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Review

A Survey on the Oscillation of First-Order Retarded Differential Equations

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Abstract: In this paper, a survey of the most interesting conditions for the oscillation of all solutions to first-order linear differential equations with a retarded argument is presented in chronological order, especially in the case when well-known oscillation conditions are not satisfied. The essential improvement and the importance of these oscillation conditions is also indicated.

Keywords: oscillation; retarded differential equations

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1. Introduction

Consider the first-order nonautonomous differential equation with a retarded argument of the form

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1.1}$$

where the functions $p, \tau \in C([t_0, \infty), \mathbb{R}^+)$, (here $\mathbb{R}^+ = [0, \infty)$), $\tau(t) \leq t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$.

By a solution of Equation (1.1), we understand a continuously differentiable function defined on $[\tau(T_0), +\infty)$ for some $T_0 \geq t$ and such that Equation (1.1) is satisfied for $t \geq T_0$. Such a solution is called *oscillatory* if it has arbitrarily large zeros, and otherwise, it is called *nonoscillatory*.

Note that a first-order linear differential equation of the form (1.1) without delay ($\tau(t) \equiv t$) *does not possess ocillatory solutions*. Indeed, it is known that all solutions of the first-order linear differential equation

$$x'(t) + p(t)x(t) = 0, \quad t \geq t_0,$$

are of the form $x(t) = Ce^{-\int p(t)dt}$, where C is an arbitrary constant. That is, all non-trivial solutions are decreasing and positive. Therefore, the investigation of oscillatory solutions is of interest for equations of the form (1.1). Furthermore, the mathematical modeling of several real-world problems leads to differential equations that depend on the past history (like equations of the form (1.1)) rather than only the current state. For the general theory, the reader is referred to [1–4].

In this paper, we present in chronological order a survey on the oscillation of this equation especially in the case where the well-known oscillation conditions

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 1 \quad \text{and} \quad \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}.$$

are not satisfied.

2. Oscillation Criteria for Equation (1.1)

Consider the scalar first-order linear nonautonomous retarded differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0,$$

where the functions $p, \tau \in C([t_{0,}\infty), \mathbb{R}^+)$, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$.

The problem of establishing sufficient conditions for the oscillation of all solutions to the retarded differential Equation (1.1) has been the subject of many investigations. See, for example, refs. [1–34] and the references cited therein.

In 1950, Myshkis [27] was the first to study the oscillation of all solutions to Equation (1.1) He proved that every solution of Equation (1.1) oscillates if

$$\limsup_{t\to\infty}[t - \tau(t)] < \infty \quad \text{and} \quad \liminf_{t\to\infty}[t - \tau(t)] \liminf_{t\to\infty} p(t) > \frac{1}{e}.$$

In 1972, Ladas, Lakshmikantham and Papadakis [23] proved that the same conclusion holds if

$$\tau \text{ is a non-decreasing function and } A := \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 1. \tag{C_1}$$

In 1979, Ladas [22] established integral conditions for the oscillation of all solutions to the equation with constant delay of the form $x'(t) + p(t)x(t - \tau) = 0$, while in 1982, Koplatadze and Canturija [19] established the following result for Equation (1.1). If

$$\mathfrak{a} := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}, \tag{C_2}$$

then all solutions of Equation (1.1) oscillate; If

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds < \frac{1}{e}, \tag{N_1}$$

then Equation (1.1) has a non-oscillatory solution.

In the special case of the retarded differential equation with a constant positive coefficient p and a constant positive delay τ, that is in the case of the equation

$$x'(t) + px(t - \tau) = 0, \quad t \geq t_0, \tag{1.1$'$}$$

a *necessary and sufficient condition* [24] for all solutions of the above equation to oscillate is

$$p\tau > \frac{1}{e} \tag{C_2)$'$ \quad ($}$$

At this point, it should be pointed out that in the case of Equation $(1.1)'$, the above-mentioned condition (C_2) reduces to the *necessary and sufficient condition* $(C_2)'$.

Observe that there is a gap between the conditions (C_1) and (C_2) when the limit $\lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds$ does not exist. How to fill this gap is an interesting problem which has been investigated by several authors in the last 35 years.

