axioms

Article

Center-like Subsets in Semiprime Rings with Multiplicative Derivations

Sarah Samah Aljohani [1,†], Emine Koç Sögütcü [2,†] and Nadeem ur Rehman [3,*,†]

1 Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia; sjohani@psu.edu.sa
2 Department of Mathematics, Faculty of Science, Sivas Cumhuriyet University, Sivas 58140, Turkey; eminekoc@cumhuriyet.edu.tr
3 Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
* Correspondence: nu.rehman.mm@amu.ac.in
† The authors contributed equally to this work.

Abstract: We introduce center-like subsets $\mathcal{Z}^*_\circ(\mathfrak{A},\mathfrak{d})$, $\mathcal{Z}^{**}_\circ(\mathfrak{A},\mathfrak{d})$, where $\mathfrak{A}$ is the ring and $\mathfrak{d}$ is the multiplicative derivation. In the following, we take a new derivation for the center-like subsets existing in the literature and establish the relations between these sets. In addition to these new sets, the theorems are generalized as multiplicative derivations instead of the derivations found in previous studies. Additionally, different proofs are provided for different center-like sets. Finally, we enrich this article with examples demonstrating that the hypotheses we use are necessary.

Keywords: center-like subset; prime ring; semiprime ring; ideal; derivation; multiplicative derivation

MSC: 16W20; 16W25; 16U70; 16U80; 16N60

check for updates

Citation: Aljohani, S.S.; Sögütcü, E.K.; Rehman, N.u. Center-like Subsets in Semiprime Rings with Multiplicative Derivations. *Axioms* **2024**, *13*, 448. https://doi.org/10.3390/axioms13070448

Academic Editor: Fabio Caldarola

Received: 6 April 2024
Revised: 14 June 2024
Accepted: 24 June 2024
Published: 2 July 2024

1. Introduction and Basic Results

Consider an associative ring $\mathfrak{A}$ with its center located at $\mathcal{Z}$. In this ring, given any elements ℓ and η, we define $[\ell,\eta]$ as the commutator $\ell\eta-\eta\ell$ and $\ell\circ\eta$ as the anti-commutator $\ell\eta+\eta\ell$. It is worth noting that a ring $\mathfrak{A}$ is classified as prime if $\ell\mathfrak{A}\eta=(0)$ implies that either ℓ or η is zero. Similarly, $\mathfrak{A}$ is termed semiprime if, for any $\ell\in\mathfrak{A}$, $\ell\mathfrak{A}\ell=(0)$ implies that ℓ itself is zero. These definitions and concepts are significant in understanding the properties and behavior of elements within the ring $\mathfrak{A}$. An additive mapping $\mathfrak{d}:\mathfrak{A}\to\mathfrak{A}$ is called a derivation if $\mathfrak{d}(\ell\eta)=\mathfrak{d}(\ell)\eta+\ell\mathfrak{d}(\eta)$ holds for all $\ell,\eta\in\mathfrak{A}$. In the paper by Daif [1], a concept known as a multiplicative derivation was introduced. A mapping $\mathfrak{d}:\mathfrak{A}\to\mathfrak{A}$ is classified as a multiplicative derivation if the condition $\mathfrak{d}(\ell\eta)=\mathfrak{d}(\ell)\eta+\ell\mathfrak{d}(\eta)$ is satisfied for all elements ℓ,η in the ring $\mathfrak{A}$. Note that these maps are not additive. Considering a ring $\mathfrak{A}=C[0,1]$ which consists of all continuous functions mapping the interval $[0,1]$ to either real or complex numbers, we define a map $\mathfrak{d}:\mathfrak{A}\to\mathfrak{A}$ as follows:

$$\mathfrak{d}(f)(\ell)=\begin{cases} f(\ell)\log|f(\ell)| & \text{if } f(\ell)\neq 0,\\ 0 & \text{otherwise.}\end{cases}$$

Although $\mathfrak{d}$ is a multiplicative derivation, it fails to be additive, meaning that it does not satisfy the complete definition of a derivation, that is, the multiplicative derivation is more general than the concept of derivation.

Numerous findings in the literature have affirmed that certain subsets of a ring $\mathfrak{A}$, determined by certain conditions of commutativity, are required to align with the center $\mathcal{Z}$. These subsets are termed center-like subsets. One example of such a set is denoted as $H(\mathfrak{A},\mathfrak{d})$, defined as the set of elements $\eta\in\mathfrak{A}$ such that $\eta\mathfrak{d}(\ell)=\mathfrak{d}(\ell)\eta$ for all $\ell\in\mathfrak{A}$, where $\mathfrak{d}$ represents a derivation. This set was introduced by Herstein, who demonstrated in [2] that if $\mathfrak{A}$ is a prime ring free of 2-torsion, then $H(\mathfrak{A},\mathfrak{d})$ coincides with the center $\mathcal{Z}$ of the ring.

In [3], Herstein stated that the hypercenter

$$S(\mathfrak{A}) = \{\eta \in \mathfrak{A} \mid \eta\ell^n = \ell^n\eta \text{ for all } \ell \in \mathfrak{A} \text{ for some } n = n(\eta,\ell) \geq 1\}$$

coincides with the center $\mathcal{Z}(\mathfrak{A})$ of $\mathfrak{A}$. In [4], Chacron stated that the cohypercenter

$$T(\mathfrak{A}) = \left\{\eta \in \mathfrak{A} \mid [\eta, \ell - \ell^2 p(\ell)] = 0 \text{ for all } \ell \in \mathfrak{A} \text{ and } p(\ell) \in [\ell] \text{ depends on } (\eta,\ell)\right\}$$

Several generalizations of the center of a ring have been introduced by Giambruno [5], who defined the enlarged hypercenter of $\mathfrak{A}$ to be the set

$$\{a \in \mathfrak{A} | a\ell^n = \ell^m a \text{ for all } \ell \in \mathfrak{A} \text{ for some } n = n(a,\ell) \geq 1 \text{ and } m = m(a,\ell) \geq 1\}$$

and showed that it is equal to $\mathcal{Z}$ when $\mathfrak{A}$ has no nonzero nil ideals. Moreover, he defined the generalized center of a ring $\mathfrak{A}$ as

$$g.c.(\mathfrak{A}) = \{a \in \mathfrak{A} | (a\ell)^n = (\ell a)^m \text{ for all } \ell \in \mathfrak{A} \text{ for some } n = n(a,\ell) \geq 1 \text{ and } m = m(a,\ell) \geq 1\}$$

and proved that $g.c.(\mathfrak{A}) = \mathcal{Z}$ if $\mathfrak{A}$ has no non-zero nil right ideals.

In [6], the authors proved that a semiprime ring must be commutative if there exits a derivation $\mathfrak{d}$ on $\mathfrak{A}$ such that $[\ell,\eta] = [\mathfrak{d}(\ell),\mathfrak{d}(\eta)]$ for all $\ell,\eta \in \mathfrak{A}$. Motivated by these results, ref. [7] defined the following subsets of a ring $\mathfrak{A}$ equipped with a derivation $\mathfrak{d} : \mathfrak{A} \to \mathfrak{A}$:

$$\begin{aligned}
\mathcal{Z}^*(\mathfrak{A},\mathfrak{d}) &= \{\eta \in \mathfrak{A} \mid [\ell,\eta] = [\mathfrak{d}(\eta),\mathfrak{d}(\ell)] \text{ for all } \ell \in \mathfrak{A}\} \\
\mathcal{Z}^{**}(\mathfrak{A},\mathfrak{d}) &= \{\eta \in \mathfrak{A} \mid [\ell,\eta] = [\mathfrak{d}(\ell),\mathfrak{d}(\eta)] \text{ for all } \ell \in \mathfrak{A}\} \\
\mathcal{Z}_1(\mathfrak{A},\mathfrak{d}) &= \{\eta \in \mathfrak{A} \mid [\mathfrak{d}(\ell),\mathfrak{d}(\eta)] = [\mathfrak{d}(\eta),\ell] + [\eta,\mathfrak{d}(\ell)] \text{ for all } \ell \in \mathfrak{A}\}.
\end{aligned}$$

Hence, it has been proved that if $\mathfrak{A}$ is a semiprime ring, then $\mathcal{Z}^*(\mathfrak{A},\mathfrak{d}) = \mathcal{Z}^{**}(\mathfrak{A},\mathfrak{d}) = \mathcal{Z}(\mathfrak{A})$. Moreover, if $\mathfrak{A}$ is a prime ring, then $\mathcal{Z}_1(\mathfrak{A},\mathfrak{d}) = \mathcal{Z}(\mathfrak{A})$. Based on these studies, Idrissi et al. [8] defined the following center-like subsets:

$$\begin{aligned}
\mathcal{Z}^+(\mathfrak{A},\mathfrak{d}) &= \{\eta \in \mathfrak{A} \mid [\ell,\eta] + [\mathfrak{d}(\eta),\mathfrak{d}(\ell)] \in \mathcal{Z}(\mathfrak{A}) \text{ for all } \ell \in \mathfrak{A}\} \\
\mathcal{Z}^-(\mathfrak{A},\mathfrak{d}) &= \{\eta \in \mathfrak{A} \mid -[\ell,\eta] + [\mathfrak{d}(\ell),\mathfrak{d}(\eta)] \in \mathcal{Z}(\mathfrak{A}) \text{ for all } \ell \in \mathfrak{A}\} \\
\mathcal{Z}^{*-}(\mathfrak{A},\mathfrak{d}) &= \{\eta \in \mathfrak{A} \mid [\mathfrak{d}(\ell),\mathfrak{d}(\eta)] - [\mathfrak{d}(\eta),\ell] - [\eta,\mathfrak{d}(\ell)] \in \mathcal{Z}(\mathfrak{A}) \text{ for all } \ell \in \mathfrak{A}\}
\end{aligned}$$

where $\mathfrak{d}$ is a derivation of $\mathfrak{A}$. Hence, they proved that if $\mathfrak{A}$ is a 2-torsion-free prime ring, then $\mathcal{Z}^+(\mathfrak{A},\mathfrak{d}) = \mathcal{Z}^-(\mathfrak{A},\mathfrak{d}) = \mathcal{Z}(\mathfrak{A})$. Moreover, if $\mathfrak{d} \neq 0$, then $\mathcal{Z}^{*-}(\mathfrak{A},\mathfrak{d}) = \mathcal{Z}(\mathfrak{A})$.

