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On Modulus Statistical Convergence in Partial Metric Spaces

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Abstract: Modulus statistical convergence has been studied in very different general settings such as topological spaces and uniform spaces. In this manuscript, modulus statistical convergence is defined and studied in partial metric spaces.

Keywords: f-statistical convergence; modulus function; strong Cesàro convergence; partial metric space

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1. Introduction

Let $\mathbb{N}$ denote the set of all natural numbers. A sequence of numbers is said to be statistically convergent to a certain number if the terms of that sequence which are far from the limit are indexed by a subset of $\mathbb{N}$ of natural density zero. The notion of statistical convergence was originally proposed by Zygmund [1] in the first edition of his 1935 monograph published in Warsaw. A few years later, Fast [2] introduced the notion of statistical convergence of number sequences via the density of subsets of $\mathbb{N}$ [3,4]. The literature of statistical convergence has, ever since, been developed and enriched in the recent past years with deep and beautiful results provided by many authors [5–13].

The notion of partial metric space was introduced by Matthews [14] as a generalization of a usual metric space in 1994, and he studied its more relevant properties. In particular, he investigated the concept of weightable quasi-metric spaces and provided a partial metric generalization of Banach's contraction principle. Later, O'Neill [15] and Heckmann [16] provided some other generalizations of partial metric spaces. Recently, the concepts of q-Cesàro and statistical convergence in partial metric spaces were introduced in [17], obtaining basic and essential results. Very recently, the authors of [18,19] introduced and studied other several types of convergence in partial metric spaces.

The purpose of this manuscript is to advance one step further on statistical convergence theory and partial metric space theory by introducing and studying f-statistical convergence in partial metric spaces, that is, statistical convergence in partial metric spaces by means of a modulus function f.

2. Materials and Methods

This section is aimed at introducing the necessary tools upon which we will base our results. It is divided into two subsections: modulus statistical convergence and partial metric spaces.

2.1. Modulus Statistical Convergence

According to [20], a function $f : [0, \infty) \to [0, \infty)$ is called a modulus when it satisfies the following:

- $f(x) = 0 \Leftrightarrow x = 0$.

- $f(x+y) \leq f(x) + f(y)\ \forall x, y \geq 0$.
- f is increasing.
- f is continuous from the right at 0.

The above properties force f to be everywhere continuous on $[0, \infty)$. Also, $f(Mx) \leq Mf(x)$ for all $M \in \mathbb{N}$ and all $x \geq 0$, and $f(\frac{x}{k}) \geq \frac{1}{k} f(x)$ for every $x \in \mathbb{R}^+$ and every $k \in \mathbb{N}$. A modulus may be unbounded or bounded. For instance, $f(x) = \frac{x}{x+1}$ is bounded, whereas $f(x) = x^p$ $(0 < p < 1)$ is unbounded.

A modulus function f is said to be compatible [21] provided that for any $\varepsilon > 0$ there can be found $\tilde{\varepsilon} > 0$ and $n_0 = n_0(\varepsilon)$ such that $\dfrac{f(n\tilde{\varepsilon})}{f(n)} < \varepsilon$ for all $n \geq n_0$. According to [21], $f(x) = x + \log(x+1)$ and $f(x) = x + \dfrac{x}{x+1}$ are compatible. However, $f(x) = \log(x+1)$ and $f(x) = W(x)$, where W is the W-Lambert function restricted to $[0, \infty)$ (in other words, the inverse of xe^x), are not compatible. For the study related to a modulus function, one may refer to [22–29] and many others.

The notion of f-density for subsets of $\mathbb{N}$ was originally coined in [30]. In this sense, the f-density of a subset A of $\mathbb{N}$ is defined by

$$d_f(A) := \lim_{n\to\infty} \frac{f(\operatorname{card}(A \cap [1,n]))}{f(n)}$$

provided that the limit exists. When f is the identity, the classical version of density [31] of subsets of $\mathbb{N}$, denoted by $d(A)$, is obtained. Some basic properties of d_f follow:

- Increasingness: $d_f(A) \leq d_f(B)$ whenever $A \subseteq B \subseteq \mathbb{N}$ and $d_f(A), d_f(B)$ exist.
- $d_f(\varnothing) = 0$.
- $d_f(\mathbb{N}) = 1$.
- $0 \leq d_f(A) \leq 1$ for every $A \subseteq \mathbb{N}$ if $d_f(A)$ exists.
- Subadditivity: $d_f(A \cup B) \leq d_f(A) + d_f(B)$ for every $A, B \subseteq \mathbb{N}$ if $d_f(A), d_f(B)$ exist.
- If $A \subseteq \mathbb{N}$ and $d_f(A) = 0$, then $d_f(\mathbb{N} \setminus A) = 1$ (the converse does not hold [30] (Example 2.1)).
- $d_f(A) = 0$ implies $d(A) = 0$ for each $A \subseteq \mathbb{N}$.
- If $A \subseteq \mathbb{N}$ is finite and f is unbounded, then $d_f(A) = 0$.