In 1988, Erbe and Zhang [13] developed new oscillation criteria by employing the upper bound of the ratio $x(\tau(t))/x(t)$ for possible non-oscillatory solutions $x(t)$ of Equation (1.1). Their result says that all the solutions of Equation (1.1) oscillate if $0 < \mathfrak{a} \leq \frac{1}{e}$ and

$$A > 1 - \frac{\mathfrak{a}^2}{4}. \tag{C_3}$$

Since then, several authors tried to obtain better results by improving the upper bound for $x(\tau(t))/x(t)$.

In 1991, Jian [17] obtained the condition

$$A > 1 - \frac{\mathfrak{a}^2}{2(1-\mathfrak{a})}, \tag{C_4}$$

while in 1992, Yu, Wang, Zhang and Qian [32] improved the above condition as follows

$$A > 1 - \frac{1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}}{2}. \tag{C_5}$$

In 1990, Elbert and Stavroulakis [10] and in 1991, Kwong [21], using different techniques, improved (C_3), in the case where $0 < \mathfrak{a} \leq \frac{1}{e}$, to the conditions

$$A > 1 - (1 - \frac{1}{\sqrt{\lambda_1}})^2 \tag{C_6}$$

and

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1}, \tag{C_7}$$

respectively, where λ_1 is the smaller real root of the exponential equation $\lambda = e^{\mathfrak{a}\lambda}$.

In 1998, Philos and Sficas [28] and in 1999, Zhou and Yu [34] and Jaroš and Stavroulakis [16] improved further the above conditions in the case where $0 < \mathfrak{a} \leq \frac{1}{e}$ as follows

$$A > 1 - \frac{\mathfrak{a}^2}{2(1-\mathfrak{a})} - \frac{\mathfrak{a}^2}{2}\lambda_1, \tag{C_8}$$

$$A > 1 - \frac{1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}}{2} - (1 - \frac{1}{\sqrt{\lambda_1}})^2, \tag{C_9}$$

and

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}}{2}, \tag{C_{10}}$$

respectively.

Consider Equation (1.1) and assume that $\tau(t)$ is continuously differentiable and that there exists $\theta > 0$ such that $p(\tau(t))\tau'(t) \geq \theta p(t)$ eventually for all t. Under this additional assumption, in 2000, Kon, Sficas and Stavroulakis [18] and in 2003, Sficas and Stavroulakis [29] established the conditions

$$A > 2\mathfrak{a} + \frac{2}{\lambda_1} - 1, \tag{C_{11}}$$

and

$$A > \frac{\ln \lambda_1 - 1 + \sqrt{5 - 2\lambda_1 + 2\mathfrak{a}\lambda_1}}{\lambda_1}, \tag{C_{12}}$$

respectively. In the case where $\mathfrak{a} = \frac{1}{e}$, then $\lambda_1 = e$, and (C_{12}) leads to

$$A > \frac{\sqrt{7 - 2e}}{e} \approx 0.459987065.$$

It is to be noted that for small values of $\mathfrak{a}$ ($\mathfrak{a} \to 0$), all the previous conditions (C_3) − (C_{11}) reduce to the condition (C_1), i.e.

$$A > 1,$$

while the condition (C_{12}) leads to

$$A > \sqrt{3} - 1 \approx 0.732,$$

which is a significant improvement. Moreover, (C_{12}) improves all the above conditions for all values of $\mathfrak{a} \in (0, \frac{1}{e}]$. Note that the value of the lower bound on A cannot be less than $\frac{1}{e} \approx 0.367879441$. Thus, the aim is to establish a condition which leads to a value *as close as possible to* $\frac{1}{e}$.

For illustrative purposes, we give the values of the lower bound on A under these conditions when (i) $\mathfrak{a} = 1/1000$ and (ii) $\mathfrak{a} = 1/e$ (Table 1).

Table 1. Values of the lower bound on A.

