(j + id)$.

(2) Suppose that there exists a connection satisfying the conditions (a), (b), and (c). From (c), we have

$$\nabla_{hX} vY - \nabla_{vY} hX = [hX, vY].$$

Multiplying by h and v and taking into account (a), we have, respectively :

$$\nabla_{hX} vY = v[hx, vY] \quad \text{and} \quad \nabla_{vY} hX = -h[hX, vY].$$

Property (b) allows us to get:

$$\nabla_{hX} hY = \nabla_{hX} jvjY = j\nabla_{hX} vjY = jv[hX, vjY].$$

$$\nabla_{vX}vY = \nabla_{vX}jhjY = j\nabla_{vX}hjY = jh[vX, hjY].$$

This proves that ∇ is uniquely defined.

Conversely, if ∇ is given by this expression, it is easy to verify (a) and (b). Also, we can verify the integrability conditions in terms of Njenhuis brackets of h, j or in terms of torsion.

It is easy to see that Chern connection is given by:

$$\nabla_X Y = h\{j[hX, jhY] + [vX, hY]\} + v\{j[vX, jvY] + [hX, vY]\}.$$

Definition 3. *Let M^{rp} be an rp-dimensional manifold. Define the almost-Grassmann structure of type (p, r) on M to be a field of isomorphismes*

$$\Phi_x : \ T_x M \to V^p \otimes V^r$$

where V^p and V^r are vector spaces of dimension p and r respectively.

An almost-Grassmann structure is integrable if and only if for all $x \in M$ there exists a neighborhood U of x such that the locally induced almost-Grassmann structure in this neighborhood is diffeomorphic to the Grassmann structure on $\mathbf{R}^{r+p}$ defined by the constant mapping $\mathbf{R}^{r+p} \to V^p \otimes V^r$, V^p, and V^r being two sub-vector spaces of $\mathbf{R}^{r+p}$ of dimension p and r, respectively.

In terms of G-structures, an almost-Grassmann structure is a $SL(p, \mathbf{R}) \otimes GL(r, \mathbf{R})$—structure.

The following theorem is due to T. HANGAN [21].

Theorem 2 (T. HANGAN)**.** *Let M be an almost-Grassmann manifold, the almost-Grassmann structure is integrable if and only if M is locally a Grassmann manifold.*

Remark 1. *We can define a web structure on $G(r+1, 1)$ as follows:*

Let $x \in \mathbb{P}^{r+1}$ and $\hat{x}$ the Schubert manifold of all straight lines intersecting at the point x. It's an $r-$ dimensional submanifold of $G(r+1, 1)$. A hypersurface V of $\mathbb{P}^{r+1}$ defines r-dimensional foliation on an open set $\mathcal{U} \subset G(r+1, 1)$: the leaves are $\hat{x}$ with $x \in V$. If we consider 3 hypersurfaces V_α $(\alpha = 1, 2, 3)$, in a general position in $\mathbb{P}^{r+1}$, they define a 3-web of dimension r on an open set $\mathcal{U}$ of $G(r+1, 1)$.

Definition 4. *A Grassmann web is the $3-$ web on $G(r+1, 1)$ defined as in the previous remark.*

A $3-$ web is Grassmannizable if it is equivalent to a Grassmann web.

In dimension 1, Akivis showed that the Grassmannizable webs generalize the linearizable webs:

Theorem 3 (Akivis). *A $3-$ web of dimension 1 on a $2-$ dimensional manifold is Grassmannizable if and only if it is linearizable.*

Indeed, if $r = 1$, the manifold $G(2, 1)$ is the dual projective space $(\mathbb{P}^2)^*$; if $x \in \mathbb{P}^2$, $\hat{x}$ is a straight line in $(\mathbb{P}^2)^*$ formed by straight lines of $\mathbb{P}^2$ with the vertex at x. Let $k : (\mathbb{P}^2)^* \to \mathbb{P}^2$ be the one-to-one mapping defined by $k(ax + by + cz) = [a : b : c]$. This mapping is well-defined since $k(\lambda(ax + by + cz)) = [a : b : c]$. The image by k of $\hat{x}$ is then a straight line in $\mathbb{P}^2$. Thus, a Grassmannizable web is linearizable.

Conversely, if a web is linearizable, the image by k^{-1} of its equivalent linear web is a Grassmann web.

3. Almost-Grassmann Structures and Isoclinic Deformations

In what follows, the word "web" will be used for a 3-web of dimension $r$on a $2r$-dimensional manifold M. We will show that an almost-Grassmann structure can be seen as a family of almost webs, knowing that an almost web is defined as three $r-$dimensional distributions on M that are piecewise transverse. In terms of Nagy's formalism [4,18], an almost web is given by a pair of $(1,1)$ fields of tensors $\{h, j\}$ satisfying the following:

(a) $h^2 = h,\ j^2 = id;$

(b) $jh = vj$ (où $v = id - h$).

Proposition 1. *For every almost web on M, there is an associated almost-Grassmann structure of type* $(2, r)$*.*

Conversely, for every almost-Grassmann structure of type $(2, r)$ *on M, there is an associated family of almost webs.*

Proof. Let $\mathcal{W}$ be an almost web. Denote by T^h, T^v, T^t the horizontal, vertical, and transversal distributions and by $\{h, j\}$, Nagy's tensors. We define the isomorphism:

$$\psi_x : \mathbf{R}^2 \otimes T_x^h \longrightarrow T_x M$$

as follows: Let $\{e_1, ..., e_r\}$ be a basis of T_x^h and $\varepsilon_1 = (1,0),\ \varepsilon_2 = (0,1)$ the trivial basis of $\mathbf{R}^2$. Set

$$\begin{array}{l} \psi_x(\varepsilon_1 \otimes e_i) = e_i \\ \psi_x(\varepsilon_2 \otimes e_i) = je_i \qquad (i = 1, ..., r) \end{array}$$

We then extend ψ_x by linearity. Since $\{e_i, je_i\}_{i=1,...,r}$ is a basis of $T_x M$, ψ_x transforms a basis into another basis; it is then an isomorphism.