From the above properties, it is not hard to infer that the collection of all subsets of $\mathbb{N}$ with f-density 0 is an ideal of $\mathcal{P}(\mathbb{N})$. Even more, if f is unbounded, then all finite subsets of $\mathbb{N}$ have null f-density, meaning that the union of all sets with null f-density is the whole of $\mathbb{N}$; therefore, under the assumption that f be unbounded, the collection of all subsets of $\mathbb{N}$ with f-density 0 is a bornology of $\mathcal{P}(\mathbb{N})$.

The next lemma can be found in [30] (Lemma 3.4) and will be exploited later on.

Lemma 1. *For each infinite subset H of $\mathbb{N}$ there is an unbounded modulus function f satisfying $d_f(H) = 1$.*

In [30], by means of the f-density of a subset of $\mathbb{N}$, the following non-matrix concept of convergence is defined: A sequence $(x_n)_{n\in\mathbb{N}}$ is said to be f-statistically convergent to x_0 if for every $\varepsilon > 0$, $\{n \in \mathbb{N} : |x_n - x_0| \geq \varepsilon\}$ has null f-density; in other words,

$$\lim_{n\to\infty} \frac{f(\operatorname{card}\{k \leq n : |x_k - x_0| \geq \varepsilon\})}{f(n)} = 0,$$

written as $f\text{-st}\lim_n x_n = x_0$.

As previously mentioned, the collection of all subsets of $\mathbb{N}$ with f-density 0 is a bornology of $\mathcal{P}(\mathbb{N})$ (for f unbounded). Therefore, f-statistical convergence is a particular case of ideal convergence.

All modulus functions considered throughout the rest of this manuscript will be assumed to be unbounded by default.

2.2. Partial Metric Spaces

This subsection is devoted to introducing some basic definitions and properties related to partial metric spaces [14,15].

Definition 1. *A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$:*

- *Indistancy implies equality: $p(x, x) = p(x, y) = p(y, y) \Leftrightarrow x = y$;*
- *Non-negativity and small self-distances: $0 \leq p(x, x) \leq p(x, y)$;*
- *Symmetry: $p(x, y) = p(x, y)$;*
- *Triangularity: $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.*

The pair (X, p) is called a partial metric space.

Every metric space is obviously a partial metric space, but the converse is not true. The following examples of non-metric partial metric spaces can be found in [14,17,32].

Example 1. *(X, p) is a non-metric partial metric space, where $X := [0, \infty)$ and $p(x, y) := \max\{x, y\}$ for all $x, y \in X$.*

Example 2. *(X, p) is a non-metric partial metric space, where $X := \mathbb{R}$ and $p(x, y) := 2^{\max\{x,y\}}$ for all $x, y \in X$.*

Example 3. *(X, p) is a non-metric partial metric space, where X is the collection of all finite sequences and all infinite sequences of a given set S and $p(x, y) := 2^{-k}$ for k the largest positive integer (possibly ∞) such that $x_i = y_i$ for each $i < k$ being $x, y \in X$ with $x = (x_0, x_1, \dots)$ and $y = (y_0, y_1, \dots)$.*

Example 4. *(X, p) is a non-metric partial metric space, where X stands for the set of all intervals $[a, b]$ for any real numbers $a \leq b$ and $p([a, b], [c, d]) := \max\{b, d\} - \min\{a, c\}$.*

Not necessarily, an element in a partial metric space has a zero distance from itself. However, if we take $p(x, x) = 0$ for every $x \in X$, then (X, p) is precisely a metric space. On the other hand, every partial metric space induces a metric space. Indeed, if (X, p) is a partial metric space, then (X, p^m) is a metric space, where

$$\begin{array}{rccl} p^m : & X \times X & \to & \mathbb{R} \\ & (x, y) & \mapsto & p^m(x, y) := 2p(x, y) - p(x, x) - p(y, y). \end{array} \tag{1}$$

It is well known that each partial metric p on X generates a T_0 topology τ_p on X for which the family of open p-balls

$$\{U_p(x, \delta) : x \in X, \delta > 0\},$$

where $U_p(x, \delta) := \{y \in X : p(x, y) < p(x, x) + \delta\}$, is a base of the topology.

Remark 1. *Let (X, p) be a partial metric space. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X and let $x_0 \in X$. Then:*

i. $(x_n)_{n \in \mathbb{N}}$ is bounded by definition whenever there exists $M > 0$ such that $p(x_n, x_m) < M$ for all $n, m \in \mathbb{N}$.
ii. $(x_n)_{n \in \mathbb{N}}$ is τ_p-convergent to x_0 if and only if $p(x_0, x_0) = \lim_{n \to \infty} p(x_0, x_n)$.
iii. $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence by definition whenever $\lim_{n,m \to \infty} p(x_n, x_m)$ exists.