	(i)	**(ii)**
(C_3):	0.999999750	0.966166179
(C_4):	0.999999499	0.892951367
(C_5):	0.999999499	0.863457014
(C_6):	0.999999749	0.845181878
(C_7):	0.999999499	0.735758882
(C_8):	0.999998998	0.709011646
(C_9):	0.999999249	0.708638892
(C_{10}):	0.999998998	0.599215896
(C_{11}):	0.999999004	0.471517764
(C_{12}):	0.733050517	0.459987065

We see that the condition (C_{12}) significantly improves all the analogous known results in the literature.

Moreover, in 1994, Koplatadze and Kvinikadze [20] improved (C_5) as follows: Assume

$$\sigma(t) := \sup_{s \le t} \tau(s), \quad t \ge 0. \tag{2.1}$$

Clearly, $\sigma(t)$ is non-decreasing and $\tau(t) \le \sigma(t)$ for all $t \ge 0$. Define

$$\psi_1(t) = 0,\ \psi_i(t) = \exp\left\{ \int_{\tau(t)}^{t} p(\xi)\psi_{i-1}(\xi)d\xi \right\},\ i = 2, 3, \ldots \text{ for } t \in \mathbb{R}^+. \tag{2.2}$$

Then, the following theorem was established in [20].

Theorem 1 ([20]). *Let* $k \in \{1, 2, \ldots\}$ *exist such that*

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left\{ \int_{\sigma(s)}^{\sigma(t)} p(\xi)\psi_k(\xi)d\xi \right\} ds > 1 - c(\mathfrak{a}), \tag{2.3}$$

where $\sigma, \psi_k, \mathfrak{a}$ *are defined by* (2.1), (2.2), (C_2) *respectively, and*

$$c(\mathfrak{a}) = \begin{cases} 0 & \text{if } \mathfrak{a} > \frac{1}{e}, \\ \frac{1}{2}\left(1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}\right) & \text{if } 0 < \mathfrak{a} \le \frac{1}{e}. \end{cases} \tag{2.4}$$

Then, all solutions of Equation (1.1) *oscillate.*

Concerning the constants 1 and $\frac{1}{e}$, which appear in the conditions (C_1), (C_2) and (N_1), in 2011, Berezansky and Braverman [7] established the following:

Theorem 2 ([7]). *For any* $\alpha \in (1/e, 1)$, *there exists a non-oscillatory equation*

$$x'(t) + p(t)x(t - \tau) = 0, \quad \tau > 0$$

with $p(t) \ge 0$ *such that*

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)ds = \alpha.$$

Also in 2011, Braverman and Karpuz [8] investigated Equation (1.1) in the case of a general argument (τ is not assumed monotone) and proved that:

Theorem 3 ([8])**.** *There is no constant $K > 0$ such that*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > K \tag{2.5}$$

implies oscillation of Equation (1.1) *for arbitrary (not necessarily non-decreasing) argument* $\tau(t) \le t$.

Remark 1. *Observe that in view of the condition* (N_1)*, the constant K in the above inequality makes sense for* $K > 1/e$.

Furthermore, in [8], condition (C_1) was improved as follows

Theorem 4 ([8])**.** *Assume that*

$$B := \limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s)\exp\left\{\int_{\tau(s)}^{\sigma(t)} p(\xi)d\xi\right\}ds > 1, \tag{2.6}$$

where $\sigma(t)$ *is defined by* (2.1)*. Then, all solutions of Equation* (1.1) *oscillate.*

In 2014, using the upper bound of the ratio $\frac{x(\tau(t))}{x(t)}$ for possible non-oscillatory solutions $x(t)$ of Equation (1.1), presented in [10,16,18,29], the above result was essentially improved in [30].

Theorem 5 ([30])**.** *Assume that* $0 < \mathfrak{a} \le \frac{1}{e}$ *and*

$$B := \limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s)\exp\left\{\int_{\tau(s)}^{\sigma(t)} p(\xi)d\xi\right\}ds > 1 - \frac{1}{2}\left(1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}\right) \tag{2.7}$$

where $\sigma(t)$ *is defined by* (2.1)*. Then, all solutions of Equation* (1.1) *oscillate.*

Remark 2 ([30])**.** *Note that as* $\mathfrak{a} \to 0$*, then condition (2.7) reduces to (2.6). However, the improvement is clear as* $\mathfrak{a} \to \frac{1}{e}$*. Actually, when* $\mathfrak{a} = \frac{1}{e}$*, the value of the lower bound on B is equal to* ≈ 0.863457014*. That is, (2.7) significantly improves (2.6).*