Nabiel [9] defined the following center-like subsets:

$$\begin{aligned}
\mathcal{Z}^{**}(\mathfrak{A},T) &= \{\eta \in \mathfrak{A} | [\ell,\eta] = [T(\ell),T(\eta)] \text{ for all } \ell \in \mathfrak{A}\} \\
\mathcal{Z}^{***}(\mathfrak{A},F,\mathfrak{d}) &= \{\eta \in \mathfrak{A} | [\ell,\eta] = [\mathfrak{d}(\eta),F(\ell)] \text{ for all } \ell \in \mathfrak{A}\} \\
\mathcal{Z}_1(\mathfrak{A},F,\mathfrak{d}) &= \{\eta \in \mathfrak{A} | [F(\ell),\mathfrak{d}(\eta)] = [\mathfrak{d}(\eta),\ell] + [\eta,F(\ell)] \text{ for all } \ell \in \mathfrak{A}\}
\end{aligned}$$

where T a is homomorphism of $\mathfrak{A}$ and $(F,\mathfrak{d})$ is a generalized derivation of $\mathfrak{A}$. He also proved the relations between these subsets and the central subset.

Recently, in [10], Zemzami, Oukhtite, and Bell introduced and studied the following new centerlike subsets:

$$\begin{aligned}
\mathcal{Z}^*(\mathfrak{A},\mathfrak{d}_1,\mathfrak{d}_2) &= \{\eta \in \mathfrak{A} \mid [\ell,\eta] = [\mathfrak{d}_1(\eta),\mathfrak{d}_2(\ell)] \text{ for all } \ell \in \mathfrak{A}\} \\
\mathcal{Z}^{**}(\mathfrak{A},\mathfrak{d}_1,\mathfrak{d}_2) &= \{\eta \in \mathfrak{A} \mid [\ell,\eta] = [\mathfrak{d}_1(\ell),\mathfrak{d}_2(\eta)] \text{ for all } \ell \in \mathfrak{A}\} \\
\mathcal{Z}_1(\mathfrak{A},\mathfrak{d}_1,\mathfrak{d}_2) &= \{\eta \in \mathfrak{A} \mid [\mathfrak{d}_1(\ell),\mathfrak{d}_2(\eta)] = [\mathfrak{d}_2(\eta),\ell] + [\eta,\mathfrak{d}_1(\ell)] \text{ for all } \ell \in \mathfrak{A}\}.
\end{aligned}$$

They proved that if $\mathfrak{A}$ is a 2-torsion-free prime ring, $\mathfrak{I}$ is a non-zero right ideal, there are $\mathfrak{d}_1,\mathfrak{d}_2$ derivations on $\mathfrak{A}$, and $A_l(\mathfrak{I}) = \{0\}$ or $\mathfrak{d}_2(\mathfrak{I})\mathfrak{I} \neq \{0\}$, then $\mathcal{Z}^*(\mathfrak{A},\mathfrak{d}_1,\mathfrak{d}_2) = \mathcal{Z}^{**}(\mathfrak{A},\mathfrak{d}_1,\mathfrak{d}_2) = \mathcal{Z}_1(\mathfrak{A},\mathfrak{d}_1,\mathfrak{d}_2) = \mathcal{Z}(\mathfrak{A})$.

In this paper, we first discuss the definition of the Jordan product based on the above center-like subsets. We change the commutator product in these sets and define the new center-like subsets as follows:

$$\mathcal{Z}_{\circ}^{*}(\mathfrak{A},\mathfrak{d}) = \{\eta \in \mathfrak{A} \mid \ell \circ \eta = \mathfrak{d}(\eta) \circ \mathfrak{d}(\ell) \text{ for all } \ell \in \mathfrak{A}\}$$
$$\mathcal{Z}_{\circ}^{**}(\mathfrak{A},\mathfrak{d}) = \{\eta \in \mathfrak{A} \mid \ell \circ \eta = \mathfrak{d}(\ell) \circ \mathfrak{d}(\eta) \text{ for all } \ell \in \mathfrak{A}\}$$

where $\mathfrak{d}$ is the derivation on $\mathfrak{A}$. We examine the relationship between the central set of these sets for both derivation and multiplicative derivation.

Various results in the literature indicate how the global structure of a ring $\mathfrak{A}$ is often tightly connected to the behavior of derivations defined on $\mathfrak{A}$. Many results in the literature have proved that some subsets of a ring $\mathfrak{A}$ defined by certain sort of commutativity condition coincide with its center $\mathcal{Z}(\mathfrak{A})$. Based on these studies, researchers have discussed the center set and sets of commutativity conditions, managing to compare each one of the above subsets with the center of $\mathfrak{A}$ for the class of prime (semiprime) rings with some additional assumptions.

Here, we establish relations that have not previously been established between the center-like subsets mentioned above. In addition, the existing relations are proven for multiplicative derivations of the semiprime ring.

There is a relationship between these sets, as follows:

$$\begin{array}{ccccc} \mathcal{Z}^{*}(\mathfrak{A},\mathfrak{d}) & \subseteq & \mathcal{Z}^{*}(\mathfrak{A},\mathfrak{d}) & \subseteq & \mathcal{Z}^{*}(\mathfrak{A},\mathfrak{d}_1,\mathfrak{d}_2) \\ \mathcal{Z}^{**}(\mathfrak{A},\mathfrak{d}) & \subseteq & \mathcal{Z}^{-}(\mathfrak{A},\mathfrak{d}) & \subseteq & \mathcal{Z}^{**}(\mathfrak{A},\mathfrak{d}_1,\mathfrak{d}_2) \\ \mathcal{Z}_1(\mathfrak{A},\mathfrak{d}) & \subseteq & \mathcal{Z}^{*-}(\mathfrak{A},\mathfrak{d}) & \subseteq & \mathcal{Z}_1(\mathfrak{A},\mathfrak{d}_1,\mathfrak{d}_2) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{Z}^{*}(\mathfrak{A},\mathfrak{d}) & \subseteq & \mathcal{Z}^{***}(\mathfrak{A},F,\mathfrak{d}) \\ \mathcal{Z}_1(\mathfrak{A},\mathfrak{d}) & \subseteq & \mathcal{Z}_1(\mathfrak{A},F,\mathfrak{d}) \end{array}.$$

Fact: Assuming that $\mathfrak{A}$ is a semiprime ring, then:

(i) No non-zero nilpotent elements are found in the center of $\mathfrak{A}$
(ii) If P is a nonzero prime ideal of $\mathfrak{A}$ and $a, b \in \mathfrak{A}$ such that $a\mathfrak{A}b \subseteq P$, then either $a \in P$ or $b \in P$
(iii) The center of $\mathfrak{A}$ contains the center of a non-zero one-sided ideal; specifically, the center contains any commutative one-sided ideal of $\mathfrak{A}$.

Without elaboration, the following fundamental identities are employed throughout this paper:

(i) $[\ell, \eta\hbar] = \eta[\ell, \hbar] + [\ell, \eta]\hbar$
(ii) $[\ell\eta, \hbar] = [\ell, \hbar]\eta + \ell[\eta, \hbar]$
(iii) $\ell\eta \circ \hbar = (\ell \circ \hbar)\eta + \ell[\eta, \hbar] = \ell(\eta \circ \hbar) - [\ell, \hbar]\eta$
(iv) $\ell \circ \eta\hbar = \eta(\ell \circ \hbar) + [\ell, \eta]\hbar = (\ell \circ \eta)\hbar + \eta[\hbar, \ell]$.