A partial metric space (X, p) is said to be a complete partial metric space if every Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ in X τ_p-converges to a certain $x_0 \in X$ such that $p(x_0, x_0) = \lim_{n,m \to \infty} p(x_n, x_m)$. According to [14], a sequence is Cauchy in the partial-metric sense precisely when it is Cauchy, in the metric sense of the word, with respect to p^m. As a consequence, a partial metric p is complete precisely when p^m is complete in the metric sense of the word.

In [33], (Corollary 3.8), completeness of uniform spaces with a countable base of entourages (like, for instance, pseudometric spaces) was characterized through the f-statistical convergence of the f-statistically Cauchy sequences.

Throughout the rest of the manuscript, whenever we talk about convergence in a partial metric space, we mean τ_p-convergence.

3. Results

In [17], the definition of statistical convergence in a partial metric space X was given as follows: A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is called statistically convergent to $x_0 \in X$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}\text{card}\{k \leq n : p(x_k, x_0) - p(x_0, x_0) \geq \varepsilon\} = 0,$$

and it is denoted as $\text{st}\lim_n p(x_n, x_0) = p(x_0, x_0)$.

Our first step is to introduce the definition of f-statistical convergence in partial metric spaces.

Definition 2. *Let X be a partial metric space, $(x_n)_{n\in\mathbb{N}} \subseteq X$, and f an unbounded modulus function. We say that the sequence $(x_n)_{n\in\mathbb{N}}$ is f-statistically convergent to $x_0 \in X$ if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{f(\text{card}\{k \leq n : p(x_k, x_0) - p(x_0, x_0) \geq \varepsilon\})}{f(n)} = 0,$$

and we denote it by $f\text{-st}\lim_n p(x_n, x_0) = p(x_0, x_0)$.

Let us display a representative example of an f-statistically convergent sequence in a non-metric partial metric space.

Example 5. *Consider the compatible unbounded modulus $f(x) = x + \dfrac{x}{1+x}$ [21]. Notice that $A := \{n^3 : n \in \mathbb{N}\}$ satisfies that $d_f(A) = 0$. Indeed, $\text{card}(A \cap [1, n]) = \lfloor \sqrt[3]{n} \rfloor$ for all $n \in \mathbb{N}$. Then*

$$\begin{aligned} 0 \quad &\leq \quad d_f(A) = \lim_{n\to\infty} \frac{f(\text{card}(A \cap [1, n]))}{f(n)} \\ &= \quad \lim_{n\to\infty} \frac{f(\lfloor \sqrt[3]{n} \rfloor)}{f(n)} \leq \lim_{n\to\infty} \frac{f(\sqrt[3]{n})}{f(n)} = \lim_{n\to\infty} \frac{\sqrt[3]{n} + \frac{\sqrt[3]{n}}{1+\sqrt[3]{n}}}{n + \frac{n}{1+n}} = 0. \end{aligned}$$

Consider the non-metric partial metric space $X := [0, \infty)$ endowed with the partial metric $p(x, y) := \max\{x, y\}$ for all $x, y \in X$. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$x_n := \begin{cases} \sqrt[3]{n}, & n \in A, \\ 0, & n \in \mathbb{N} \setminus A. \end{cases}$$

Notice that $(x_n)_{n\in\mathbb{N}}$ is f-statistically convergent to 1 in X. Indeed,

$$p(x_n, 1) = \begin{cases} \sqrt[3]{n}, & n \in A, \\ 1 & n \in \mathbb{N} \setminus A. \end{cases}$$

Therefore, for every $\varepsilon > 0$, $\{n \in \mathbb{N} : p(x_n, 1) - p(1, 1) \geq \varepsilon\} \subseteq A$, so

$$d_f(\{n \in \mathbb{N} : p(x_n, 1) - p(1, 1) \geq \varepsilon\}) \leq d_f(A) = 0.$$

The following remark establishes the relation between convergence and f-statistical convergence in partial metric spaces.

