axioms

Article

C_0–Semigroups Approach to the Reliability Model Based on Robot-Safety System

Ehmet Kasim *,† and Aihemaitijiang Yumaier †

College of Mathematics and Systems Science, Xinjiang University, Urumqi 830046, China; 107552100677@stu.xju.edu.cn

* Correspondence: ehmetkasim@xju.edu.cn

† These authors contributed equally to this work.

Abstract: This paper considers a system with one robot and n safety units (one of which works while the others remain on standby), which is described by an integro-deferential equation. The system can fail in the following three ways: fails with an incident, fails safely and fails due to the malfunction of the robot. Using the C_0–semigroups theory of linear operators, we first show that the system has a unique non-negative, time-dependent solution. Then, we obtain the exponential convergence of the time-dependent solution to its steady-state solution. In addition, we study the asymptotic behavior of some time-dependent reliability indices and present a numerical example demonstrating the effects of different parameters on the system.

Keywords: robot-safety system; C_0–semigroups; well-posedness; exponential stability; reliability indices

MSC: 47D06; 47A10; 90B25

Citation: Kasim, E.; Yumaier, A. C_0–Semigroups Approach to the Reliability Model Based on Robot-Safety System. *Axioms* **2024**, *13*, 423. https://doi.org/10.3390/axioms13070423

Academic Editors: Cheng-Cheng Zhu, Hari Mohan Srivastava and Feliz Manuel Minhós

Received: 8 May 2024
Revised: 6 June 2024
Accepted: 13 June 2024
Published: 24 June 2024

1. Introduction

Robots are increasingly being used for a variety of tasks, including welding, forging, resource exploration and development, disaster relief and evacuation, complex surgery, bomb disposal, and machining. Because robots use electronic, electrical, mechanical, hydraulic, and pneumatic components, reliability issues are quite difficult to address due to the diversity of failure factors in robotic systems. However, the reliability of robots, as the main technical indicator for measuring the quality of industrial robots, is receiving unprecedented attention. The unreliable robot can bring a range of issues, including human damage. Thus, it is important to carefully consider the reliability of the robot. The robots must be both reliable and safe, so it is equipped with several safety devices. To analyze the reliability of a robot, we need to consider the relationship between the reliability and the safety device.

In reliability theory, there are many methods to analyze the reliability of engineering systems. In particular, the use of supplementary variables technique to establish models and analyze the reliability of robot systems has been widely studied. In 1955, Cox [1] first proposed the "supplementary variable technique (SVT)" and established the M/G/1 queuing system. Gaver [2] was the first to apply this technique for reliability models, and subsequently, other authors followed this line of research, such as Linton [3], Gupta and Gupta [4], Shi and Li [5], Chung [6], Oliveira et al. [7], Zhang and Wu [8], Shakuntla et al. [9], Singh et al. [10], Ke et al. [11], Shekhar et al. [12], Gao and Wang [13].

There is a considerable amount of research literature on robot-safety systems, yet the research on system reliability remained limited. Most articles focus on repeatability and accuracy [14,15]. Dillon and Yang [16] first studied a system with a robot and its safety device. The mathematical model was established by introducing SVT, and the steady-state solution (S-SS) was examined by the Laplace transforms. They then investigated a system with two robots and a safety device, one working and the other in storage [17].

Many researchers subsequently used the SVT to establish various robot-safety systems and studied the steady-state reliability indices of these systems, see [18–21] and references therein. All of these researchers studied the reliability models of robot-safety systems under the assumption that the dynamic solution converges to its S-SS. The S-SS is well known to depends on the time-dependent solution (T-DS), and the T-DS can clearly reflect the operating trend of the system. As a result, it is necessary to investigate the existence and uniqueness of T-DS, as well as their asymptotic behavior and the instantaneous reliability index. In 2001, Gupur first introduced the dynamic analysis for the study of reliability models by the C_0–semigroups theory [22,23]. Guo and Xu [24] studied a system composed of one robot and one safety device, and determined the existence and uniqueness of the system's T-DS as well as its asymptotic behavior. Chen and Xu [25] introduced the repair rate as a periodic function for the above system and analyzed the exponentially stability. Gupur [26] considered a human–machine system and demonstrated the well posedness of the system and the asymptotic behavior of the T-DS, proved the quasi-compactness of the C_0-semigroup, determined that the C_0-semigroup exponentially converges to a projection operator [27], and finally obtained an expression for the projection operator using the residual theorem [28]. Zhang [29] considered a system consisting two robots and one safety unit and investigated the exponential stability of the T-DS. Qiao and Ma [30] discussed the system composed of a safety component and two redundant robots. Zhou and Wei [31] have further investigated the system studied in [24].

Based on the above literature, we found that the results of dynamic analysis of robot-safety systems are few and limited to special cases, i.e., simple systems consisting of one robot or two robots and one safety device. Recent advances have allowed robot-safety systems to become more and more complex to improve their performances. However, these complex systems have strong applicability in engineering. Thus, the reliability of complex robot-safety systems has become a serious and urgent problem. In this paper, we consider a system with one robot and n safety units and perform dynamic analysis on the system.

The robot safety system, according to Cheng and Dhillon [32], is described by the following integro-differential equations:

$$
\begin{aligned}
&\frac{dP_0(t)}{dt} = -a_0\,P_0(t) + \mu_s P_1(t) + \sum_{k=n+1}^{n+3}\int_0^\infty P_k(y,t)\mu_k(y)dy,\\
&\frac{dP_j(t)}{dt} = \lambda_s P_{j-1}(t) - a_j P_j(t) + \mu_s P_{j+1}(t), \quad 1 \leqslant j \leqslant n-1,\\
&\frac{dP_n(t)}{dt} = \lambda_s P_{n-1}(t) - a_n\,P_n(t),\\
&\frac{\partial P_k(y,t)}{\partial t} + \frac{\partial P_k(y,t)}{\partial y} = -\mu_k(y)P_k(y,t),\\
&P_{n+1}(0,t) = \lambda_{ss}P_n(t),\\
&P_{n+2}(0,t) = \lambda_{si}P_n(t),\\
&P_{n+3}(0,t) = \lambda_r \sum_{j=0}^{n-1} P_j(t).\\
&P_0(0) = 1,\ P_i(0) = 0, \quad 1 \leqslant i \leqslant n, \quad P_k(y,0) = 0, \quad n+1 \leqslant k \leqslant n+3.
\end{aligned}
\tag{1}
$$

where $a_0 = \lambda_s + \lambda_r$, $a_j = \lambda_s + \lambda_r + \mu_s$ $(1 \leqslant j \leqslant n-1)$, $a_n = \lambda_{ss} + \lambda_{si} + \mu_s$, and $(y,t) \in [0,\infty) \times [0,\infty)$. $P_i(t)$ denotes the probability that the system is in state i $(1 \leqslant i \leqslant n)$ at time t; $P_k(y,t)\ dx$ denotes the probability that at time t, the system is in state k $(n+1 \leqslant k \leqslant n+3)$ and the elapsed repair time is in $(y, y+\Delta y)$; λ_s/λ_r represents the robot/the safety unit's failure rate; $\lambda_{ss}/\lambda_{si}$ represents the failure rate of the system failing safely/failing with an incident; the repair rate of the safety unit is denoted by μ_s;

and $\mu_k(y)(n+1 \leqslant k \leqslant n+3)$ represents the system's repair rate when it is in state k and satisfies

$$\mu_k(y) \geq 0, \quad \int_0^\infty \mu_k(y)dy = \infty.$$

The organization of the remainder of this paper is outlined below. Section 2 introduces the transformation of the given system into an abstract Cauchy problem. Section 3 examines the system's well posedness. Section 4 investigates the exponential convergence of the T-DS to its S-SS. Section 5 discusses the asymptotic behavior of instantaneous reliability indices. Section 6 uses numerical examples to illustrate the sensitivity of reliability indices to system parameter variations. Section 7 concludes with a summary of findings and suggestions for future research.

2. The Abstract Cauchy Problem

We begin by introducing the following notation:

$$\Upsilon = \begin{pmatrix} \Gamma_0 & \mathbf{0} \\ \Gamma_1 & \mathbf{0} \end{pmatrix}$$

where $\mathbf{0}$ is a zero matrix. $\Gamma_0 = [(a_{i,j})_0]$ is an $n \times n$ matrix whose elements are

$$(a_{i,j})_0 = \begin{cases} 1, & \text{if} \quad 0 \leqslant i = j \leqslant n, \\ 0, & \text{other.} \end{cases}$$

$\Gamma_1 = [(a_{l,m})_1]$ is a $3 \times n$ matrix whose elements are

$$(a_{l,m})_1 = \begin{cases} \lambda_{ss}, & \text{if} \quad l = 1,\ m = n, \\ \lambda_{si}, & \text{if} \quad l = 2,\ m = n, \\ \lambda_r, & \text{if} \quad l = 3,\ 1 \leqslant m \leqslant n-1, \\ 0, & \text{other.} \end{cases}$$

Take the following state space

$$\mathcal{X} = \left\{ \mathbf{P} \in \mathbf{R}^n \times \left(L^1[0,\infty)\right)^3 \middle| \|\mathbf{P}\| = \sum_{i=0}^{n} |P_i| + \sum_{k=n+1}^{n+3} \|P_k\|_{L^1[0,\infty)} < \infty \right\}.$$

Clearly, $\mathcal{X}$ is a Banach space. Following that, operators and their domains are defined as follows

$$\begin{aligned}
\mathbb{A}\mathbf{P} &= \begin{pmatrix} A_0 & \mathbf{0} \\ \mathbf{0} & A_1 \end{pmatrix} \mathbf{P}, \\
\mathcal{D}(\mathbb{A}) &= \left\{ \mathbf{P} \in \mathcal{X} \middle| \begin{array}{l} \frac{dP_k(y)}{dy} \in L^1[0,\infty),\ P_k(y)(n+1 \leqslant k \leqslant n+3) \\ \text{are absolutely continuous and } P(0) = \Upsilon P(y) \end{array} \right\}, \\
\mathbb{U}\mathbf{P} &= \begin{pmatrix} U_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{P}, \quad D(\mathbb{U}) = \mathcal{X}, \\
\mathbb{E}\mathbf{P} &= \begin{pmatrix} E_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{P} + E_1, \quad D(\mathbb{E}) = \mathcal{X},
\end{aligned}$$

where

$$A_0 = \begin{pmatrix} -a_0 & 0 & \cdots & 0 & 0 \\ 0 & -a_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -a_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & -a_n \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -\frac{d}{dy} - \mu_{n+1}(y) & 0 & 0 \\ 0 & -\frac{d}{dy} - \mu_{n+2}(y) & 0 \\ 0 & 0 & -\frac{d}{dy} - \mu_{n+3}(y) \end{pmatrix},$$

$$U_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \lambda_s & 0 & \cdots & 0 & 0 \\ 0 & \lambda_s & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_s & 0 \end{pmatrix},$$

$$E_0 = \begin{pmatrix} 0 & \mu_s & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mu_s & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \mu_s \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad E_1 = \begin{pmatrix} \sum_{k=n+1}^{n+3} \int_0^\infty P_j(y,t)\mu_j(y)dy \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

Thus, the above system of Equation (1) can be rewritten as an abstract Cauchy problem in $\mathcal{X}$:

$$\begin{cases} \frac{d\,\mathbf{P}(t)}{dt} = (\mathbb{A} + \mathbb{U} + \mathbb{E})\mathbf{P}(t), \quad t \in (0,\infty), \\ \mathbf{P}(0) = \left(1,0,0,0\right)^T. \end{cases} \tag{2}$$

3. Well Posedness of (2)

We begin by demonstrating that $\mathbb{A} + \mathbb{U} + \mathbb{E}$ generates a positive contraction C_0-semigroup $\mathfrak{T}(t)$ on $\mathcal{X}$.

Theorem 1. *If* $\overline{\mu_k} = \sup_{x\in[0,\infty)} \mu_k(x) < \infty$ *for* $n+1 \leqslant k \leqslant n+3$, *then* $\mathbb{A} + \mathbb{U} + \mathbb{E}$ *generates a positive contraction* C_0-*semigroup* $\mathfrak{T}(t)$.

Proof. We will estimate $\|(\gamma I - \mathbb{A})^{-1}\|$ as a first step. To do this, consider $(\gamma I - \mathbb{A})\mathbf{P} = Z$, for $Z \in \mathcal{X}$, that is,

$$(\gamma + a_j)P_j = Z_j, \quad j = 0,1,\ldots,n, \tag{3}$$

$$\frac{dP_k(y)}{\partial d} = -(\gamma + \mu_k(y))P_k(y) - Z_k(y), \quad n+1 \leqslant k \leqslant n+3, \tag{4}$$

$$P_{n+1}(0) = \lambda_{ss}P_n, \tag{5}$$

$$P_{n+2}(0) = \lambda_{si}P_n, \tag{6}$$

$$P_{n+3}(0) = \lambda_r \sum_{j=0}^{n-1} P_j. \tag{7}$$

Solving (3) and (4), we have

$$P_j = \frac{1}{\gamma + a_j} Z_j, \quad j = 0,1,\ldots,n, \tag{8}$$

$$
\begin{aligned}
P_k(y) =& P_k(0)e^{-\int_0^y(\gamma+\mu_k(\tau))d\tau} \\
&+ e^{-\int_0^y(\gamma+\mu_k(\tau))d\tau}\int_0^y Z_k(\xi)e^{\int_0^\xi(\gamma+\mu_k(\tau))d\tau}d\xi, \quad n+1 \leqslant k \leqslant n+3.
\end{aligned} \tag{9}
$$

For (5)–(7) together with (8) and (9), we can get that

$$P_{n+1}(0) = \lambda_{ri}P_n = \frac{\lambda_{ss}}{\gamma+a_n}Z_n, \tag{10}$$

$$P_{n+2}(0) = \lambda_{rs}P_n = \frac{\lambda_{si}}{\gamma+a_n}Z_n, \tag{11}$$

$$P_{n+3}(0) = \lambda_r\sum_{j=0}^{n-1}P_j = \sum_{j=0}^{n-1}\frac{\lambda_r}{\gamma+a_j}Z_j. \tag{12}$$