Extending by linearity, we get

$$\psi_x\big((\alpha, \beta) \otimes hX\big) = \alpha hX + \beta jhX.$$

Let's calculate $\psi_x^{-1} : T_x M \to \mathbf{R}^2 \otimes T_x^h$

We have

$$\begin{array}{l} \psi_x^{-1}(e_i) = \varepsilon_1 \otimes e_i \\ \psi_x^{-1}(je_i) = \varepsilon_2 \otimes e_i \qquad (i = 1, ..., r) \end{array}$$

Then, if $X \in T_x M,\ X = X^i e_i + Y^i je_i$, we have:

$$\psi_x^{-1}(X) = X^i \varepsilon_1 \otimes e_i + Y^i \varepsilon_2 \otimes e_i = \big(X^i(1,0) + Y^i(0,1)\big)e_i = (X^i, Y^i) \otimes e_i.$$

By identifying ψ_x^{-1} to a mapping

$$\varphi_x : (\mathbf{R}^2)^* \times T_x M \to T_x^h$$

we get for $(\alpha, \beta) \in (\mathbf{R}^2)^*$

$$\varphi_x\big((\alpha, \beta), X\big) = \alpha hX + \beta hjX.$$

Thus, we define an almost-Grassmann structure of type $(2, r)$ on M.

Note that

$$\begin{array}{rcl} T_x^h & = & \psi_x\big((1,0) \otimes T_x^h\big), \\ T_x^v & = & \psi_x\big((0,1) \otimes T_x^h\big), \\ T_x^t & = & \psi_x\big((1,-1) \otimes T_x^h\big). \end{array}$$

Conversely, suppose that M is endowed with an almost-Grassmann structure of type $(2, r)$, and let

$$\psi_x : \mathbf{R}^2 \otimes \mathbf{R}^r \longrightarrow T_x M$$

be the vector space isomorphism defined by this structure. Let $[\alpha : \beta] \in \mathbb{P}^1(\mathbf{R})$ and

$$\Delta_{[\alpha:\beta]} = \psi_x\big((\alpha, \beta) \otimes \mathbf{R}^r\big).$$

$\Delta_{[\alpha:\beta]}$ is an r-dimensional sub vector space of T_xM. Since

$$(\lambda\alpha, \lambda\beta) \otimes \mathbf{R}^r = (\alpha, \beta) \otimes \mathbf{R}^r$$

$\Delta_{[\alpha:\beta]}$ is well defined.

On the other hand, if $X \in \Delta_{[\alpha:\beta]} \cap \Delta_{[\alpha':\beta']}$, with $[\alpha : \beta] \neq [\alpha', \beta']$, we have: $X = \psi(Y)$ with

$$Y \in (\alpha, \beta) \otimes \mathbf{R}^r \ \cap \ (\alpha', \beta') \otimes \mathbf{R}^r.$$

then there exist $v, w \in \mathbf{R}^r$ such that

$$Y = (\alpha, \beta) \otimes v \ = \ (\alpha', \beta') \otimes w.$$

This implies:

– either $v = w = 0$, which means $Y = 0$ then $X = 0$
– or v and w are nonzero collinear vectors, and $[\alpha : \beta] = [\alpha' : \beta']$, which is excluded. Then, necessarily $X = 0$. Thus, the distributions $(\Delta_{[\alpha:\beta]})_{[\alpha:\beta]\in\mathbb{P}^1(\mathbf{R})}$ are piecewise transverse. Consequently, every choice of 3 functions ρ_1, ρ_2, ρ_3 from M to $\mathbb{P}^1$, defines an almost 3-web on M.

□

Definition 5. *We say that two almost-webs are isoclinicly equivalent if they define the same tensorial structure.*

Consider now the particular case of 2 almost webs $\mathcal{W}$, $\mathcal{W}'$ of type $\mathcal{W} = \{T^h, T^v, T^t\}$ and $\mathcal{W}' = \{T^h, T^v, T^{t'}\}$ (Only the transversal distributions are different). Let

$$\psi_x : \mathbf{R}^2 \otimes T_x^h \longrightarrow T_xM$$

be the associated almost-Grassmann structure, then

$$T_x^h = \psi_x\big((1,0) \otimes T_x^h\big), \quad T_x^v = \psi_x\big((0,1) \otimes T_x^h\big), \quad T_x^t = \psi_x\big((1,-1) \otimes T_x^h\big).$$

$\mathcal{W}'$ will define the same tensorial structure if and only if there exists $[\alpha : \beta] \in \mathbb{P}^1$ such that $T_x^{t'} = \psi_x([\alpha : \beta] \otimes T_x^h)$, which means

$$T^{t'} = \mathrm{Im}(\alpha\, h \, + \, \beta\, jh)$$

Naturally, we get $[\alpha : \beta] \neq [1 : 0]$ and $[\alpha : \beta] \neq [0, 1]$ since the values $[1 : 0]$ and $[0 : 1]$ define the distributions T^h and T^v.

Definition 6. *We say that $T^{t'}$ is an isoclinic deformation of T^t.*

Thus, $\mathcal{W}$ and $\mathcal{W}'$ are equivalent if and only if $\mathcal{W}'$ can be obtained from $\mathcal{W}$ by a "deformation" with respect to a (projective) parameter of the transversal distribution T^t. (If the parameter is $[1 : -1]$ we get T^t).

In general, two almost webs are isoclinically equivalent if the 3 distributions of $\mathcal{W}'$ are obtained by an isoclinic "deformation" of those of $\mathcal{W}$.

Notation 1. *If $\mathcal{W} = \{T^h, T^v, T^t\}$, we set, for $[\alpha : \beta] \in \mathbb{P}^1$, with $\alpha \neq 0$, $\beta \neq 0$*

$$\Delta_{[\alpha:\beta]} = \mathrm{Im}(\alpha\, h \, + \, \beta\, jh).$$

We denote by $\mathcal{W}_{[\alpha:\beta]}$ the web $\{T^h, T^v, \Delta_{[\alpha:\beta]}\}$.

We also set

$$b = -\frac{\alpha}{\beta}$$

and $\mathcal{W}_{[\alpha:\beta]}$, will be denoted $\mathcal{W}_b$.