2. Center-like Subsets in Semiprime and Prime Rings

Lemma 1 ([11],Lemma 2 (b))**.** *If $\mathfrak{A}$ is a semiprime ring, then the center of a nonzero ideal of $\mathfrak{A}$ is contained in the center of $\mathfrak{A}$.*

Lemma 2 ([12], Lemma 3.1)**.** *Let $\mathfrak{A}$ be a 2-torsion-free semiprime ring and $\Im$ a left ideal of $\mathfrak{A}$. If $a, b \in \mathfrak{A}$, then the relation $a\ell b + b\ell a = 0$ for all $\ell \in \Im$ implies that $a\ell b = b\ell a = 0$ for all $\ell \in \Im$.*

In all previous studies, $\mathcal{Z}^{**}(\Im, \mathfrak{d})$ has been proven based on $\mathcal{Z}^{*}(\Im, \mathfrak{d})$. In this paper, it is instead proved by considering the set itself. In addition, the following theorem is a generalization of Theorem 1 from ([9]) and Theorem 2.1 from ([6]).

Theorem 1. *Let $\mathfrak{A}$ be a semiprime ring, $\Im$ a nonzero ideal of $\mathfrak{A}$ and $\mathfrak{d}$ a multiplicative derivation of $\mathfrak{A}$. Then, $\mathcal{Z}^{**}(\Im, \mathfrak{d}) = \mathcal{Z}$.*

Proof. We can easily show that $\mathcal{Z} \subseteq \mathcal{Z}^{**}(\Im,\mathfrak{d})$. Hence, we need only prove that $\mathcal{Z}^{**}(\Im,\mathfrak{d}) \subseteq \mathcal{Z}$. Let $\eta \in \mathcal{Z}^{**}(\Im, \mathfrak{d})$. Per the hypothesis, we obtain

$$[\mathfrak{d}(\ell), \mathfrak{d}(\eta)] = [\ell, \eta] \text{ for all } \ell \in \Im.$$

Replacing ℓ in this equation with $\ell\eta$, we obtain

$$[\mathfrak{d}(\ell)\eta + \ell\mathfrak{d}(\eta), \mathfrak{d}(\eta)] = [\ell\eta, \eta], \tag{1}$$

thus,

$$[\mathfrak{d}(\ell), \mathfrak{d}(\eta)]\eta + \mathfrak{d}(\ell)[\eta, \mathfrak{d}(\eta)] + \ell[\mathfrak{d}(\eta), \mathfrak{d}(\eta)] + [\ell, \mathfrak{d}(\eta)]\mathfrak{d}(\eta) = [\ell, \eta]\eta.$$

Using Equation (1), we have

$$\mathfrak{d}(\ell)[\eta, \mathfrak{d}(\eta)] + [\ell, \mathfrak{d}(\eta)]\mathfrak{d}(\eta) = 0.$$

Writing ℓ for $\mathfrak{A}\ell$, $\mathfrak{A} \in \mathfrak{A}$ in the last equation and above equation, respectively, we have

$$\mathfrak{d}(\mathfrak{A})\ell[\eta, \mathfrak{d}(\eta)] + [\mathfrak{A}, \mathfrak{d}(\eta)]\ell\mathfrak{d}(\eta) = 0 \text{ for all } \ell \in \Im, \mathfrak{A} \in \mathfrak{A}. \tag{2}$$

Taking $\mathfrak{A}$ for $\mathfrak{d}(\eta)$ in (2), we can see that

$$\mathfrak{d}^2(\eta)\ell[\eta, \mathfrak{d}(\eta)] = 0 \text{ for all } \ell \in \Im,$$

thus,

$$\mathfrak{d}^2(\eta)\Im[\eta, \mathfrak{d}(\eta)] = (0),$$

That is,

$$\mathfrak{d}^2(\eta)\mathfrak{A}\Im[\eta, \mathfrak{d}(\eta)] = (0).$$

Because $\mathfrak{A}$ is semiprime, we must have a family $\wp = \{P_\alpha \mid \alpha \in \Lambda\}$ of prime ideals such that $\cap P_\alpha = \{0\}$. If P is a typical member of $\wp$, then we have

$$\mathfrak{d}^2(\eta) \in P \text{ or } \Im[\eta, \mathfrak{d}(\eta)] \subseteq P$$

from Fact (ii). Assuming that $\mathfrak{d}^2(\eta) \in P$ and using Equation (1), for all $\ell \in \Im$ we obtain

$$\begin{aligned} [\ell\mathfrak{d}(\eta), \eta] &= [\mathfrak{d}(\ell\mathfrak{d}(\eta)), \mathfrak{d}(\eta)] \\ [\ell, \eta]\mathfrak{d}(\eta) + \ell[\mathfrak{d}(\eta), \eta] &= [\mathfrak{d}(\ell)\mathfrak{d}(\eta) + \ell\mathfrak{d}^2(\eta), \mathfrak{d}(\eta)] \\ [\ell, \eta]\mathfrak{d}(\eta) + \ell[\mathfrak{d}(\eta), \eta] &= [\mathfrak{d}(\ell), \mathfrak{d}(\eta)]\mathfrak{d}(\eta) + [\ell\mathfrak{d}^2(\eta), \mathfrak{d}(\eta)]. \end{aligned}$$

Using Equation (1), we find that

$$\ell[\mathfrak{d}(\eta), \eta] = [\ell\mathfrak{d}^2(\eta), \mathfrak{d}(\eta)] \text{ for all } \ell \in \Im.$$

Using the fact that $\mathfrak{d}^2(\eta) \in P$ and that P is the ideal of R, we have $\ell[\mathfrak{d}(\eta), \eta] \in P$ for all $\ell \in \Im$, that is, $\Im[\mathfrak{d}(\eta), \eta] \subseteq P$. Either of these conditions implies that $\Im[\mathfrak{d}(\eta), \eta] \in P$ for any $P \in \wp$. Thus, we can conclude that

$$[\mathfrak{d}(\eta), \eta]\Im \subseteq \cap P_\alpha = (0),$$

and consequently that

$$[\mathfrak{d}(\eta), \eta]\Im = (0),$$

That is,

$$[\mathfrak{d}(\eta), \eta]\mathfrak{A}\ell = 0, \text{ for all } \ell \in \Im, \ \mathfrak{A} \in \mathfrak{A}.$$

Replacing ℓ in this equation with $[\mathfrak{d}(\eta), \eta]$, we have

$$[\mathfrak{d}(\eta), \eta]\mathfrak{A}[\mathfrak{d}(\eta), \eta] = 0 \text{ for all } \mathfrak{A} \in \mathfrak{A},$$

thus,

$$[\mathfrak{d}(\eta), \eta]\mathfrak{A}[\mathfrak{d}(\eta), \eta] = (0).$$

Because $\mathfrak{A}$ is semiprime ring, we obtain $[\mathfrak{d}(\eta), \eta] = 0$. Using this equation in (2), we arrive at

$$[\mathfrak{r}, \mathfrak{d}(\eta)]\ell\mathfrak{d}(\eta) = 0 \text{ for all } \ell \in \Im, \mathfrak{r} \in \mathfrak{A}.$$

Replacing ℓ in this equation with $\ell\mathfrak{r}$, we find that

$$[\mathfrak{r}, \mathfrak{d}(\eta)]\ell\mathfrak{r}\mathfrak{d}(\eta) = (0), \text{ for all } \ell \in \Im, \mathfrak{r} \in \mathfrak{A}.$$

Multiplying the left-hand side of the previous equation by $\mathfrak{r}$, we have

$$[\mathfrak{r}, \mathfrak{d}(\eta)]\ell\mathfrak{d}(\eta)\mathfrak{r} = 0 = 0 \text{ for all } \ell \in \Im, \mathfrak{r} \in \mathfrak{A}.$$

Subtracting the last two equalities, we arrive at

$$[\mathfrak{r}, \mathfrak{d}(\eta)]\ell[\mathfrak{r}, \mathfrak{d}(\eta)] = 0 \text{ for all } \ell \in \Im, \mathfrak{r} \in \mathfrak{A},$$

hence,

$$[\mathfrak{r}, \mathfrak{d}(\eta)]\Im\mathfrak{A}[\mathfrak{r}, \mathfrak{d}(\eta)]\Im = (0) \text{ for all } \mathfrak{r} \in \mathfrak{A}.$$

Because $\mathfrak{A}$ is semiprime ring, we have

$$[\mathfrak{r}, \mathfrak{d}(\eta)]\Im = (0) \text{ for all } \mathfrak{r} \in \mathfrak{A},$$

thus,

$$[\mathfrak{r}, \mathfrak{d}(\eta)]\mathfrak{A}\Im = (0) \text{ for all } \mathfrak{A} \in \mathfrak{A}.$$

which implies that

$$[\ell, \mathfrak{d}(\eta)]\mathfrak{A}[\ell, \mathfrak{d}(\eta)] = 0 \text{ for all } \ell \in \Im.$$

The semiprime of $\mathfrak{A}$ indicates that $[\ell, \mathfrak{d}(\eta)] = 0$ for all $\ell \in \Im$; thus, $\mathfrak{d}(\eta) \in \mathcal{Z}(\Im)$. From Lemma 1, we have $\mathfrak{d}(\eta) \in \mathcal{Z}$, while using Equation (1) we have $\eta \in \mathcal{Z}(\Im)$. Again from Lemma 1, we have $\eta \in \mathcal{Z}$. Thus, we can conclude that $\mathcal{Z}^{**}(\mathfrak{A}, \mathfrak{d}) = \mathcal{Z}$. □