Remark 2. *Let X be a partial metric space and let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be convergent to some $x_0 \in X$. Take $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $p(x_n, x_0) - p(x_0, x_0) < \varepsilon$. Therefore, $\operatorname{card}\{k \leq n : p(x_k, x_0) - p(x_0, x_0) \geq \varepsilon\} \leq n_0$ for all $n \in \mathbb{N}$. Since for all f unbounded, the f-density of any finite set is zero, this means that $(x_n)_{n\in\mathbb{N}}$ is f-statistically convergent to x_0. Therefore, we obtain that*

$$\lim_n p(x_n, x_0) = p(x_0, x_0) \Rightarrow \forall f \text{ unbounded } f\text{-st}\lim_n p(x_n, x_0) = p(x_0, x_0). \tag{2}$$

Conversely, suppose that $(x_n)_{n\in\mathbb{N}}$ is not convergent to x_0. In this case, there exists $\varepsilon > 0$ for which the set $H := \{k \in \mathbb{N} : p(x_k, x_0) - p(x_0, x_0) \geq \varepsilon\}$ is infinite. In view of Lemma 1, there exists an unbounded modulus function f such that $d_f(H) = 1$. In other words, $(x_n)_{n\in\mathbb{N}}$ is not f-statistically convergent to x_0. As a consequence, we have that

$$\forall f \text{ unbounded } f\text{-st}\lim_n p(x_n, x_0) = p(x_0, x_0) \Rightarrow \lim_n p(x_n, x_0) = p(x_0, x_0). \tag{3}$$

The following remark establishes the relation between statistical convergence and f-statistical convergence in partial metric spaces.

Remark 3. *Let X be a partial metric space and let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be f-statistically convergent to some $x_0 \in X$. For every $\varepsilon > 0$ and $r > 0$, there exists $n_r \in \mathbb{N}$ such that*

$$\frac{f(\operatorname{card}\{k \leq n : p(x_k, x_0) - p(x_0, x_0) \geq \varepsilon\})}{f(n)} \leq \frac{1}{r}$$

for $n \geq n_r$, hence

$$f(\operatorname{card}\{k \leq n : p(x_k, x_0) - p(x_0, x_0) \geq \varepsilon\}) \leq \frac{f(n)}{r} \leq f\left(\frac{n}{r}\right).$$

By relying on the increasingness of f, we obtain that

$$\frac{1}{n}\operatorname{card}\{k \leq n : p(x_k, x_0) - p(x_0, x_0) \geq \varepsilon\} \leq \frac{1}{r}$$

for $n \geq n_r$. This means that $\operatorname{st}\lim_n p(x_n, x_0) = p(x_0, x_0)$. Therefore, we have that

$$\exists f \text{ unbounded } f\text{-st}\lim_n p(x_n, x_0) = p(x_0, x_0) \Rightarrow \operatorname{st}\lim_n p(x_n, x_0) = p(x_0, x_0). \tag{4}$$

Conversely, assume that f is a compatible modulus function and that $(x_n)_{n\in\mathbb{N}}$ is statistically convergent to x_0. Take an arbitrary $\varepsilon > 0$. Note that f is compatible; thus, we can find $\widetilde{\varepsilon} > 0$ and $n_0 = n_0(\varepsilon)$ such that $\frac{f(n\widetilde{\varepsilon})}{f(n)} < \varepsilon$ for all $n \geq n_0$. Fix another arbitrary $\varepsilon_1 > 0$. Since $\operatorname{st}\lim_n p(x_n, x_0) = p(x_0, x_0)$, there exists $n_1 = n_1(\varepsilon)$ such that if $n > n_1$, then $\operatorname{card}\{k \leq n : p(x_k, x_0) - p(x_0, x_0) \geq \varepsilon_1\} \leq n\widetilde{\varepsilon}$. From the increasingness of f, we have

$$\frac{f(\operatorname{card}\{k \leq n : p(x_k, x_0) - p(x_0, x_0) \geq \varepsilon_1\})}{f(n)} \leq \frac{f(n\widetilde{\varepsilon})}{f(n)} < \varepsilon$$

for $n \geq \max\{n_0, n_1\}$. As a consequence, $f\text{-st}\lim_n p(x_n, x_0) = p(x_0, x_0)$. Therefore, we have that

$$\operatorname{st}\lim_n p(x_n, x_0) = p(x_0, x_0) \Rightarrow \forall f \text{ compatible } f\text{-st}\lim_n p(x_n, x_0) = p(x_0, x_0). \tag{5}$$

The following definition serves to introduce the notion of proper f-statistical convergence in partial metric spaces. This notion is specific for non-metric partial metric spaces.

Definition 3. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a partial metric space X, f an unbounded modulus function, and $x_0 \in X$. If $f\text{-st}\lim_n x_n = x_0$ in (X, p^m), then we say that the sequence $(x_n)_{n\in\mathbb{N}}$ is properly f-statistically convergent to x_0 and it is denoted by $f_{\mathtt{pr}}\text{-st}\lim_n x_n = x_0$.*

The next step is to relate proper f-statistical convergence with f-statistical convergence in partial metric spaces.