The objective of the following paragraphs is to give invariants of an almost-Grassmann structure of type $(2, r)$, considering them as isoclinic "deformations" of an almost web, we will then give geometric interpretations of these invariants.

4. Invariants of an Almost-Grassmann Structure

Proposition 2. *Let $\mathcal{W} = (T^h, T^v, T^t)$ be an almost web, $\{h, j\}$ Nagy's tensors, and*

$$\mathcal{W}_b = \{T^h, T^v, \Delta_b = \mathrm{Im}(h - bjh)\}$$

an isoclinic deformation of $\mathcal{W}$. Then, Nagy's tensors of $\mathcal{W}_b$ are:

$$\begin{cases} h' &= h \\ j' &= bjh + \frac{1}{b}jv \end{cases}$$

Indeed, let $\{h', j'\}$ be Nagy's tensors of $\mathcal{W}_b$. It's clear that $h' = h$. On the other hand, $j'|_{T^h}$ is the projection on T^v in the decomposition $TM = T^h \oplus \Delta_b$. Then, if $X \in TM$, there exists $Y \in TM$ such that

$$j'(hX) = vY \quad \text{and} \quad hX - vY \in \Delta_b.$$

There exists then $Z \in TM$ such that

$$hX - vY = hZ - bjhZ.$$

Thus

$$hX = hZ \quad \text{and} \quad vY = bjhZ.$$

Hence, $vY = bjhX$ and so

$$j'hX = bjhX \tag{1}$$

for all $X \in TM$. On the other hand, $j'|_{T^v}$ is the projection on T^h in the decomposition $TM = T^v \oplus \Delta_b$. Consequently, if $X \in TM$, there exists $Y \in TM$ such that

$$j'vX = hY \quad \text{and} \quad vX - hY \in \Delta_b.$$

Then, there exists $Z \in TM$ such that

$$vX - hY = hZ - bjhZ.$$

Thus

$$vX = -bjhZ \quad \text{and} \quad hY = -hZ.$$

Hence $hY = -(-\frac{1}{b}jvX)$ and then

$$j'vX = \frac{1}{b}jvX. \tag{2}$$

From (1) and (2) we have $j' = bjh + \frac{1}{b}jv$.

Proposition 3. *(Invariant of torsion)—Let $\mathcal{W}$ be an almost web, T the torsion in Chern's connection, and*

$$\omega = \frac{TrT}{r-1},$$

where $Tr : X \mapsto Trace\ T(X,\cdot)$. Then the tensor $\Im \in \Lambda^2 T^*M \otimes TM$ defined by

$$\Im = T - (i_h\omega \wedge h + i_v\omega \wedge v)$$

is invariant by isoclinic deformations, and thus an invariant of the almost-Grassmann structure.

Proof. Let ∇' be Chern's connection associated with $\{h', j'\}$. From [4], we have:

$$\begin{cases} \nabla'_{h'X} h'Y = h'j'[h'X, j'h'Y] = \frac{1}{b} jv[hX, bjhY] = \frac{1}{b} hX(b)hY + \nabla_{hX} hY \\ \nabla'_{v'X} v'Y = v'j'[v'X, j'v'Y] = bjh[vX, \frac{1}{b} jvY] = -\frac{1}{b} vX(b)vY + \nabla_{vX} vY \\ \nabla'_{h'X} v'Y = v'[h'X, v'Y] = v[hX, vY] = \nabla_{hX} vY \\ \nabla'_{v'X} v'Y = h'[v'X, h'Y] = h[vX, hY] = \nabla_{vX} hY \end{cases}$$

then

$$\nabla'_X Y = \nabla_X Y + \frac{1}{b} hX(b)hY - \frac{1}{b} vX(b)vY$$

thus

$$\nabla' = \nabla + \frac{1}{b}(d_h b \otimes h - d_v b \otimes v)$$

It follows that:

$$T' = T + \frac{1}{b}(d_h b \wedge h - d_v b \wedge v) \quad (3)$$

where T' is the torsion of ∇'. Using the trace, we get:

$$Tr(T') = Tr(T) + \frac{1}{b}\{(r-1)d_h b - (r-1)d_v b\},$$

which means

$$\frac{r-1}{b} d_{h-v} b = Tr(T') - Tr(T). \quad (4)$$

Applying i_h on (4), we get:

$$\frac{1}{b} d_h b = \frac{1}{r-1}\big(i_h Tr(T') - i_h Tr(T)\big). \quad (5)$$

Applying i_v:

$$\frac{1}{b} d_v b = \frac{1}{r-1}\big(i_v Tr(T) - i_v Tr(T')\big). \quad (6)$$

By substitution of (5) and (6) in (3), we get:

$$T' = T + \frac{1}{r-1}(i_h Tr(T') - i_h Tr(T)) \wedge h - \frac{1}{r-1}(i_v Tr(T) - i_v Tr(T')) \wedge v,$$

which proves that the tensor $\Im = T - (i_h\omega \wedge h + i_v\omega \wedge v)$ is invariant. □

Proposition 4. *(Invariant of curvature)—Let $\mathcal{W}$ be an almost web, R the curvature of Chern's connection, and $\rho \in \Lambda^2 T^*M$ the scalar 2-form given by*

$$\rho(X,Y) = \frac{1}{r} Trace\big(R(X,Y) : Z \longmapsto R(X,Y)Z.\big)$$

Then the tensor $\Re \in \Lambda^2 T^* \otimes T^*M \otimes TM$ given by

$$\Re = R - \rho \otimes I$$

is invariant by isoclinic deformations, and thus an invariant of the almost Grassmann structure.