Theorem 2. *Let $\mathfrak{A}$ be a semiprime ring, $\Im$ a nonzero ideal of $\mathfrak{A}$ and $\mathfrak{d}$ a multiplicative derivation of $\mathfrak{A}$; then, $\mathcal{Z}^{*}(\Im, \mathfrak{d}) = \mathcal{Z}$.*

Proof. We have

$$\begin{aligned}
\mathcal{Z}^{*}(\Im, \mathfrak{d}) &= \{\eta \in \Im \mid [\ell, \eta] = [\mathfrak{d}(\eta), \mathfrak{d}(\ell)] \text{ for all } \ell \in \Im\} \\
&= \{\eta \in \Im \mid [\ell, \eta] = -[\mathfrak{d}(\ell), \mathfrak{d}(\eta)] \text{ for all } \ell \in \Im\} \\
&= \{\eta \in \Im \mid [\ell, \eta] = [(-\mathfrak{d})(\eta), (-\mathfrak{d})(\ell)] \text{ for all } \ell \in \Im\} \\
&= \mathcal{Z}^{**}(\Im, -\mathfrak{d}).
\end{aligned}$$

Because $\mathfrak{d}$ is multiplicative derivation of $\mathfrak{A}$, $(-\mathfrak{d})$ is a multiplicative derivation of $\mathfrak{A}$. From Theorem 1, we can conclude that $\mathcal{Z}^{*}(\Im, \mathfrak{d}) = \mathcal{Z}$. □

Corollary 1. *Let $\mathfrak{A}$ be a semiprime ring, $\Im$ a nonzero ideal of $\mathfrak{A}$, and $\mathfrak{d}$ a nonzero derivation; then:*

(i) $\mathcal{Z}^{**}(\Im, \mathfrak{d}) = \mathcal{Z}$

(ii) $\mathcal{Z}^{*}(\Im, \mathfrak{d}) = \mathcal{Z}$.

Proof. Every derivation is a multiplicative derivation. Therefore, from Theorems 1 and 2 we can prove that $\mathcal{Z}^{**}(\Im, \mathfrak{d}) = \mathcal{Z}^{*}(\Im, \mathfrak{d}) = \mathcal{Z}$. □

Theorem 3. *Let $\mathfrak{A}$ be a semiprime ring, $\Im$ a nonzero ideal of $\mathfrak{A}$ and $\mathfrak{d}$ a multiplicative derivation of $\mathfrak{A}$; then, $\mathcal{Z}^{**}_{\circ}(\Im, \mathfrak{d}) = \mathcal{Z}$.*

Proof. We can easily show that $\mathcal{Z} \subseteq \mathcal{Z}^{**}_{\circ}(\Im, \mathfrak{d})$. Hence, we need only prove that $\mathcal{Z}^{**}_{\circ}(\Im, \mathfrak{d}) \subseteq \mathcal{Z}$.

Let $\eta \in \mathcal{Z}^{**}_{\circ}(\Im, \mathfrak{d})$. Per the hypothesis, we have

$$\mathfrak{d}(\ell) \circ \mathfrak{d}(\eta) = \ell \circ \eta \text{ for all } \ell, \eta \in \Im. \tag{3}$$

Replacing ℓ in the last equation with $\ell\eta$, we obtain

$$(\mathfrak{d}(\ell)\eta + \ell\mathfrak{d}(\eta)) \circ \mathfrak{d}(\eta) = (\ell \circ \eta)\eta,$$

thus,

$$(\mathfrak{d}(\ell) \circ \mathfrak{d}(\eta))\eta + \mathfrak{d}(\ell)[\eta, \mathfrak{d}(\eta)] + (\ell \circ \mathfrak{d}(\eta))\mathfrak{d}(\eta) + \ell[\mathfrak{d}(\eta), \mathfrak{d}(\eta)] = (\ell \circ \eta)\eta.$$

Using the hypothesis, we can see that

$$\mathfrak{d}(\ell)[\eta, \mathfrak{d}(\eta)] + (\ell \circ \mathfrak{d}(\eta))\mathfrak{d}(\eta) = 0.$$

Taking ℓ for $\mathfrak{r}\ell$, $\mathfrak{r} \in \mathfrak{A}$ in the last equation and this equation, respectively, we find that

$$\mathfrak{d}(\mathfrak{r})\ell[\eta, \mathfrak{d}(\eta)] + \mathfrak{r}\mathfrak{d}(\ell)[\eta, \mathfrak{d}(\eta)] + \mathfrak{r}(\ell \circ \mathfrak{d}(\eta))\mathfrak{d}(\eta) - [\mathfrak{r}, \mathfrak{d}(\eta)]\ell\mathfrak{d}(\eta) = 0, \tag{4}$$

thus,

$$\mathfrak{d}(\mathfrak{r})\ell[\eta, \mathfrak{d}(\eta)] - [\mathfrak{r}, \mathfrak{d}(\eta)]\ell\mathfrak{d}(\eta) = 0.$$

Replacing r with $\mathfrak{d}(\eta)$ in (4), we have

$$\mathfrak{d}^2(\eta)\ell[\eta, \mathfrak{d}(\eta)] = 0 \text{ for all } \ell \in \Im,$$

$$\mathfrak{d}^2(\eta)\Im[\eta, \mathfrak{d}(\eta)] = (0).$$

Hence,

$$\mathfrak{d}^2(\eta)\mathfrak{A}\Im[\eta, \mathfrak{d}(\eta)] = (0).$$

Because $\mathfrak{A}$ is semiprime, it must contain a family $\wp = \{P_\alpha \mid \alpha \in \Lambda\}$ of prime ideals such that $\cap P_\alpha = \{0\}$. If P is a typical member of $\wp$, from Fact (ii) we have

$$\mathfrak{d}^2(\eta) \in P \text{ or } \Im[\eta, \mathfrak{d}(\eta)] \subseteq P.$$

Assuming that $\mathfrak{d}^2(\eta) \in P$, we can use Equation (3) to find that for all $\ell \in \Im$ we have

$$\begin{aligned} \ell\mathfrak{d}(\eta) \circ \eta &= \mathfrak{d}(\ell\mathfrak{d}(\eta)) \circ \mathfrak{d}(\eta) \\ (\ell \circ \eta)\mathfrak{d}(\eta) + \ell[\mathfrak{d}(\eta), \eta] &= (\mathfrak{d}(\ell)\mathfrak{d}(\eta) + \ell\mathfrak{d}^2(\eta)) \circ \mathfrak{d}(\eta) \\ (\ell \circ \eta)\mathfrak{d}(\eta) + \ell[\mathfrak{d}(\eta), \eta] &= (\mathfrak{d}(\ell) \circ \mathfrak{d}(\eta))\mathfrak{d}(\eta) + (\ell\mathfrak{d}^2(\eta)) \circ \mathfrak{d}(\eta). \end{aligned}$$

Using Equation (3), we have

$$\ell[\mathfrak{d}(\eta), \eta] = (\ell\mathfrak{d}^2(\eta)) \circ \mathfrak{d}(\eta) \text{ for all } \ell \in \Im.$$

If we use the fact that $\mathfrak{d}^2(\eta) \in P$ and that P is the ideal of $\mathfrak{A}$, we have $\ell[\mathfrak{d}(\eta), \eta] \in P$ for all $\ell \in \Im$, that is, $\Im[\mathfrak{d}(\eta), \eta] \subseteq P$. Either of these conditions implies that $\Im[\mathfrak{d}(\eta), \eta] \in P$ for any $P \in \wp$. Thus, we can conclude that

$$[\mathfrak{d}(\eta), \eta]\Im \subseteq \cap P_\alpha = (0).$$

The rest of the proof is the same as Theorem 1. □

Theorem 4. *Let $\mathfrak{A}$ be a semiprime ring, $\Im$ a nonzero ideal of $\mathfrak{A}$ and $\mathfrak{d}$ a multiplicative derivation of $\mathfrak{A}$; then, $\mathcal{Z}^*_\circ(\Im, \mathfrak{d}) = \mathcal{Z}$.*

Proof. Using the fact that $\mathcal{Z}^*_\circ(\Im, \mathfrak{d}) = \mathcal{Z}^{**}_\circ(\Im, -\mathfrak{d})$ and applying Theorem 3, we obtain $\mathcal{Z}^*_\circ(\Im, \mathfrak{d}) = \mathcal{Z}$. $\square$

Example 1. *Let $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2$, where $\mathfrak{A}_1$ is a ring with a non-zero multiplicative derivation δ such that $\mathfrak{d}_2(\mathfrak{r}_1, \mathfrak{r}_2) = (0, \mathfrak{r}_2)$ and $\mathfrak{A}_2$ is a noncommutative ring. Then, it is easy to verify that $\mathfrak{A}$ is not semiprime. For $\mathfrak{r} = (0, \mathfrak{r}_2)$, we have $\mathfrak{r} \in \mathcal{Z}^*_\circ(\mathfrak{A}, \mathfrak{d})$; however, $\mathfrak{r} \notin \mathcal{Z}$, that is, $\mathcal{Z}^*_\circ(\mathfrak{A}, \mathfrak{d}) \neq \mathcal{Z}$.*

Corollary 2. *Let $\mathfrak{A}$ be a semiprime ring, $\Im$ a nonzero ideal of $\mathfrak{A}$, and $\mathfrak{d}$ a nonzero derivation. Then:*

(i) $\mathcal{Z}^*_\circ(\Im, \mathfrak{d}) = \mathcal{Z}$
(ii) $\mathcal{Z}^{**}_\circ(\Im, \mathfrak{d}) = \mathcal{Z}$.