Remark 3 ([30])**.** *Observe that under the additional assumption that* $\tau(t)$ *is continuously differentiable and that there exists* $\theta > 0$ *such that* $p(\tau(t))\tau'(t) \ge \theta p(t)$ *eventually for all t, (see [18,29]) the condition (2.7) of Theorem 5 reduces to*

$$B > 1 - \frac{1}{2}\left(1 - \mathfrak{a} - \sqrt{(1-\mathfrak{a})^2 - 4M}\right), \tag{2.7$'$}$$

where M is given by

$$M = \frac{e^{\lambda_1\theta\mathfrak{a}} - \lambda_1\theta\mathfrak{a} - 1}{(\lambda_1\theta)^2}$$

and λ_1 *is the smaller root of the equation* $\lambda = e^{\lambda\mathfrak{a}}$*. In the case that* $\theta = 1$ *from [30], it follows that*

$$M = \frac{e^{\lambda_1\theta\mathfrak{a}} - \lambda_1\theta\mathfrak{a} - 1}{(\lambda_1\theta)^2}$$

and λ_1 is the smaller root of the equation $\lambda = e^{\lambda \mathfrak{a}}$. When $\theta = 1$, from [30], it follows that

$$\frac{1}{2}\left(1-\mathfrak{a}-\sqrt{(1-\mathfrak{a})^2-4M}\right)=1-\mathfrak{a}-\frac{1}{\lambda_1}$$

and in the case that $\mathfrak{a} = \frac{1}{e}$, then $\lambda_1 = e$ and $(2.7)'$ leads to

$$B > 1-\left(1-\frac{2}{e}\right)=\frac{2}{e}\approx 0.735758882.$$

That is, condition $(2.7)'$ significantly improves (2.7) but of course under the additional (stronger) assumptions on $\tau(t)$ and $p(t)$.

In 2015 , Infante, Koplatadze and Stavroulakis [15] proved that all solutions of Equation (1.1) oscillate if one of the following conditions is satisfied:

$$\limsup_{t\to\infty}\int_{g(t)}^{t} p(s)\,\mathrm{e}^{\int_{\tau(s)}^{g(t)} p(u)\,\mathrm{e}^{\int_{\tau(u)}^{u} p(v)dv} du} ds > 1, \tag{2.8}$$

or

$$\limsup_{\epsilon\to 0^+}\left(\limsup_{t\to\infty}\int_{g(t)}^{t} p(s)\mathrm{e}^{(\lambda(k)-\epsilon)\int_{\tau(s)}^{g(t)} p(u)du} ds\right) > 1, \tag{2.9}$$

where $g(t)$ is a non-decreasing function satisfying that $\tau(t) \le g(t) \le t$ for all $t \ge t_1$ and some $t_1 \ge t_0$.

In 2016, El-Morshedy and Attia [12] proved that Equation (1.1) is oscillatory if there exists a positive integer n such that

$$\limsup_{t\to\infty}\left(\int_{g(t)}^{t} q_n(s)ds + c(k^*)\mathrm{e}^{\int_{g(t)}^{t}\sum_{i=0}^{n-1} q_i(s)ds}\right) > 1, \tag{2.10}$$

where

$$k^* := \liminf_{t\to\infty}\int_{g(t)}^{t} p(s)\,ds,$$

c, g are defined as before and the sequence $\{q_n(t)\}$ is given by

$$q_0(t) = p(t),\ q_1(t) = q_0(t)\int_{\tau(t)}^{t} q_0(s)\mathrm{e}^{\int_{\tau(s)}^{t} q_0(u)du} ds,$$

$$q_n(t) = q_{n-1}(t)\int_{g(t)}^{t} q_{n-1}(s)\mathrm{e}^{\int_{g(s)}^{t} q_{n-1}(u)du} ds, \quad n = 2, 3, \ldots .$$

In 2018, Chatzarakis, Purnaras and Stavroulakis [9] improved the above conditions as follows.