By using (9), the Fubini theorem and the following inequalities

$$e^{-\int_\xi^y\mu_k(\tau)d\tau} \leq 1, \quad \text{for } y \geq \xi \geq 0; \quad e^{-\int_0^y\mu_k(\tau)d\tau} \leq 1, \quad \text{for } y \in [0,\infty),$$

we deduce (without loss of generality, assume $\gamma > 0$)

$$
\begin{aligned}
\|P_k\|_{L^1[0,\infty)} \leq& |P_j(0)|\int_0^\infty e^{-\gamma y}dy \\
&+ \int_0^\infty e^{-\gamma y}\int_0^y |Z_k(\xi)|e^{\int_\xi^y(\gamma+\mu_k(\tau))d\tau}d\xi dy \\
\leq& |P_k(0)|\int_0^\infty e^{-\gamma y}dy + \int_0^\infty e^{-\gamma y}\int_0^y |Y_k(\xi)|e^{\gamma y}d\xi dy \\
=& \frac{1}{\gamma}|P_k(0)| + \int_0^\infty |Z_k(\xi)|e^{\gamma\xi}\int_\xi^\infty e^{-\gamma y}dyd\xi \\
=& \frac{1}{\gamma}|P_k(0)| + \frac{1}{\gamma}\|Z_k\|_{L^1[0,\infty)}, \quad n+1 \leqslant k \leqslant n+3.
\end{aligned} \tag{13}
$$

Combinging (13), (12), (11), (10) and (8), we deduce

$$
\begin{aligned}
\|\mathbf{P}\| \leq& \sum_{j=0}^n \frac{1}{\gamma+a_j}|Z_j| + \frac{1}{\gamma}\sum_{k=n+1}^{n+3}|P_k(0)| + \frac{1}{\gamma}\sum_{k=n+1}^{n+3}\|Z_k\|_{L^1[0,\infty)} \\
=& \sum_{j=0}^n \frac{1}{\gamma+a_j}|Z_j| + \frac{1}{\gamma}\left\{\frac{\lambda_{ss}}{\gamma+a_n}|Z_n| + \frac{\lambda_{si}}{\gamma+a_n}|Z_n| + \sum_{j=0}^{n-1}\frac{\lambda_r}{\gamma+a_j}|Z_j|\right\} \\
&+ \frac{1}{\gamma}\sum_{k=n+1}^{n+3}\|Z_k\|_{L^1[0,\infty)} \\
=& \frac{\gamma+\lambda_r}{\gamma(\gamma+a_0)}|Z_0| + \sum_{j=1}^{n-1}\frac{\gamma+\lambda_r}{\gamma(\gamma+a_j)}|Z_j| \\
&+ \frac{\gamma+\lambda_{ss}+\lambda_{si}}{\gamma(\gamma+a_n)}|Z_n| + \frac{1}{\gamma}\sum_{k=n+1}^{n+3}\|Z_k\|_{L^1[0,\infty)} \\
<& \frac{1}{\gamma}\left\{\sum_{j=0}^n|Z_j| + \sum_{k=n+1}^{n+3}\|Z_k\|_{L^1[0,\infty)}\right\} \\
=& \frac{1}{\gamma}\|Z\|.
\end{aligned} \tag{14}
$$

Equation (14) shows that

$$(\gamma I - \mathbb{A})^{-1} : \mathcal{X} \to \mathcal{D}(\mathbb{A}), \quad \|(\gamma I - \mathbb{A})^{-1}\| \leq \frac{1}{\gamma}.$$

The second step will be to demonstrate that $\mathcal{D}(\mathbb{A})$ is dense in $\mathcal{X}$. Let

$$L = \left\{ \mathbf{P} \middle| \begin{array}{l} \mathbf{P}(y) = (P_0, P_1, \ldots, P_n, P_{n+1}(y), P_{n+2}(y), P_{n+3}(y)), \\ P_k(y) \in C_0^\infty[0,\infty), \quad n+1 \leqslant k \leqslant n+3 \end{array} \right\},$$

then $\overline{L} = \mathcal{X}$. Take

$$S = \left\{ \mathbf{P} \middle| \begin{array}{l} \mathbf{P}(y) = (P_0, P_1, \ldots, P_n, P_{n+1}(y), P_{n+2}(y), P_{n+3}(y)), \\ P_k(y) \in C_0^\infty[0,\infty), \exists\, \alpha_k > 0, \text{such that } P_k(y) = 0, \\ \text{for } y \in [0, \alpha_k], \quad n+1 \leqslant k \leqslant n+3 \end{array} \right\},$$

then $\overline{S} = L$ by Adams [33]. As a result, proving $\overline{\mathcal{D}(\mathbb{A})} = \mathcal{X}$ suffices to show that $S \subset \overline{\mathcal{D}(\mathbb{A})}$. Hence, if $S \subset \overline{\mathcal{D}(\mathbb{A})}$, then $X = \overline{L} = \overline{\overline{S}} = \overline{S} \subset \overline{\overline{\mathcal{D}(\mathbb{A})}} = \overline{\mathcal{D}(\mathbb{A})} \subset \mathcal{X}$ gives $\mathcal{X} = \overline{\mathcal{D}(\mathbb{A})}$.

Take any $\mathbf{P} \in S$, such that, $P_k(y) = 0$, for all $y \in [0, \alpha_k]$, that is $P_k(y) = 0$, for all $y \in [0, \varsigma]$ $(n+1 \leqslant k \leqslant n+3)$, here $0 < \varsigma < \min\{\alpha_{n+1}, \alpha_{n+2}, \alpha_{n+3}\}$. Define

$$\begin{aligned} \mathcal{F}^\varsigma(y) =& (\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_n, \mathcal{F}_{n+1}(x), \mathcal{F}_{n+2}(x), \mathcal{F}_{n+3}(x)), \\ \mathcal{F}^\varsigma(0) =& \left(P_0, P_1, \ldots, P_n, \lambda_{ss} P_n, \lambda_{si} P_n \sum_{j=0}^{n-1} \lambda_r P_j \right), \end{aligned}$$

where

$$\mathcal{F}_k^\varsigma(y) = \begin{cases} \mathcal{F}_k^\varsigma(0)(1 - \frac{y}{\varsigma})^2, & y \in [0, \varsigma), \\ P_k(y), & y \in [\varsigma, \infty). \end{cases} \quad n+1 \leqslant k \leqslant n+3.$$

Then, $\mathcal{F}^\varsigma \in \mathcal{D}(\mathbb{A})$. Moreover,

$$\begin{aligned} \|\mathbf{P} - F^\varsigma\| &= \sum_{k=n+1}^{n+3} \int_0^\varsigma |\mathcal{F}_k^\varsigma(0)| \left(1 - \frac{y}{\varsigma}\right)^2 dy \\ &= \sum_{k=n+1}^{n+3} \int_0^\varsigma |\mathcal{F}_k^\varsigma(0)| \frac{\varsigma}{3} \to 0, \quad \text{as } \varsigma \to 0. \end{aligned}$$

This implies $S \subset \overline{\mathcal{D}(\mathbb{A})}$, thus, $\mathcal{D}(\mathbb{A})$ is dense in $\mathcal{X}$.

We can conclude that $\mathbb{A}$ generates a C_0-semigroup based on the preceding two steps and the Hille–Yosida Theorem. Furthermore, we can deduce that $\mathbb{A} + \mathbb{U} + \mathbb{E}$ generates a C_0-semigroup $\mathfrak{T}(t)$ using perturbation theory of C_0-semigroup (see Gupur et al. [34]).

In the final step, we show that $\mathbb{A} + \mathbb{U} + \mathbb{E}$ is a dispersive operator. Choosing, for $\mathbf{P} \in \mathcal{D}(\mathbb{A})$,

$$\Psi(y) = \left(\frac{[P_0]^+}{P_0}, \frac{[P_1]^+}{P_1}, \ldots, \frac{[P_n]^+}{P_n}, \frac{[P_{n+1}(y)]^+}{P_{n+1}(y)}, \frac{[P_{n+2}(y)]^+}{P_{n+2}(y)}, \frac{[P_{n+3}(y)]^+}{P_{n+3}(y)} \right),$$

where

$$\begin{aligned} [P_i]^+ =& \begin{cases} P_i, & \text{if} \quad P_j > 0, \\ 0, & \text{if} \quad P_j \leq 0, \end{cases} \quad 0 \leqslant j \leqslant n, \\ [P_k(y)]^+ =& \begin{cases} P_k(y), & \text{if} \quad P_k(y) > 0, \\ 0, & \text{if} \quad P_k(y) \leq 0, \end{cases} \quad n+1 \leqslant k \leqslant n+3. \end{aligned}$$

Let $W_j = \{y \in [0,\infty) \big| P_k(y) > 0\}$ and $\widetilde{W}_k = \{y \in [0,\infty) \big| P_k(y) \le 0\}$, then we get

$$\int_0^\infty \frac{dP_k(y)}{dy}\frac{[P_k(y)]^+}{P_k(y)}dy = \int_{W_k}\frac{dP_k(y)}{dy}\frac{[P_k(y)]^+}{P_k(y)}dy + \int_{\widetilde{W}_k}\frac{dP_k(y)}{dy}\frac{[P_k(y)]^+}{P_k(y)}dy$$
$$= \int_{W_k}\frac{d[P_k(y)]^+}{dy}dy = -[P_k(0)]^+, \quad n+1 \leqslant k \leqslant n+3. \tag{15}$$

Using (15) for such Ψ and boundary conditions, we deduce

$$\begin{aligned}
&\langle(\mathbb{A}+\mathbb{U}+\mathbb{E})\mathbf{P},\Psi\rangle \\
&= \left[-a_0P_0 + \mu_sP_1 + \sum_{k=n+1}^{n+3}\int_0^\infty P_k(y)\mu_k(y)dy\right]\frac{[P_0]^+}{P_0} \\
&\quad + \sum_{j=1}^{n-1}\left[\lambda_sP_{j-1} - a_jP_j + \mu_sP_{j+1}\right]\frac{[P_j]^+}{P_j} \\
&\quad + \left[\lambda_sP_{n-1} - a_nP_n\right]\frac{[P_n]^+}{P_n} \\
&\quad + \sum_{k=n+1}^{n+3}\int_0^\infty\left[-\frac{dP_k(y)}{dy} - \mu_k(y)P_k(y)\right]\frac{[P_k(y)]^+}{P_k(y)}dy \\
&\le -a_0[P_0]^+ + \mu_s[P_1]^+ + \frac{[P_0]^+}{P_0}\sum_{k=n+1}^{n+3}\int_0^\infty[P_k(y)]^+\mu_k(y)dy \\
&\quad + \sum_{j=1}^{n-1}\lambda_s[P_{j-1}]^+ - \sum_{j=1}^{n-1}a_j[P_j]^+ + \sum_{j=1}^{n-1}\mu_s[P_{j+1}]^+ \\
&\quad + \lambda_s[P_{n-1}]^+ - a_n[P_n]^+ + \lambda_{ss}[P_n]^+ + \lambda_{si}[P_n]^+ \\
&\quad + \lambda_r\sum_{j=0}^{n-1}[P_j]^+ - \sum_{k=n+1}^{n+3}\int_0^\infty[P_k(y)]^+\mu_k(y)dy \\
&= \left(\frac{[P_0]^+}{P_0} - 1\right)\sum_{k=n+1}^{n+3}\int_0^\infty[P_k(y)]^+\mu_k(y)dy \le 0.
\end{aligned} \tag{16}$$

The conclusion follows from (16).

Therefore, from above results together with Fillips theorem, we deduce that $\mathbb{A}+\mathbb{U}+\mathbb{E}$ generates a positive contraction C_0-semigroup, and it is just $\mathfrak{T}(t)$ by the uniqueness theorem of the semigroup. $\square$

The following is the dual space of $\mathcal{X}$.

$$\mathcal{X}^* = \left\{\mathbf{Q}^* \in R^n \times (L^\infty[0,\infty))^3 \,\middle|\, |||\mathbf{Q}^*||| = \sup\left\{\sup_{0\leqslant j\leqslant n}|Q_j^*|, \sup_{n+1\le k\le n+3}\|Q_k^*\|_{L^\infty[0,\infty)}\right\} < \infty\right\},$$

obviously, it is a Banach space.

Define

$$Y = \left\{\mathbf{P} \in X \,\middle|\, \begin{array}{l}\mathbf{P}(y) = (P_0, P_1, \ldots, P_n, P_{n+1}(y), P_{n+2}(y), P_{n+3}(y)) \\ P_j \ge 0, \text{ for } 0 \leqslant j \leqslant n; P_k(y) \ge 0, \ n+1 \leqslant k \leqslant n+3, \ y \in [0,\infty)\end{array}\right\} \subset \mathcal{X}.$$

Then, by Theorem 1, it follows that $T(t)Y \subset Y$. Choose $Q^* = \|\mathbf{P}\|(\overbrace{1,1,\ldots,1}^{n},1,1,1)$, for $\mathbf{P} \in D(\mathbb{A}) \cap Y$, thus, we have $Q^* \in \mathcal{X}^*$ and

$$\begin{aligned}
\langle(\mathbb{A}+\mathbb{U}+\mathbb{E})\mathbf{P}, Q^*\rangle =& \left[-a_0P_0 + \mu_s P_1 + \sum_{k=n+1}^{n+3}\int_0^\infty P_k(x)\mu_k(y)dy\right] \\
&+ \sum_{j=1}^{n-1}\left[\lambda_s P_{j-1} - a_j P_j + \mu_s P_{j+1}\right]\|\mathbf{P}\| + \left[\lambda_s P_{n-1} - a_n P_n\right]\|\mathbf{P}\| \\
&+ \sum_{k=n+1}^{n+3}\int_0^\infty\left[-\frac{dP_k(y)}{dy} - \mu_k(y)P_k(y)dy\right]\|\mathbf{P}\| \\
=& \left[\sum_{j=0}^{n-1}\lambda_s P_j - \sum_{j=0}^{n} a_j P_j + \sum_{j=1}^{n}\mu_s P_{j+1}\right]\|\mathbf{P}\| \\
&- \sum_{k=n+1}^{n+3}\int_0^\infty \frac{dP_k(y)}{dy}dy\|\mathbf{P}\| \\
=& \left[\sum_{j=0}^{n-1}\lambda_s P_j - (\lambda_s+\lambda_r)P_0 - \sum_{j=1}^{n-1}(\lambda_s+\lambda_r+\mu_s)P_j\right. \\
&\left.- (\lambda_{ss}+\lambda_{si}+\mu_s)P_n + \sum_{j=1}^{n}\mu_s P_{j+1}\right]\|\mathbf{P}\| \\
&+ \left[\lambda_{ss}P_n + \lambda_{si}P_n + \lambda_r\sum_{j=0}^{n-1}P_j\right]\|\mathbf{P}\| = 0.
\end{aligned}$$

As a result, $\mathbb{A}+\mathbb{U}+\mathbb{E}$ is conservative with respect to the set

$$\Lambda(\mathbf{P}) = \left\{Q^* \in \mathcal{X}^* \mid \langle \mathbf{P}, Q^*\rangle = \|P\|^2 = \||Q^*\||^2\right\},$$

and we can now deduce the following result from the Fattorini theorem [35] (p. 155).