Proof. Let R' be the curvature of the connection ∇'. We have:

$$\begin{aligned} R'(X,Y)Z &= R(X,Y)Z - \frac{1}{b}((d_h b)[X,Y])hZ - \frac{1}{b}((d_v b)[X,Y])vZ \\ &+ \left[-\frac{1}{b^2}(db(X))(d_h b)(Y)) + \frac{1}{b}X(d_h b(Y))\right]hZ \\ &+ \left[\frac{1}{b^2}(db(X))(d_v b)(Y)) - \frac{1}{b}X(d_v b(Y))\right]vZ \\ &+ \left[\frac{1}{b^2}(db(Y))(d_h b)(X)) - \frac{1}{b}Y(d_h b(X))\right]hZ \\ &+ \left[-\frac{1}{b^2}(db(Y))(d_v b)(X)) + \frac{1}{b}Y(d_v b(X))\right]vZ. \end{aligned}$$

But

$$\begin{aligned} d_v b([hX,hY]) &= -(dd_v b)(hX,hY) = dd_h b(hX,hY) \\ d_h b([vX,vY]) &= -dd_h b(vX,vY) \\ dd_h b(hX,vY) &= -vY \cdot hX \cdot b - d_h b([hX,vY]) \\ dd_h b(vX,hY) &= vX \cdot hY \cdot b - d_h b([vX,hY]) \\ dd_h b(hX,vY) &= -dd_v b(hX,vY) = -hX \cdot vY \cdot b + d_v b([hX,vY]) \\ dd_h b(vX,hY) &= -dd_v b(vX,hY) = hY \cdot vX \cdot b + d_b b(vX,hY) \end{aligned}$$

Then

$$\begin{aligned} R'(X,Y)Z \quad = R(X,Y)Z \quad & + \tfrac{1}{b} dd_h b(hX,hY)Z + \tfrac{1}{b} dd_h b(vX,vY)Z \\ & \tfrac{1}{b} dd_h b(hX,vY)hZ + \tfrac{1}{b} dd_h b(vX,hY)hZ \\ & \tfrac{1}{b} dd_h b(hX,vY)vZ + \tfrac{1}{b} dd_h b(vX,hY)vZ \\ & \tfrac{1}{b^2}(d_h b) \wedge (d_v b)(X,Y)hZ + \tfrac{1}{b^2}(d_h b) \wedge (d_v b)(X,Y)vZ \end{aligned},$$

which means

$$R' = R + \left(\frac{1}{b} dd_h b + \frac{1}{b^2} d_h b \wedge d_v b\right) \otimes I$$

then, using the trace:

$$\rho' = \rho + \left(\frac{1}{b} dd_h b + \frac{1}{b^2} d_h b \wedge d_v b\right)$$

so

$$R' - \rho' \otimes I = R - \rho \otimes I,$$

which shows that $\Re$ is an invariant of the almost-Grassmann structure. □

We will now construct a third invariant, using Grifone's formalism (cf. [20]), for connection theory. Recall that a connection ∇ on M can be characterized by the horizontal projector H, which is a tensors field on TM. The torsion T and the curvature R of ∇ can be seen as 2 semi-basic tensors of type (12) defined respectively by:

$$t = [J,H] \quad \text{and} \quad K = -\frac{1}{2}[H,H].$$

which are related to T and R by the following formulas:

$$\begin{aligned} t(X,Y) &= T^v(X,Y) \\ K(X,Y) &= R^V(X,Y)\,S \end{aligned}$$

for $X, Y \in TTM$, where T^v and R^V are respectively the vertical liftings of T and R, and the vector S an arbitrary spray (cf. [19,20]).

Proposition 5. *(Derived invariant of curvature)—The semi-basic tensor of type (13) on TM*

$$\Re^{(1)} = [J, K] - \rho \wedge J$$

is invariant by isoclinic deformations, and thus is an invariant of the almost-Grassmann structure.

Lemma 1. *The semi-basic tensor of type (12) on TM $K - \rho \otimes C$ is invariant by isoclinic deformations.*

Indeed, we have $I^V = J$

$$\Re^V = R^V - \rho \otimes J$$

then

$$\Re^V(X, Y)\, S = R^V(X, Y)\, S - \rho(X, Y)\, C = K(X, Y) - \rho(X, Y) \otimes C$$

then the semi-basic tensor on TM, $K - \rho \otimes C$ is invariant by isoclinic deformations.

The tensor of the previous lemma is in fact the lifting of $\Re$ and does not give a new invariant. But $[J, K - \rho \otimes C]$ is a new non trivial invariant. We have:

$$[J, K - \rho \otimes C] = [J, K] - d_J\rho \otimes C + d\rho \otimes JC - \rho \wedge [J, C] = [J, K] - \rho \wedge J$$

since $d_J\rho = 0$, ρ being a basic form. Hence $[J, K] - \rho \wedge J$ is invariant.

REMARK—If we apply the same calculation, starting with the invariant $\Im$, we do not get a new invariant because $\Im^V$ is a basic tensor and then $[J, \Im^V] = 0$.

5. Invariants by Isoclinic Deformation of the Other Distributions

Let $\mathcal{W} = \{T^h,\ T^v,\ T^t\}$ be an almost web. We calculated invariants by isoclinic deformations of the transverse bundle. Naturally, we can also deform the other bundles and obtain other invariants. The following proposition states that if $\mathcal{W}$ is a web, the nullity of the invariants does not depend on the choice of bundle.

We will denote by $\Im_t$, $\Re_t$, $\Re_t^{(1)}$ the invariants constructed in the previous paragraph, and $\Im_h$, $\Re_h$, $\Re_h^{(1)}$ the invariants obtained by variating T^h while keeping the same Chern's connection (the invariants obtained by variating T^v are equal to those obtained by variation of T^h since T^h and T^v are symmetrical with respect to Chern's connection).

Proposition 6. *Let $\mathcal{W} = \{T^h,\ T^v,\ T^t\}$ be an almost web and*

$$\begin{aligned} \tilde{T} \quad = \quad & T(X, Y) + T\bar{\wedge} jh(X, Y) + 2(jh)^* jT(X, Y) + 2vj[jhX, jhY] \\ & -4h[jhX, jhY] - 2jh[jhX, vY] + 2jh[jhY, vX] - 2jh[vX, vY]. \end{aligned}$$

Then the tensor

$$\Im_h = \tilde{T} - \frac{1}{r-1}\{i_{h-jh}Tr(\tilde{T}) \wedge (h - jh) + i_{v+jh}Tr(\tilde{T}) \wedge (v + jh)\}$$

is invariant by isoclinic deformations of the bundle T^h.