Theorem 6 ([9]). *Assume that for some $j \in \mathbb{N}$*

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{\sigma(t)} P_j(u)du\right) ds > 1, \tag{2.11}$$

or

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{\sigma(t)} P_j(u)du\right) ds > 1-\frac{1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}}{2}, \tag{2.12}$$

or

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} P_j(u)du\right) ds > \frac{2}{1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}}, \tag{2.13}$$

or

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{\sigma(s)} P_j(u)du\right)ds > \frac{1+\ln\lambda_1}{\lambda_1} - \frac{1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}}{2}, \tag{2.14}$$

where

$$P_j(t) = p(t)\left[1+\int_{\tau(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} P_{j-1}(\xi)d\xi\right)du\right)ds\right], \tag{2.15}$$

with $P_0(t) = p(t)$. $0 < \mathfrak{a} \le \frac{1}{e}$, *and* λ_1 *is the smaller root of the equation* $\lambda = e^{\mathfrak{a}\lambda}$. *Then, all solutions of Equation* (1.1) *oscillate.*

Theorem 7 ([9])**.** *Assume that for some* $j \in \mathbb{N}$

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{\sigma(s)} P_j(u)du\right)ds > \frac{1}{e}, \tag{2.16}$$

where P_j *is defined by* (2.15). *Then, all solutions of Equation* (1.1) *oscillate.*

It is easy to see that the conditions (2.11), (2.12), (2.14), and (2.16) substantially improve the conditions (C_1), (2.6), (2.7), (C_{10}) and (C_2). That improvement can immediately be observed if we compare the corresponding parts on the left-hand side of these conditions.

In 2019, Bereketoglu et al. [6] proved that all solutions of Equation (1.1) oscillate if for some $\ell \in \mathbb{N}$, the following condition holds

$$\limsup_{t\to\infty}\int_{g(t)}^{t} p(s)e^{\int_{\tau(s)}^{g(t)} P_\ell(u)du}ds > 1 - c(k^*), \tag{2.17}$$

where

$$P_\ell(t) = p(t)\left[1+\int_{g(t)}^{t} p(s)e^{\int_{\tau(s)}^{t} P_{\ell-1}(u)du}ds\right],\ P_0(t) = p(t).$$

In 2020, Attia, El-Morshedy and Stavroulakis [5] obtained new sufficient criteria of recursive type for the oscillation of Equation (1.1),

Assume that c, g, k^*, λ, t_1 are defined as above and $g^i(t)$ stands for the ith composition of g. For fixed $n \in \mathbb{N}$, define $\{R_{m,n}(t)\}$, $\{Q_{m,n}(t)\}$, eventually, as follows:

$$R_{m,n}(t) = 1 + \int_{\tau(t)}^{t} p(s)e^{\int_{\tau(s)}^{t} p(u)Q_{m-1,n}(u)du}ds, \quad m = 1, 2, \ldots,$$

$$Q_{i,j}(t) = e^{\int_{\tau(t)}^{t} p(s)Q_{i,j-1}(s)ds}, \quad i = 1, 2, \ldots, m-1,\ j = 1, 2, \ldots, n$$

where

$$Q_{0,0}(t) = (\lambda(k^*) - \epsilon)\left(1 + (\lambda(k^*) - \epsilon)\int_{\tau(t)}^{g(t)} p(s)ds\right),$$

$$Q_{0,r}(t) = e^{\int_{\tau(t)}^{t} p(s)Q_{0,r-1}(s)ds}, \quad r = 1, 2, \ldots, n$$

$$Q_{i,0}(t) = R_{i,n}, \quad i = 1, 2, \ldots, m-1$$

and $\epsilon \in (0,\ \lambda(k^*))$.

Theorem 8 ([5])**.** *Assume that* $k^* \le \frac{1}{e}$ *and* $m, n \in \mathbb{N}$ *such that*

$$\limsup_{t\to\infty}\int_{g(t)}^{t} p(s)e^{\int_{\tau(s)}^{g(t)} p(u)e^{\int_{\tau(u)}^{u} p(v)R_{m,n}(v)dv}du}ds > 1 - c(k^*). \tag{2.18}$$

Then, all solutions of Equation (1.1) oscillate.