Theorem 2. *$\mathfrak{T}(t)$ is isometric for $\mathbf{P}(0) = (1,0,0,0)^T$, i.e., $\|\mathfrak{T}(t)\mathbf{P}(0)\| = \|\mathbf{P}(0)\|$, $\forall t \in [0,\infty)$.*

This section's main result is derived from Theorems 1 and 2.

Theorem 3. *Equation (2) has a unique positive T-DS $\mathbf{P}(y,t)$ satisfying*

$$\|\mathbf{P}(\cdot,t)\| = 1, \quad \forall t \in [0,\infty).$$

Proof. Since $\mathbf{P}(0) \in D(\mathbb{A}^2) \cap Y$, Theorem 1 and Theorem 1.81 in [34] show that the system (2) has a unique positive T-DS $\mathbf{P}(x,t)$, i.e.,

$$\mathbf{P}(x,t) = \mathfrak{T}(t)\mathbf{P}(0), \quad t \in [0,\infty).$$

We can deduce

$$\|\mathbf{P}(\cdot,t)\| = \|\mathbf{P}(0)\| = 1, \quad \forall t \in [0,\infty).$$

□

4. Asymptotic Behavior of the T-DS of (2)

Analysis show that, similar to the proof of Theorem 1, operator $\mathbb{A}$ generates a positive contraction C_0-semigroup $\mathfrak{T}_0(t)$. Therefore, we will demonstrate quasi-compactness of $\mathfrak{T}(t)$ by showing that $\mathfrak{T}_0(t)$ is a quasi-compact operator.

Lemma 1. *If $\mathbf{P}(y,t) = (\mathfrak{T}_0(t)\, v)(y)$ is a solution of the system*

$$\begin{aligned} &\frac{d\,\mathbf{P}(t)}{dt} = \mathbb{A}\mathbf{P}(t),\ t \in [0,\infty), \\ &\mathbf{P}(0) = v \in \mathcal{D}(\mathbb{A}). \end{aligned} \tag{17}$$

Then,

$$\mathbf{P}(y,t) = \begin{cases} \begin{pmatrix} v_0 e^{-a_0 t} \\ v_1 e^{-a_1 t} \\ \vdots \\ v_n e^{-a_n t} \\ P_{n+1}(0,t-y)e^{-\int_0^y \mu_{n+1}(\sigma)d\sigma} \\ P_{n+2}(0,t-y)e^{-\int_0^y \mu_{n+2}(\sigma)d\sigma} \\ P_{n+3}(0,t-y)e^{-\int_0^y \mu_{n+3}(\sigma)d\sigma} \end{pmatrix} & \text{as } y < t, \\ \begin{pmatrix} v_0 e^{-a_0 t} \\ v_1 e^{-a_1 t} \\ \vdots \\ v_n e^{-a_n t} \\ v_{n+1}(y-t)e^{-\int_{y-t}^y \mu_{n+1}(\sigma)d\sigma} \\ v_{n+2}(y-t)e^{-\int_{y-t}^y \mu_{n+2}(\sigma)d\sigma} \\ v_{n+3}(y-t)e^{-\int_{y-t}^y \mu_{n+3}(\sigma)d\sigma} \end{pmatrix} & \text{as } y > t, \end{cases}$$

where $P_j(0,t-y)$ are given by (4).

Proof. Because $\mathbf{P}(y,t) = (\mathfrak{T}_0(t)\, v)(y)$ is a solution of (17), it satisfies

$$\frac{dP_j(t)}{dt} = -a_j P_0(t), \quad 0 \leqslant j \leqslant n, \tag{18}$$

$$\frac{\partial P_k(y,t)}{\partial t} + \frac{\partial P_j(y,t)}{\partial y} = -\mu_k(y)P_k(y,t), \quad n+1 \leqslant k \leqslant n+3, \tag{19}$$

$$P_{n+1}(0,t) = \lambda_{ri}P_n(t), \tag{20}$$

$$P_{n+2}(0,t) = \lambda_{rs}P_n(t), \tag{21}$$

$$P_{n+3}(0,t) = \lambda_r \sum_{j=0}^{n-1} P_j(t), \tag{22}$$

$$P_j(0) = v_j, \quad 0 \le j \le n, \quad P_k(y,0) = v_k(y), \quad n+1 \leqslant k \leqslant n+3. \tag{23}$$

Take $z = y - t$ and $\Pi_k(t) = P_k(z+t,t),\ n+1 \leqslant k \leqslant n+3$, then from (19), we get

$$\frac{d\Pi_k(t)}{dt} = -\,\mu_k(z+t)\Pi_k(t), \quad n+1 \leqslant k \leqslant n+3. \tag{24}$$

If $z < 0$ (i.e., $y < t$), then using $\Pi_k(-z) = P_k(0,-z) = P_k(0,t-y),\ n+1 \leqslant k \leqslant n+3$ and integrating (24) from $-z$ to t separately, we deduce

$$\begin{aligned} P_k(y,t) = \Pi_k(t) = \Pi_k(-z)e^{-\int_{-z}^t \mu_k(z+\sigma)d\sigma} &\overset{y=z+\sigma}{=\!=\!=} P_k(0,t-y)e^{-\int_0^{z+t}\mu_k(y)dy} \\ &= P_k(0,t-y)e^{-\int_0^y \mu_k(\sigma)d\tau}, \quad n+1 \leqslant k \leqslant n+3. \end{aligned} \tag{25}$$

Soving (18) and applying (23) gives

$$P_j(t) = v_j e^{-a_j t}, \quad 0 \leqslant j \leqslant n. \tag{26}$$

If $z > 0$ (i.e., $y > t$), using the relations $\Pi_k(0) = P_k(z,0) = v_k(t-y)$, $n+1 \leqslant k \leqslant n+3$ and integrating (19) from 0 to t, as well as a similar argument to (25), we obtain

$$\begin{aligned} P_k(y,t) &= \Pi_k(t) = \Pi_k(0)e^{-\int_0^t \mu_k(z+\sigma)d\sigma} \\ &\xlongequal{\delta=z+\sigma} v_k(y-t)e^{-\int_z^{z+t}\mu_k(\delta)d\delta} \\ &= v_k(y-t)e^{-\int_{y-t}^{y}\mu_k(\sigma)d\sigma}, \quad n+1 \leqslant k \leqslant n+3. \end{aligned} \tag{27}$$

Equations (25)–(27) complete the proof. □

Define

$$(\mathfrak{S}(t)\mathbf{P})(y) = \begin{cases} 0, & y \in [0,t), \\ (\mathfrak{T}_0(t)\mathbf{P})(y), & y \in [t,\infty), \end{cases} \quad (\mathfrak{V}(t)\mathbf{P})(y) = \begin{cases} (\mathfrak{T}_0(t)\mathbf{P})(y), & y \in [0,t), \\ 0, & y \in [t,\infty). \end{cases}$$

Clearly,

$$(\mathfrak{T}_0(t)\mathbf{P})(y) = (\mathfrak{S}(t)\mathbf{P})(y) + (\mathfrak{V}(t)\mathbf{P})(y), \quad \forall \mathbf{P} \in \mathcal{X}.$$

From Theorem 1.35 in [34], we can conclude the following Lemma.

Lemma 2. *If and only if the following two conditions are satisfied, a bounded and closed subset $S \subset \mathcal{X}$ is relatively compact.*

$$\begin{aligned} &(1) \ \sum_{k=n+1}^{n+3} \lim_{\tau\to 0}\int_0^\infty |\psi_k(y+\tau) - \psi_k(x)|dx = 0, \\ &\qquad \textit{uniformly for } \psi = (\psi_0,\psi_1,\ldots,\psi_n,\psi_{n+1},\psi_{n+2},\psi_{n+3}) \in S. \\ &(2) \ \sum_{k=n+1}^{n+3} \lim_{\tau\to\infty}\int_\tau^\infty |\psi_k(y)|dy = 0, \\ &\qquad \textit{uniformly for } \psi = (\psi_0,\psi_1,\ldots,\psi_n,\psi_{n+1},\psi_{n+2},\psi_{n+3}) \in S. \end{aligned}$$

Theorem 4. *$\mathfrak{V}(t)$ is a compact operator on $\mathcal{X}$.*

Proof. We only need to prove condition (1) in Lemma 2 by definition of $\mathfrak{V}(t)$. Take $\mathbf{P}(y,t) = (\mathfrak{T}_0(t)v)(y), y \in [0,t)$, for bounded $v \in \mathcal{X}$, then $\mathbf{P}(y,t)$ is a solution of (17). Hence, from Lemma 1, we deduce, for $y \in [0,t), \tau \in (0,t], y+\tau \in [0,t)$

$$\begin{aligned} &\sum_{k=n+1}^{n+3}\int_0^t |P_k(y+\tau,t) - P_k(y,t)|dy \\ &= \sum_{k=n+1}^{n+3}\int_0^t \left|P_k(0,t-y-\tau)e^{-\int_0^{y+\tau}\mu_k(\sigma)d\sigma} - P_k(0,t-y)e^{-\int_0^y \mu_k(\sigma)d\sigma}\right|dy \\ &= \sum_{k=n+1}^{n+3}\int_0^t \left|P_k(0,t-y-\tau)e^{-\int_0^{y+\tau}\mu_k(\sigma)d\sigma} - P_k(0,t-y-\tau)e^{-\int_0^y \mu_k(\sigma)d\sigma}\right. \\ &\quad \left. + P_k(0,t-y-\tau)e^{-\int_0^y \mu_k(\sigma)d\sigma} - P_k(0,t-y)e^{-\int_0^y \mu_k(\sigma)d\sigma}\right|dy \\ &\leq \sum_{k=n+1}^{n+3}\int_0^t |P_k(0,t-y-\tau)|\left|e^{-\int_0^{y+\tau}\mu_k(\sigma)d\sigma} - e^{-\int_0^y \mu_k(\sigma)d\sigma}\right|dy \\ &\quad + \sum_{k=n+1}^{n+3}\int_0^t |P_k(0,t-y-\tau) - P_k(0,t-y)|e^{-\int_0^y \mu_k(\sigma)d\sigma}dy. \end{aligned} \tag{28}$$

The procedure is to estimate each term of (28). Applying the boundary conditions, we get

$$|P_{n+1}(0,t-y-\tau)| = \lambda_{ss}|v_n e^{-a_n(t-y-\tau)}| \le \lambda_{ss}|v_n| \le \lambda_{ss}\|v\|_{\mathcal{X}}, \tag{29}$$

$$|P_{n+2}(0,t-y-\tau)| = \lambda_{si}|v_n e^{-a_n(t-y-\tau)}| \le \lambda_{rs}|v_n| \le \lambda_{si}\|v\|_{\mathcal{X}}, \tag{30}$$

$$|P_{n+3}(0,t-y-\tau)| \le \lambda_r \sum_{j=0}^{n-1}\left|v_j e^{-a_j(t-y-\tau)}\right| \le \lambda_r \sum_{j=0}^{n-1}|v_j| \le \lambda_r\|v\|_{\mathcal{X}}. \tag{31}$$

We can estimate the first term of (28) using (29)–(31).

$$\begin{aligned}
&\sum_{k=n+1}^{n+3}\int_0^t |P_k(0,t-y-\tau)|\left|e^{-\int_0^{y+\tau}\mu_k(\sigma)d\sigma} - e^{-\int_0^{y}\mu_k(\sigma)d\sigma}\right|dy \\
&\le \{\lambda_{ss}+\lambda_{si}+\lambda_r\}\|v\|_{\mathcal{X}}\sum_{k=n+1}^{n+3}\int_0^t\left|e^{-\int_0^{y+\tau}\mu_k(\sigma)d\sigma} - e^{-\int_0^{y}\mu_k(\sigma)d\sigma}\right|dy \\
&\longrightarrow 0, \text{ as } \quad \tau \to 0, \quad \text{uniformly for } v.
\end{aligned} \tag{32}$$

Now, we will estimate the second term in (28). Using Lemma 1 and boundary conditions, we calculate

$$\begin{aligned}
|P_{n+1}(0,t-y-\tau) - P_{n+1}(0,t-y)| &\le |\lambda_{ss}(v_n e^{-a_n(t-y-\tau)} - v_n e^{-a_n(t-y)})| \\
&\le \lambda_{ss}\|v\|_{\mathcal{X}}|e^{-a_n(t-y-\tau)} - e^{-a_n(t-y)}| \\
&\longrightarrow 0 \quad \text{as } \tau \to 0, \quad \text{uniformly for } v,
\end{aligned} \tag{33}$$

$$\begin{aligned}
|P_{n+2}(0,t-y-\tau) - P_{n+2}(0,t-y)| &\le |\lambda_{si}(v_n e^{-a_n(t-y-\tau)} - v_n e^{-a_n(t-y)})| \\
&\le \lambda_{si}\|v\|_{\mathcal{X}}|e^{-a_n(t-y-\tau)} - e^{-a_n(t-y)}| \\
&\longrightarrow 0 \quad \text{as } \tau \to 0, \quad \text{uniformly for } v,
\end{aligned} \tag{34}$$

$$\begin{aligned}
|P_{n+3}(0,t-y-\tau) - P_{n+3}(0,t-y)| &\le \left|\lambda_r\sum_{j=0}^{n-1} v_j(e^{-a_j(t-y-\tau)} - e^{-a_j(t-y)})\right| \\
&\le \lambda_r\|v\|_{\mathcal{X}}\sum_{j=0}^{n-1}\left|e^{-a_j(t-y-\tau)} - e^{-a_j(t-y)}\right| \\
&\longrightarrow 0 \quad \text{as } \tau \to 0, \quad \text{uniformly for } v.
\end{aligned} \tag{35}$$

From (33)–(35), we deduce

$$\sum_{k=n+1}^{n+3}\int_0^t |P_k(0,t-y-\tau) - P_j(0,t-y)|e^{-\int_0^y\mu_k(\sigma)d\sigma} \longrightarrow 0, \text{ as } \tau \to 0, \text{ uniformly for } v. \tag{36}$$

Combining (32) with (36), we obtain, for $y \in [0,t), \tau \in (0,t], y+\tau \in [0,t)$

$$\sum_{k=n+1}^{n+3}\int_0^t |P_k(y+\tau,t) - P_k(y,t)|dy \longrightarrow 0, \text{ as } \tau \to 0, \quad \text{uniformly for } v. \tag{37}$$