Theorem 1. *Let X be a partial metric space, $(x_n)_{n\in\mathbb{N}} \subseteq X$, f an unbounded modulus function, and $x_0 \in X$. Then*

$$f_{\mathtt{pr}}\text{-st}\lim_n x_n = x_0 \Leftrightarrow f\text{-st}\lim_n p(x_n, x_0) = f\text{-st}\lim_n p(x_n, x_n) = p(x_0, x_0).$$

Proof. First off, notice that, in accordance with [33] (Theorem 3.6), for any sequence $(y_n)_{n\in\mathbb{N}}$ and any y_0 in any metric space Y, f-st $\lim_n y_n = y_0$ if and only if there exists $A \subseteq \mathbb{N}$ such that $d_f(A) = 0$ and $\lim_{n\in\mathbb{N}\setminus A} y_n = y_0$.

$\Rightarrow$ Suppose first that $(x_n)_{n\in\mathbb{N}}$ is properly f-statistically convergent to x_0. Since (X, p^m) is a metric space, according to [33] (Theorem 3.6), we can take $A \subseteq \mathbb{N}$ such that $d_f(A) = 0$ and $\lim_{n\in\mathbb{N}\setminus A} x_n = x_0$. Fix an arbitrary $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that if $n \in \mathbb{N} \setminus A$ and $n \geq n_0$, then $p^m(x_n, x_0) < \varepsilon$. Then

$$p(x_n, x_0) - p(x_0, x_0) \leq 2p(x_n, x_0) - p(x_0, x_0) - p(x_n, x_n) = p^m(x_n, x_0) < \varepsilon$$

and

$$|p(x_n, x_n) - p(x_0, x_0)| \leq 2p(x_n, x_0) - p(x_0, x_0) - p(x_n, x_n) = p^m(x_n, x_0) < \varepsilon$$

for all $n \in \mathbb{N} \setminus A$ with $n \geq n_0$, meaning that f-st $\lim_n p(x_n, x_0) = p(x_0, x_0)$ and f-st $\lim_n p(x_n, x_n) = p(x_0, x_0)$ in view again of [33] (Theorem 3.6).

$\Leftarrow$ Conversely, suppose next that f-st $\lim_n p(x_n, x_0) = f$-st $\lim_n p(x_n, x_n) = p(x_0, x_0)$. By relying again on [33] (Theorem 3.6), we may assume the existence of $A \subseteq \mathbb{N}$ such that $d_f(A) = 0$ and $\lim_{n\in\mathbb{N}\setminus A} p(x_n, x_0) = \lim_{n\in\mathbb{N}\setminus A} p(x_n, x_n) = p(x_0, x_0)$. Take an arbitrary $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that if $n \in \mathbb{N} \setminus A$ and $n \geq n_0$, then $p(x_n, x_0) - p(x_0, x_0) < \frac{\varepsilon}{3}$ and $p(x_n, x_n) - p(x_0, x_0) < \frac{\varepsilon}{3}$. Then for each $n \in \mathbb{N} \setminus A$ with $n \geq n_0$, we have that

$$\begin{aligned} p^m(x_n, x_0) &= 2p(x_n, x_0) - p(x_0, x_0) - p(x_n, x_n) \\ &= 2(p(x_n, x_0) - p(x_0, x_0)) + (p(x_0, x_0) - p(x_n, x_n)) \\ &< 2\frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

As a consequence, by again applying [33] (Theorem 3.6), f-st $\lim_n p^m(x_n, x_0) = 0$, that is, $f_{\mathtt{pr}}$-st $\lim_n x_n = x_0$.

□

In [17] (Definition 4.1), the definition of strong q-Cesàro summability in a partial metric space X was given as follows: A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is called strong q-Cesàro summable to $x_0 \in X$ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (p(x_k, x_0) - p(x_0, x_0))^q = 0,$$

and it is denoted as $[\mathrm{Ces}^q]$- $\lim_n p(x_n, x_0) = p(x_0, x_0)$.

The following definition introduces the notion of f-strong q-Cesàro summability in partial metric spaces.

Definition 4. *Let X be a partial metric space, $(x_n)_{n\in\mathbb{N}} \subseteq X$, f an unbounded modulus function, and q a positive real number. We say that the sequence $(x_n)_{n\in\mathbb{N}}$ is f-strongly q-Cesàro summable to $x_0 \in X$ provided that*

$$\lim_{n\to\infty} \frac{f(\sum_{k=1}^{n}(p(x_k, x_0) - p(x_0, x_0))^q)}{f(n)} = 0$$

and it is denoted as $[\mathrm{Ces}_f^q]$- $\lim_n p(x_n, x_0) = p(x_0, x_0)$.

Next, we discuss the corresponding notion of proper f-strong q-Cesàro summability in partial metric spaces.

Definition 5. *Let X be a partial metric space, $(x_n)_{n\in\mathbb{N}} \subseteq X$, f an unbounded modulus function, and q a positive real number. We say that the sequence $(x_n)_{n\in\mathbb{N}}$ is properly f-strongly q-Cesàro summable to $x_0 \in X$ provided that*

$$\lim_{n\to\infty} \frac{f(\sum_{k=1}^{n} p^m(x_k, x_0)^q)}{f(n)} = 0$$

and we write $[\text{Ces}^q_{f_{\text{pr}}}]\text{-}\lim_n x_n = x_0$.