Proof. To calculate $\Im_h$, it is enough to exchange T^h and T^t and use the result of Proposition 3. □

Lemma 2. *If we exchange T^h and T^t, Nagy's tensors become*

$$\begin{cases} \tilde{h} = h - jh, \\ \tilde{j} = v - h - hj. \end{cases}$$

Proof. $\tilde{h}$ is the projection on T^t for the $T^t \oplus T^v$ decomposition. We then have $\tilde{h} = h - jh$. Indeed, $T^t = \mathrm{Im}(h - jh)$, and $h - jh$ is a projector with kernel T^v.

On the other hand, $\tilde{j}$ exchanges the transversal and vertical vectors: for $X \in TM$, there exists $Y \in TM$ such that $\tilde{j}\tilde{h}X = vY$; Y is such that $\tilde{h}X - vY \in T^h$. Thus, there exists $Z \in TM$ such that $(hX - jhX) - vY = hZ$. We deduce that $hX = hZ$ and $-jhX - vY = 0$, which means that $vY = -jhX$ and

$$\tilde{j}\tilde{h} = -jh.$$

In a similar way, for all $X \in TM$, there exists $Y \in TM$ such that $\tilde{j}\tilde{v}X = \tilde{h}Y$ and Y is such that $\tilde{h}Y - \tilde{v}X \in T^h$. Then, there exists $Z \in TM$ such that $\tilde{h}Y - \tilde{v}X = \tilde{h}Z$. Hence $(hY - jhY) - (vX + jhX) = hZ$, which means that $-vx - jhX - jhY = 0$ and $jhY = -(vX + jhX)$, then $\tilde{h}Y = -(jvX + hX) + (vX + jhX)$, and consequently:

$$\tilde{j}\tilde{v} = -jv - h + v + jh,$$

thus

$$\tilde{j} = v - h - hj.$$

□

Lemma 3. *The torsion of Chern's connection calculated by exchanging T^h and T^t is*

$$\begin{aligned} \tilde{T} &= T(X,Y) + T\bar{\wedge} jh(X,Y) + 2(jh)^*jT(X,Y) + 2vj[jhX, jhY] \\ &\quad -4h[jhX, jhY] - 2jh[jhX, vY] + 2jh[jhY, vX] - 2jh[vX, vY]. \end{aligned}$$

Indeed, we have:

$$\tilde{\nabla}_{\tilde{h}X}\tilde{h}Y = \tilde{h}\tilde{j}[\tilde{h}X, \tilde{j}\tilde{h}Y] = \nabla_{hX}hY - \nabla_{hX}jhY + v[jhX, jhY] - \nabla_{jhY}hX - h[jhX, jhY] + \nabla_{jhY}jhX + jh[jhX, jhY] - hj[jhX, jhY].$$
$$\tilde{\nabla}_{\tilde{h}X}\tilde{v}Y = \tilde{v}[\tilde{h}X, \tilde{v}Y] = \nabla_{hX}vY + \nabla_{hX}jhY - v[jhX, vY] - v[jhX, jhY] - \nabla_{vY}jhX - \nabla_{jhY}jhX - jh[jhX, vY] - jh[jhX, jhY].$$
$$\tilde{\nabla}_{\tilde{v}X}\tilde{h}Y = \tilde{h}[\tilde{v}X, \tilde{h}Y] = \nabla_{vX}hY - h[vX, jhY] + \nabla_{jhX}hY - h[jhX, jhY] - \nabla_{vX}jhY + jh[vX, jhY] - \nabla_{jhX}jhY + jh[jhX, jhY].$$
$$\tilde{\nabla}_{\tilde{v}V}\tilde{v}Y = \tilde{v}\tilde{j}[\tilde{v}X, \tilde{j}\tilde{v}Y] = -jh[vX, vY] + \nabla_{vX}jhY - jh[vX, jhy] + \nabla_{vX}vY - jh[jhX, vY] + \nabla_{jhX}jhY - jh[jhX, jhY] + \nabla_{jhX}vY.$$

Thus

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + T(jhX, vY) + h[jhX, vY] + jT(jhX, jhY) + vj[jhX, jhY] \\ &\quad -2h[jhX, jhY] - 2jh[jhX, vY] - h[vX, jhY] - jh[vX, vY]. \end{aligned}$$

And the torsion $\tilde{T}$ is given by:

$$\begin{aligned} \tilde{T} &= \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] \\ &= T(X,Y) + T\bar{\wedge} jh(X,Y) + 2(jh)^*jT(X,Y) + 2vj[jhX, jhY] \\ &\quad -4h[jhX, jhY] - 2jh[jhX, vY] + 2jh[jhY, vX] - 2jh[vX, vY]. \end{aligned}$$

Using now Proposition 3, we directly get the invariant $\Im_h$.

Proposition 7. *If $\mathcal{W}$ is a web, (the distributions T^h, T^v, T^t are integrable), then*

$$\Im_t = 0 \Longleftrightarrow \Im_h = 0$$

In particular, for a web, if one of the torsion invariants vanishes, all other torsion invariants also vanish.

Proof. If $\mathcal{W}$ is a web, then

$$\tilde{T} = T(X,Y) + T\bar{\wedge} jh(X,Y) + 2(jh)^*jT(X,Y).$$

Since T^t is integrable, we then get $j^*T = -jT$, and

$$\tilde{T} = -\gamma T + T\bar{\wedge} jh.$$

It follows that $Tr(\tilde{T}) = i_{jh-\gamma}Tr(T)$ since $Tr(\gamma T) = i_\gamma Tr(T)$ and $Tr(jhT) = 0$. Then, if $\Im_h = 0$, which means

$$-\gamma T + T\barwedge jh + \frac{1}{r-1}\{i_h trT \wedge (h - jh) + i_\gamma TrT \wedge (v + jh)\} = 0 \tag{7}$$

this is equivalent to

$$hT = \frac{1}{r-1} i_h Tr(T) \wedge h.$$

Indeed, the necessary condition is obvious. Suppose that

$$hT = \frac{1}{r-1} i_h Tr(T) \wedge h, \tag{8}$$

$\mathcal{W}$ being a web. The Equation (8) is equivalent to

$$vT = -\frac{1}{r-1} i_{hj} Tr(T) \wedge v. \tag{9}$$

But for the same reason, we have, $i_j Tr(T) = -Tr(jT)$ then the Equation (9) can be written

$$vT = \frac{1}{r-1} i_v Tr(T) \wedge v, \tag{10}$$

which gives from (8) and (10):

$$T = \frac{1}{r-1}(i_h Tr(T) \wedge h + i_v Tr(T) \wedge v).$$

Then

$$T\barwedge jh = \frac{1}{r-1}(i_{jh} Tr(T) \wedge v + i_v Tr(T) \wedge jh). \tag{11}$$