The same conclusion can be drawn for $\tau \in [-t,0),\ \ y+\tau \in [0,t)$. This finishes the proof. □

Theorem 5. *If* $0 < \underline{\mu_k} \le \mu_k(y) \le \overline{\mu_k} < \infty$, *for* $n+1 \leqslant k \leqslant n+3$, *then* $\mathfrak{S}(t)$ *satisfies*

$$\|\mathfrak{S}(t)\,v\|_{\mathcal{X}} \le e^{-\min\{a_0,a_n,\underline{\mu_{n+1}},\underline{\mu_{n+2}},\underline{\mu_{n+3}}\}t}\|v\|_{\mathcal{X}}.$$

Proof. For any $v \in \mathcal{X}$, we estimate

$$\begin{aligned}||\mathfrak{S}(t)\,v(\cdot)||_{\mathcal{X}} &= \sum_{j=0}^{n}|P_j(t)| + \sum_{k=n+1}^{n+3}\int_t^\infty |P_k(y,t)|dy \\ &\leq \sum_{j=0}^{n}|v_j|e^{-a_j t} + \sum_{k=n+1}^{n+3}\int_0^\infty |v_k(y-t)|e^{-\int_{y-t}^{y}\mu_k(\sigma)d\sigma}dy \\ &\leq e^{-\min\{a_0,a_n\}t}\sum_{j=0}^{n}|v_j| + e^{-\min\{\underline{\mu_{n+1}},\underline{\mu_{n+2}},\underline{\mu_{n+3}}\}\,t}\sum_{k=n+1}^{n+3}\|v_k\|_{L^1[0,\infty)} \\ &\leq e^{-\min\{a_0,a_n,\underline{\mu_{n+1}},\underline{\mu_{n+2}},\underline{\mu_{n+3}}\}t}\|v\|_{\mathcal{X}}.\end{aligned}$$

□

Theorems 4 and 5 give

$$\|\mathfrak{T}_0(t) - \mathfrak{V}(t)\| = \|\mathfrak{S}(t)\| \leq e^{-\min\{a_0,a_n,\underline{\mu_{n+1}},\underline{\mu_{n+2}},\underline{\mu_{n+3}}\}t} \to 0, \quad t \to \infty.$$

Hence, we can obtain the following result by Definition 1.85 in [34].

Theorem 6. *If* $0 < \underline{\mu_k} \leq \mu_k(y) \leq \overline{\mu_k} < \infty$, *for* $n+1 \leqslant k \leqslant n+3$, *then* $\mathfrak{T}_0(t)$ *is a quasi-compact operator on* $\mathcal{X}$.

We get the following result by combining Theorem 6 and Proposition [36] (p. 215), as well as the compactness of the $\mathbb{U}$ and $\mathbb{E}$ on $\mathcal{X}$.

Corollary 1. *If* $0 < \underline{\mu_k} \leq \mu_k(y) \leq \overline{\mu_k} < \infty$, *for* $n+1 \leqslant k \leqslant n+3$, *then* $\mathfrak{T}(t)$ *is a quasi-compact operator on* $\mathcal{X}$.

Lemma 3. $0 \in \sigma_p(\mathbb{A}+\mathbb{U}+\mathbb{E})$, *and geometric multiplicity of* 0 *is one.*

Proof. Take $(\mathbb{A}+\mathbb{U}+\mathbb{E})\mathbf{P} = 0$. Hence,

$$a_0P_0 = \mu_1P_1 + \sum_{k=n+1}^{n+3}\int_0^\infty P_k(y)\mu_k(y)dy, \tag{38}$$
$$a_jP_j = \lambda_sP_{j-1} + \mu_sP_{j+1}, \quad 1 \leqslant j \leqslant n-1, \tag{39}$$
$$a_nP_n = \lambda_sP_{n-1}, \tag{40}$$
$$\frac{dP_k(y)}{\partial y} = -\mu_k(y)P_k(y), \quad n+1 \leqslant k \leqslant n+3, \tag{41}$$
$$P_{n+1}(0) = \lambda_{ss}P_n, \tag{42}$$
$$P_{n+2}(0) = \lambda_{si}P_n, \tag{43}$$
$$P_{n+3}(0) = \lambda_r\sum_{j=0}^{n-1}P_j. \tag{44}$$

Solving (41), we have

$$P_k(y) = P_k(0)e^{-\int_0^y\mu_k(\sigma)d\sigma}, \quad n+1 \leqslant k \leqslant n+3. \tag{45}$$

If we take

$$b_n = a_n, \quad b_l = a_l - \frac{\lambda_s\mu_s}{b_{l+1}}, \quad l = 1,2,\ldots,n-1.$$

Then, from (38)–(40) we can show that

$$
\begin{aligned}
P_n &= \frac{\lambda_s P_{n-1}}{a_n} = \frac{\lambda_s}{b_n} P_{n-1}, \\
a_{n-1} P_{n-1} &= \lambda_s P_{n-2} + \mu_n P_n = \lambda_s P_{n-2} + \frac{\lambda_s \mu_s}{b_n} P_{n-1} \\
&\Longrightarrow \\
P_{n-1} &= \frac{\lambda_s P_{n-2}}{a_{n-1} - \frac{\lambda_s \mu_s}{b_n}} = \frac{\lambda_s}{b_{n-1}} P_{n-2}, \\
a_{n-2} P_{n-2} &= \lambda_s P_{n-3} + \mu_s P_{n-1} = \lambda_s P_{n-3} + \frac{\lambda_s \mu_s P_{n-2}}{b_{n-1}} \\
&\Longrightarrow \\
P_{n-2} &= \frac{\lambda_s P_{n-3}}{a_{n-2} - \frac{\lambda_s \mu_s}{b_{n-1}}} = \frac{\lambda_s}{b_{n-2}} P_{n-3}, \\
\cdots & \\
a_2 P_2 &= \lambda_s P_1 + \mu_s P_3 = \lambda_s P_1 + \frac{\lambda_s \mu_s}{b_3} P_2 \\
&\Longrightarrow \\
P_2 &= \frac{\lambda_s}{a_2 - \frac{\lambda_s \mu_s}{b_3}} P_1 = \frac{\lambda_s}{b_2} P_1, \\
a_1 P_1 &= \lambda_s P_0 + \mu_s P_2 = \lambda_s P_0 + \frac{\lambda_s \mu_2}{b_2} P_1 \\
&\Longrightarrow \\
P_1 &= \frac{\lambda_s}{a_1 - \frac{\lambda_s \mu_2}{b_2}} P_0 = \frac{\lambda_s}{b_1} P_0, \\
&\Longrightarrow \\
P_n &= \frac{\lambda_s}{b_n} \frac{\lambda_s}{b_{n-1}} P_{n-2} = \cdots = \frac{\lambda_s^n}{\prod\limits_{l=1}^{n} b_l} P_0.
\end{aligned}
$$

Thus,

$$
P_j = \frac{\lambda_s^j}{\prod\limits_{l=1}^{j} b_l} P_0, \quad j = 1, 2, \ldots, n. \tag{46}
$$

Combining (42)–(44) and (45) with (46), we estimate

$$
\begin{aligned}
\|P\| &= \sum_{j=0}^{n} |P_j| + \sum_{k=n+1}^{n+3} \int_0^\infty |P_k(0)| e^{-\int_0^y \mu_k(\tau) d\tau} dy \\
&\leq \left[1 + \sum_{j=1}^{n} \lambda_s^j \left(\prod_{l=1}^{j} b_j \right)^{-1} \right] |P_0| + \sum_{k=n+1}^{n+3} |P_k(0)| \int_0^\infty e^{-\int_0^y \mu_k(\tau) d\tau} dy \\
&= \left[1 + \sum_{j=1}^{n} \lambda_s^j \left(\prod_{l=1}^{j} b_j \right)^{-1} \right] |P_0| \\
&\quad + \lambda_{ss} \lambda_s^n \left(\prod_{l=1}^{n} b_l \right)^{-1} \int_0^\infty e^{-\int_0^y \mu_{n+1}(\tau) d\tau} dy |P_0|
\end{aligned}
$$

$$+\lambda_{si}\lambda_s^n\left(\prod_{l=1}^{n}b_l\right)^{-1}\int_0^\infty e^{-\int_0^y \mu_{n+2}(\tau)d\tau}dx|P_0|$$
$$+\lambda_r\left[1+\lambda_s^n\sum_{j=0}^{n-1}\left(\prod_{l=1}^{j}b_l\right)^{-1}\right]\int_0^\infty e^{-\int_0^y \mu_{n+3}(\tau)d\tau}dy|P_0|<\infty. \tag{47}$$

Equation (47) show that $0\in\sigma_p(\mathbb{A}+\mathbb{U}+\mathbb{E})$, i.e., the point spectrum of $\mathbb{A}+\mathbb{U}+\mathbb{E}$, and from (46), it can be seen that the geometric multiplicity of 0 is one. □

Lemma 4. *The adjoint operator of $\mathbb{A}+\mathbb{U}+\mathbb{E}$ is given by*

$$(\mathbb{A}+\mathbb{U}+\mathbb{E})^*\mathbf{Q}^* =(\mathcal{M}+\mathcal{N}+\mathcal{L})\mathbf{Q}^*,$$

where

$$\mathcal{M}=\begin{pmatrix}\mathcal{M}_0 & \mathbf{0}_{(n+1)\times 3}\\ \mathbf{0}_{3\times(n+1)} & \mathcal{M}_1\end{pmatrix},\quad \mathcal{N}=\begin{pmatrix}\mathbf{0}_{(n+1)\times(n+1)} & \mathbf{0}_{(n+1)\times 3}\\ \mathcal{N}_0 & \mathbf{0}_{3\times 3}\end{pmatrix},\quad \mathcal{L}=\begin{pmatrix}\mathbf{0}_{(n+1)\times(n+1)} & \mathcal{L}_0\\ \mathbf{0}_{3\times(n+1)} & \mathbf{0}_{3\times 3}\end{pmatrix},$$

and

$$\mathcal{M}_0=\begin{pmatrix}-a_0 & \lambda_s & 0 & \cdots & 0 & 0\\ \mu_s & -a_1 & \lambda_s & \cdots & 0 & 0\\ 0 & \mu_s & -a_2 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & -a_{n-1} & \lambda_s\\ 0 & 0 & 0 & \cdots & \mu_s & -a_n\end{pmatrix},$$

$$\mathcal{M}_1=\begin{pmatrix}-\frac{d}{dy}-\mu_{n+1}(y) & 0 & 0\\ 0 & -\frac{d}{dy}-\mu_{n+2}(y) & 0\\ 0 & 0 & -\frac{d}{dy}-\mu_{n+3}(y)\end{pmatrix},$$

$$\mathcal{N}_0=\begin{pmatrix}\mu_{n+1}(y) & 0 & \cdots & 0\\ \mu_{n+2}(y) & 0 & \cdots & 0\\ \mu_{n+3}(y) & 0 & \cdots & 0\end{pmatrix},\quad \mathcal{L}_0=\begin{pmatrix} & & \lambda_r\\ \mathbf{0}_{2\times(n+1)} & & \lambda_r\\ & & \\ & & \lambda_r\\ \lambda_{ss} & \lambda_{si} & 0\end{pmatrix},$$

$\mathbf{0}_{m\times n}$ denotes the $m\times n-$ zero matrix.

$$D((\mathbb{A}+\mathbb{U}+\mathbb{E})^*)=\left\{\frac{dQ_k^*(y)}{dy}\text{ exists and }Q_k^*(\infty)=\alpha,\quad n+1\leqslant k\leqslant n+3\right\},$$

and the constant α in $D((\mathbb{A}+\mathbb{U}+\mathbb{E})^*)$ is independent of j.

Proof. For $\mathbf{Q}^*\in D((\mathbb{A}+\mathbb{U}+\mathbb{E})^*)$, we have

$$\begin{aligned}\langle(\mathbb{A}+\mathbb{U}+\mathbb{E})\mathbf{P},\mathbf{Q}^*\rangle=&\,Q_0^*\left[-a_0P_0+\mu_sP_1+\sum_{k=n+1}^{n+3}\int_0^\infty P_k(y)\mu_k(y)dy\right]\\ &+\sum_{j=1}^{n-1}Q_j^*\left[\lambda_sP_{j-1}-a_jP_j+\mu_sP_{j+1}\right]+Q_n^*\left[\lambda_sP_{n-1}-a_nP_n\right]\\ &+\sum_{k=n+1}^{n+3}\int_0^\infty Q_k^*(y)\left[-\frac{dP_k(y)}{dy}-\mu_k(y)P_k(y)\right]dy\\ =&\,P_0[-a_0Q_0^*+\lambda_sQ_1^*]+\sum_{j=1}^{n-1}P_j\left[\mu_jQ_{j-1}^*-a_jQ_j^*+\lambda_sQ_{j+1}^*\right]\end{aligned}$$

$$
\begin{aligned}
&+ P_n\left[\mu_s Q_{n-1}^* - a_n Q_n^*\right] + \lambda_{ss} P_n Q_{n+1}^*(0) + \lambda_{si} P_n Q_{n+2}^*(0) \\
&+ \lambda_r \sum_{j=0}^{n-1} P_j Q_{n+3}^*(0) \\
&+ \sum_{k=n+1}^{n+3} \int_0^\infty P_k(y)\left[\frac{dQ_k^*(y)}{dy} - \mu_k(y) Q_k^*(y) + \mu_k(y) Q_0^*\right] dy \\
=\;& P_0\left[-a_0 Q_0^* + \lambda_s Q_1^* + \lambda_r Q_{n+3}^*(0)\right] \\
&+ \sum_{j=1}^{n-1}\left[\mu_j Q_{j-1}^* - a_j Q_j^* + \lambda_s Q_{j+1}^* + \lambda_r Q_{n+3}^*(0)\right] \\
&+ P_n\left[\mu_s Q_{n-1}^* - a_n Q_n^* + \lambda_{ss} Q_{n+1}^*(0) + \lambda_{si} Q_{n+2}^*(0)\right] \\
&+ \sum_{k=n+1}^{n+3} \int_0^\infty P_k(y)\left[\frac{dQ_k^*(y)}{dy} - \mu_k(y) Q_k^*(y) + \mu_k(y) Q_0^*\right] dy \\
=\;& \langle \mathbf{P}, (\mathcal{M} + \mathcal{N} + \mathcal{L})\mathbf{Q}^* \rangle.
\end{aligned}
$$

□

Lemma 5. *$0 \in \sigma_p((\mathbb{A} + \mathbb{U} + \mathbb{E})^*)$ and geometric multiplicity of 0 is one.*