The following theorem serves to characterize proper f-strong q-Cesàro summability via f-strong q-Cesàro summability in partial metric spaces.

Theorem 2. *Let X be a partial metric space, $(x_n)_{n\in\mathbb{N}} \subseteq X$, f an unbounded modulus function, q a positive real number, and $x_0 \in X$. Then*

$$[\text{Ces}^q_{f_{\text{pr}}}]\text{-}\lim_n x_n = x_0 \Rightarrow [\text{Ces}^q_f]\text{-}\lim_n p(x_n, x_0) = [\text{Ces}^q_f]\text{-}\lim_n p(x_n, x_n) = p(x_0, x_0).$$

Proof. The following inequalities hold for all $n \in \mathbb{N}$:

$$\begin{aligned} p^m(x_n, x_0) &= 2p(x_n, x_0) - p(x_0, x_0) - p(x_n, x_n) \\ &= 2(p(x_n, x_0) - p(x_0, x_0)) + (p(x_0, x_0) - p(x_n, x_n)) \\ &\leq 2(p(x_n, x_0) - p(x_0, x_0)) + |p(x_0, x_0) - p(x_n, x_n)|, \end{aligned}$$

$$p(x_n, x_0) - p(x_0, x_0) \leq 2p(x_n, x_0) - p(x_0, x_0) - p(x_n, x_n) = p^m(x_n, x_0),$$

and

$$|p(x_n, x_n) - p(x_0, x_0)| \leq 2p(x_n, x_0) - p(x_0, x_0) - p(x_n, x_n) = p^m(x_n, x_0).$$

As a consequence,

$$\frac{f(\sum_{k=1}^{n}(p(x_k, x_0) - p(x_0, x_0))^q)}{f(n)} \leq \frac{f(\sum_{k=1}^{n} p^m(x_k, x_0)^q)}{f(n)}$$

and

$$\frac{f(\sum_{k=1}^{n} |p(x_k, x_k) - p(x_0, x_0)|^q)}{f(n)} \leq \frac{f(\sum_{k=1}^{n} p^m(x_k, x_0)^q)}{f(n)}$$

for each $n \in \mathbb{N}$. □

The next theorem relates f-strong q-Cesàro summability with strong q-Cesàro summability in partial metric spaces.

Theorem 3. *Let X be a partial metric space, $(x_n)_{n\in\mathbb{N}} \subseteq X$, f an unbounded modulus function, and q a positive real number. If $(x_n)_{n\in\mathbb{N}}$ is f-strongly q-Cesàro summable to some $x_0 \in X$, then it is strongly q-Cesàro summable to x_0 and f-statistically convergent to x_0.*

Proof. Since $(x_n)_{n\in\mathbb{N}}$ is f-strongly q-Cesàro summable to $x_0 \in X$, there exists $n_r \in \mathbb{N}$ for every $r \in \mathbb{N}$, satisfying that

$$\frac{f(\sum_{k=1}^{n}(p(x_n, x_0) - p(x_0, x_0))^q)}{f(n)} \leq \frac{1}{r}$$

for every $n \geq n_r$. From the properties of the modulus function f,

$$f\left(\sum_{k=1}^{n}(p(x_n, x_0) - p(x_0, x_0))^q\right) \leq \frac{1}{r} f(n) \leq f\left(\frac{n}{r}\right).$$

for every $n \geq n_r$. From the increasingness of f, we obtain the following inequality:

$$\frac{1}{n}\sum_{k=1}^{n}(p(x_n, x_0) - p(x_0, x_0))^q \leq \frac{1}{r}$$

for every $n \geq n_r$, which gives that $(x_n)_{n\in\mathbb{N}}$ is strongly q-Cesàro summable to x_0. Next, let us prove that $(x_n)_{n\in\mathbb{N}}$ is f-statistically convergent to x_0. Let $\varepsilon > 0$ (of the form $1/r$ for $r \in \mathbb{N}$ sufficiently large) and denote

$$K_n := \{k \leq n : (p(x_k, x_0) - p(x_0, x_0))^q \geq \varepsilon\}$$

for every $n \in \mathbb{N}$. The following inequality holds for every $n \in \mathbb{N}$:

$$\begin{aligned}
& f\left(\sum_{k=1}^{n}(p(x_n, x_0) - p(x_0, x_0))^q\right) \\
= \; & f\left(\sum_{k\in K_n}^{n}(p(x_n, x_0) - p(x_0, x_0))^q + \sum_{k\notin K_n}^{n}(p(x_n, x_0) - p(x_0, x_0))^q\right) \\
\geq \; & f\left(\sum_{k\in K_n}^{n}(p(x_n, x_0) - p(x_0, x_0))^q\right) \\
\geq \; & f\left(\sum_{k\in K_n}^{n}\varepsilon\right) \geq \varepsilon f\left(\sum_{k\in K_n}^{n}1\right) \\
= \; & \varepsilon f(\mathrm{card}\{k \leq n : (p(x_k, x_0) - p(x_0, x_0))^q \geq \varepsilon\}).
\end{aligned}$$