Using $i_h trT = -i_{jh} trT$, the Equations (8), (10) and (11) show that the Equation (7) is verified, which means $\Im_h = 0$. In conclusion, it is enough to notice from (10) that $\Im = 0$ if and only if the Equation (8) is verified. □

6. Interpretation of the Invariants of an Almost-Grassmann Structure

1. Interpretation of $\Im = 0$:

Proposition 8. *Let $\mathcal{W}$ be an almost web such that $\Im_t = 0$; then the distributions T^h and T^v are integrable. In particular, if $\Im_t = 0$ and $\Im_h = 0$, then $\mathcal{W}$ is a web.*

Indeed, suppose that $\Im_t = 0$, which means that $T = i_h\omega \wedge h + i_v\omega \wedge v$. We get $vT(hX, hY) = 0$ *and* $hT(vX, vY) = 0$, for all $X,\ Y \in TM$. On the other hand:

$$T(hX, hY) = \nabla_{hX} hY - \nabla_{hY} hX - [hX, hY]$$

and

$$T(vX, vY) = \nabla_{vX} vY - \nabla_{vY} vX - [vX, vY],$$

which gives $v[hX, hY] = h[vX, vY] = 0$, then T^h and T^v are integrable.

Theorem 4. *Let $\mathcal{W} = \{T^h,\ T^v,\ T^t\}$ be a 3-web on a $2r-$ dimensional manifold M, $(r \geq 3)$, and $p \in M$. Then, for every $b_0 \in R$, there exists a neighborhood U of p and a function b defined on U such that $b(p) = b_0$ and the distribution $\Delta_b = \mathrm{Im}(h - bjh)$ is integrable if and only if $\Im = 0$.*

Proof. This theorem is essentially due to Akivis, [5] where the webs verifying this property are called *isoclinic webs* (cf. for example [6]). We just have to prove that $\Im = 0$ if and only if there exists a 1-form $a \in (T^h)^*$ such that

$$h^*T = a \wedge h.$$

Suppose that $\Im = 0$; we then have $T = i_h\omega \wedge h + i_v\omega \wedge v$. By setting

$$a(X) = i_h\omega(X) \qquad \text{for all } X \in T^h,$$

we have $h^*T = a \wedge h$.

Conversely, suppose that there exists $a \in T^{h^*}$ such that $h^*T = a \wedge h$ and let's show that *if $\mathcal{W}$ is a web*, then

$$T = a \wedge h - i_j a \wedge v.$$

Indeed, we have $h^*T = a \wedge h$, which means that for all $X, Y \in TM$, we have: $T(hX, hY) = a(X)hY - a(Y)hX$, then

$$jT(hX, hY) = a(X)jhY - a(Y)jhX. \tag{12}$$

From the integrability identities if ∇ is Chern's connection associated with an almost-web, defined by Nagy's tensores $\{h,\ j\}$ then $\nabla h = 0$, $\nabla j = 0$, $T(hX, vY) = 0$, where T is the torsion of ∇. The almost-web is a web if the integrability conditions are satsfied: $vT(hX, hY) = 0$, $hT(vX, vY) = 0$, and $jT(X, Y) = -T(jX, jY)$, the Equation (12) can be written:

$$-T(jhX, jhY) = a(x)jhY - a(Y)jhX,$$

which means

$$-T(vX, vY) = a(jX)vY - a(jY)vX$$

thus:

$$v^*T = -i_j a \wedge v$$

then

$$T = a \wedge h - i_j a \wedge v.$$

If we set

$$\omega = i_{h-hj} a,$$

we then have $T = i_h\omega \wedge h + i_v\omega \wedge v$. By calculating the trace, we find $\omega = \frac{Tr\,T}{r-1}$, and then $\Im = 0$. □

2. Interpretation of $\Im = 0$, $\Re = 0$:

Theorem 5. *Let $\mathcal{W}$ be a web such that $\Im = 0$ and $\Re = 0$. Then $\mathcal{W}$ is parallelizable.*

Proof. Since $\Re = 0$, we have, from Lemma 1,

$$K = \rho \otimes C.$$

Consider Bianchi's first identity; we have $[J, K] = [H, t]$ (cf. [20]). Then

$$[J, K] = [J, \rho \otimes C] = d_J\rho \otimes C + \rho \wedge [J, C]$$

But ρ is basic, then $d_J\rho = 0$ and $[J, C] = J$, and consequently

$$(i) \qquad [J, K] = \rho \wedge J.$$

On another hand, if $\Im = 0$, we have $T = i_h\omega \wedge h + i_v\omega \wedge v$, then

$$t = i_h\omega \wedge h^V + i_v\omega \wedge v^V,$$

thus

$$[H,t] = d_H i_h\omega \wedge h^V - i_h\omega \wedge [H,h^V] + d_H i_v\omega \wedge v^V - i_v\omega \wedge [H,v^V].$$

Since $i_h\omega$ is a basic form, we have $d_H i_h\omega = di_h\omega$, and in a similar way $d_H i_v\omega = di_v\omega$,, then

$$[H,t] = di_h\omega \wedge h^V - i_h\omega \wedge [H,h^V] + di_v\omega \wedge v^V - i_v\omega \wedge [H,v^V].$$

But for every tensor L of type (11) on M, we have:

$$[H,L^V] = (\mathcal{A}\nabla L)^V + (LT)^V$$

where $\mathcal{A}$ denotes the anti-symmetrization. Since $\nabla h = 0$, we have

$$[H,h^V] = (hT)^V = i_h\omega \wedge h^V.$$

And in a similar way,

$$[H,v^V] = i_v\omega \wedge v^V.$$

Then

$$[H,t] = di_h\omega \wedge h^V - i_h\omega \wedge i_h\omega \wedge h^V + di_v\omega \wedge v^V - i_v\omega \wedge i_v\omega \wedge v^V = di_h\omega \wedge h^V + di_v\omega \wedge v^V$$