Proof. Consider $(\mathbb{A} + \mathbb{U} + \mathbb{E})^* \mathbf{Q}^* = 0$, i.e.,

$$
\begin{aligned}
& -a_0 Q_0^* + \lambda_s Q_1^* + \lambda_r Q_{n+3}^*(0) = 0, && (48)\\
& \mu_j Q_{j-1}^* - a_j Q_j^* + \lambda_s Q_{j+1}^* + \lambda_r Q_{n+3}^*(0) = 0, \quad 1 \leqslant j \leqslant n-1, && (49)\\
& \mu_n Q_{n-1}^* - a_n Q_n^* + \lambda_{ss} Q_{n+1}^*(0) + \lambda_{si} Q_{n+2}^*(0) = 0, && (50)\\
& \frac{dQ_k^*(y)}{dy} = \mu_k(y) Q_k^*(y) - \mu_k(y) Q_0^*, \quad k = n+1, n+2, n+3. && (51)\\
& Q_k^*(\infty) = \alpha, \quad n+1 \leqslant k \leqslant n+3. && (52)
\end{aligned}
$$

Solving (51), we deduce

$$
Q_k^*(y) = b_k e^{\int_0^y \mu_k(\tau) d\tau} - e^{\int_0^y \mu_k(\tau) d\tau} \int_0^y Q_0^* \mu_k(\xi) e^{-\int_0^\xi \mu_k(\tau) d\tau} d\xi, \quad n+1 \leqslant k \leqslant n+3. \tag{53}
$$

Multiplying $e^{-\int_0^y \mu(\tau)) d\tau}$ to the both side of (53), we have

$$
b_k = \int_0^\infty \mu_k(\xi) Q_0^* e^{-\int_0^\xi \mu_k(\tau) d\tau} d\xi, \quad n+1 \leqslant k \leqslant n+3. \tag{54}
$$

Substituting (54) into (53), we get

$$
\begin{aligned}
Q_k^*(y) &= Q_0^*\, e^{\int_0^y \mu_k(\tau) d\tau} \int_y^\infty \mu_k(\xi) e^{-\int_0^\xi \mu_k(\tau) d\tau} d\xi \\
&= Q_0^*\, e^{\int_0^y \mu_k(\tau) d\tau} \left(-e^{-\int_0^\xi \mu_k(\tau) d\tau} \Big|_y^\infty \right) = Q_0^*, \quad n+1 \leqslant k \leqslant n+3. \qquad (55)
\end{aligned}
$$

Substituting (55) into (48)–(50), we have

$$
Q_j^* = Q_0^*, \quad 1 \leqslant j \leqslant n. \tag{56}
$$

Equations (55)–(56) give

$$|||\mathbf{Q}^*||| = \max\Big\{ \max_{1\leqslant j\leqslant n} |Q_j^*|, \max_{n+1\leqslant k\leqslant n+3} \|Q_k^*\|_{L^\infty[0,\infty)} \Big\} = |Q_0^*| < \infty,$$

which imply $0 \in \sigma_p((\mathbb{A}+\mathbb{U}+\mathbb{E})^*)$. Furthermore, from (55) and (56), it is easy to verify that the geometric multiplicity of 0 is one. □

By using Lemmas 3 and 5 and Theorem 3, we can deduce that the algebraic multiplicity of 0 is one and the spectral bound $s(\mathbb{A}+\mathbb{U}+\mathbb{E}) = 0$. Finally, the conditions of Theorem 1.90 in [34] are fulfilled. Therefore, we get the following result.

Theorem 7. *If* $0 < \underline{\mu_k} \leqslant \mu_k(y) \leqslant \overline{\mu_k} < \infty$, *for* $n+1 \leqslant k \leqslant n+3$, *then there exist a spectral projection* $\mathcal{P}$ *with rank one such that*

$$\| \mathfrak{T}(t) - \mathcal{P} \| \leqslant Ke^{-\varrho t},$$

where $\mathcal{P} = \frac{1}{2\pi i}\int_\Gamma (\eta I - \mathbb{A} - \mathbb{U} - \mathbb{E})^{-1} \mathrm{d}z$ *and* Γ *is a circle with a radius of sufficiently small and a center of* 0.

It is evident that $\{\eta \in \sigma(\mathbb{A}+\mathbb{U}+\mathbb{E}) \mid Re\eta = 0\} = \{0\}$ by Theorem 3, Corollary 1 and Lemma 3. In other words, the resolvent set of $\mathbb{A}+\mathbb{U}+\mathbb{E}$ includes all points on the imaginary axis except zero.

Remark 1. *Based on the analysis above, we can conclude that the system's T-DS strongly converges to its S-SS, i.e.,* $\lim_{t\to\infty} \mathbf{P}(y,t) = \zeta\mathbf{P}(y)$, *where* $\mathbf{P}(y)$ *is the eigenvector corresponding to* 0.

In the following, we investigate exponential convergence of system's T-DS. For this goal, we first determine the explicit expression of $\mathcal{P}$ by the growth bound of $\mathfrak{T}(t)$.

Lemma 6. *For any* $\eta \in \rho(\mathbb{A}+\mathbb{U}+\mathbb{E})$, *we get*

$$(\eta I - \mathbb{A} - \mathbb{U} - \mathbb{E})^{-1}\begin{pmatrix} \mathbb{y}_0 \\ \mathbb{y}_1 \\ \vdots \\ \mathbb{y}_n \\ \mathbb{y}_{n+1} \\ \mathbb{y}_{n+2} \\ \mathbb{y}_{n+3} \end{pmatrix} = \begin{pmatrix} \mathbb{b}_0 \\ \mathbb{b}_1 \\ \vdots \\ \mathbb{b}_n \\ \mathbb{b}_{n+1} \\ \mathbb{b}_{n+2} \\ \mathbb{b}_{n+3} \end{pmatrix}, \quad \forall \mathbb{b} \in \mathcal{X},$$

where

$$\begin{aligned}
\mathbb{y}_j =& \frac{|G_i(\eta)|}{|G(\eta)|}, \quad j = 0,1,2,\ldots,n, \\
\mathbb{y}_{n+1}(y) =& \lambda_{ss}\, \frac{|G_n(\eta)|}{|G(\eta)|}\, e^{-\int_0^y (\eta+\mu_{n+1}(\xi))d\xi} \\
&+ e^{-\int_0^y (\eta+\mu_j(\xi))d\xi} \int_0^y \mathbb{b}_{n+1}(\tau) e^{\int_0^y (\eta+\mu_{n+1}(\xi))d\xi} d\tau, \\
\mathbb{y}_{n+2}(y) =& \lambda_{si}\, \frac{|G_n(\eta)|}{|G(\eta)|}\, e^{-\int_0^y (\eta+\mu_{n+2}(\xi))d\xi} \\
&+ e^{-\int_0^y (\eta+\mu_{n+2}(\xi))d\xi} \int_0^y \mathbb{b}_{n+2}(\tau) e^{\int_0^y (\eta+\mu_{n+2}(\xi))d\xi} d\tau, \\
\mathbb{y}_{n+3}(y) =& \lambda_r \sum_{j=0}^{n-1} \frac{|G_j(\eta)|}{|G(\eta)|}\, e^{-\int_0^y (\eta+\mu_{n+3}(\xi))d\xi}
\end{aligned}$$

$$+ e^{-\int_0^x (\eta+\mu_{n+3}(\xi))d\xi} \int_0^y \mathbb{b}_{n+3}(\tau) e^{\int_0^y (\eta+\mu_{n+3}(\xi))d\xi} d\tau,$$

$$G(\eta) = \begin{pmatrix} \eta + a_0 - \lambda_r\beta_{n+3} & -\mu_s - \lambda_r\beta_{n+3} & -\lambda_r\beta_{n+3} & \cdots & -\lambda_r\beta_{n+3} & -\lambda_{ss}\beta_{n+1} - \lambda_{si}\beta_{n+2} \\ -\lambda_s & \eta + a_1 & -\mu_s & \cdots & 0 & 0 \\ 0 & -\lambda_s & \eta + a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \eta + a_{n-1} & -\mu_s \\ 0 & 0 & 0 & \cdots & -\lambda_s & \eta + a_n \end{pmatrix},$$

$$\beta_k = \int_0^\infty \mu_k(\tau) e^{-\int_0^y (\eta+\mu_k(\tau))d\tau} dy, \quad n+1 \leqslant k \leqslant n+3,$$

and $|G(\eta)|$ denotes the determinant of $G(\eta)$, and $G_i(\eta)$ is the same matrix $G(\eta)$ such that ith column is replaced with constants.

Proof. Consider the equation $(\eta I - \mathbb{A} - \mathbb{U} - \mathbb{E})\mathbb{y} = \mathbb{b}$, for $\mathbb{b} \in \mathcal{X}$. Hence,

$$(\eta + a_0)\mathbb{y}_0 - \mu_s \mathbb{y}_1 - \sum_{k=n+1}^{n+3} \int_0^\infty \mathbb{y}_k(y)\mu_k(y)dy = \mathbb{b}_0, \tag{57}$$

$$-\lambda_s \mathbb{y}_{j-1} + (\eta + a_j)\mathbb{y}_j - \mu_s \mathbb{y}_{j+1} = \mathbb{b}_j, \quad 1 \leqslant j \leqslant n-1, \tag{58}$$

$$-\lambda_s \mathbb{y}_{n-1} + (\eta + a_n)\mathbb{y}_n = \mathbb{b}_n, \tag{59}$$

$$\frac{d\mathbb{y}_k(y)}{\partial y} = -(\eta + \mu_k(y))\mathbb{y}_k(y) + \mathbb{b}_k(y), \quad n+1 \leqslant k \leqslant n+3, \tag{60}$$

$$\mathbb{y}_{n+1}(0) = \lambda_{ss}\mathbb{y}_n, \tag{61}$$

$$\mathbb{y}_{n+2}(0) = \lambda_{si}\mathbb{y}_n, \tag{62}$$

$$\mathbb{y}_{n+3}(0) = \lambda_r \sum_{j=0}^{n-1} \mathbb{y}_j. \tag{63}$$

Solving (60) and using (61)–(63), we have

$$\begin{aligned} \mathbb{y}_{n+1}(y) =& \lambda_{ss}\, \mathbb{y}_n\, e^{-\int_0^y (\eta+\mu_{n+1}(\xi))d\xi} \\ &+ e^{-\int_0^y (\eta+\mu_{n+1}(\xi))d\xi} \int_0^y \mathbb{b}_{n+1}(\tau) e^{\int_0^y (\eta+\mu_{n+1}(\xi))d\xi} d\tau, \end{aligned} \tag{64}$$

$$\begin{aligned} \mathbb{y}_{n+2}(y) =& \lambda_{si}\, \mathbb{y}_n\, e^{-\int_0^y (\eta+\mu_{n+2}(\xi))d\xi} \\ &+ e^{-\int_0^y (\eta+\mu_{n+2}(\xi))d\xi} \int_0^y \mathbb{b}_{n+2}(\tau) e^{\int_0^y (\eta+\mu_{n+2}(\xi))d\xi} d\tau, \end{aligned} \tag{65}$$

$$\begin{aligned} \mathbb{y}_{n+3}(y) =& \lambda_r \sum_{j=0}^{n-1} \mathbb{y}_j\, e^{-\int_0^y (\eta+\mu_{n+3}(\xi))d\xi} \\ &+ e^{-\int_0^y (\eta+\mu_{n+3}(\xi))d\xi} \int_0^y \mathbb{b}_{n+3}(\tau) e^{\int_0^y (\eta+\mu_{n+3}(\xi))d\xi} d\tau. \end{aligned} \tag{66}$$

Substituting (64)–(66) into (57)–(59), we get

$$\begin{aligned} &\left(\eta + a_0 - \lambda_r \int_0^\infty \mu_{n+3}(y)\, e^{-\int_0^y (\eta+\mu_{n+3}(\xi))d\xi} dy\right)\mathbb{y}_0 \\ &- \left(\mu_s + \lambda_r \int_0^\infty \mu_{n+3}(y)\, e^{-\int_0^y (\eta+\mu_{n+3}(\xi))d\xi} dy\right)\mathbb{y}_1 \\ &- \lambda_r \int_0^\infty \mu_{n+3}(y)\, e^{-\int_0^y (\eta+\mu_{n+3}(\xi))d\xi} dy \sum_{j=2}^{n-1} \mathbb{y}_j \end{aligned}$$

$$-\left(\lambda_{ss}\int_0^\infty \mu_{n+1}(y)\,e^{-\int_0^y(\eta+\mu_{n+1}(\xi))d\xi}dy+\lambda_{si}\int_0^\infty \mu_{n+2}(y)\,e^{-\int_0^y(\eta+\mu_{n+2}(\xi))d\xi}dy\right)\mathbb{y}_n$$
$$=\sum_{k=n+1}^{n+3}\int_0^\infty \mu_k(y)\,e^{-\int_0^y(\eta+\mu_k(\xi))d\xi}\int_0^y \mathbb{b}_k(\tau)e^{\int_0^y(\eta+\mu_k(\xi))d\xi}d\tau dy+\mathbb{b}_0, \tag{67}$$
$$-\lambda_s\mathbb{y}_{j-1}+(\eta+a_j)\mathbb{y}_j-\mu_s\mathbb{y}_{j+1}=\mathbb{b}_j,\quad j=1,2,\cdots,n-1, \tag{68}$$
$$-\lambda_s\mathbb{y}_{n-1}+(\eta+a_n)\mathbb{y}_n=\mathbb{b}_n. \tag{69}$$

Equations (67)–(69) give

$$G(\eta)\overline{\mathbb{y}}=\overline{\mathbb{b}},$$

where

$$\overline{\mathbb{y}}=\left(\mathbb{y}_0,\mathbb{y}_1,\ldots,\mathbb{y}_{n-1},\mathbb{y}_n\right)^T,$$
$$\overline{\mathbb{b}}=\left(\sum_{k=n+1}^{n+3}\mathbb{B}_k(\eta)+\mathbb{b}_0,\mathbb{b}_1,\mathbb{b}_2,\ldots,\mathbb{b}_{n-1},\mathbb{b}_n\right)^T,$$
$$\mathbb{B}_k=\int_0^\infty \mu_k(\xi)e^{-\int_0^y(\eta+\mu_k(\xi))d\xi}\int_0^y \mathbb{b}_k(\tau)e^{\int_0^y(\eta+\mu_k(\xi))d\xi}d\tau dy.$$

Using Cramer's rule, we derive

$$\mathbb{y}_j=\frac{|G_j(\eta)|}{|G(\eta)|},\quad j=0,1,2,\ldots,n. \tag{70}$$

Substituting (70) into (64), (65), and (66) separately, we get the rest of the Lemma's results. □

In summary, we present the following main results.