Corollary 3. *Let $\mathfrak{A}$ be a prime ring, $\Im$ a nonzero ideal of $\mathfrak{A}$, and $\mathfrak{d}$ a nonzero multiplicative derivation. Then:*

(i) $\mathcal{Z}^{**}(\Im, \mathfrak{d}) = \mathcal{Z}$
(ii) $\mathcal{Z}^{*}(\Im, \mathfrak{d}) = \mathcal{Z}$
(iii) $\mathcal{Z}^*_\circ(\Im, \mathfrak{d}) = \mathcal{Z}$
(iv) $\mathcal{Z}^{**}_\circ(\Im, \mathfrak{d}) = \mathcal{Z}$.

By removing the conditions $A_l(\Im) = (0)$ or $\mathfrak{d}_2(\Im)\Im \neq (0)$ in ([10], Theorem 1), the study is generalized to the semiprime ring. Moreover, the following theorem generalizes Theorem 2.

Theorem 5. *Let $\mathfrak{A}$ be a semiprime ring, $\Im$ a nonzero ideal of $\mathfrak{A}$ and $\mathfrak{d}_1, \mathfrak{d}_2$ two multiplicative derivations of $\mathfrak{A}$. Then, $\mathcal{Z}^*(\Im, \mathfrak{d}_1, \mathfrak{d}_2) = \mathcal{Z}$.*

Proof. We can easily show that $\mathcal{Z} \subseteq \mathcal{Z}^*(\Im, \mathfrak{d}_1, \mathfrak{d}_2)$. We want to prove that $\mathcal{Z}^*(\Im, \mathfrak{d}_1, \mathfrak{d}_2) \subseteq \mathcal{Z}$. Letting $\eta \in \mathcal{Z}^*(\Im, \mathfrak{d}_1\mathfrak{d}_2)$, we can obtain that

$$[\ell, \eta] = [\mathfrak{d}_1(\eta), \mathfrak{d}_2(\ell)] \text{ for all } \ell \in \Im. \tag{5}$$

Replacing ℓ in this equation with ℓw, $w \in \Im$, we obtain

$$[\ell, \eta]w + \ell[w, \eta] = [\mathfrak{d}_1(\eta), \mathfrak{d}_2(\ell)]w + \mathfrak{d}_2(\ell)[\mathfrak{d}_1(\eta), w] + [\mathfrak{d}_1(\eta), \ell]\mathfrak{d}_2(w) + \ell[\mathfrak{d}_1(\eta), \mathfrak{d}_2(w)].$$

Using Equation (5), it can be seen that

$$\mathfrak{d}_2(\ell)[\mathfrak{d}_1(\eta), w] + [\mathfrak{d}_1(\eta), \ell]\mathfrak{d}_2(w) = 0. \tag{6}$$

Taking ℓ for ℓt, $t \in \Im$ in the last equation, we have

$$\mathfrak{d}_2(\ell)t[\mathfrak{d}_1(\eta), w] + \ell\mathfrak{d}_2(t)[\mathfrak{d}_1(\eta), w] + \ell[\mathfrak{d}_1(\eta), t]\mathfrak{d}_2(w) + [\mathfrak{d}_1(\eta), \ell]t\mathfrak{d}_2(w) = 0.$$

Using Equation (6), we can see that

$$\mathfrak{d}_2(\ell)t[\mathfrak{d}_1(\eta), w] + [\mathfrak{d}_1(\eta), \ell]t\mathfrak{d}_2(w) = 0.$$

Replacing w in the above equation with ℓ, we obtain

$$\mathfrak{d}_2(\ell)t[\mathfrak{d}_1(\eta), \ell] + [\mathfrak{d}_1(\eta), \ell]t\mathfrak{d}_2(\ell) = 0 \text{ for all } \ell, t \in \Im.$$

From Lemma 2, we have

$$\mathfrak{d}_2(\ell)t[\mathfrak{d}_1(\eta), \ell] = 0 \text{ for all } \ell \in \Im,$$

That is,
$$\mathfrak{d}_2(\ell)\mathfrak{A}\Im[\ell, \mathfrak{d}_1(\eta)] = (0).$$

Because $\mathfrak{A}$ is semiprime, it must contain a family $\wp = \{P_\alpha \mid \alpha \in \Lambda\}$ of prime ideals such that $\cap P_\alpha = \{0\}$. If P is a typical member of $\wp$ and $\ell \in \Im$, then we have

$$\Im[\ell, \mathfrak{d}_1(\eta)] \subseteq P \text{ or } \mathfrak{d}_2(\ell) \in P$$

from Fact (ii). Assuming that there exists $\ell \in \Im$ such that $\Im[\ell, \mathfrak{d}_1(\eta)] \nsubseteq P$, $\mathfrak{d}_2(\ell) \in P$. Using Equation (5), we have

$$[\ell, \mathfrak{d}_1(\eta)] = [\mathfrak{d}_1(\mathfrak{d}_1(\eta)), \mathfrak{d}_2(\ell)].$$

Multiplying the left-hand side of the last equation by $\mathcal{Z} \in \Im$, we can see that

$$\mathcal{Z}[\ell, \mathfrak{d}_1(\eta)] = \mathcal{Z}[\mathfrak{d}_1(\mathfrak{d}_1(\eta)), \mathfrak{d}_2(\ell)].$$

Using $\mathfrak{d}_2(\ell) \in P$, we arrive at $\mathcal{Z}[\ell, \mathfrak{d}_1(\eta)] \in P$ for all $\mathcal{Z} \in \Im$, that is, $\Im[\ell, \mathfrak{d}_1(\eta)] \subseteq P$. Either of these conditions implies that $\Im[\ell, \mathfrak{d}_1(\eta)] \subseteq P$, which is a contradiction; thus, $\Im[\ell, \mathfrak{d}_1(\eta)] \subseteq P$ for any $P \in \wp$. Therefore,

$$\Im[\ell, \mathfrak{d}_1(\eta)] \subseteq \cap P_\alpha = (0),$$

hence,
$$\Im[\ell, \mathfrak{d}_1(\eta)] = (0).$$
meaning that
$$\Im\mathfrak{A}[\ell, \mathfrak{d}_1(\eta)] = (0)$$
and consequently that

$$[\ell, \mathfrak{d}_1(\eta)]\mathfrak{A}[\ell, \mathfrak{d}_1(\eta)] = (0) \text{ for all } \ell \in \Im$$

Because $\mathfrak{A}$ is semiprime ring, we can see that $[\ell, \mathfrak{d}_1(\eta)] = 0$ for all $\ell \in \Im$. Then, $\mathfrak{d}_1(\eta) \in \mathcal{Z}(\Im)$. From Lemma 1, we have $\mathfrak{d}_1(\eta) \in \mathcal{Z}$. Using Equation (5), we obtain $\eta \in \mathcal{Z}(\Im)$. Again from Lemma 1, we have $\eta \in \mathcal{Z}$. Thus, we can conclude that $\mathcal{Z}^*(\Im, \mathfrak{d}_1, \mathfrak{d}_2) = \mathcal{Z}$. □

Example 2. *Let $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2$, where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are rings. It is easy to verify that $\mathfrak{A}$ is not a semiprime ring with multiplicative derivations $\mathfrak{d}_1$ provided by $\mathfrak{d}_1(\mathfrak{r}_1, \mathfrak{r}_2) = (\mathfrak{r}_1, 0)$ and $\mathfrak{d}_2(\mathfrak{r}_1, \mathfrak{r}_2) = (0, \mathfrak{r}_2)$. For $\mathfrak{r} = (0, \mathfrak{r}_2)$, we have $\mathfrak{r} \in \mathcal{Z}^*(\mathfrak{A}, \mathfrak{d}_1, \mathfrak{d}_2)$; however, $\mathfrak{r} \notin \mathcal{Z}$, that is, $\mathcal{Z}^*(\mathfrak{A}, \mathfrak{d}_1, \mathfrak{d}_2) \neq \mathcal{Z}$.*

Theorem 6. *Let $\mathfrak{A}$ be a semiprime ring, $\Im$ a nonzero ideal of $\mathfrak{A}$ and $\mathfrak{d}_1, \mathfrak{d}_2$ two multiplicative derivations of $\mathfrak{A}$. Then, $\mathcal{Z}^{**}(\Im, \mathfrak{d}_1, \mathfrak{d}_2) = \mathcal{Z}$.*

Proof. We have

$$\begin{aligned}
\mathcal{Z}^{**}(\Im, \mathfrak{d}_1, \mathfrak{d}_2) &= \{\eta \in \mathfrak{A} \mid [\ell, \eta] = [\mathfrak{d}_1(\ell), \mathfrak{d}_2(\eta)] \text{ for all } \ell \in \mathfrak{A}\} \\
&= \{\eta \in \Im \mid [\ell, \eta] = -[\mathfrak{d}_2(\eta), \mathfrak{d}_1(\ell)], \text{ for all } \ell \in \Im\} \\
&= \{\eta \in \Im \mid [\ell, \eta] = [(-\mathfrak{d}_2)(\eta), \mathfrak{d}_1(\ell)], \text{ for all } \ell \in \Im\} \\
&= \mathcal{Z}^*(\Im, -\mathfrak{d}_2, \mathfrak{d}_1).
\end{aligned}$$