Hence:

$$[H,t] = \left(di_h\omega \wedge h + di_v\omega \wedge v\right)^V$$

From (i), *Bianchi's first identity gives:*

$$\rho \wedge J = \left(di_h\omega \wedge h + di_v\omega \wedge v\right)^V,$$

which means

$$(\rho \wedge I)^V = \left(di_h\omega \wedge h + di_v\omega \wedge v\right)^V$$

or, in an equivalent way:

(ii) $$\rho \wedge I = di_h\omega \wedge h + di_v\omega \wedge v.$$

Calculating this expression's trace, we find

(iii) $$2(r-1)\rho = rd\omega - i_h di_h\omega - i_v di_v\omega.$$

On another hand, we have $R(hX,hY)Z = 0$ for all X,Y,Z, then

$$h^*\rho = 0,$$

hence

$$rh^*d\omega - h^*i_h di_h\omega = 0,$$

which means

$$rh^*d\omega - 2h^*di_h\omega = 0.$$

But, for every tensor L of type (11) such that $[L,L] = 0$ we have:

$$L^*d\omega = d_L i_l\omega.$$

So, we have: $rd_h i_h\omega - 2d_h i_h\omega = 0$,, which means $d_h i_h\omega = 0$ and then

(iv) $$i_h d_h\omega = di_h\omega.$$

The equation (iii) can be written

$$2(r-1)\rho = rd\omega - d\omega = (r-1)d\omega$$

hence:

$$(v) \qquad \rho = \frac{1}{2}d\omega.$$

On another hand, multiplying (ii) by h, we get

$$\rho \wedge h = (di_h\omega) \wedge h$$

taking the trace and using (iv):

$$r\rho - i_h\rho = rdi_h\omega - i_hdi_h\omega = (r-1)di_h\omega$$

Then

$$(r-2)\rho = (r-1)d\omega.$$

If $r = 2$ we get $d\omega = 0$ so, from (v), $\rho = 0$.
If $r \neq 2$,

$$\rho = \frac{r-1}{r-2}d\omega$$

and, by comparison with (v), $\rho = 0$. We then have $\rho = 0$ for all $r \geq 2$ and then $R = 0$, which means $\mathcal{W}$ is parallelizable. □

3. Interpretation of $\Re^{(1)} = 0$:

Before studying the third invariant, recall the definition of subwebs, and transversally geodesic webs (cf. [8,9,18,23]).
Let M be a $2r$-dimensional manifold, $r \geq 1$,

Definition 7. *Let $\mathcal{W}$ be a web on M and S a submanifold of M. A web $\overline{\mathcal{W}}$ on S is a sub-web of $\mathcal{W}$ if it leaves are the intersection of S with the leaves of $\mathcal{W}$.*

Definition 8. *A 2-dimensional sub vector space $\mathcal{P}$ of T_pM is a transversal plan if $\dim \mathcal{P} \cap T_p^h = 1$ and if it is invariant with respect to j_p.*

A $2-$ dimensional surface S of M is said to be transversally geodesic if, for all $q \in S$, the tangent space T_qS is a transversal plan.

Let $X \in \mathcal{P} \cap T_p^h$, $X \neq 0$. We have $jX \in \mathcal{P} \cap T_p^v$; then $\mathcal{P}$ is spanned by X and jX. Conversely, for every non-zero horizontal X, the plan spanned by X and jX is invariant by j and consequently is a transversal plan. Thus, a transversal plan is a plan of type Vect(hX, jhX) with $X \in T_pM$, $X \neq 0$.

We easily see that a 2-dimensional surface S is endowed with a $\mathcal{W}$-subweb structure if and only if it's transversally geodesic.

Indeed, let S be a transversally geodesic 2-dimensional surface. It's clear that Nagy's tensors $\{h, j\}$ can be restricted on TS and consequently the surface S can be endowed with a 3-web structure.

Conversely, if S is a subweb of M and $p \in S$, then its horizontal, vertical, and transversal distributions are contained in the tangent space T_pS and this plan is invariant by j_p. It follows that S is transversally geodesic.

We can also verify that a transversally geodesic surface is an autoparallel submanifold with respect to Chern's connection. We deduce that the leaves of the subweb induced on a transversally geodesic surface are geodesic of Chern's connection associated with $\mathcal{W}$, which justify the terminology.

Definition 9. *A 3-web $\mathcal{W}$ is said to be transversally geodesic in a point $p \in M$ if, for every horizontal vector $X_p \in T^h_p M$, there exists a neighborhood U of p and a field of horizontal vectors X on U extending X_p, such that the field of transversal plans spanned by X and jX is integrable (U is then a partition of transversally geodesic leaves).*

Theorem 6. *A web $\mathcal{W}$ is transversally geodesic if and only if $\Re^{(1)} = 0$.*

Proof. It is a result proved by Akivis (cf. [23]) and showed again in an intrinsic way by Nagy (cf. [18]). What is new here is the interpretation in terms of invariants of almost-Grassmann structure. For the proof, it is enough to notice that $\Re^{(1)} = 0$ if and only if $[J, K] = \rho \wedge J$. But writing this equation locally, we get

$$R^{\mu}_{\alpha\beta,\gamma} + R^{\mu}_{\beta\gamma,\alpha} + R^{\mu}_{\gamma\alpha,\beta} = \rho_{\alpha\beta}\delta^{\mu}_{\gamma} + \rho^{\mu}_{\beta\gamma}\delta^{\mu}_{\alpha} + \rho_{\gamma\alpha}\delta^{\mu}_{\beta},$$

which is Akivis' condition for a web to be transversally geodesic. □

We deduce the following result:

Corollary 1. *A 3-web is Grassmannian if and only if $\Im = 0$ and $\Re^{(1)} = 0$.*

7. Conclusions

A 3- web of dimension r on a manifold M of dimension $2r$ is given by 3 foliations in general position. Two webs are equivalent at a point p of M if there is a germ of local diffeomorphisms at p that exchanges them.