Theorem 8. *If* $0<\underline{\mu_k}\leqslant\mu_k(y)\leqslant\overline{\mu_k}<\infty$, *for* $n+1\leqslant k\leqslant n+3$, *then*

$$\|\mathbf{P}(\cdot,t)-\mathbf{P}(\cdot)\|\leq Ke^{-\varrho t},\quad t>0.$$

that is, the system's T-DS exponentially converges to its S-SS.

Proof. Obviously,

$$\|\mathfrak{T}_0(t)-\mathfrak{V}(t)\|=\|\mathfrak{S}(t)\|\leq e^{-\min\{a_0,a_n,\underline{\mu_{n+1}},\underline{\mu_{n+2}},\underline{\mu_{n+3}}\}t}$$
$$\Longrightarrow$$
$$\lim_{t\to\infty}\frac{\ln\|\mathfrak{T}_0(t)-\mathfrak{V}(t)\|}{t}\leq-\min\{a_0,a_n,\underline{\mu_{n+1}},\underline{\mu_{n+2}},\underline{\mu_{n+3}}\}.$$

Thus, the essential growth bound of $\mathfrak{T}_0(t)$ satisfies $\omega_{ess}(\mathfrak{T}_0(t))\leq-\min\{a_0,a_n,\underline{\mu_{n+1}},\underline{\mu_{n+2}},\underline{\mu_{n+3}}\}$ by Proposition 2.10 in Engel and Nagel [37] (p. 258).

Since $\mathbb{E}$ and $\mathbb{U}$ are compact operators, by Proposition 2.12 in [37], we have

$$\omega_{ess}(\mathbb{A}+\mathbb{U}+\mathbb{E})=\omega_{ess}(\mathfrak{T}(t))=\omega_{ess}(\mathfrak{T}_0(t))\leq-\min\{a_0,a_n,\underline{\mu_{n+1}},\underline{\mu_{n+2}},\underline{\mu_{n+3}}\}.$$

Thus, 0 is a pole of $(\eta I-\mathbb{A}-\mathbb{U}-\mathbb{E})^{-1}$ of order 1 by Corollary 2.11 in [37]. Moreover, from Theorem 8, Lemma 6 and residue theorem, we have

$$\mathcal{P}\begin{pmatrix} \mathbb{b}_0 \\ \mathbb{b}_1 \\ \vdots \\ \mathbb{b}_n \\ \mathbb{b}_{n+1} \\ \mathbb{b}_{n+2} \\ \mathbb{b}_{n+3} \end{pmatrix} = \lim_{\eta\to 0} \eta(\eta I - \mathbb{A} - \mathbb{U} - \mathbb{E})^{-1} \begin{pmatrix} \mathbb{b}_0 \\ \mathbb{b}_1 \\ \vdots \\ \mathbb{b}_n \\ \mathbb{b}_{n+1} \\ \mathbb{b}_{n+2} \\ \mathbb{b}_{n+3} \end{pmatrix} = \lim_{\eta\to 0} \eta \begin{pmatrix} \mathbb{y}_0 \\ \mathbb{y}_1 \\ \vdots \\ \mathbb{y}_n \\ \mathbb{y}_{n+1} \\ \mathbb{y}_{n+2} \\ \mathbb{y}_{n+3} \end{pmatrix}.$$

By calculating the above limit, we can now determine the projection operator. Let

$$\alpha_k = \int_0^\infty e^{-\eta y} e^{-\int_0^y \mu_k(\tau)d\tau} dy, \quad \overline{\alpha}_k = \int_0^\infty e^{-\int_0^y \mu_k(\tau)d\tau} dy,$$
$$\beta_k = 1 - \eta \int_0^\infty e^{-\eta y} e^{-\int_0^y \mu_k(\tau)d\tau} dy = 1 - \eta\alpha_k, \quad n+1 \leqslant k \leqslant n+3.$$

Then, we can simplify $|G(\eta)|$ as

$$|G(\eta)| = \eta \begin{vmatrix} 1+\lambda_r\alpha_{n+3} & 1+\lambda_r\alpha_{n+3} & 1+\lambda_r\alpha_{n+3} & \cdots & 1+\lambda_r\alpha_{n+3} & 1+\lambda_{ss}\alpha_{n+1}+\lambda_{si}\alpha_{n+2} \\ -\lambda_s & \eta+a_1 & -\mu_s & \cdots & 0 & 0 \\ 0 & -\lambda_s & \eta+a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \eta+a_{n-1} & -\mu_s \\ 0 & 0 & 0 & \cdots & -\lambda_s & \eta+a_n \end{vmatrix},$$

and

$$\lim_{\eta\to 0} \eta \frac{1}{|G(\eta)|}$$
$$= \frac{1}{\begin{vmatrix} 1+\lambda_r\overline{\alpha_{n+3}} & 1+\lambda_r\overline{\alpha_{n+3}} & 1+\lambda_r\overline{\alpha_{n+3}} & \cdots & 1+\lambda_r\overline{\alpha_{n+3}} & 1+\lambda_{ss}\overline{\alpha_{n+1}}+\lambda_{si}\overline{\alpha_{n+2}} \\ -\lambda_s & a_1 & -\mu_s & \cdots & 0 & 0 \\ 0 & -\lambda_s & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & -\mu_s \\ 0 & 0 & 0 & \cdots & -\lambda_s & a_n \end{vmatrix}} \doteq \frac{1}{\mathbb{M}}.$$

By Lemma 6, the Fubini theorem and

$$\int_0^\infty \mu_k(y) e^{-\int_0^y \mu_k(\tau)d\tau} dy = 1, \quad n+1 \leqslant k \leqslant n+3,$$
$$\int_0^\infty \mu_k(\xi) e^{-\int_0^y \mu_k(\xi)d\xi} \int_0^k \mathbb{b}_k(\tau) e^{\int_0^y \mu_k(\xi)d\xi} d\tau dy = \int_0^\infty \mathbb{b}_k(y) dy, \quad n+1 \leqslant k \leqslant n+3.$$

We obtain that

$$\lim_{\eta\to 0} |G_0(\eta)| = \begin{vmatrix} \sum_{k=n+1}^{n+3} \int_0^\infty \mathbb{b}_k(y)dy + \mathbb{b}_0 & -\mu_s-\lambda_r & -\lambda_r & \cdots & -\lambda_r & -\lambda_{ss}-\lambda_{si} \\ \mathbb{b}_1 & a_1 & -\mu_s & \cdots & 0 & 0 \\ \mathbb{b}_2 & -\lambda_s & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{b}_{n-1} & 0 & 0 & \cdots & a_{n-1} & -\mu_s \\ \mathbb{b}_n & 0 & 0 & \cdots & -\lambda_s & a_n \end{vmatrix}$$

$$
= \begin{vmatrix} \sum_{k=n+1}^{n+3} \int_0^\infty \mathbb{b}_k(y)dy + \sum_{j=0}^{n} \mathbb{b}_j & 0 & 0 & \cdots & 0 & 0 \\ \mathbb{b}_1 & a_1 & -\mu_s & \cdots & 0 & 0 \\ \mathbb{b}_2 & -\lambda_s & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{b}_{n-1} & 0 & 0 & \cdots & a_{n-1} & -\mu_s \\ \mathbb{b}_n & 0 & 0 & \cdots & -\lambda_s & a_n \end{vmatrix}
$$

$$
= \begin{vmatrix} a_1 & -\mu_s & \cdots & 0 & 0 \\ -\lambda_s & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & -\mu_s \\ 0 & 0 & \cdots & -\lambda_s & a_n \end{vmatrix} \doteq \mathbb{M}_0,
$$

$$
\lim_{\eta \to 0} |G_1(\eta)| = \begin{vmatrix} a_0 - \lambda_r & \sum_{k=n+1}^{n+3} \int_0^\infty \mathbb{b}_k(y)dy + \mathbb{b}_0 & -\lambda_r & \cdots & -\lambda_r & -\lambda_{ss} - \lambda_{si} \\ -\lambda_s & \mathbb{b}_1 & -\mu_s & \cdots & 0 & 0 \\ 0 & \mathbb{b}_2 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \mathbb{b}_{n-1} & 0 & \cdots & a_{n-1} & -\mu_s \\ 0 & \mathbb{b}_n & 0 & \cdots & -\lambda_s & a_n \end{vmatrix}
$$

$$
= \begin{vmatrix} 0 & \sum_{k=n+1}^{n+3} \int_0^\infty \mathbb{b}_k(y)dy + \sum_{j=0}^{n} \mathbb{b}_j & 0 & \cdots & 0 & 0 \\ -\lambda_s & \mathbb{b}_1 & -\mu_s & \cdots & 0 & 0 \\ 0 & \mathbb{b}_2 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \mathbb{b}_{n-1} & 0 & \cdots & a_{n-1} & -\mu_s \\ 0 & \mathbb{b}_n & 0 & \cdots & -\lambda_s & a_n \end{vmatrix}
$$

$$
= \begin{vmatrix} a_2 & -\mu_s & \cdots & 0 & 0 \\ -\lambda_s & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & -\mu_s \\ 0 & 0 & \cdots & -\lambda_s & a_n \end{vmatrix} \doteq \mathbb{M}_1,
$$

$$
\lim_{\eta \to 0} |G_i(\eta)|
$$

$$
= \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & \sum_{k=n+1}^{n+3} \int_0^\infty \mathbb{b}_k(y)dy + \sum_{j=0}^{n} \mathbb{b}_j & 0 & \cdots & 0 & 0 & 0 \\ -\lambda_s & a_1 & -\mu_s & \cdots & 0 & \mathbb{b}_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\lambda_s & a_2 & \cdots & 0 & \mathbb{b}_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\lambda_s & \cdots & 0 & \mathbb{b}_3 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{i-1} & \mathbb{b}_{i-1} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -\lambda_s & \mathbb{b}_i & -\mu_s & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \mathbb{b}_{i+1} & a_{i+1} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \mathbb{b}_{n-2} & 0 & \cdots & a_{n-2} & -\mu_s & 0 \\ 0 & 0 & 0 & \cdots & 0 & \mathbb{b}_{n-1} & 0 & \cdots & -\lambda_s & a_{n-1} & -\mu_s \\ 0 & 0 & 0 & \cdots & 0 & \mathbb{b}_n & 0 & \cdots & 0 & -\lambda_s & a_n \end{vmatrix}
$$

$$= (-\lambda_s)^i \times \begin{vmatrix} a_{i+1} & -\mu_s & 0 & \cdots & 0 & 0 & 0 \\ -\lambda_s & a_{i+2} & -\mu_s & \cdots & 0 & 0 & 0 \\ 0 & -\lambda_s & a_{i+3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-2} & -\mu_s & 0 \\ 0 & 0 & 0 & \cdots & -\lambda_s & a_{n-1} & -\mu_s \\ 0 & 0 & 0 & \cdots & 0 & -\lambda_s & a_n \end{vmatrix}_{(n-i)\times(n-i)} \doteq \mathbb{M}_i, \quad i = 2, 3, \cdots, n-1,$$

$$\lim_{\eta\to 0} |G_n(\eta)| = (-\lambda_s)^n \doteq \mathbb{M}_n.$$

Finally, we derive

$$\lim_{\eta\to 0} \eta \mathrm{y}_j = \lim_{\eta\to 0} \eta \frac{|G_j(\eta)|}{|G(\eta)|} = \frac{\mathbb{M}_j}{\mathbb{M}} \doteq P_i, \quad j = 0, 1, 2, \cdots, n, \tag{71}$$

$$\begin{aligned} \lim_{\eta\to 0} \eta \mathrm{y}_{n+1}(y) =& \lambda_{ss}\, e^{-\int_0^y \mu_{n+1}(\xi)d\xi} \lim_{\eta\to 0} \eta \mathrm{y}_n \\ =& \lambda_{ss}\, e^{-\int_0^y \mu_{n+1}(\xi)d\xi} \frac{\mathbb{M}_n}{\mathbb{M}} \doteq P_{n+1}(y), \end{aligned} \tag{72}$$

$$\begin{aligned} \lim_{\eta\to 0} \eta \mathrm{y}_{n+2}(y) =& \lambda_{si}\, e^{-\int_0^y \mu_{n+2}(\xi)d\xi} \lim_{\eta\to 0} \eta \mathrm{y}_n \\ =& \lambda_{si}\, e^{-\int_0^y \mu_{n+2}(\xi)d\xi} \frac{\mathbb{M}_n}{\mathbb{M}} \doteq P_{n+2}(y), \end{aligned} \tag{73}$$

$$\begin{aligned} \lim_{\eta\to 0} \eta \mathrm{y}_{n+3}(y) =& \lambda_r\, e^{-\int_0^y (\eta+\mu_{n+3}(\xi))d\xi} \sum_{j=0}^{n-1} \lim_{\eta\to 0} \eta \mathrm{y}_j \\ =& \lambda_r\, e^{-\int_0^y (\eta+\mu_{n+3}(\xi))d\xi} \sum_{j=0}^{n-1} \frac{\mathbb{M}_j}{\mathbb{M}} \doteq P_{n+3}(y). \end{aligned} \tag{74}$$

Combining (71)–(74) with Lemma 6 we obtain

$$\mathcal{P}\mathbf{P}(0) = \mathbf{P}(y). \tag{75}$$

Thus, we conclude by (75), Theorems 3 and 7 that

$$\begin{aligned} \|\mathbf{P}(\cdot, t) - \mathbf{P}(\cdot)\| &\le \|\mathfrak{T}(t) - \mathcal{P}\| \|\mathbf{P}(0)\| \\ &\le Ke^{-\varrho t} \|\mathbf{P}(0)\| = Ke^{-\varrho t}, \quad t \ge 0. \end{aligned}$$

□

5. Reliability Indices

Some reliability indices is discussed briefly in this section. For detailed discussion, we refer the reader to [34] (p. 256). From the Remark 1, we have

$$\lim_{t\to\infty} P_j(t) = P_j, \quad j = 0, 1, \cdots, n, \tag{76}$$

$$\lim_{t\to\infty} \int_0^\infty |P_k(y,t) - P_k(y)| dx = 0, \quad n+1 \leqslant k \leqslant n+3. \tag{77}$$

Equation (77) implies

$$\lim_{t\to\infty} \int_0^\infty |\mu_k(y) P_k(y,t) - \mu_k(y) P_k(y)| dx = 0, \quad n+1 \leqslant k \leqslant n+3. \tag{78}$$