From Theorem 6, we have $\mathcal{Z}^{**}(\Im, \mathfrak{d}_1, \mathfrak{d}_2) = \mathcal{Z}$. □

Example 3. *Let $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2$, where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are rings. Then, it is easy to verify that $\mathfrak{A}$ is not a semiprime ring with multiplicative derivations $\mathfrak{d}_1$ provided by $\mathfrak{d}_1(\mathfrak{r}_1, \mathfrak{r}_2) = (\mathfrak{r}_1, 0)$ and $\mathfrak{d}_2(\mathfrak{r}_1, \mathfrak{r}_2) = (0, \mathfrak{r}_2)$. For $\mathfrak{r} = (0, \mathfrak{r}_2)$, we have $\mathfrak{r} \in \mathcal{Z}^{**}(\mathfrak{A}, \mathfrak{d}_1, \mathfrak{d}_2)$; however, $\mathfrak{r} \notin \mathcal{Z}$, that is, $\mathcal{Z}^{**}(\mathfrak{A}, \mathfrak{d}_1, \mathfrak{d}_2) \neq \mathcal{Z}$.*

Corollary 4. *Let $\mathfrak{A}$ be a prime ring, $\Im$ a nonzero ideal of $\mathfrak{A}$ and $\mathfrak{d}_1, \mathfrak{d}_2$ two derivations of $\mathfrak{A}$. Then:*

(i) $\mathcal{Z}^*(\Im, \mathfrak{d}_1, \mathfrak{d}_2) = \mathcal{Z}$

(ii) $\mathcal{Z}^{**}(\Im, \mathfrak{d}_1, \mathfrak{d}_2) = \mathcal{Z}$.

The following theorems are not true for a semiprime ring. An example of this has already has been cleared. Thus, the last two theorems are proved for a prime ring.

Theorem 7. *Let $\mathfrak{A}$ be a prime ring, $\Im$ a nonzero ideal of $\mathfrak{A}$ and $\mathfrak{d}$ a nonzero multiplicative derivation of $\mathfrak{A}$. Then, $\mathcal{Z}_1(\Im, \mathfrak{d}) = \mathcal{Z}$.*

Proof. It is clear to see that $\mathcal{Z} \subseteq \mathcal{Z}_1(\Im, \mathfrak{d})$; hence, we only need to show that $\mathcal{Z}_1(\Im, \mathfrak{d}) \subseteq \mathcal{Z}$. Let $\eta \in \mathcal{Z}_1(\Im, \mathfrak{d})$. Then, we have

$$[\mathfrak{d}(\ell), \mathfrak{d}(\eta)] = [\mathfrak{d}(\eta), \ell] + [\eta, \mathfrak{d}(\ell)] \text{ for all } \ell \in \Im. \tag{7}$$

Replacing ℓ in the above equation with $\ell\eta$, we obtain

$$\begin{aligned}&\mathfrak{d}(\ell)[\eta, \mathfrak{d}(\eta)] + [\mathfrak{d}(\ell), \mathfrak{d}(\eta)]\eta + \ell[\mathfrak{d}(\eta), \mathfrak{d}(\eta)] + [\ell, \mathfrak{d}(\eta)]\mathfrak{d}(\eta)\\ &= [\mathfrak{d}(\eta), \ell]\eta + \ell[\mathfrak{d}(\eta), \eta] + [\eta, \mathfrak{d}(\ell)]\eta + \ell[\eta, \mathfrak{d}(\eta)] + [\eta, \ell]\mathfrak{d}(\eta).\end{aligned}$$

Using Equation (7), we can see that

$$\mathfrak{d}(\ell)[\eta, \mathfrak{d}(\eta)] + [\ell, \mathfrak{d}(\eta)]\mathfrak{d}(\eta) = [\eta, \ell]\mathfrak{d}(\eta). \tag{8}$$

Writing ℓ for $\mathfrak{r}\ell$, $\mathfrak{r} \in \mathfrak{A}$, we obtain

$$\mathfrak{d}(\mathfrak{r})\ell[\eta, \mathfrak{d}(\eta)] + \mathfrak{r}\mathfrak{d}(\ell)[\eta, \mathfrak{d}(\eta)] + [\mathfrak{r}, \mathfrak{d}(\eta)]\ell\mathfrak{d}(\eta) + \mathfrak{r}[\ell, \mathfrak{d}(\eta)]\mathfrak{d}(\eta) = \mathfrak{r}[\eta, \ell]\mathfrak{d}(\eta) + [\eta, \mathfrak{r}]\ell\mathfrak{d}(\eta).$$

Using Equation (8), we can see that

$$\mathfrak{d}(\mathfrak{r})\ell[\eta, \mathfrak{d}(\eta)] + [\mathfrak{r}, \mathfrak{d}(\eta)]\ell\mathfrak{d}(\eta) = [\eta, \mathfrak{r}]\ell\mathfrak{d}(\eta). \tag{9}$$

Replacing $\mathfrak{r}$ with η in (9), we obtain

$$\mathfrak{d}(\eta)\ell[\eta, \mathfrak{d}(\eta)] + [\eta, \mathfrak{d}(\eta)]\ell\mathfrak{d}(\eta) = 0.$$

From Lemma 2, we have

$$\mathfrak{d}(\eta)\ell[\eta, \mathfrak{d}(\eta)] = 0 \text{ for all } \ell, \eta \in \Im. \tag{10}$$

Replacing ℓ with $\eta\ell$ in (10), we find that

$$\mathfrak{d}(\eta)\eta\ell[\eta, \mathfrak{d}(\eta)] = (0) \text{ for all } \ell, \eta \in \Im. \tag{11}$$

Multiplying the left-hand side of (10) by η, we obtain

$$\eta\mathfrak{d}(\eta)\ell[\eta, \mathfrak{d}(\eta)] = 0 \text{ for all } \ell, \eta \in \Im. \tag{12}$$

Subtracting (11) from (12), we arrive at

$$[\eta, \mathfrak{d}(\eta)]\ell[\eta, \mathfrak{d}(\eta)] = 0 \text{ for all } \ell \in \Im.$$

Because $\mathfrak{A}$ is a prime ring, we obtain

$$[\eta, \mathfrak{d}(\eta)] = 0.$$

Using Equation (9), we have

$$[\mathfrak{r}, \mathfrak{d}(\eta)]\ell\mathfrak{d}(\eta) = [\eta, \mathfrak{r}]\ell\mathfrak{d}(\eta),$$

thus,

$$[\mathfrak{r}, \mathfrak{d}(\eta)]\ell\mathfrak{d}(\eta) + [\mathfrak{r}, \eta]\ell\mathfrak{d}(\eta) = 0.$$

that is,

$$[\mathfrak{r}, \mathfrak{d}(\eta) + \eta]\ell\mathfrak{d}(\eta) = 0.$$

This implies that

$$[\mathfrak{r}, \mathfrak{d}(\eta) + \eta]\mathfrak{A}\Im\mathfrak{d}(\eta) = (0) \text{ for all } \mathfrak{r} \in \mathfrak{A}.$$

Because $\mathfrak{A}$ is prime ring, we have $\mathfrak{d}(\eta) + \eta \in \mathcal{Z}$ or $\Im\mathfrak{d}(\eta) = 0$. Because $\Im$ is a non-zero ideal of $\mathfrak{A}$, we obtain $\mathfrak{d}(\eta) + \eta \in \mathcal{Z}$ or $\mathfrak{d}(\eta) = 0$. Assuming that $\mathfrak{d}(\eta) + \eta \in \mathcal{Z}$, using Equation (7) we have $[\mathfrak{d}(\ell), \mathfrak{d}(\eta)] + [\mathfrak{d}(\ell), \eta] = [\mathfrak{d}(\eta), \ell]$, that is, $[\mathfrak{d}(\eta), \ell] = 0$ for all $\ell \in \Im$. We can conclude that $\mathfrak{d}(\eta) \in \mathcal{Z}(\Im)$. From Lemma 1, we have $\mathfrak{d}(\eta) \in \mathcal{Z}$. Using Equation (9), we obtain

$$[\eta, \mathfrak{r}]\ell\mathfrak{d}(\eta) = 0 \text{ for all } \ell \in \Im,\ \mathfrak{r} \in \mathfrak{A},$$

that is,

$$[\eta, \mathfrak{r}]\mathfrak{A}\Im\mathfrak{d}(\eta) = (0).$$

Because $\mathfrak{A}$ is a prime ring, we have $\eta \in \mathcal{Z}$ or $\Im\mathfrak{d}(\eta) = 0$. Because $\Im$ is a non-zero ideal of $\mathfrak{A}$, we have $\eta \in \mathcal{Z}$ or $\mathfrak{d}(\eta) = 0$. Either of these conditions implies that $\mathfrak{d}(\eta) = 0$. Using Equation (7), we obtain

$$[\eta, \mathfrak{d}(\ell)] = 0 \text{ for all } \ell \in \Im.$$

Replacing ℓ by $\ell\hbar$, $\hbar \in \Im$, we can see that

$$[\eta, \mathfrak{d}(\ell)\hbar + \ell\mathfrak{d}(\hbar)] = 0,$$

thus,

$$\mathfrak{d}(\ell)[\eta, \hbar] + [\eta, \ell]\mathfrak{d}(\hbar) = 0.$$

Taking ℓ for ℓw, $w \in \Im$ when using this equation, we obtain

$$\mathfrak{d}(\ell)w[\eta, \hbar] + [\eta, \ell]w\mathfrak{d}(\mathcal{Z}\hbar) = 0.$$