If both sides of the above inequality are divided by $f(n)$ and by taking the limit as $n \to \infty$, we obtain that the sequence $(x_n)_{n\in\mathbb{N}}$ is f-statistically convergent to x_0. □

With some additional conditions, the converse of the above theorem is also satisfied.

Theorem 4. *Let X be a partial metric space, $(x_n)_{n\in\mathbb{N}} \subseteq X$, f a compatible modulus function, and q a positive real number. Let $x_0 \in X$. If $(x_n)_{n\in\mathbb{N}}$ is strongly q-Cesàro summable to x_0 or f-statistically convergent to x_0 and bounded, then it is f-strongly q-Cesàro summable to x_0.*

Proof. Let us assume first that $(x_n)_{n\in\mathbb{N}}$ is strongly q-Cesàro summable to x_0. Fix an arbitrary $\varepsilon > 0$. Since f is a compatible modulus function, there exist $\widetilde{\varepsilon} > 0$ and $n_0 = n_0(\varepsilon)$ such that $\frac{f(n\widetilde{\varepsilon})}{f(n)} < \varepsilon$ for all $n \geq n_0$. On the other hand, $(x_n)_{n\in\mathbb{N}}$ is strongly q-Cesàro summable to x_0, meaning that there exists $n_1 = n_1(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{k=1}^{n}(p(x_n, x_0) - p(x_0, x_0))^q \leq n\widetilde{\varepsilon}$$

for every $n \geq n_1$. By the increasingness of f, we have

$$f\left(\sum_{k=1}^{n}(p(x_n, x_0) - p(x, x_0))^q\right) \leq f(n\widetilde{\varepsilon}).$$

for every $n \geq n_1$. By dividing both sides of the above inequality by $f(n)$, we obtain

$$\frac{f(\sum_{k=1}^{n}(p(x_n, x_0) - p(x_0, x_0))^q)}{f(n)} \leq \frac{f(n\widetilde{\varepsilon})}{f(n)} \leq \varepsilon$$

for every $n \geq \max\{n_0, n_1\}$. This shows that f-strongly q-Cesàro is summable to x_0. Next, let us assume that $(x_n)_{n\in\mathbb{N}}$ is f-statistically convergent to x_0 and bounded. Fix again an arbitrary $\varepsilon > 0$. Since $(x_n)_{n\in\mathbb{N}}$ is bounded, there exists $M \in \mathbb{N}$ sufficiently large for which $(p(x_n, x_0) - p(x_0, x_0))^q < M$ for each $n \in \mathbb{N}$. Also, by hypothesis, f is a compatible modulus function, so, again, there are $\widetilde{\varepsilon} > 0$ and $n_0 = n_0(\varepsilon)$ such that $\frac{f(n\widetilde{\varepsilon})}{f(n)} < \varepsilon$ for all $n \geq n_0$. Denote

$$K_n := \{k \leq n : (p(x_k, x_0) - p(x_0, x_0))^q \geq \widetilde{\varepsilon}\}$$

for every $n \in \mathbb{N}$ and let $H_n := \mathbb{N} \setminus K_n$. The properties satisfied by f allow the following inequalities:

$$
\begin{aligned}
& \frac{f\left(\sum_{k=1}^{n}(p(x_n,x_0)-p(x_0,x_0))^q\right)}{f(n)} \\
= \; & \frac{f\left(\sum_{k\in K_n}(p(x_n,x_0)-p(x_0,x_0))^q+\sum_{k\in H_n}(p(x_n,x_0)-p(x_0,x_0))^q\right)}{f(n)} \\
\leq \; & \frac{f\left(M\sum_{k\in K_n}1+\sum_{k\in H_n}(p(x_n,x_0)-p(x_0,x_0))^q\right)}{f(n)} \\
\leq \; & M\frac{f(\mathrm{card}\{k\leq n:(p(x_k,x_0)-p(x_0,x_0))^q\geq\widetilde{\varepsilon}\})}{f(n)}+\frac{f(n\widetilde{\varepsilon})}{f(n)} \\
< \; & M\frac{f(\mathrm{card}\{k\leq n:(p(x_k,x_0)-p(x_0,x_0))^q\geq\widetilde{\varepsilon}\})}{f(n)}+\varepsilon
\end{aligned}
$$

for every $n \geq n_0$. Since $(x_n)_{n\in\mathbb{N}}$ is f-statistically convergent to x_0,

$$\lim_{n\to\infty}\frac{f(\mathrm{card}\{k\leq n:(p(x_k,x_0)-p(x_0,x_0))^q\geq\widetilde{\varepsilon}\})}{f(n)}=0.$$