Theorem 9 ([5]). *Assume that* $k^* \leq \frac{1}{e}$ *and*

$$\limsup_{t\to\infty} \int_{g(t)}^{t} p(s)e^{(\lambda(k^*)-\epsilon)\int_{\tau(s)}^{g(t)} p(u)du+(\lambda(k^*)-\epsilon)^2 \int_{\tau(s)}^{g(t)} p(u)\int_{\tau(u)}^{g(u)} p(v)dvdu} ds > 1 - c(k^*), \quad (2.19)$$

where $\epsilon \in (0, \lambda(k^*))$. *Then, all solutions of Equation (1.1) oscillate.*

Theorem 10 ([5]). *Assume that* $k^* \leq \frac{1}{e}$ *and* $m, n \in \mathbb{N}$ *such that*

$$\limsup_{t\to\infty} \int_{g(t)}^{t} p(s)e^{\int_{\tau(s)}^{g(t)} p(u)R_{m,n}(u)du} ds > 1 - c(k^*). \quad (2.20)$$

Then, all solutions of Equation (1.1) oscillate..

Theorem 11 ([5]). *Assume that* $k^* \leq \frac{1}{e}$ *and* $m, n \in \mathbb{N}$ *such that*

$$\limsup_{t\to\infty} \left(\int_{g(t)}^{t} p(s) + (\lambda(k^*) - \epsilon) \int_{g(t)}^{t} p(s) \int_{\tau(s)}^{g(t)} p(u)e^{\int_{\tau(u)}^{g^2(t)} p(v)R_{m,n}(v)dv} duds \right) > 1 - c(k^*), \quad (2.21)$$

where $\epsilon \in (0, \lambda(k^*))$. *Then, all solutions of Equation (1.1) oscillate.*

Theorem 12 ([5]). *Let* $A^* := \limsup_{t\to\infty} \int_{g(t)}^{t} p(s)ds < 1,\ 0 < k^* \leq \frac{1}{e}$

$$\int_{g(s)}^{g(t)} p(u)du \geq \int_{s}^{t} p(u)du, \quad \text{for all } s \in [g(t), t], \quad (2.22)$$

and

$$\alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{g(t)} p(s)ds. \quad (2.23)$$

If one of the following conditions is satisfied:

(i) $\quad A^* > \frac{-1-\alpha\lambda(k^*)+\sqrt{2+(1+\alpha\lambda(k^*))^2+2k^*\lambda(k^*)}}{\lambda(k^*)}$

(ii) $\ A^* > 1 + k^* + \frac{1}{\lambda(k^*)} + \alpha - \sqrt{\left(1 + k^* + \frac{1}{\lambda(k^*)} + \alpha\right)^2 - 2\left(k^* + \frac{1}{\lambda(k^*)}\right)}$

then all solutions of Equation (1.1) oscillate.

Remark 4 ([5]).

(i) *Condition (2.22) is satisfied if (see [6,18])*

$$p(g(t))g'(t) \geq p(t) \quad \text{eventually for all } t.$$

(ii) *It is easily shown that the conclusion of Theorem 12 is valid if* $p(t) > 0$ *and condition (2.22) is replaced by*

$$\liminf_{t\to\infty} \frac{p(g(t))g'(t)}{p(t)} = 1.$$

Corollary 1 ([5]). *Assume that* $0 < \mathfrak{a} \leq 1/e$, $A < 1$ *and* $\tau(t)$ *is a non-decreasing continuous function such that*

$$\int_{\tau(s)}^{\tau(t)} p(u)du \geq \int_{s}^{t} p(u)du, \quad \text{for all } s \in [\tau(t), t].$$

If

$$A > \min\left\{\frac{-1+\sqrt{3+2a\lambda(a)}}{\lambda(a)}, 1+\mathfrak{a}+\frac{1}{\lambda(a)} - \sqrt{1+\left(\mathfrak{a}+\frac{1}{\lambda(a)}\right)^2}\right\} \tag{2.24}$$

then all solutions of Equation (1.1) oscillate.