Replacing $\hbar$ in the above equation with ℓ, we have

$$\mathfrak{d}(\ell)w[\eta, \ell] + [\eta, \ell]w\mathfrak{d}(\ell) = 0.$$

From Lemma 2, we have

$$[\eta, \ell]\mathfrak{A}\Im\mathfrak{d}(\ell) = 0.$$

Because $\mathfrak{A}$ is a prime ring and $\Im$ is a non-zero ideal of $\mathfrak{A}$, we can see that $[\eta, \ell] = 0$ or $\mathfrak{d}(\ell) = 0$. We can now define the following two additive subgroups:

$$A = \{\ell \in \Im \mid [\eta, \ell] = 0\} \text{ and } B = \{\ell \in \Im \mid \mathfrak{d}(\ell) = 0\}.$$

It is clear that $\Im = A \cup B$. Because a group cannot be a union of two of its subgroups, it must be the case that either $A = \Im$ or $B = \Im$. If $B = \Im$, then $\mathfrak{d}(\ell) = 0$ for all $\ell \in \Im$. Replacing ℓ in this equation with $\ell\mathfrak{r}$, $\mathfrak{r} \in \mathfrak{A}$, we arrive at $\ell\mathfrak{d}(\mathfrak{r}) = 0$, that is, $\Im\mathfrak{A}\mathfrak{d}(\mathfrak{r}) = (0)$ for all $\mathfrak{r} \in \mathfrak{A}$. Because $\Im$ is a non-zero ideal of $\mathfrak{A}$, we obtain $\mathfrak{d} = 0$, which is a contradiction; thus, $A = \Im$ and we can conclude that $\eta \in \mathcal{Z}(\Im)$. From Lemma 1, we have $\eta \in \mathcal{Z}$. □

Corollary 5 ([3], Theorem 2.5)**.** *"Let $\mathfrak{A}$ be a prime ring and $\mathfrak{d}$ a nonzero derivation of $\mathfrak{A}$. Then, $\mathcal{Z}_1(\mathfrak{A}, \mathfrak{d}) = \mathcal{Z}$."*

In semiprime ring, we cannot prove that $\mathcal{Z}_1(\mathfrak{A}, \mathfrak{d}) \nsubseteq \mathcal{Z}$, as the following example shows.

Example 4. *Let $\mathfrak{A} = M_2(\mathbb{Z}) \times \mathbb{Z}[\ell]$ be a semiprime ring and let $\mathfrak{d} : \mathfrak{A} \to \mathfrak{A}$, $\mathfrak{d}(A, p(\ell)) = (0, p'(\ell))$ be a multiplicative derivation, where $p'(\ell)$ is the derivation of $p(\ell)$. Then, it is easy to verify that $\mathfrak{r} = \left(\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, 0 \right) \notin \mathcal{Z}$ but $\mathfrak{r} \in \mathcal{Z}_1(\mathfrak{A}, \mathfrak{d})$, that is, $\mathcal{Z}_1(\mathfrak{A}, \mathfrak{d}) \nsubseteq \mathcal{Z}$.*

The following theorem generalizes Theorem 6.

Theorem 8. *Let $\mathfrak{A}$ be a prime ring, $\Im$ a nonzero ideal of $\mathfrak{A}$, and $\mathfrak{d}_1, \mathfrak{d}_2$ two multiplicative derivations of $\mathfrak{A}$. Then, $\mathcal{Z}_1(\Im, \mathfrak{d}_1, \mathfrak{d}_2) = \mathcal{Z}$.*

Proof. We can easily show that $\mathcal{Z} \subseteq \mathcal{Z}_1(\Im, \mathfrak{d}_1, \mathfrak{d}_2)$. Hence, we need only prove that $\mathcal{Z}_1(\Im, \mathfrak{d}_1, \mathfrak{d}_2) \subseteq \mathcal{Z}$. Let $\eta \in \mathcal{Z}_1(\Im, \mathfrak{d}_1, \mathfrak{d}_2)$. We can find that

$$[\mathfrak{d}_1(\ell), \mathfrak{d}_2(\eta)] = [\mathfrak{d}_2(\eta), \ell] + [\eta, \mathfrak{d}_1(\ell)] \text{ for all } \ell \in \Im. \tag{13}$$

Replacing ℓ in the above equation with $\ell\mathfrak{r}, \mathfrak{r} \in \Im$, we obtain

$$\begin{aligned}&[\mathfrak{d}_1(\ell), \mathfrak{d}_2(\eta)]\mathfrak{r} + \mathfrak{d}_1(\ell)[\mathfrak{r}, \mathfrak{d}_2(\eta)] + \ell[\mathfrak{d}_1(\mathfrak{r}), \mathfrak{d}_2(\eta)] + [\ell, \mathfrak{d}_2(\eta)]\mathfrak{d}_1(\mathfrak{r}) \\ &= [\mathfrak{d}_2(\eta), \ell]\mathfrak{r} + \ell[\mathfrak{d}_2(\eta), \mathfrak{r}] + [\eta, \mathfrak{d}_1(\ell)]\mathfrak{r} + \mathfrak{d}_1(\ell)[\eta, \mathfrak{r}] + [\eta, \ell]\mathfrak{d}_1(\mathfrak{r}) + \ell[\eta, \mathfrak{d}_1(\mathfrak{r})].\end{aligned}$$

Using Equation (13), we obtain

$$\mathfrak{d}_1(\ell)[\mathfrak{r}, \mathfrak{d}_2(\eta)] + [\ell, \mathfrak{d}_2(\eta)]\mathfrak{d}_1(\mathfrak{r}) = \mathfrak{d}_1(\ell)[\eta, \mathfrak{r}] + [\eta, \ell]\mathfrak{d}_1(\mathfrak{r}),$$

that is,

$$\mathfrak{d}_1(\ell)([\mathfrak{r}, \mathfrak{d}_2(\eta)] + [\mathfrak{r}, \eta]) = ([\eta, \ell] + [\mathfrak{d}_2(\eta), \ell])\mathfrak{d}_1(\mathfrak{r}),$$

thus,

$$\mathfrak{d}_1(\ell)[\mathfrak{r}, \mathfrak{d}_2(\eta) + \eta] = [\eta + \mathfrak{d}_2(\eta), \ell]\mathfrak{d}_1(\mathfrak{r}). \tag{14}$$

Replacing $\mathfrak{r}$ by $\mathfrak{r}\hbar$, $\hbar \in \Im$ in (14), it can be seen that

$$\mathfrak{d}_1(\ell)[\mathfrak{r}, \mathfrak{d}_2(\eta) + \eta]\hbar + \mathfrak{d}_1(\ell)\mathfrak{r}[\hbar, \mathfrak{d}_2(\eta) + \eta] = [\eta + \mathfrak{d}_2(\eta), \ell]\mathfrak{d}_1(\mathfrak{r})\hbar + [\eta + \mathfrak{d}_2(\eta), \ell]\mathfrak{r}\mathfrak{d}_1(\hbar).$$

Using Equation (14), we have

$$\mathfrak{d}_1(\ell)\mathfrak{r}[\hbar, \mathfrak{d}_2(\eta) + \eta] = [\eta + \mathfrak{d}_2(\eta), \ell]\mathfrak{r}\mathfrak{d}_1(\hbar).$$

Taking $\hbar$ for ℓ in the last equation, we have

$$\mathfrak{d}_1(\ell)\mathfrak{r}[\ell, \mathfrak{d}_2(\eta) + \eta] + [\ell, \mathfrak{d}_2(\eta) + \eta]\mathfrak{r}\mathfrak{d}_1(\ell) = 0.$$

From Lemma 2, we have

$$\mathfrak{d}_1(\ell)\mathfrak{r}[\ell, \mathfrak{d}_2(\eta) + \eta] = 0 \text{ for all } \ell, \mathfrak{r} \in \Im,$$

thus,

$$\mathfrak{d}_1(\ell)r\Im[\ell, \mathfrak{d}_2(\eta) + \eta] = (0) \text{ for all } \ell, \mathfrak{r} \in \Im.$$

Because $\mathfrak{A}$ is prime ring and $\Im$ is a non-zero ideal of $\mathfrak{A}$, we arrive at

$$\mathfrak{d}_1(\ell) = 0 \text{ or } [\ell, \mathfrak{d}_2(\eta) + \eta] = 0.$$