Let $x \in \mathbb{P}^{r+1}$ and $\hat{x}$ the Schubert manifold of all straight lines intersecting at the point x. $\hat{x}$ is an $r-$ dimensional submanifold of $G(r+1,1)$. A hypersurface V of $\mathbb{P}^{r+1}$ defines an r-dimensional foliation on an open set $\mathcal{U} \subset G(r+1,1)$: the leaves are $\hat{x}$ with $x \in V$. If we consider 3 hypersurfaces V_α $(\alpha = 1,2,3)$, in a general position in $\mathbb{P}^{r+1}$, they define a 3-web of dimension r on an open set $\mathcal{U}$ of $G(r+1,1)$ that we call the Grassmann web. A web is said to be Grassmannizable if it is equivalent to a Grassmannian web.

Akivis has shown that Grassmannizable webs are webs that are both isoclinic and transversally geodesic. In this paper, we find three invariants $\Im, \Re, \Re^1$, of type $(2,r)$ of Hangan tensorial structures expressed in terms of torsion and curvature of the unique Chern connection associated with the web as well as in terms of the Nagy's tensors that define the foliations; we show the following results:

1. $\mathcal{W}$ is isoclinic if and only if $\Im = 0$.
2. $\mathcal{W}$ is transversally geodesic if and only if $\Re^1 = 0$.
3. $\mathcal{W}$ is Grassmannizable if and only if $\Im = 0$ and $\Re^1 = 0$.
4. If $\Im = 0$ and $\Re = 0$, then $\mathcal{W}$ is parallelizable.

Author Contributions: Conceptualization, J.S.; Methodology, J.S. and R.A.; Validation, J.S. and R.A.; Formal analysis, J.S.; Investigation, J.S.; Resources, R.A.; Writing – original draft, J.S.; Visualization, R.A.; Supervision, R.A.; Project administration, R.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Gronwall, T.H. Sur les équations entre trois variables représentables par des nomogrammes à points aligné. *J. Liouville* **1912** , *8*, 59–102.
2. Blaschke,W. *Einführung in die Geometrie der Waben*; Birkhäuser: Stuttgart, Germany, 1955; p. 108.
3. Akivis, M.A.; Shelkhov, A.M. *Geometry of Algebra of Multi Dimensional Three-Webs*; Kulwer Academic Publishers: Dordrecht, The Netherlands, 1992.

4. Nagy, P.T. Invariant tensorfields and the canonical connection of a 3-web. *Aequationes Math.* **1988**, *35*, 31–44. [CrossRef]
5. Akivis, M.A. On isoclinic three-webs and their interpretation in a ruled space of projective connection. *Sibirsk. Mat. Zh.* **1974**, *15*, 1–9+3–15. [CrossRef]
6. Goldberg, V.V. *Theory of Multicodimensional* $(n+1)$*-Webs*; Kulwer Academic Publishers: Dordrecht, The Netherlands, 1988.
7. Goldberg, V.V. On a linearizability condition for a three-web on a two-dimensional manifold. In *Differential Geometry, (Peniscola, 1988)*; Lecture Notes in Math. 1410; Springer: Berlin, Germany; New York, NY, USA, 1989; pp. 223–239.
8. Goldberg, V.V. The almost-Grassmann manifold that is connected with an $(n+1)$-web of multidimensional surfaces. *Izv. Vyssh. Uchebn. Zaved. Mat.* **1975**, *159*, 29–38. Erratum in *Soviet Math. (Iz. VUZ)* **1975**, *19*, 23–31.
9. Goldberg, V.V. Theory of Multicodimensional $(n+1)$-webs of maximum 2-rank. *Proc. Am. Math. Soc.* **1987**, *100*, 701–708. [CrossRef]
10. Grifone, J.; Muszsnay, Z.; Saab, J. On the linearizability of 3-webs. *Nonlinear Anal.* **2001**, *47*, 2643–2654 [CrossRef]
11. Goldberg, V.V.; Lychagin, V.V. On the Blashke conjecture for 3 webs. *J. Geom. Anal.* **2006**, *16*, 69–115. [CrossRef]
12. Goldberg, V.V.; Lychagin, V.V. On the linearization of planer three-webs and Blashke's conjecture. *C. R. Acad. Sci. Paris Ser.* **2005**, *1*, 341.
13. Muzsnay, Z. On the problem of linearizability of a 3-web. *Nonlinear Anal. Theory Methods Appl.* **2008**, *68*, 1595–1602. [CrossRef]
14. Muzsnay, Z. On the linearizability of 3-webs: End of controversy. *C. R. Acad. Sci. Paris Ser.* **2018**, *356*, 97–99. [CrossRef]
15. Wang, J. On the Gronwall conjecture. *J. Geom. Anal.* **2012**, *22*, 38–73. [CrossRef]
16. Aganov, S.I. Gronwall's conjecture for 3-webs with infinitesimal symmetries. *Commun. Annal. Geom.* **2020**, *28*, 519–545. [CrossRef]
17. Aganov, S.I. Gronwall's Conjecture for 3-webs with two pencils of lines. *Differ. Geom. Its Appl.* **2023**, *91*, 102071. [CrossRef]
18. Nagy, P.T.; Strambch, K. *Loops in Group Theory and Lie Theory*; Walter de Gruyter: Berlin, Germany; New York, NY, USA, 2002.
19. Grifone, J.; Muzsnay, Z. *Variational Principles for Second-Order Differential Equations*; Word Scientific: Singapore, 2000.
20. Grifone, J. Structure presque-tangent et connexions I, II. *Ann. Inst. Fourier* **1972**, *XXII*, 287–334+291–338.
21. Hangan, T.H. Géométrie diffé rentielle Grassmannienne. *Rev. Roumaine Math. Pures Appl.* **1966**, *11*, 519–531.
22. Hangan, T.H. Tensor-Product tangent Bundles. *Arch. Math* **1968**, *19*, 436–440. [CrossRef]
23. Akivis, M.A. Three-webs of mutidimensional surfaces. *Trudy Geometr. Sem.* **1969**, *2*, 7–31.