Remark 5 ([5]). *1. Condition (2.18), with $n = 1$ and $n = 2$, improves conditions (C_1), (2.6), (2.7) and* (2.8) *respectively.*
2. *Condition (2.19) improves condition (2.9).*
3. *Condition (2.20) with $n = 1$ improves condition (2.17) with $\ell = 1$.*
4. *It is easy to see that*

$$\frac{-1+\sqrt{3+2a\lambda(a)}}{\lambda(a)} \le \frac{\ln\lambda(a) - 1 + \sqrt{5-2\lambda(a)+2a\lambda(a)}}{\lambda(a)}$$

for all $\lambda(a) \in [1, e]$. Therefore, condition (2.24) improves condition (C_{12}).

3. Discussion

In this survey paper, the first-order linear non-autonomous retarded differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \ge t_0,$$

is considered. The most interesting oscillation conditions since 1950 are presented in chronological order, especially in the case where the well-known oscillation conditions

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds > 1 \quad \text{and} \quad \liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}.$$

are not satisfied. The improvement and significance of the presented conditions is indicated in detail in several remarks.

As it has been mentioned above, the lower bound on $A = \limsup_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds$ cannot be less than $\frac{1}{e} \approx 0.367879441$. Therefore, it would be of paramount importance to establish a condition which leads to a value of A (cf. values on Table 1) *as close as possible to* $\frac{1}{e}$. Thus, the following very interesting open problem arises.

4. Open Problem

Does the condition

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}, \text{ where } \tau \text{ is a non-decreasing function} \tag{$(C_1)'$}$$

(without additional assumption on $p(t)$) imply that all solutions of Equation (1.1) *oscillate?*

Observe that, in view of condition (N_1), the above condition $(C_1)'$ would be *a necessary and sufficient condition for the oscillation of all solutions to Equation* (1.1).

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References

1. Erbe, L.H.; Kong, Q.; Zhang, B.G. *Oscillation Theory for Functional Differential Equations*; Marcel Dekker: New York, NY, USA, 1995.
2. Gopalsamy, K. *Stability and Oscillations in Delay Differential Equations of Population Dynamics*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1992.
3. Gyori, I.; Ladas, G. *Oscillation Theory of Delay Differential Equatiosn with Applications*; Clarendon Press: Oxford, UK, 1991.