Assuming that $\mathfrak{d}_1(\ell) = 0$, using Equation (14) we have

$$[\eta + \mathfrak{d}_2(\eta), \ell]\mathfrak{d}_1(\mathfrak{r}) = 0 \text{ for all } \mathfrak{r} \in \Im.$$

Replacing $\mathfrak{r}$ in this equation with $\mathfrak{r}t$, $t \in \mathfrak{A}$, it can be seen that

$$[\eta + \mathfrak{d}_2(\eta), \ell]\mathfrak{r}\mathfrak{d}_1(t) = 0 \text{ for all } t \in \mathfrak{A},\ \mathfrak{r} \in \Im.$$

Because $\mathfrak{A}$ is prime ring, we have $[\eta + \mathfrak{d}_2(\eta), \ell] = 0$ or $\mathfrak{d}_1(t) = 0$. Because $\mathfrak{d}_1$ is non-zero, we have $[\eta + \mathfrak{d}_2(\eta), \ell] = 0$. Either of these conditions implies that $[\eta + \mathfrak{d}_2(\eta), \ell] = 0$ for all $\ell \in \Im$, that is.

$$[\eta, \ell] + [\mathfrak{d}_2(\eta), \ell] = 0,$$

thus,

$$-[\mathfrak{d}_2(\eta), \ell] = [\eta, \ell]$$

Using Equation (13), we have

$$[\eta, \mathfrak{d}_1(\ell)] = -[\eta, \ell] + [\eta, \mathfrak{d}_1(\ell)]$$

Therefore, we obtain $[\eta, \ell] = 0$ for all $\ell \in \Im$, proving that $\eta \in \mathcal{Z}(\Im)$. From Lemma 1, we have $\eta \in \mathcal{Z}$. □

Example 5. *Let $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2$, where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are prime rings. Then, it is easy to verify that $\mathfrak{A}$ is a prime ring with multiplicative derivation $\mathfrak{d}_1$ provided by $\mathfrak{d}_1(\mathfrak{r}_1, \mathfrak{r}_2) = (\mathfrak{r}_1, 0)$ and $\mathfrak{d}_2(\mathfrak{r}_1, \mathfrak{r}_2) = (0, \mathfrak{r}_2)$. We can conclude that $\mathcal{Z}_1(\mathfrak{A}, \mathfrak{d}_1, \mathfrak{d}_2) = \mathcal{Z}$.*

Corollary 6 ([11], Theorem 2)**.** *Let $\mathfrak{A}$ be a prime ring and let $\mathfrak{d}_1, \mathfrak{d}_2$ be two derivations of $\mathfrak{A}$. Then, $\mathcal{Z}_1(\mathfrak{A}, \mathfrak{d}_1, \mathfrak{d}_2) = \mathcal{Z}$.*

In a semiprime ring, we cannot prove that $\mathcal{Z}_1(\mathfrak{A}, \mathfrak{d}_1, \mathfrak{d}_2) \not\subseteq \mathcal{Z}$, as the following example shows.

Example 6. *Let $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2$, where $\mathfrak{A}_1$ is a commutative domain with nonzero derivation δ and $\mathfrak{A}_2$ is a noncommutative prime ring. Then, it is easy to verify that $\mathfrak{A}$ is semiprime with multiplicative derivation $\mathfrak{d}_1$ provided by $\mathfrak{d}_1(\mathfrak{r}_1, \mathfrak{r}_2) = (\delta(\mathfrak{r}_1), 0)$ and $\mathfrak{d}_2(\mathfrak{r}_1, \mathfrak{r}_2) = (0, \delta(\mathfrak{r}_2))$. For $\mathfrak{r} = (0, \mathfrak{r}_2)$, we have $\mathfrak{r} \in \mathcal{Z}_1(\Im, \mathfrak{d}_1, \mathfrak{d}_2)$; however,, $\mathfrak{r} \notin \mathcal{Z}$, that is, $\mathcal{Z}_1(\mathfrak{A}, \mathfrak{d}_1, \mathfrak{d}_2) \not\subseteq \mathcal{Z}$.*

Open Problem: Our hypotheses are addressed to center-like sets on prime and semiprime rings. More general results can be provided when all hypotheses regarding semiprime rings are taken into account. In this study, the new center-like set was produced using commutativity conditions. In future studies, the results can be generalized based on the change conditions in the literature by taking new center-like sets as derivations, generalized derivations, semi-derivations, and homoderivations in semiprime and prime rings and by taking the center-like sets provided here as Lie ideals instead of ideals. Center-like sets can be defined as well. In addition, in previous studies the relations of center-like sets with each other have been examined under the conditions of the derivatives and new structures we have provided above. In addition to these studies, if articles [13–15] on rings and semi-rings are taken into consideration, center-like sets can be studied in these rings as well.

3. Conclusions

In this study, the relationship between center-like sets and the centering set of a differentiated and multiplicatively differentiated semiprime ring is established. New sets have been defined and previous sets have been discussed with a different derivative structures. Additionally, taking into account the existing studies in the literature, results have been examined for the multiplicative derivatives of semiprime and prime rings. Examples are provided in the context of each theorem to show that the given conditions are necessary. In our future work, we plan to generalize the clusters discussed here by taking new derivatives and Lie ideals.

Author Contributions: These materials are the result of the joint efforts of S.S.A., E.K.S., and N.u.R. Conceptualization, E.K.S.; Methodology, E.K.S. and N.u.R.; Validation, E.K.S., N.u.R. and S.S.A.; Writing—original draft, E.K.S.; Writing—review & editing, N.u.R.; Funding acquisition, S.S.A. All authors have read and agreed to the published version of the manuscript.

Funding: The authors would like to acknowledge the support of Prince Sultan University for paying the article processing charges (APC) of this publication and for their financial support.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: All data required for this article are included within this article.

Acknowledgments: The authors are greatly indebted to the referee for valuable suggestions and comments which have immensely improved this article.

Conflicts of Interest: The authors declare no conflicts of interest.

Notations

$\mathfrak{A}$	ring	$\mathcal{Z}$	center
$\mathfrak{I}$	ideal	$[,]$	commutator product
$S(\mathfrak{A})$	hypercenter	$\ell, \eta, \hbar$	elements in the ring
$T(\mathfrak{A})$	cohypercenter	$A_l(\mathfrak{A})$	left annihilator of $\mathfrak{A}$
$\mathfrak{d}, \mathfrak{d}_1, \mathfrak{d}_2$	derivation and multiplicative derivation	$\circ$	Jordan product
$C[0,1]$	all continuous functions in the interval $[0,1]$		

References

1. Daif, M.N. When is a multiplicative derivation additive. *Int. J. Math. Math. Sci.* **1991**, *14*, 615–618. [CrossRef]
2. Herstein, I.N. A note on derivations II. *Canad. Math. Bull.* **1979**, *22*, 509–511. [CrossRef]
3. Herstein, I.N. On the hypercenter of a ring. *J. Algebra* **1975**, *36*, 151–157. [CrossRef]
4. Chacron, M. A commutativity theorem for rings. *Proc. Am. Math. Soc.* **1976**, *59*, 211–216. [CrossRef]
5. Giambruno, A. Some generalizations of the center of a ring. *Rend. Circ. Mat. Palermo* **1978**, *27*, 270–282. [CrossRef]
6. Bell, H.E.; Daif, M.N. On commutativity and strong commutativity-preserving maps. *Can. Math. Bull.* **1994**, *37*, 443–447. [CrossRef]
7. Bell, H.E.; Daif, M.N. Center-like subsets in rings with derivations or epimorphisms. *Bull. Iran. Math. Soc.* **2016**, *42*, 873–878.
8. Idrissi, M.A.; Oukhtite, L.; Muthana, N. Center-like subsets in rings with derivations or endomorphisms. *Commun. Algebra* **2019**, *47*, 3794–3799. [CrossRef]
9. Nabiel, H. Ring subsets that be center-like subsets. *J. Algebra Appl.* **2018**, *17*, 1850048. [CrossRef]
10. Zemzami, O.; Oukhtite, L.; Bell, H.E. Center-like subsets in prime rings with derivations and endomorphisms. *Aequat. Math.* **2021**, *95*, 589–598. [CrossRef]
11. Daif, M.N.; Bell, H.E. Remarks on derivations on semiprime rings. *Intern. J. Math. Math. Sci.* **1992**, *15*, 205–206. [CrossRef]
12. Bell, H.E. Some commutativity results involving derivations. In *Trends in Theory of Rings and Modules*; Rizvi, S.T., Zaidi, S.M.A., Eds.; Anamaya Publishers: New Delhi, India, 2005; pp. 11–16.
13. Mehmood, F.; Shi, F.; Hayat, K. A new approach to the fuzzification of rings. *J. Nonlinear Convex Anal.* **2021**, *21*, 2637–2646.
14. Mehmood, F.; Shi, F.; Hayat, K.; Yang, X. The Homomorphism Theorems of M-Hazy Rings and Their Induced Fuzzifying Convexities. *Mathematics* **2020**, *8*, 411. [CrossRef]
15. Mahmood, T.; Hayat, K. Characterizations of Hemi-Rings by their Bipolar-Valued Fuzzy h-Ideals. *Inf. Sci. Lett.* **2015**, *4*, 51–59.