Therefore, by taking the limit as $n \to \infty$ in the above inequality and from the arbitrariness of $\varepsilon > 0$, we obtain that

$$\lim_{n\to\infty}\frac{f\left(\sum_{k=1}^{n}(p(x_n,x_0)-p(x_0,x_0))^q\right)}{f(n)}=0,$$

which implies that $(x_n)_{n\in\mathbb{N}}$ is f-strongly q-Cesàro summable to x_0. $\square$

4. Discussion

Recently, in [33], f-statistical convergence was transported to the scope of uniform spaces. It is well known that pseudometric spaces are uniform spaces. However, partial metric spaces need not necessarily be uniform spaces.

Uniformities provide the right structure to define notions such as uniform continuity, uniform convergence, Cauchy sequences or nets, and completeness. For instance, a function $f : X \to Y$ between uniform spaces X, Y is said to be uniformly continuous provided that for every entourage V of Y there exists an entourage U of X such that $f(U) \subseteq V$, where $f(U) := \{(f(x_1), f(x_2) \in Y \times Y : (x_1, x_2) \in U\}$. A sequence $(x_n)_{n\in\mathbb{N}}$ in a uniform space X is said to be a Cauchy sequence provided that for every entourage U in X there exists $n_U \in \mathbb{N}$ in such a way that $(x_p, x_q) \in U$ for all $p, q \in \mathbb{N}$ with $p, q \geq n_U$. A net $(f_\lambda)_{\lambda\in\Lambda}$ of functions from a given set I, endowed with a bornology $\mathcal{G}$, to a uniform space X converges to some $f_0 \in X^I$ if and only if for every $G \in \mathcal{G}$ and every entourage V of X there exists $\lambda_{G,U} \in \Lambda$ such that $(f_\lambda(i), f_0(i)) \in V$ for all $i \in G$ and all $\lambda \in \Lambda$ with $\lambda \geq \lambda_{G,U}$ (this is precisely the topology of uniform convergence on elements of $\mathcal{G}$).

As mentioned before, every pseudometric space X is a uniform space, where a base of entourages is given by $U_\delta := \{(x, y) \in X \times X : d(x, y) < \delta\}$, for each $\delta > 0$. Next, suppose that X is a partial metric space. If we define

$$U_\delta := \{(x,y) \in X \times X : p(x,y) < \delta\}$$

for every $\delta > 0$, then there is no guarantee that the diagonal Δ_X of X will be contained in U_δ because it might occur that $p(x, x) > \delta$. A way to overcome this issue is by setting

$$U_\delta := \{(x,y) \in X \times X : 2p(x,y) < p(x,x) + p(y,y) + \delta\},$$

but this is precisely the metric uniformity derived from the metric space (X, p^m). Observe that the metric topology of (X, p^m) is not necessarily the same as the partial metric topology of X.

The next proposition is another way to see that non-metric partial metric spaces are not necessarily uniform spaces.

Proposition 1. *Let X be a partial metric space. If there exist $x, y \in X$ such that $x \neq y$ and $p(x,y) = p(y,y)$, then X is not Hausdorff; hence, it is not regular. As a consequence, X is not a uniform space.*

Proof. For every $\delta > 0$, $p(x,y) = p(y,y) < p(y,y) + \delta$, meaning that $x \in U_p(y,\delta)$. Therefore, no disjoint open sets contain x and y separately. Notice that T_0 and regular imply Hausdorff, therefore X cannot be regular (recall that it was mentioned in Section 2 that partial metric topologies are T_0). Finally, it is well known that every uniform space is regular; hence, X cannot be a uniform space. □

Notice that, under the settings of the previous proposition, it must necessarily occur that $p(x,x) < p(x,y)$. The settings of the above proposition are satisfied by most of the non-metric partial metric spaces, in particular, by the partial metric given by $p(x,y) := \max\{x,y\}$ in $X := [0,\infty)$.

As a consequence, non-metric partial metric spaces are not necessarily uniform spaces; hence, the results provided in [33] on f-statistical convergence do not necessarily apply to partial metric spaces.

5. Conclusions

The field of summability and convergence is constantly being enriched with extensions of statistical convergence by moduli to very different ambiences. This manuscript takes one leap further in this trend by transporting statistical convergence by moduli to general partial metric spaces. As we discussed in the previous section, non-metric partial metric spaces need not necessarily be uniform spaces, which shows the relevance and importance of developing statistical convergence by moduli in partial metric spaces.

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