4. Hale, J.K. *Theory of Functional Differential Equations*; Springer: New York, NY, USA, 1977.
5. Attia, E.R.; El-Morshedy, H.A.; Stavroulakis, I.P. Oscillation Criteria for First Order Differential Equations with Non-Monotone Delays. *Symmetry* **2020**, *12*, 718. [CrossRef]
6. Bereketoglu, H.; Karakoc, F.; Oztepe, G.S.; Stavroulakis, I.P. Oscillation of first order differential equations with several non-monotone retarded arguments. *Georgian Math. J.* **2020**, *27*, 341–350. [CrossRef]
7. Berezansky, L.; Braverman, E. On some constants for oscillation and stability of delay equations. *Proc. Am. Math. Soc.* **2011**, *139*, 4017–4026. [CrossRef]
8. Braverman, E.; Karpuz, B. On oscillation of differential and difference equations with non-monotone delays. *Appl. Math. Comput.* **2011**, *58*, 766–775. [CrossRef]
9. Chatzarakis, G.E.; Purnaras, I.K.; Stavroulakis, I.P. Oscillation tests for differential equations with deviating arguments. *Adv. Math. Sci. Appl.* **2018**, *27*, 1–28.
10. Elbert, A.; Stavroulakis, I.P. Oscillations of first order differential equations with deviating arguments. In *Recent Trends in Differential Equations*; World Sci. Publishing Co.: Hackensack, NJ, USA, 1992; pp. 163–178.
11. Elbert, A.; Stavroulakis, I.P. Oscillation and non-oscillation criteria for delay differential equations. *Proc. Am. Math. Soc.* **1995**, *123*, 1503–1510. [CrossRef]
12. El-Morshedy, H.A.; Attia, E.R. New oscillation criterion for delay differential equations with non-monotone arguments. *Appl. Math. Lett.* **2016**, *54*, 54–59. [CrossRef]
13. Erbe, L.H.; Zhang, B.G. Oscillation of first order linear differential equations with deviating arguments. *Differ. Integral Equ.* **1988**, *1*, 305–314. [CrossRef]
14. Fukagai, N.; Kusano, T. Oscillation theory of first order functional differential equations with deviating arguments. *Ann. Mat. Pura Appl.* **1984**, *136*, 95–117. [CrossRef]
15. Infante, G.; Koplatadze, R.; Stavroulakis, I.P. Oscillation criteria for differential equations with several retarded arguments. *Funkc. Ekvac.* **2015**, *58*, 347–364. [CrossRef]
16. Jaroš, J.; Stavroulakis, I.P. Oscillation tests for delay equations. *Rocky Mt. J. Math.* **1999**, *29*, 139–145.
17. Jian, C. Oscillation of linear differential equations with deviating argument. *Math. Pract. Theory*, **1991**, *1*, 32–41. (In Chinese)
18. Kon, M.; Sficas, Y.G.; Stavroulakis, I.P. Oscillation criteria for delay equations. *Proc. Am. Math. Soc.* **2000**, *128*, 2989–2997. [CrossRef]
19. Koplatadze, R.G.; Chanturija, T.A. On the oscillatory and monotonic solutions of first order differential equations with deviating arguments. *Differ. Uravn.* **1982**, *18*, 1463–1465.
20. Koplatadze, R.G.; Kvinikadze, G. On the oscillation of solutions of first order delay differential inequalities and equations. *Georgian Math. J.* **1994**, *1*, 675–685. [CrossRef]
21. Kwong, M.K. Oscillation of first order delay equations. *J. Math. Anal. Appl.* **1991**, *156*, 274–286. [CrossRef]
22. Ladas, G. Sharp conditions for oscillations caused by delay. *Appl. Anal.* **1979**, *9*, 93–98. [CrossRef]
23. Ladas, G.; Laskhmikantham, V.; Papadakis, J.S. Oscillations of higher-order retarded differential equations generated by retarded arguments. In *Delay and Functional Differential Equations and Their Applications*; Academic Press: New York, NY, USA, 1972; pp. 219–231.
24. Ladas, G.; Stavroulakis, I.P. On delay differential inequalities of first order. *Funkc. Ekvac.* **1982**, *25*, 105–113.
25. Ladde, G.S.; Lakshmikantham, V.; Zhang, B.G. *Oscillation Theory of Differential Equations with Deviating Arguments*; Marcel Dekker: New York, NY, USA, 1987.
26. Li, B. Oscillations of first order delay differential equations. *Proc. Am. Math. Soc*. **1996**, *124*, 3729–3737. [CrossRef]
27. Myshkis, A.D. Linear homogeneous differential equations of first order with deviating arguments. *Uspekhi Mat. Nauk* **1950**, *5*, 160–162. (In Russian)
28. Philos, C.G.; Sficas, Y.G. An oscillation criterion for first-order linear delay differential equations. *Canad. Math. Bull.* **1998**, *41*, 207–213. [CrossRef]
29. Sficas, Y.G.; Stavroulakis, I.P. Oscillation criteria for first-order delay equations. *Bull. Lond. Math. Soc.* **2003**, *35*, 239–246. [CrossRef]
30. Stavroulakis, I.P. Oscillation criteria for delay and difference equations with non-monotone arguments. *Appl. Math. Comput.* **2014**, *226*, 661–672. [CrossRef]
31. Wang, Z.C.; Stavroulakis, I.P.; Qian, X.Z. A Survey on the oscillation of solutions of first order linear differential equations with deviating arguments. *Appl. Math. E-Notes* **2002**, *2*, 171–191.
32. Yu, J.S.; Wang, Z.C.; Zhang, B.G.; Qian, X.Z. Oscillations of differential equations with deviating arguments. *Pan-Am. Math. J.* **1992**, *2*, 59–78.
33. Zhou, D. On some problems on oscillation of functional differential equations of first order. *J. Shandong Univ.* **1990**, *25*, 434–442.
34. Zhou, Y.; Yu, Y.H. On the oscillation of solutions of first order differential equations with deviating arguments. *Acta Math. Appl. Sin.* **1999**, *15*, 288–302.