We know

$$A(t) = \sum_{j=0}^{n-1} P_j(t).$$

Which together with (46) and (76), we obtain

$$A = \lim_{t\to\infty} A(t) = \sum_{j=0}^{n-1} P_j = \left[1 + \sum_{j=1}^{n-1}\left(\prod_{k=1}^{j} b_k\right)^{-1} \lambda_s^j\right] P_0.$$

We have

$$m_f = \lim_{t\to\infty} m_f(t) = \lambda_r \sum_{i=0}^{n-1} P_i + (\lambda_{ss} + \lambda_{si})P_n = \lambda_r A + (\lambda_{ss} + \lambda_{si})P_n.$$

The system's time-dependent renewal frequency indicates that the frequency of system's state returns to the initial states, and we have

$$\begin{aligned} m_r = \lim_{t\to\infty} m_r = \lim_{t\to\infty} &\left\{\mu_s P_1(t) + \sum_{k=n+1}^{n+3}\int_0^\infty \mu_k(y)P_k(t,y)dy\right\} \\ =&\mu_s P_1 + \sum_{k=n+1}^{n+3}\int_0^\infty \mu_k(y)P_k(y)dy \\ =&\frac{\lambda_s\mu_s}{b_1}P_0 + (\lambda_{ss} + \lambda_{si})\lambda_s^n\left(\prod_{j=1}^{n} b_j\right)^{-1} P_0 \\ &+ \lambda_r\left[1 + \lambda_s^n \sum_{j=0}^{n-1}\left(\prod_{k=1}^{j} b_k\right)^{-1}\right] P_0. \end{aligned}$$

If we set $\mu_k(y) = 0$, (for $\mathrm{n} + 1 \leqslant k \leqslant \mathrm{n} + 3$), then we get the new system shown below.

$$\begin{aligned} \frac{d\widetilde{P}_0(t)}{dt} =& a_0\widetilde{P}_0(t) + \mu_s\widetilde{P}_1(t), \\ \frac{d\widetilde{P}_j(t)}{dt} =& -a_j\widetilde{P}_j(t) + \mu_s\widetilde{P}_{j+1}(t) + \lambda_s\widetilde{P}_{j-1}(t), \quad 1 \leqslant j \leqslant n-1, \\ \frac{d\widetilde{P}_n(t)}{dt} =& -a_n\widetilde{P}_n(t) + \lambda_s\widetilde{P}_{n-1}(t), \\ \frac{d\widetilde{P}_{n+1}(t)}{dt} =& \lambda_{ss}\widetilde{P}_n(t), \\ \frac{d\widetilde{P}_{n+2}(t)}{dt} =& \lambda_{si}\widetilde{P}_n(t), \\ \frac{d\widetilde{P}_{n+3}(t)}{dt} =& \lambda_r \sum_{j=0}^{n-1}\widetilde{P}_j(t). \end{aligned}$$

As a result, by a similar argument, the system's time-dependent reliability converges to a constant number

$$\lim_{t\to\infty} R(t) = \lim_{t\to\infty}\sum_{j=0}^{n+3}\widetilde{P}_j(t) = \sum_{j=0}^{n+3}\widetilde{P}_j = R.$$

Remark 2. *It is clear from the above results that we can obtain the results in [32] by the normalizing condition* $\sum\limits_{j=0}^{n} P_j(t) + \sum\limits_{k=n+1}^{n+3}\int\limits_0^\infty P_k(y,t)dy = 1$. *Hence, our results generalize those in [32].*

6. Numerical Results

This section provides numerical examples to investigate how changes in system parameters affect the reliability indices, using Matlab 2017a for calculations. To begin, we assume that $n = 3$ without losing generality and the system's repair time is Gamma distributed with $\mu_k(y) = \mu_k$, $(n+1 \leqslant k \leqslant n+3)$. The system parameters are fixed as

$$\lambda_s = 0.001, \quad \lambda_r = 0.0009, \quad \lambda_{ss} = 0.0006, \quad \lambda_{si} = 0.0004$$
$$\mu_s = 0.007, \quad \mu_{n+1} = 0.0018, \quad \mu_{n+2} = 0.0012, \quad \mu_{n+3} = 0.003.$$

For different values of β, the variations in the system's time-dependent availability (Figure 1a), failure frequency (Figure 1b), and renewal frequency (Figure 1c) are plotted with respect to t in Figure 1. In each case, the $A(t)$ and $m_f(t)$ decrease rapidly as time increases, eventually becoming constant at some value. As time passes, the $m_r(t)$ increases rapidly in the early stages, then becomes constant at some value after a long run.

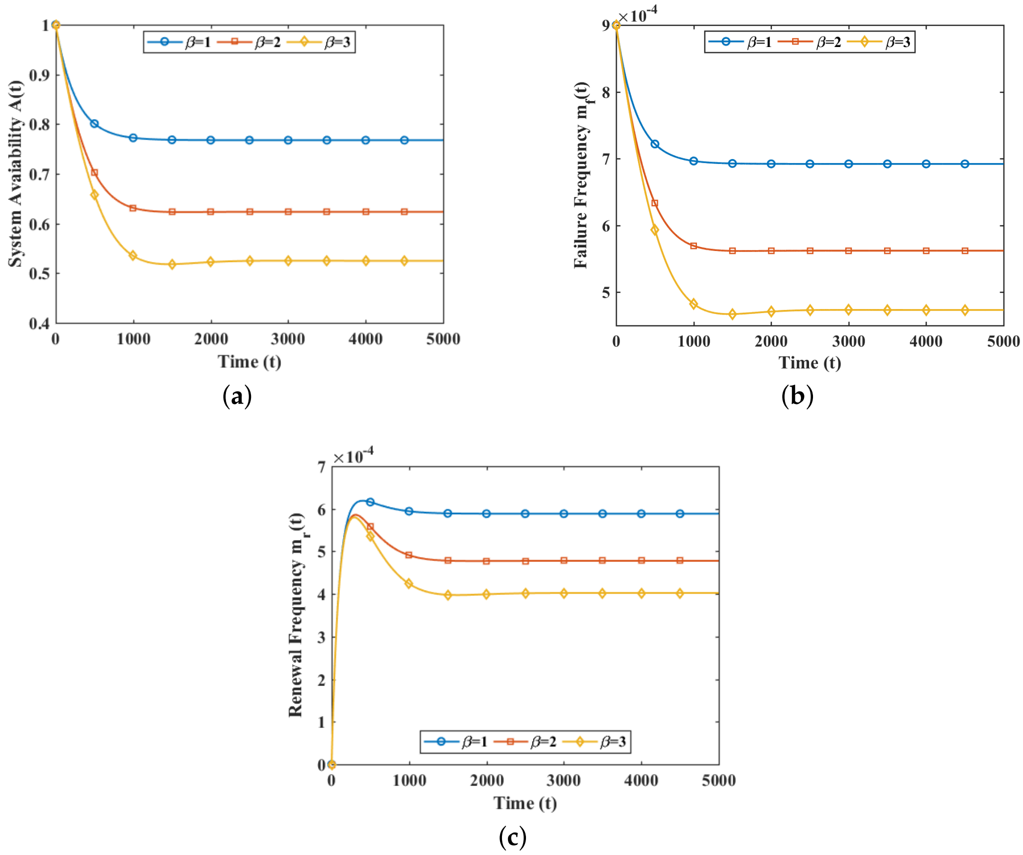


Figure 1. Time-dependent reliability indices for different β. (**a**) A(t) for for different β; (**b**) $m_f(t)$ for different β; (**c**) $m_f(t)$ for different β.

Furthermore, we observe that as β increases, the system's time-dependent availability, failure frequency, and renewal frequency decrease.

In the following, we further analyze the effect of different values of the failure and repair rates on the system's reliability indices for $\beta = 1$ (i.e., the repair time is exponential distributed). Figure 2 shows that as time increases, these reliability indices converge to some fixed value. As expected, $A(t)$ decreases with increasing λ_s (Figure 2a) and λ_r (Figure 2b). $m_f(t)$ decreases with increasing λ_s, but its effect on the failure frequency is not evident (Figure 2c), and $m_f(t)$ increases as λ_r (Figure 2d) increases. $m_r(t)$ increases with increasing λ_s (Figure 2e) and decreases with increasing λ_r (Figure 2f). Furthermore, changes in the system parameters λ_{ss} and λ_{si} had almost no effect on the system reliability indices. In Table 1, we only list the effect of λ_{ss} and λ_{si} on the time-dependent availability $A(t)$.

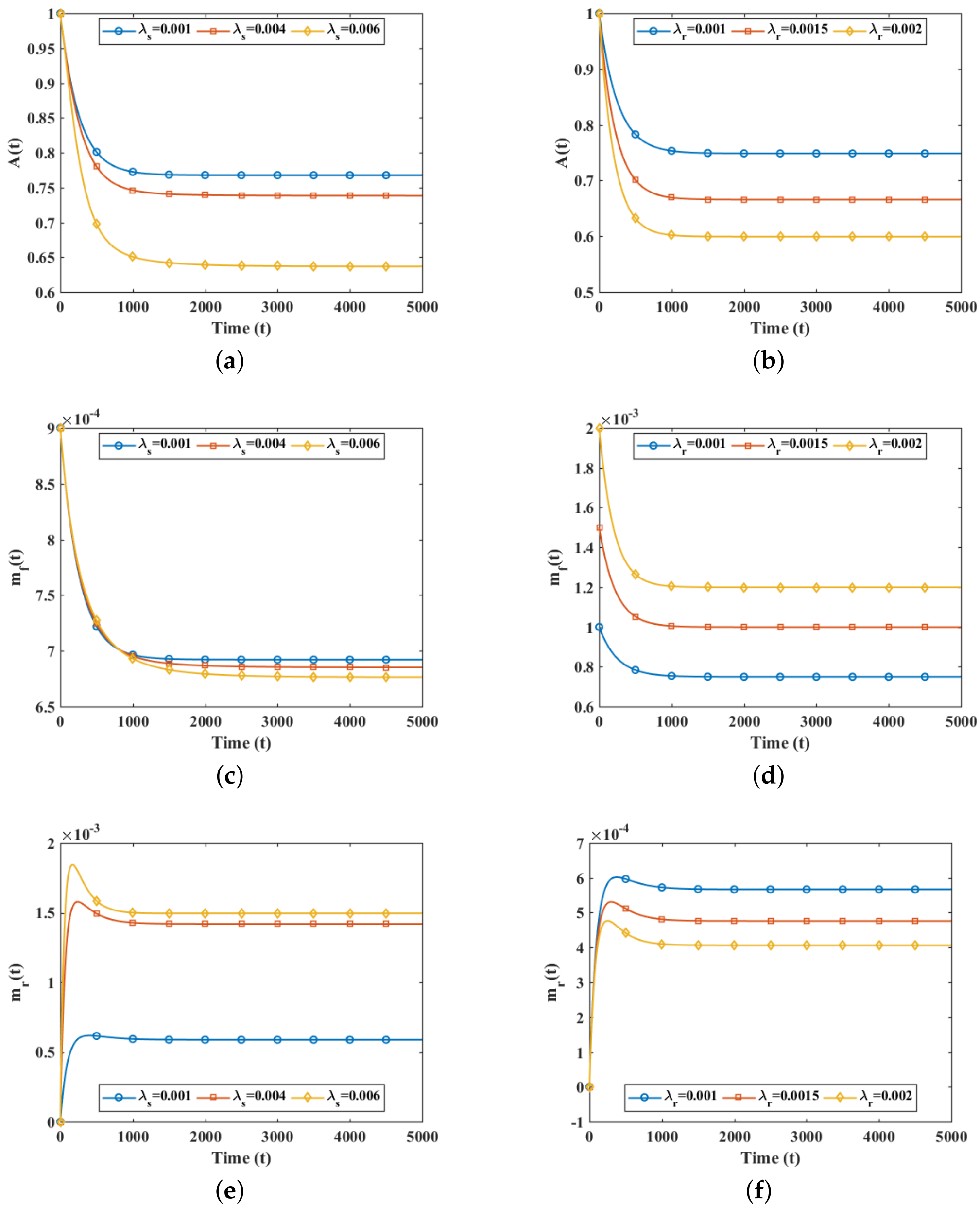


Figure 2. Time-dependent reliability indices for different failure rates. (**a**) A(t) for for different λ_s; (**b**) A(t) for for different λ_r; (**c**) $m_f(t)$ for different λ_s; (**d**) $m_f(t)$ for different λ_r; (**e**) $m_r(t)$ for different λ_s; (**f**) $m_r(t)$ for different λ_r.

Table 1. Time-dependent system availability $A(t)$ for different λ_{ss} and λ_{si}.

Time t	λ_{ss}			λ_{si}		
	0.0005	**0.001**	**0.0015**	**0.0003**	**0.0006**	**0.0009**
500	0.800906	0.800872	0.800840	0.800908	0.800882	0.800858
1000	0.772421	0.772313	0.772217	0.772430	0.772339	0.772254
2000	0.767730	0.767579	0.767447	0.767751	0.767598	0.767456
3000	0.767603	0.767448	0.767313	0.767629	0.767459	0.767301
4000	0.767592	0.767437	0.767301	0.767620	0.767444	0.767282
5000	0.767589	0.767434	0.767298	0.767617	0.767440	0.767276

The behavior of the reliability indices for different repair rates is depicted in Figure 3, showing that these indices increase as μ_s and μ_{n+3} increase. From this figure, we conclude that the changes in parameter μ_s have little effect on $A(t)$ and $m_f(t)$. Moreover, Table 2 reveals that the effect of the changes of the parameters μ_{n+1} and μ_{n+2} on $A(t)$ are not significant. It is also observed that the changes in the parameter μ_{n+1} and μ_{n+2} on $m_f(t)$ and $m_r(t)$ are not significant. Furthermore, these indices approach a constant value that time goes to infinity.

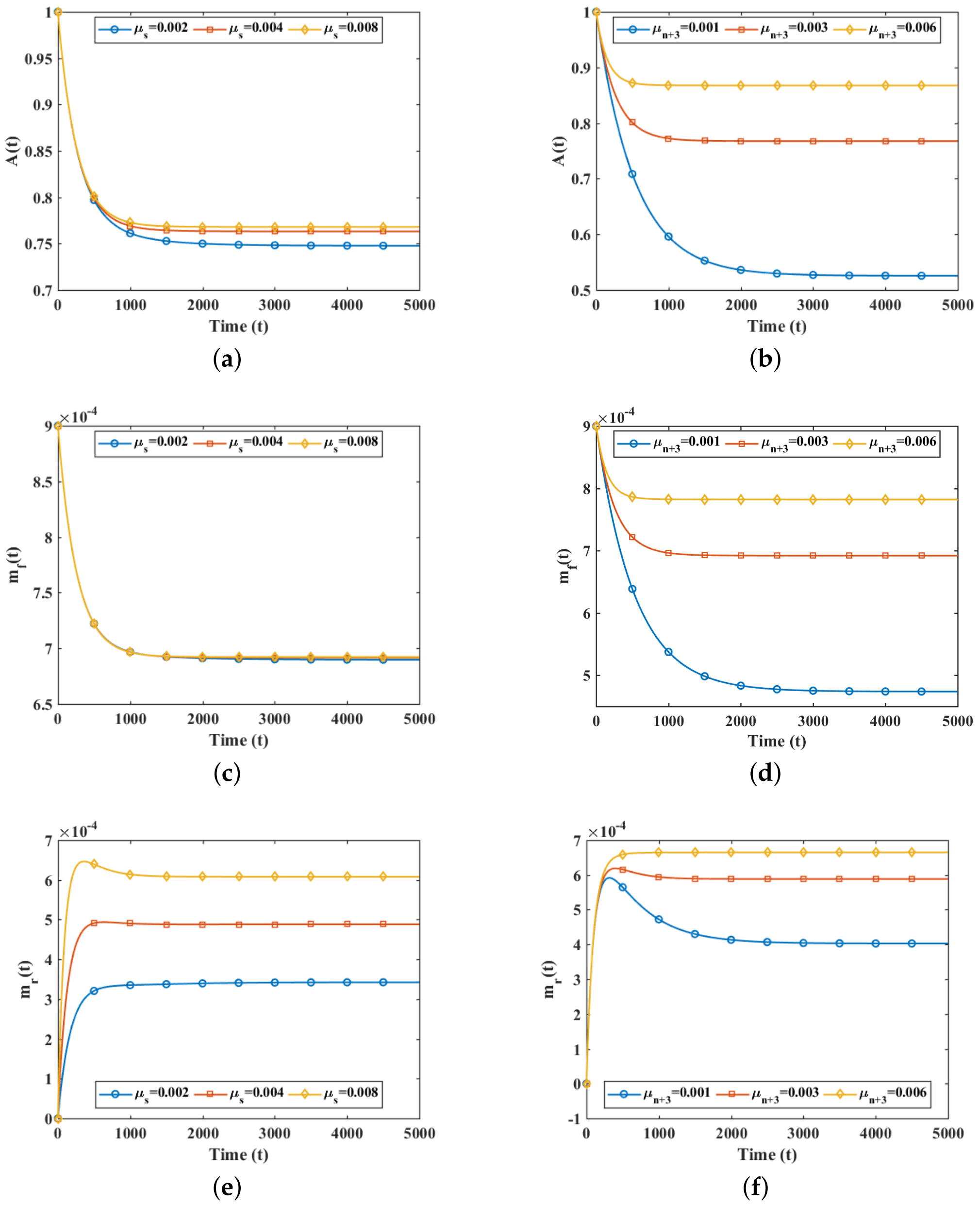
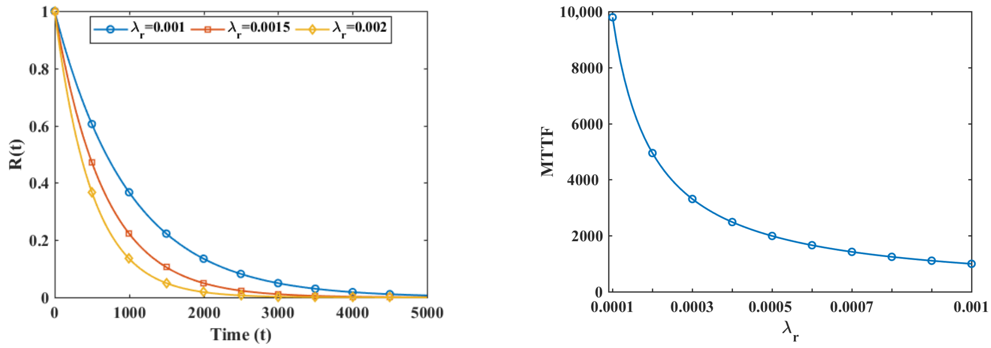


Figure 3. Time-dependent reliability indices for different repair rates. (**a**) A(t) for for different μ_s; (**b**) A(t) for for different μ_{n+3}; (**c**) $m_f(t)$ for different μ_s; (**d**) $m_f(t)$ for different μ_{n+3}; (**e**) $m_r(t)$ for different μ_s; (**f**) $m_r(t)$ for different μ_{n+3}.

Table 2. Time-dependent system availability $A(t)$ with different μ_{n+1} and μ_{n+2}.

Time t	μ_{n+1}			μ_{n+2}		
	0.0018	**0.0036**	**0.0072**	**0.0012**	**0.0024**	**0.0048**
500	0.800899	0.800921	0.800949	0.800897	0.800929	0.800947
1000	0.772399	0.772490	0.772564	0.772385	0.772519	0.772558
2000	0.767698	0.767854	0.767938	0.767660	0.767914	0.767955
3000	0.767571	0.767736	0.767820	0.767517	0.767813	0.767854
4000	0.767559	0.767726	0.767810	0.767498	0.767810	0.767850
5000	0.767557	0.767724	0.767807	0.767492	0.767810	0.767850

Figure 4 illustrates the effect of λ_r on system's time-dependent reliability and MTTF. We note that $R(t)$ decreases as λ_r increases and vanishes as time goes to infinity (Figure 4a). The MTTF decreases as λ_r increases (Figure 4b).



(**a**) $R(t)$ for different λ_r (**b**) $MTTF$ versus λ_r

Figure 4. Reliability and MTTF for different λ_r. (**a**) R(t) for different λ_r; (**b**) Effect of λ_r on MTTF.

Finally, in Table 3, we show the effect of the number of safety units in the system on the system transient availability. The availability increases as the number of safety units increases. However, having too many safety units does not contribute as much to the availability of this system.

Table 3. Time-dependent system availability $A(t)$ with different numbers of safety units.

$\lambda_s = 0.001,\ \lambda_r = 0.0009,\ \lambda_{ss} = 0.0006,\ \lambda_{si} = 0.0004$ $\mu_s = 0.007,\ \mu_{n+1} = 0.0018,\ \mu_{n+2} = 0.0012,\ \mu_{n+3} = 0.003$	
n	$A(t)$
2	0.755816691376763
3	0.767557102387178
5	0.769204802928852
8	0.769230719866391
10	0.769230769237539
20	0.769230770014985

7. Conclusions

In this paper, a robot-safety system consisting of one robot and n safety units with perfect switching is studied. We converted the model into an abstract Cauchy problem in Banach space and did dynamic analysis by the operator semigroup theory of linear operators. We proved that the system has unique nonnegative T-DS and that T-DS exponentially converges to its S-SS. Furthermore, we discussed the asymptotic property of the system's instantaneous reliability indices and showed that they all converge to some constant. In order to investigate the impact of parameter changes on system reliability indices, some numerical examples are also presented.

We concluded that the increase in the number of safety units for this system does not necessarily improve the system's instantaneous availability from the above numerical results. Thus, in the future, we can further study the robot-safety system consisting of n robots and m standby safety units.

Author Contributions: Conceptualization, E.K. and A.Y.; methodology, E.K. and A.Y.; validation, E.K.; writing—original draft preparation, E.K.; writing—review and editing, E.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Natural Science Foundation of Xinjiang Uygur Autonomous Region, No. 2022D01C46.

Data Availability Statement: Data is contained within the article.

Acknowledgments: The authors would like to thank the editor and referees for their valuable comments.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

SVT	Supplementary variable technique
T-DS	Time-dependent solution
S-SS	Steady-state solutions

References

1. Cox, D.R. The analysis of non-markovian stochastic processes by the inclusion of supplementary variables. *Math. Proc. Camb. Phil. Soc.* **1955**, *51*, 433–441. [CrossRef]
2. Gaver, D.P. Time to failure and availability of parallel redundant systems with repair. *IEEE Trans. Reliab.* **1963**, *R-12*, 30–38. [CrossRef]
3. Linton, D.G. Some advancements in the analysis of two-unit parallel redundant systems. *Microelectron. Reliab.* **1976**, *15*, 39–46. [CrossRef]
4. Gupta, P.P.; Gupta, R.K. Cost analysis of an electronic repairable redundant system with critical human errors. *Microelectron. Reliab.* **1986**, *26*, 417–421. [CrossRef]
5. Shi, D.H.; Li, W. Availability analysis of a two-unit series system with shut-off rule and "first-fail, first-repaired". *Acta. Math. Appl.* **1993**, *9*, 88–91. [CrossRef]
6. Chung, W.K. Reliability of imperfect switching of cold stanby systems with multiple non-critical and critical errors. *Microelectron. Reliab.* **1995**, *35*, 1479–1482. [CrossRef]
7. Oliveira, E.A.; Alvim, A.C.M.; Frutuoso e Melo, P.F. Unavailability analysis of safety systems under aging by supplementary variables with imperfect repair. *Ann. Nucl. Energy* **2005**, *32*, 241–252. [CrossRef]
8. Zhang, Y.L.; Wu, S.M. Reliability analysis for a $k/n(f)$ system with repairable repair-equipment. *Appl. Math. Model.* **2009**, *33*, 3052–3067. [CrossRef]
9. Shakuntla, S.; Lal, A.K.; Bhatia, S.S.; Singh, J. Reliability analysis of polytube industry using supplementary variable technique. *Appl. Math. Comput.* **2011**, *218*, 3981–3992. [CrossRef]
10. Singh, V.V.; Ram, M.; Rawal, D.K. Cost analysis of an engineering system involving subsystems in series configuration. *IEEE. Autom. Sci. Eng.* **2013**, *10*, 1124–1130. [CrossRef]
11. Ke, J.C.; Liu, T.H.; Yang, D.Y. Modeling of machine interference problem with unreliable repairman and standbys imperfect switchover. *Reliab. Eng. Syst. Saf.* **2018**, *174*, 12–18. [CrossRef]
12. Shekhar, C.; Kumar, A.; Varshney, S. Fault-tolerant redundant repairable system with different failures and delays. *Eng. Comput.* **2020**, *37*, 1043–1071. [CrossRef]
13. Gao, S.; Wang, J. Reliability and availability analysis of a retrial system with mixed standbys and an unreliable repair facility. *Reliab. Eng. Syst. Saf.* **2021**, *205*, 107240. [CrossRef]
14. Dhillon, B.; Fashandi, A.R.M. Robot systems reliability and safety: A review. *J. Qual. Maint. Eng.* **2002**, *8*, 170–212. [CrossRef]
15. Kulić, D.; Croft, E.A. Real-time safety for human-robot interaction. *Rob. Autom. Syst.* **2006**, *54*, 1–12.
16. Dhillon, B.; Yang, N.F. Availability analysis of a robot with safety system. *Microelectron. Reliab.* **1996**, *36*, 169–177. [CrossRef]
17. Dhillon, B.S.; Yang, N.F. Formulas for analyzing a redundant robot configuration with a built-in safety system. *Microelectron. Reliab.* **1997**, *37*, 557–563. [CrossRef]
18. Dhillon, B.S.; Fashandi, A.R.M. Robotic systems probabilistic analysis. *Microelectron. Reliab.* **1997**, *37*, 211–224. [CrossRef]

19. Dhillon, B.S.; Li, Z.J. Stochastic analysis of a system with redundant robots, one built-in safety unit, and common-cause failures. *J. Intell. Rob. Syst.* **2006**, *45*, 137–155. [CrossRef]
20. Jain, M.; Preeti. Performance analysis of a repairable robot safety system with standby, imperfect coverage and reboot delay. *Int. J. Eng. Trans. C Aspects* **2013** *26*, 1077–1088. [CrossRef]
21. Vanderperre, E.J.; Makhanov, S.S. Overall availability and risk analysis of a general robot csafety device system. *Int. J. Syst. Sci.* **2015**, *46*, 1889–1896. [CrossRef]
22. Gupur, G. Well-posedness of the system consisting of two repairable units. *Acta Anal. Funct. Appl.* **2001**, *3*, 188–192.
23. Gupur, G.; Li, X.Z. Semigroup method for a mathematical model in reliability analysis. *J. Syst. Sci. Syst. Eng.* **2001**, *10*, 137–147.
24. Guo, W.H.; Xu, G.Q. Stability analysis of the system consising of a robot and its associated safety mechanism. *Math. Pract. Theory* **2003**, *33*, 116–122.
25. Chen, Y.; Xu, G.Q. Analysis of exponential stability to the robot system with periodic repair function. *Acta. Anal. Funct. Appl.* **2010**, *12*, 170–179. [CrossRef]
26. Gupur, G. Well-posedness of the model describing a repairable, standby, human & machine system. *J. Syst. Sci. Complex.* **2003**, *16*, 483–493.
27. Gupur, G. Asymptotic property of the solution of a repairable, standby, human and machine system. *Int. J. Pure Appl. Math.* **2006**, *28*, 35–54.
28. Aili, M.; Gupur, G. Further result on a repairable, standby human and machine system. *Int. J. Pure Appl. Math.* **2016**, *101*, 571–594.
29. Zhang, X. Exponential stability analysis of a redundant robot configuration with a built-in safety systemfurther result on a repairable, standby human and machine system. *J. Syst. Sci. Inf.* **2019**, *131*, 289–293.
30. Qiao, X.; Ma, D. Reliability and numerical analysis of a robot safety system. *J Syst. Sci. Complex.* **2019**, *32*, 1072–1092. [CrossRef]
31. Li, Z.; Wei, L. Reliability and controllability of the system consisting of a robot and its associated safety mechanism. *J. Math. Res. Appl.* **2020**, *40*, 9.
32. Dhillon, B.S.; Shen, C. Reliability and availability analysis of a robot-safety system. *J. Qual. Maint. Eng.* **2011**, *17*, 203–232.
33. Adams, R.A. *Sobolev Spaces*; Academic Press: New York, NY, USA, 1975.
34. Gupur, G. *Functional Analysis Methods for Reliability Models*; Springer: Basel, Switzerland, 2011.
35. Fattorini, H.O. *The Cauchy Problem*; Cambridge University Press: Cambridge, UK, 1984.
36. Nagel, R. *One-Parameter Semigroups of Positive Operators (LNM 1184)*; Springer: Berlin/Heidelberg, Germany, 1986.
37. Engel, K.J.; Nagel, R. *One-Parameter Semigroups for Linear Evolution Equations*; Springer: New York, NY, USA, 2000.